DESCRIPTORS OF GENERALIZATION IN PRIMARY SCHOOL MATHEMATICALLY GIFTED STUDENTS

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Abstract. *Generalization has been identified in several investigations as a capacity characteristic of mathematical giftedness, mainly observable in problem solving. We present part of a research focused on analyzing data about students' capacity of algebraic generalization, operationalized by a set of descriptors of different aspects of generalization, observed in the evidence of the problem-solving processes by Olympic primary school students. We have selected several types of problems that favor the observation of these descriptors. We present some solutions showing ways the descriptors are evidenced in solutions by mathematically gifted students.*

Key words: *Descriptors, generalization, mathematical giftedness, pattern problems, primary school.*

INTRODUCTION

In the study of mathematically giftedness, different researchers have focused on identifying and analyzing specific characteristics of mathematically gifted students (MG students hereafter). To do it, they have focused on experimenting with sets of specific problems and analyzing the cognitive processes evidenced by the students when solving them (Freiman, 2006; Jaime & Gutiérrez, 2014; Krutetskii, 1976). Several types of problems have proved to be useful, when solved by MG students, to induce the emergence of some specific traits of mathematical giftedness; or, when solved also by regular students, to compare the solutions by MG and regular students, to decide which observed characteristics differ in both groups of students (Fritzlar & Karpinski-Siebold, 2012; Heinze, 2005).

Mathematical generalization is frequently reported as a distinctive characteristic of MG students (Benedicto, Jaime, & Gutiérrez, 2015; Freiman, 2006; Fritzlar & Karpinski-Siebold, 2012; Krutetskii, 1976; Sriraman, 2003). In particular, problems requiring making algebraic generalizations are a potential and effective tool to develop students' mathematical capacities, by bringing into play a variety of aspects such as generalization, algebraic thinking, visualization, and creativity (Amit & Neria, 2008; Vale et al., 2012).

We are conducting research focused on analyzing several MG students' capacities, algebraic generalization being one of them. We have operationalized the analysis of students' outcomes by creating a set of descriptors of different aspects of generalization and applying the descriptors to the observed evidences of problem-solving processes by Olympic primary school students. We have collected data from the solutions by students in grades 2, 4, 5, and 6 to some problems posed in the Mathematics Olympics for Primary Education of Costa Rica.

The research objective of this paper is to present a summary of the theoretical component of our investigation on MG students' capacity of algebraic generalization, formed by a set of descriptors that have proved to be useful in the analysis of problem-solving processes. We also present some examples of applications of the descriptor to analyze solutions of a geometric pattern problem. An innovative contribution of our research is that we have integrated several theoretical elements about generalization into a methodological tool that facilitates the analysis of students' solutions to different types of generalization problems.

THEORETICAL FRAMEWORK: DESCRIPTORS OF ALGEBRAIC GENERALIZATION

In this paper, we follow Radford (2006) and define *algebraic generalization* of a pattern (graphical or numerical) as "the capability of grasping a commonality noticed on some elements of a sequence S, being aware that this commonality applies to all the terms of S and being able to use it to provide a direct expression of whatever term of S" (p. 5). When solving algebraic generalization problems, students may evidence different styles of reasoning, which we have differentiated by means of descriptors, aiming to establish guidelines to operationalize the analysis of students' solutions. We have incorporated the definitions of descriptors from previous research (Arbona, 2016; Radford, 2006) and have grouped them according to the *kinds of activities* that students may have to perform when solving algebraic generalization problems: handling of the data of a problem, calculation strategies for direct questions, forms of generalization developed, and calculation strategies for inverse questions:

DG1. *Data handling.* Descriptors of the ways students understand and handle the data provided by a problem.

- DG1.1. *Geometric handling of the data.*
- DG1.2. *Arithmetic handling of data.*

DG2. *Calculation of the value of specific terms (direct questions).* Descriptors of the ways students calculate the value of the term in a given position of the sequence.

- DG2.1. *Calculation by counting.*
- DG2.2. *Recursive calculation.*
- DG2.3. *Functional calculation.*
- DG2.4. *Proportional calculation.*

DG3. *Levels of generalization.* Descriptors of the ways students generalize the relationship defining a given sequence.

- DG3.1. *Naive induction.*
- DG3.2. *Arithmetic generalization.*
- DG3.3. *Factual algebraic generalization.*
- DG3.4. *Contextual algebraic generalization.*
- DG3.5. *Symbolic algebraic generalization.*

DG4. *Calculation of the position of specific terms (inverse questions).* Descriptors of the ways students calculate the position in the sequence of the term with a given value.

DG4.1. *Recursive calculation.*

DG4.2. *Trial and error.*

DG4.3. *Wrong inversion of operations.*

DG4.4. *Incomplete inversion.*

DG4.5. *Inversion with wrong order of operations.*

DG4.6. *Correct inversion of operations.*

DG4.7. *Solving an equation.*

PROBLEMS OF GENERALIZATION AND EXAMPLES OF EMERGENCE OF DESCRIPTORS

Different types of problems can mobilize algebraic generalization. To analyze MG students' solutions and the descriptors evidenced, we designed problems of the types (a) geometric pattern showing several terms graphically, (b) incomplete table of values, (c) verbal statement, (d) geometric pattern showing one term graphically, and (e) geometric pattern with two simultaneous patterns. Table 1 shows, for each type of problems, how many times we observed each descriptor in the students' solutions, grouped according to the kinds of activities performed in the generalization process.

Problems	Type a	Type b	Type c	Type d	Type e
Amount of problems	13	1	1	1	2
Data handling					
DG1.1	71			13	10
DG1.2	126			1	15
Calculation of the values of specific terms					
DG2.1	42			11	4
DG2.2	94	10	13		10
DG2.3	54		2	3	7
DG2.4					
Levels of generalization					
DG3.1, DG3.3	1, 14				
DG3.2, DG3.4	56, 12		11, 1		
DG3.5	15		2		4
Calculation of the position of specific terms					
DG4.1, DG4.3, DG4.4, DG4.5	27, 3, 6, 1				
DG4.2, DG4.6, DG4.7	8, 18, 4			1, 4, 1	

Table 1. Number of times each descriptors was identified in each type of problems.

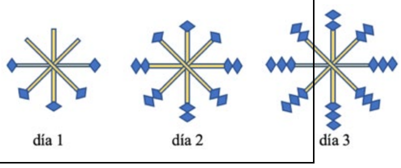
Some descriptors were rarely evidenced: in some cases, like DG3.5 and DG4.7, because they require mathematical knowledge that most primary school students still did not have; in other cases, like DG2.4 and DG3.1, the descriptors correspond to basic or erroneous ways of reasoning, overcome by most Olympic students. Other descriptors were often evidenced: recursivity was the most used method to calculate both values of given terms (DG2.2) and positions of terms with given values (DG4.1); furthermore, it was more common for students to do arithmetic (DG3.2) than algebraic (DG3.3-5) generalizations, because most of them did not have previous experience in expressing algebraically their ideas.

Analysis of solutions to a problem of type a)

Next, we present several solutions to a problem of type a) by potential MG students who got good results in the mentioned Olympics, to exemplify the way we analyze the solutions and identify descriptors of generalization.

Adela is building a crib mobile for a baby. She has the structure of the mobile and adds some pieces every day, following a rule she has decided, which can be seen in the figures.

a) How many pieces will the mobile have on day 7?
b) Explain the rule that Adela has used to decorate the mobile.
c) How many days will Adela need to decorate the mobile if she has exactly 181 pieces?



This type of problems admits different resolution paths, so the descriptors can be evidenced in various ways. For our problem, student S1 answered question a) (Figure 1), by drawing the terms from the first to the seventh (DG1.1) and counting the pieces of the seventh figure (DG2.1). On the contrary, student S2 transformed the geometric pattern into a numerical sequence (DG1.2), organized the numerical values in a table (Figure 1), identified the difference between consecutive terms, and obtained the requested value after writing the necessary terms to the table, by adding the difference (DG2.2).

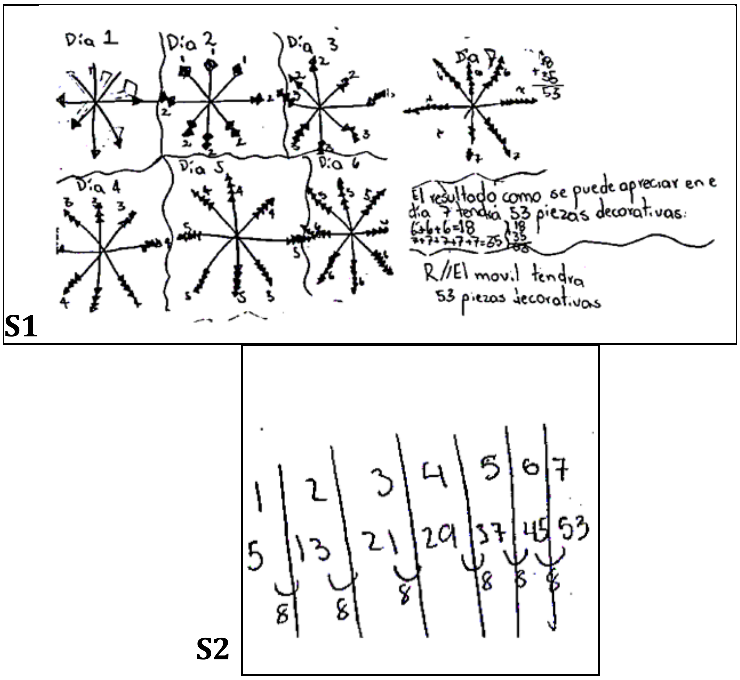


Figure 1. Solutions of S1 and S2 to question a).

In question b), we analyze the ways students expressed the requested generalization (Figure 2) and the level of generalization, to identify the descriptors showed. Student S3 wrote: [she] *adds one more piece per rod each day. And the 3 above were started on day 2* [then he drew the first three terms] *And* [the mobile] *enlarges one piece per day at the end of each rod*. Later he explained: *Adela used a pattern. It consists of adding one piece per day to each end of the rods. Only that* [she] *decided to start the three rods above the next day (day 2)*. The student related the number of pieces of the mobile to the number of days (DG2.3), showing an arithmetic generalization in which the requested value can be found recursively by adding the difference between consecutive terms (DG3.2).

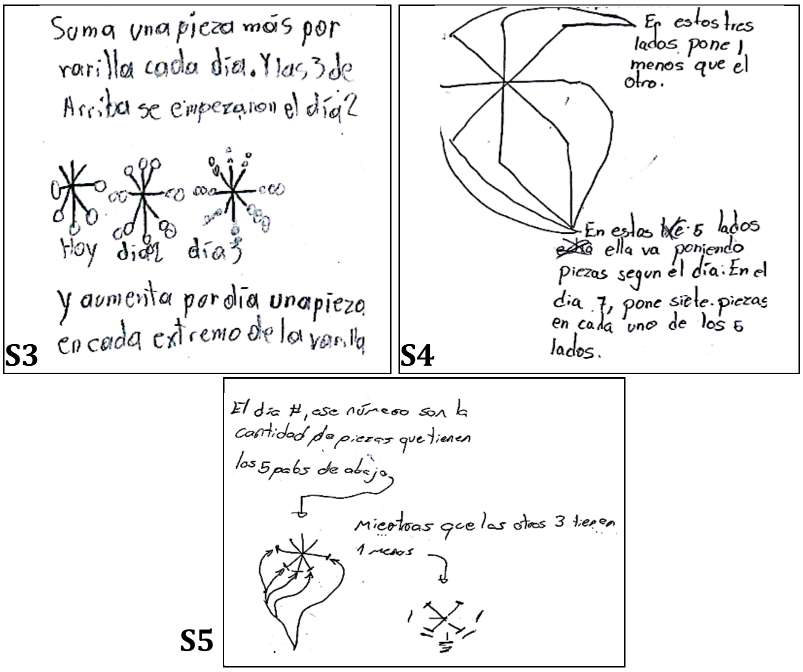


Figure 2. Solutions of S3, S4, y S5 to the second section of problem 1.

Student S4 drew a picture of the rods (Figure 2), pointed to the three upper rods, and wrote: *on these three sides* [rods] *she puts 1 less than* [on] *the other* [rods]. Then he pointed to the five lower rods and wrote: *on these 5 sides* [rods] *she places pieces according to the day. On day 7, she puts seven pieces on each of the 5 sides*. Thus, S4 established two partial relationships between the term, the pieces to make the mobile (DG2.3), and the calculations involved, expressing them by specific actions with particular numbers (DG3.3).

Student S5 wrote: *on day #, that number* [of the day] *is the number of pieces that the 5 sticks below have,* and he drew the mobile in day 1 (Figure 2). Then, he continued: *while the other 3* [rods] *have 1* [piece] *less* and drew the mobile in day 2. Later, he clarified: *each number of a day is the quantity of the pieces that the 5 bottom sticks have, and the other 3 top sticks have one piece less*. S5 showed a functional way of calculation (DG2.3) and a contextual algebraic generalization (DG3.4), by relating the position of any term (day #), the number of pieces that day, and the calculations involved; he expressed the generalization naming the variables verbally, with the only exception of using "#" for a generic day.

For question c), we present solutions by four MG students and analyze the descriptors of inverse question showed (Figure 3). The most frequent strategy was that of S6, who recursively calculated the consecutive terms from the first one to the term having the desired number of pieces. He explained: *She will take 23 days to complete the mobile, because when she reaches this day, she will run out of pieces*. This is a recursive process where the values of all the terms, starting from term 1, were calculates by adding the constant difference (DG4.1).

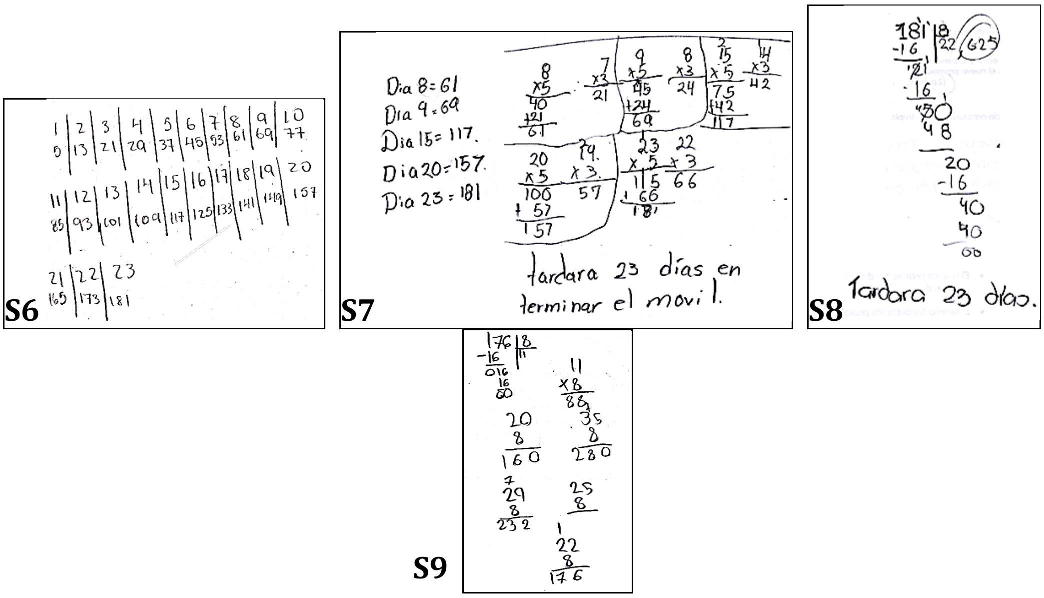


Figure 3. Solutions of S6, S7, S8, and S9 to question c).

Student S7 related several days (positions of terms) with their number of elements in the mobile (Figure 3) by using the relationship 5n+3(n-1) to calculate the values. She calculated the values of different non-consecutive, days starting on day 7, until she arrived at day 23, the one sought. This is a typical trial and error process to find the solution (DG4.2).

Student S8 did not get the exact answer but determined the solution by a wrong procedure of rounding (Figure 3): he knew that he had to invert the calculations, so he divided 181 by 8, because 8 pieces are added each day, but he did not consider that he had first to add the three missing pieces on day 1, so the division was not exact. This solution is an example of DG4.4 since S8 tried to invert the arithmetic operations, but he did not invert all of them.

On the contrary, student S9, who had used the relationship 8(n-1)+5 in questions a) and b), correctly inverted the operations in question c), as showed in Figure 3. He explained: *she will take 23 days decorating the mobile because, as the first day she decorated 5* [pieces], *I subtracted 5 from 181 and 176 are left, and 22x8=176 plus the one* [day] *that was removed give 23 days*. This solution was based on the correct inversion of the operations (DG4.6).

CONCLUSIONS

We have presented part of a research aimed to identify descriptors of different characteristics of MG students' behavior when solving Olympic problems. We have presented a structured theoretical framework for the operational analysis of MG students' capacity of algebraic generalization, as demonstrated when solving diverse types of problems selected to mobilize that capacity. Among the various types of problems we have used, the geometric pattern problems have proved to be the most useful, because they have induced the emergence of more diversity of descriptors.

We have presented a diversity of answers to a same problem, showing that, even in special groups of students like those participating in a mathematical competition, it is possible to differentiate grades of acquisition of a capacity.

A limitation of our research is that the sample is not big enough to generalize results. The definitions of the descriptors of generalization and the results presented are basis for us to continue working in this research direction, observing the presence of the descriptors in different groups of students and using them to identify MG students in ordinary classrooms.

Acknowledgements

This publication is part of the research projects PID2020-117395RB-I00, funded by MCIN/ AEI/10.13039/501100011033, and 820-CO-081, funded by Universidad de Costa Rica.

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