Discriminating proof abilities of secondary school students with different mathematical talent

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A characteristic differentiating mathematically talented students from average students is their ability to solve problems, in particular proof problems. Many publications analyzed mathematically talented students' ways to solve problems, but there is a lack of data about the ways those students learn to make proofs. We present results from a study were we posed some geometry proof problems to secondary school students having different degrees of mathematical ability. We have classified their answers into categories of proofs. Our results suggest that the ability to make proofs of the mathematically talented secondary school students is better than that of the average students in their grade and also that mathematically talented students could be ready to begin learning to make deductive proofs even at secondary school grade 1.

Keywords: Mathematically talented and gifted students, learning proof, secondary school, geometry education.

INTRODUCTION

During last decades there is being an increasing number of publications informing on research on several components of teaching and learning proof (Hanna & de Villiers, 2012; Reid & Knipping, 2010; Harel & Sowder, 2007; Mariotti, 2006). Another important area of research in mathematics education is related to mathematically talented students (MT students hereafter), including, in particular, mathematically gifted students. The literature pays attention to their identification and to several aspects of those students' learning, reasoning, problem solving styles, behaviour, affectivity, etc. (Greenes, 1981; Leikin, 2010; Sriraman, 2008). A frequent methodology to identify MT students' characteristics is to compare the ways they and average students solve the same tasks (Heinze, 2005).

We are interested in the link between both research programs: The ways MT secondary school students learn to make mathematical proofs. There is a general agreement that MT students' learning processes are different from their age peers' ones (Sriraman, 2004). To better understand MT students and to design adequate ways to teach them proof, teachers and researchers work on identifying those differences. In particular, two research questions still needing an answer are:

Are secondary school MT students different from their classmates when solving geometry proof problems?

Are secondary school MT students in various school grades different when solving geometry proof problems?

To get information on these questions, we carried out a research experiment aimed to analyze proofs produced by secondary school students and to find possible differences among students with different levels of mathematical ability or in different school grades. We posed several geometry proof problems to a sample of 1st and 4th grade students and we classified their answers according to the categories of proofs in Marrades & Gutiérrez (2000). Due to the limited number of MT students in our experiment, we do not pretend to get general conclusions, but to bring data from this case study that might point at some differences.

LITERATURE REVIEW

Most of the literature on teaching and learning proof pays attention to whole class groups or, when they are based on case studies, to ordinary students. On the other side, many publications on MT or gifted students pay attention to their ways to solve problems, in particular proof problems, but they do not inform on characteristics of proofs produced by MT students nor on MT students' progress in learning to prove.

Leikin (2010) reflects on some aspects of classes for groups of MT students. She suggests that classes must be challenging for the students and a way to get it is by means of problem solving. Then, Leikin analyzes different types of challenging problem solving tasks, namely inquiry-based, multiple-solution, and proof tasks. The paper also presents an example of a lesson for 9th grade MT students, including examples of students solving proof problems, but Leikin's analysis does not enter into the characteristics of the proofs produced.

Koichu & Berman (2005) present a study based on MT students trained to solve olympiad-style problems. They analyze students' preferences for algebraic or geometric ways to solve geometry problems. Their conclusions show that MT students choose or value proofs taking into consideration effectiveness and elegance, and also that, when students are deciding how to solve problems, they may experience a conflict due to their preference for effectiveness or for elegance.

Housman & Porter (2003) analyzed the types of proof schemes (Harel & Sowder, 1998) produced by undergraduate mathematics majors who had received none, one or two proof-oriented courses, and had earned only A's and B's in their university mathematics subjects. The authors considered that these students were above-average mathematically talented. The students were presented 7 conjectures, and they were asked to state whether the conjectures are true or false and write the proof. Some conjectures are false, so one would expect empirical proofs (show a counter-example) for them. Housman and Porter's results demonstrate that we should expect a variety of types of proofs in any group of MT students, but they did not inform on students' processes to learn to prove nor included average students solving the same tasks.

Sriraman (2003) compared average and MT 9th grade students' solutions to non-routine combinatorial problems. His results show that the MT students were able to get generalized solutions, while the average students used particular cases. Although Sriraman's problems are not proof problems, the ability of his MT students to generalize is an indicator that they could give deductive answers to proof problems.

Sriraman (2004) analyzed the answers to a proof problem on triangles by 9th grade gifted students with no previous contact with proof. Students' processes of solution began with a checking of examples; they concluded that the conjecture is true only for equilateral triangles; then the students looked for counter-examples for non-equilateral triangles and, finally, they tried to prove the conjecture for the equilateral triangle. Sriraman did not give information about the types of proof produced by the students, but it seems that their proofs were empirical although near to deductive.

All the authors referenced in this section have used proof problems as part of their experiments, but none of them has analyzed students' processes of learning to make proofs. Only Housman & Porter (2003) paid attention to the types of proofs produced by MT students, although they did not compare MT and average students.

THEORETICAL FRAMEWORK

The literature shows a diversity of definitions of (mathematically) able, talented or gifted students, where some authors consider the terms as equivalent while others consider them as different (Leung, 1994; Sriraman, 2004), but entering into the analysis of different definitions is not our objective. For us, MT students are those who, when doing mathematics, show certain traits of mathematical ability higher or more developed than other students with the same age, experience, or school grade. Behaviour traits of MT students have been identified, among others, by Freiman (2006), Greenes (1981), Krutetskii (1976) and Miller (1990). In this context, we consider mathematically gifted students as an extreme case of MT students (that, in many countries, got more than 130 points in an IQ test).

In the context of secondary school mathematics, a mathematics education research line focuses on students' process of learning to prove, including both the ability to make a proof and the understanding of the characteristics of mathematical proofs. There is, among teachers and researchers, a diversity of positions respect to the concept of mathematical proof (Cabassut et al., 2012). For some of them, the term proof refers to the logic-formal proofs, and they use terms like justification, argument or explanation to refer to non-formal ways to warrant the truthfulness of a mathematical conjecture. Others define a proof as any mathematical argument created to convince somebody (oneself or an interlocutor) of the truthfulness of a conjecture. We agree with the later, since it has proved to be very fruitful to consider as proofs both authoritarian or ritual arguments, empirical arguments, informal deductive arguments, and logic-formal arguments (Harel & Sowder, 2007). In this framework, the process of learning to prove can be seen as a continuous progress, along the years, starting with authoritarian proofs and ending with logic-formal proofs.

Marrades & Gutiérrez (2000) presented a structure of categories of empirical and deductive proofs. We have used a variation of that structure (Figure 1) to organize the proofs made by our students. This consists of: i) Remove the "intellectual" category, since it really can only be matched to generic example proofs, so it is unnecessary. ii) Add the category of "informal deductive proofs" to discriminate deductive proofs lacking the formal style of language from formal proofs; those proofs are typical of students beginning to understand the need of deductive proofs (3rd Van Hiele level). We also add a "No answer" category, addressed to those outcomes that were either blank or providing no information at all about students' reasoning.

The categories cannot be linearly ordered according to their quality, but there is a perception that empirical categories are more elaborated from top (failed) to bottom (generic example). The same can be said for deductive categories of proofs.

METHODOLOGY

The research experiment took place in a secondary school in a big city in Spain. The students were a convenience sample consisting on several class groups of pupils of a mathematics teacher willing to collaborate. Table 1 shows the number of students in each group. We are reporting here results from students in grades 1 (aged 12–13) and 4 (aged 15–16). A part of the students participating in the experiment were in average class groups (without MT students), while the others were MT students attending a workshop devoted, mainly, to problem solving. The students participating in the experiment had not worked before on the stated problems nor on other similar ones.

To inform on the two questions stated in the introduction, we present data to compare the answers i) by grade 1 average and MT students and ii) by grade 1 and grade 4 MT students. We do not use here data from the grade 4 average students.

We selected several paper-and-pencil geometry proof problems to pose two problems to each group of students participating in the experiment. We did not use the term "prove" in the statements to avoid the possibility of a misunderstanding by some students that could not be habituated to it. To select the problems we had to take into account i) that the students should know the geometric contents necessary to solve them

	Grade 1	Grade 4
Average students	13	41
MT students	3	4

Table 1: Distribution of the students in the sample

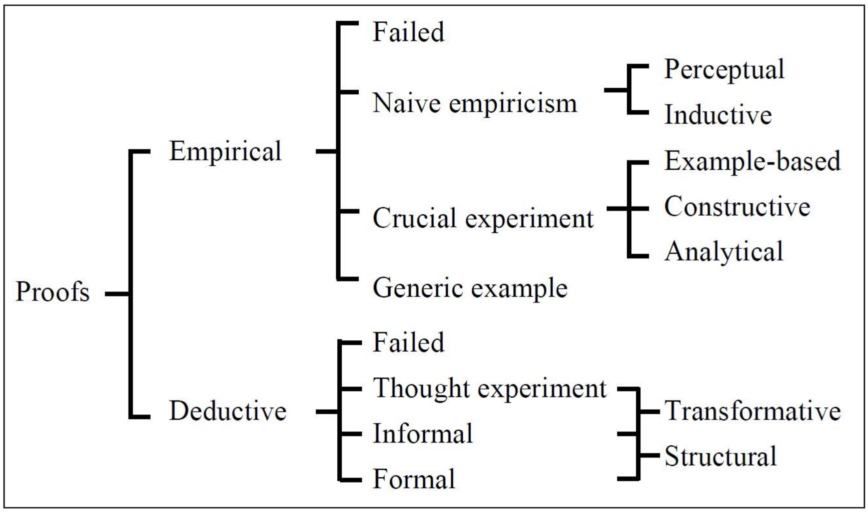


Figure 1: Structure of categories of mathematical proofs

but they had not solved the problems previously, and ii) that the topics of the problems should be related to the one being studied by the average groups at the time of the administration. The second condition to be fulfilled impeded us to pose the same problems to all students. Our aim is to identify the types of proofs produced by the students so data may suggest differences among average and MT students in the same grade or among MT students in different grades.

The problems were posed to each group of students in an ordinary class session of about 50 minutes. They worked alone and the teacher answered questions about the meaning of the statements but he did not give clues for the solution. Our data are the students' written answers.

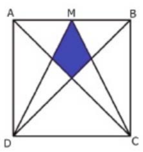
In grade 1, the two problems posed both to the average and the MT groups where:

1) How many diagonals does an n-sided polygon have? Justify your answer.

2) How much is the sum of the internal angles of an *n*-sided polygon? Justify your answer.

The two problems posed to the grade 4 MT group were problem 1 and:

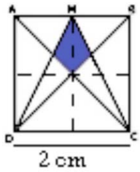
3) In square *ABCD* (figure on the right), *M* is the midpoint of side *AB*. We draw the two diagonals and segments *CM* and *DM*. Which fraction of the total area is the area of the shaded shape? Justify your answer.



To facilitate the answers to less able students and to students who did not come up with a way to solve a problem, each problem had two parts, labelled A and B. Part A was the statements given above. Part B included a clue aimed to help students to start solving the problem or to find a way to the answer. Problem 1B was:

1B) How many diagonals can be drawn from a vertex of a pentagon? How many diagonals can be drawn from a vertex of an *n*-sided polygon? How many diagonals does an *n*-sided polygon have? Justify your answer.

Problem 2B asked for the sum of the internal angles of a quadrilateral, a pentagon and an n-sided polygon. Problem 3B had the same statement as problem 3A but the figure was the one on the right.



To prevent the possibility that students could answer part A of a problem after having read the statement of part B, students were given part B of the problems only after they had completed part A and had given it back to the teacher.

RESULTS

We have classified students' answers according to the categories of proofs displayed in Figure 1. In the following paragraphs, we include information about

	No answer		Empir. failed		Naive empiricism				Crucial experiment						Generic ex.	
					Percep.		Induct.		Examp. -based		Constr.		Analyt.			
	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT
Problem 1A	3		3		1	2	1		4					1	1	
Problem 1B	6		1			1	1		2		1		1		1	1
Problem 2A	6		2		1	2	2								1	1
Problem 2B	7		2				1		1					1	1	1

	Deduct. failed		Thought experiment				Informal				Formal			
			Transf.		Struct.		Transf.		Struct.		Transf.		Struct.	
	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT
Problem 1A														
Problem 1B						1								
Problem 2A	1													
Problem 2B	1					1								

Table 2: Proofs made by grade 1 average (AV, 13 students) and MT students (3 students)

	No answer		Empir. failed		Naive empiricism				Crucial experiment						Generic ex.	
					Percep.		Induct.		Examp. - based		Constr.		Analyt.			
	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4
Problem 1A					2					1			1	1		1
Problem 1B					1									1	1	1

	Deduct. failed		Thought experiment				Informal				Formal			
			Transf.		Struct.		Transf.		Struct.		Transf.		Struct.	
	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4
Problem 1A						1								
Problem 1B					1	1				1				

Table 3: Proofs made by grade 1 (3 students) and grade 4 (4 students) MT students in problem 1

the results of the experiment respect to the question stated in the introduction.

Do average and MT students in the same grade make different proofs?

Table 2 presents the types of proofs made by the grade 1 students in the sample.

All 12 possible MT students' answers (100%) were meaningful proofs. Respect to the categories of proofs produced, 5 out of 12 possible MT students' proofs (41,7% of the answers) were basic empirical proofs (naive empiricism perceptual) and 7 proofs (58,3%) were in the most elaborated types of empirical proofs and in the basic type of deductive proofs (2 crucial experiment analytic, 3 generic example, and 2 thought experiment structural proofs).

On the other hand, 20 out of 52 possible average students' answers (38,5%) were meaningful proofs. All these proofs were in the different empirical types, with more presence in the basic empirical categories (70% of the answers; 7 naive empiricism and 7 crucial experiment example-based proofs) than in the more elaborated categories (30%; 1 crucial experiment constructive, 1 crucial experiment analytic, and 4 generic example proofs). The average students did not produce any deductive proof.

Do MT students in grade 1 and grade 4 make different proofs?

Both MT students in grades 1 and 4 solved problem 1, so we can give an answer to this question based on their proofs in this problem. Table 3 presents the types of proofs made in problem 1 by the MT students in grades 1 and 4.

In problem 1A, all MT students in grade 1 (100%) and 3 MT students in grade 4 (75%) made empirical proofs. In problem 1B, 2 MT students in grade 1 (67%) and 2 MT students in grade 4 (50%) made empirical proofs. All but one deductive proofs made were the type thought experiment structural and one MT student in grade 4 made an informal structural proof for problem 1B. Figure 2 shows the answer of this student to problem 1A. He started by checking some specific polygons (4, 5, 6, 8 sides); this let him identify a relationship that he succeeded in expressing as a (correct) formula, although he was not able to prove its truthfulness. This incomplete proof has the characteristics of a thought experiment structural, since the student uses some examples to get an abstract relationship.

This student demonstrates the usefulness of parts A and B of the problems, since the help in problem 1B allowed him to write the deductive proof that he was not able to imagine when he was working in problem 1A.

This complete proof has the characteristics of a deductive informal structural proof, since the student does not use examples to write the deductive proof of the formula.

CONCLUSIONS

Related to the first research question (about differences in proof abilities among MT and average students in the same grade), our data show a clear difference in the ability to write mathematical proofs in favour of secondary school grade 1 MT students (100% of their answers) respect to their average peers (38,5% of their answers). Although the number of MT students does not allow a valid statistical comparison with the average students, the data from our experiment suggest a

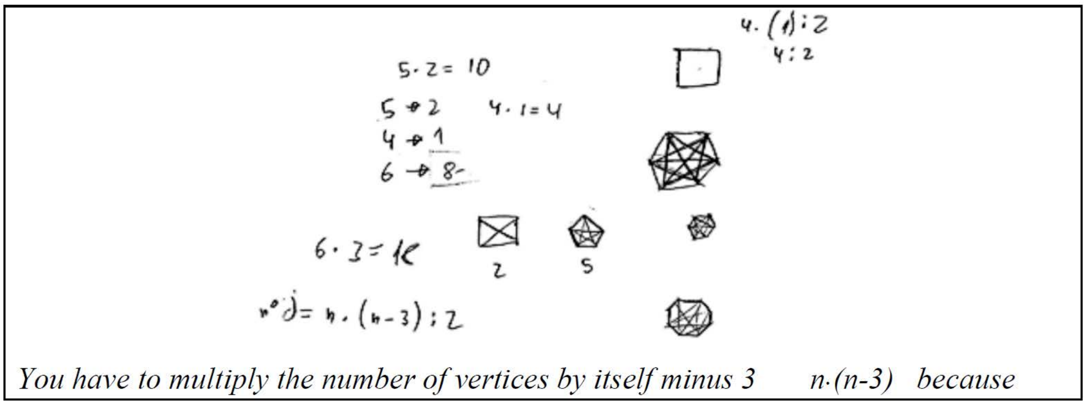


Figure 2: Answer by a grade 4 MT student to problem 1A

1. [From each vertex] *ever can be drawn 3* [diagonals] *less that the number* [of sides] *of the polygon because it is not possible to join* [a vertex] *with itself nor with the two* [vertices] *next to it.*
In the previous page I wrote the formula n·(n-3):2 because you multiply the number of vertices times the [the number of] *other non-consecutive vertices, and then you divide by* [2] *because there are repeated diagonals.*

Figure 3: Answer by the same grade 4 MT student to problem 1B

clear difference in the capabilities of MT and average students to produce mathematical proofs, in favour of the former.

It is not surprising that very few deductive proofs were produced by the 1st graders, nor that all of them were in the basic deductive category (thought experiment). The deductive proofs were made by one MT student in part B of the problems. This suggests that it would be worth study more in detail whether, as early as in grade 1, MT students could be introduced into deductive reasoning and, with some help from the teacher, some of them could write simple deductive proofs.

Related to the second research question (about differences in proof abilities among MT students in several grades), the data show that the empirical proofs made by the MT 4th graders were better than those made by the MT 1st graders, since 4th graders did not made naive empiricism proofs and most of their empirical proofs were in the types crucial experiment analytic and generic example.

MT 4th graders made more deductive proofs than MT 1st graders, and the proofs made by MT students in grade 4 were in the same type or better than those made by MT grade 1 students. These data suggest that secondary school MT students are able to advance in the learning of mathematical proof when they are allowed to gain experience in solving proof problems and they have guidance by their teachers. In our case, the ordinary classes provided such experience and, mainly, the workshop they were attending.

As a final summary, we can conclude that MT secondary school students seem to be more capable of producing mathematical proofs (either empirical or deductive) than the average students in the same grades, and also that MT students are able to begin learning mathematical proofs from grade 1, and they can learn to express their justifications in an organized way, progressing along the grades in their ability to make more elaborated proofs, even deductive proofs.

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