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Assessing Reasoning Abilities in Geometry

John Pegg (The University of New England, Australia)
Angel Gutiérrez (Universidad de Valencia, Spain)
Pedro Huerta (Universidad de Valencia, Spain)

Introduction

Assessment practices have long been an issue in the context of teaching and learning mathematics. It is taken almost as an axiom that what, and how, teachers assess informs learners of what is really valued and important in formal education (Cole 1990). Historical analyses, such as those carried out by Romberg (1987) in the United States and Griffiths and Howson (1974) in the United Kingdom, have clearly documented some roots of the assessment movement and many of its evolutionary paths. These paths mirror changing perceptions of authorities, teachers and researchers about the nature and aims of teaching and learning mathematics, the increased sophistication of the tools available to teachers to undertake assessment, and a growing awareness of how learners acquire competencies. Nevertheless, apart from excising some excesses of the past, most of the substance and philosophy of current assessment practice by teachers around the world has shown little change over time. Sometimes, purported changes are simply old ideas wrapped up with new vocabulary and technology. At other times, the changes proposed are so dramatic that teachers are not ready or willing to accept or implement them, hence they ignore the proposals and continue with their current practice. Complicating the situation further, and also impeding the implementation of new initiatives, is the linking of traditional assessment methods to the maintenance of 'standards' and traditions.

Despite these impediments, there is evidence during the 1990's of an increased global focus on assessment issues in mathematics by researchers. In many ways this movement represents an evaluation of the assessment role and its many forms of practice. Reasons for this trend can only be surmised, but a plausible explanation is that the failure of many innovative curriculum and teaching reforms in the period between the 1960's and 1990's can be traced back to a lack of synchronisation between the intended curricular changes and teachers' assessment practices at that time (Niss, 1993b). Hence, more information about different forms of assessment and how new approaches may be implemented, within day-to-day teaching constraints, may have many benefits.

As a possible reaction to this situation, a number of research projects in recent years have been developed throughout the world to address general and/or specific issues associated with assessment in mathematics. The essence of this global focus has been captured by an ICMI Study, held at Calonge (Spain) in April 1991 with the theme Assessment in Mathematics Education and its Effects. At that conference, the focus on assessment was broad and a raft of issues was canvassed. The reader is directed to two publications that arose from the conference (Niss, 1993a and b).

From this work, it emerges that the application of new assessment practices must serve the perceived needs of society. The key question here is: Who decides? There are many stakeholders, such as Governments, education authorities, employers, schools, parents, teachers, learners and researchers. It is often here that conflict occurs as different interest groups strive for recognition of a particular point of view. Hence, tensions are inevitable as important questions are posed in terms of (i) where does one find the resources for assessment (in terms of money and time), (ii) who should be the beneficiary of the assessment activity, e.g. the learner, teacher or some potential employer, (iii) how accurate are assessment practices, i.e. their validity and reliability, and (iv) how appropriate are certain assessment tasks in addressing specified goals of learning.

In this section we consider in a broad way all four of these points, referred to the context of teaching and learning geometry. We are particularly interested in assessment practices that (i) inform the learner and the teacher about the quality of geometrical knowledge acquisition and the learning experience, and (ii) offer advice on appropriate ways for instruction in this area of mathematics to proceed. At the same time, if assessment is linked to verifiable theoretical perspectives that mirror learners' cognitive growth, then issues concerning the accuracy of assessment and the appropriateness of tasks can be addressed more easily.

1. Background

As stated in the introduction to Section I, we distinguish between "assessment" and "evaluation". We adopt the more common stance that the word "assessment" relates to the process of collecting evidence on, and making judgements about, learners' needs, strengths, abilities and achievements. This definition sets it apart from the meaning of the word "evaluation", which is best described as the process of collecting evidence on, and making judgements about, the effectiveness of curricula initiatives, educational policies, and teaching programs and procedures. Assessment procedures can be either *formative*, i.e. they can occur during, and inform, the learning process, or they can be *summative*, i.e.

they can occur at the end of some topic, and provide an indication of what has been learnt. Sometimes assessment has a general character, dealing with learner performance across several topics while at other times it is domain specific, dealing with a specific topic or part of mathematics. Clearly, the more topics one attempts to assess at a given time the greater the pressure to quantify the results. Hence, the information available from the assessment process to improve the learning environment for a given learner is less than it otherwise might be. This tension is but one further dilemma in the assessment process.

Another way to consider assessment practices, which offer a related concern to that described above, is from an ideological point of view. Here one can identify distinct paradigms which bring with them competing philosophies. At its simplest one can identify two paradigms which can be interpreted as lying at ends of an assessment continuum. As one moves from one end to the other, features of both can be identified in varying proportions. At one end the approach is often referred to as traditional and it can be seen to dominate much of current practice all over the world, e.g. most large-population examinations and standardized achievement tests belong here. This method has a quantitative feel, is usually used to provide summative data and tends to focus upon single marks, which allow learners to be ranked. There is little advice in this form of assessment for the learner except, maybe, "work harder" if the scores are low and "keep up the good work" if the scores are high. Also, there is a reverence, by many in society, associated with such scores of an implied high degree of accuracy that does not mirror reality.

At the other end of the continuum, the paradigm is qualitative in nature. It represents a departure from an assessment focused only on learning skills, facts and algorithms to one interested in identifying and interpreting learners' understanding and reasoning. As a result, the information provided is usually in a non-numeric form that can be directly linked to the specifics of what has been taught or learnt. This means that this form of assessment is better able to inform the direction of the instruction process for both teachers and learners.

Both perspectives of assessment have their place, depending on the purposes to be addressed. In fact, although pure quantitative assessment methods dominate common practice in schools it is not uncommon to see qualitative aspects emerging to provide support or additional information. In such cases, both understanding and memory recall are usually being assessed. For instance, diagnostic assessment (related to diagnostic teaching) is interested in knowing what deficiencies in understanding limit learners' performances within a certain mathematics domain, and what are the origins of those difficulties. This approach to assessment attempts to identify what aspects of the concepts

are understood by learners, and which pre-requisite skills, definitions and procedures are well understood or wrongly applied.

An important issue for the teaching and learning of geometry is its almost total reliance on the quantitative paradigm. Geometry has traditionally been considered as the part of mathematics where learners would learn formal mathematical reasoning but, in spite of this fact, teachers usually assess their learners' knowledge of geometry in ways that encourage rote learning. As a balance to this situation and with the purpose of encouraging the use of alternative methods, it is the qualitative perspective that underscores this section, within the context of the learning of geometry.

Our aim in this section is to address two qualitative approaches to assessment, which have been used either as tool for formative or summative assessment within the context of learning geometry. Specifically, our focus is on forms of assessment which allow for the interpretation of learners' responses within a framework of cognitive growth. This allows teachers to see where their learners are on some developmental ladder and, at the same time, provide advice on possible pathways for future teaching endeavours. There are possibly many models that may satisfy such a qualitative approach to the assessment of teaching and learning of geometry, but two have received attention from researchers in recent years. They have proved to be very valuable aids in assisting the development of forms of assessment of the teaching and learning of geometry that can be used by classroom teachers. They are the Van Hiele Theory and the SOLO Taxonomy. The purpose of the remainder of this section is to describe and analyze these frameworks for assessment, their similarities and differences, and their potential as assessment tools.

2. The van Hiele Theory

Since the 1980's the van Hiele theory has been the catalyst for considerable research into the learning and teaching of geometry. This theory drew upon the work of Gestalt psychology in an attempt to address the difficulties of teaching geometry. In addition, an important part of the roots of van Hiele's work is based upon Piaget's ideas and perceived problems with Piaget's theory. In particular, Piaget's *structures d'ensemble*, where learners are seen to pass through a series of stages that are closely linked to age parameters, was discounted. Instead, van Hiele attributes growth to strong social-cultural features which are linked to a teaching/learning plan that has more sympathy with Vygotsky's (1978) ideas than with the biological maturation basis of Piaget (Van Hiele, 1986).

In general, the main focus of research on the van Hiele theory has been directed at the notion of a hierarchy of five levels of thinking. The levels have been described and discussed extensively in the literature (Burger, Shaughnessy, 1986; Clements, Battista, 1992; Fuys, Geddes, Tischler, 1988; Hoffer, 1983; Jaime & Gutiérrez, 1990; Pegg, 1995). A brief summary is provided below for completeness.

Level 1: Figures are judged by their appearance and recognised by their form or shape. The mathematical properties of a figure play no explicit role in its identification.

Level 2: Figures are identified by their mathematical properties. The properties, however, are seen to be independent of one another. They are discovered and generalized from the observation of a few examples.

Level 3: The properties of figures are no longer seen as independent. There is seen to be an ordering of the properties, with one property preceding or following from other properties. Relationships between different figures are also understood.

Level 4: The roles of the elements of an axiomatic system and of deduction are understood. Necessary and sufficient conditions can be employed. Proofs can be developed, not simply learned by rote.

Level 5: Comparison of various deductive systems can be undertaken. Different geometries can be explored based upon various systems of postulates.

In addition, the theory advocates a learning cycle of five teaching phases which encourage learner development from one level to the next.

Phase 1: *Information (Inquiry)*. This part of the process allows learners to discuss the nature of the area to be investigated. It also allows teachers to be aware of their learners' previous knowledge and thinking level(s) in the topic.

Phase 2: *Directed Orientation*. Learners begin to look at the area to be studied by means of completing a number of simple tasks. The teachers' role is to direct their classes to explore the object of study.

Phase 3: *Explicitation*. As a result of the manipulation of materials and the completion of tasks set by the teacher, the need to talk about the subject matter becomes important. During the early part of the process learners use their own language. However, over time teachers assist their learners to refine their language and gradually incorporate, where appropriate, correct technical terms.

Phase 4: *Free Orientation*. Here learners are expected to find their own way to solve or undertake a variety of tasks and activities, relevant to the topic of study, which can have multiple solution paths. The teachers' role is to encourage different solutions to the problems as well as the inventiveness of learners' approaches.

Phase 5: *Integration*. The learners achieve an overview of the area of study. They are now clear of the purposes of instruction and have reached the next level. Teachers are expected to assist with the process.

While there has been little research associated with the five phases, an impressive number of studies has considered aspects related to the first four of the van Hiele levels. This work has resulted in a substantial body of evidence providing empirical evidence for the levels, although the support for other features of the theory has been more ambiguous. Some researchers, such as Burger and Shaughnessy (1986), and Gutiérrez et al. (1991), have even challenged what van Hiele refers to as one of the most "distinctive features" of the levels, namely, their discontinuity. As a result, a deeper knowledge of the structure of the levels has been gained.

It would seem that van Hiele followed Piaget, maybe unknowingly, in one important aspect, namely, he sought a uni-dimensional path of cognitive development. It is as if he was trying to encompass variability within a single universal construct. From this he was able to identify and draw out features of the levels. Some of these aspects of the original characterization of his theory only became contentious when there were attempts to provide too broad a generalisation by characterising learners' understandings over a range of concepts or when the analysis of a learner's understanding was explored in too fine a detail. The implications of this being that (i) learners may be on different van Hiele levels for different concepts, and (ii) the van Hiele levels are not particularly suited for fine-grained analyses of a learner's understanding.

Four issues stand out that need to be considered if the van Hiele levels are used in assessment practices. First, van Hiele provided an important starting point for consideration of learner growth in geometry. By identifying a broad framework and describing a set of possible invariant structures, he has created a background from which other research can be judged. This is a necessary first step towards fully understanding learner cognitive growth.

Second, the model, as defined by van Hiele, is uni-dimensional. It offers a single universal pathway to understanding, where there are no alternative directions. Any differences in understanding can only be attributed to the rate of progress along this single path. In effect, the model proposed might be more accurately thought of as an 'ideal' or 'average' path of cognitive growth in geometry.

However, as suggested above, the results of recent research have offered a more detailed and realistic view of the van Hiele levels. In particular, it seems we are now in a position to comment upon when the transition from one level to the next is possible, and that a learner might progress simultaneously in the acquisition of understandings related to two consecutive levels (Gutiérrez et al., 1991). This new vision of the van Hiele levels is, in some ways, bi-dimensional, since learners can be assigned degrees of acquisition for a number of levels. As a necessary complement to this view of the model, new tools for the assessment of learners' acquisition of the levels have also been designed (Gutiérrez et al., 1991; Jaime & Gutiérrez, 1994; Gutiérrez & Jaime, 1995).

Third, while the assumption that all learners progress through a single path of cognitive growth is a potential disadvantage for the model, it does not suffer the same problems as other frameworks of assessment which have fixed content outcomes. Instead, for the van Hiele levels, actual specific details of the levels are not pre-determined and there is opportunity both for flexibility and diversity. The levels depend upon, and are sensitive to, the context of the learning/teaching program.

Fourth, having assessed learners' understanding, there exists a structure in van Hiele theory upon which to base a program of instruction. This structure is provided by the teaching phases which describe a process of moving learners in an instructional setting from one level to the next.

This last aspect is significant as it offers a way forward for a teacher following the outcomes of assessment. Despite this importance, the teaching phases have not been used often, at least explicitly, to implement teaching sequences. One example of such an implementation can be seen in the papers of Jaime (1993) and Jaime and Gutiérrez (1995). These authors designed a set of units for teaching plane isometries in grades 3 to 12, starting with activities of van Hiele level 1 (aimed at the recognition of each isometry as a movement and the performance of isometries with manipulatives) and ending with activities of level 4 (such as those involving deduction and proof of both properties identified in activities related to previous levels and new complex properties, acquisition of a global view of the algebraic structure of plane isometries, and learning to do formal proofs in this context). For each van Hiele level, the activities were organized according to the phases of teaching.

For instance, when learning about rotations at level 3, a typical activity of phase 2 for this level is to ask pupils to deduce and justify that

(i) the product of two rotations is equivalent to a translation when the sum of the rotation angles is a multiple of 360°, or alternatively

(ii) the product is equivalent to another rotation whose angle is the sum of the rotation angles.

Pupils deduced the results by experimentation and observation on some examples, and they tried to validate their conclusions in an abstract way, trying to produce arguments independent of the specific examples used. Later, when pupils had learned about the relationships among rotations and products of reflections, pupils at level 3 were asked, in a typical activity of phase 4, to prove that the product of two rotations is either a rotation or a translation. Here the context is more formal, approaching the kind of formal reasoning applicable to level 4, and pupils were expected to put together several pieces of knowledge to produce an acceptable proof.

Overall, it is the integration of a consistent learning cycle in which entry conditions and exit level boundaries are determined, that sets the van Hiele theory apart from other models. It is also this feature which makes it so useful as an assessment process.

3. The SOLO Taxonomy

To interpret individual learner understandings in geometry more clearly, there appears to be a need to move beyond the broad five-level system proposed by van Hiele. This can be achieved by employing a model, consistent with van Hiele's ideas, that allows for finer-grain categorisations. Such a model is referred to as the SOLO (Structure of the Observed Learning Outcome) Taxonomy and was developed by John Biggs and Kevin Collis. Since the early 1980s it has undergone continual refinement and these developments have been documented in Biggs and Collis (1991) and Pegg (1992a). In brief, the SOLO Taxonomy is a categorisation system designed to evaluate the quality of a response to some stimulus. It has much in common with a number of post- or neo-piagetian formulations of writers such as Case (1992) and Fischer and Knight (1990).

The SOLO Taxonomy has been developed independently of both the van Hiele theory and research into geometrical understanding. Nevertheless, this broader, more detailed system has been shown to be particularly effective in helping interpret learners' understanding in geometry (see, Pegg & Woolley, 1994; Pegg & Faithfull, 1995).

The SOLO Taxonomy allows learner responses to be coded in terms of two aspects. The first describes the type of thinking involved and is referred to by one of five modes, and the second describes the quality of the response within a mode by using the notion of levels. The modes are:

Sensori-motor : Associated with motor activity and can be described as tacit knowledge.

Ikonic : Associated with imaging, imagination and language development and can be described as intuitive knowledge.

Concrete symbolic : Associated with the use and manipulation of written symbol systems and is a form of declarative knowledge.

Formal : Associated with abstract constructs no longer restricted to a real world referent, the knowledge as theoretical.

Post formal : Associated with challenging and extending the theoretical constructs developed in the formal mode.

While these modes have much in common with Piaget's stages there are at least two important differences. First, a newly developing mode does not subsume or replace earlier modes. Instead, earlier modes continue to evolve and provide support for later acquired modes. Second, these later developed modes can assist further growth in earlier developed modes.

The SOLO levels associated with each mode have the same broad description for the different modes. They are:

Pre-structural : Where the response is based on an irrelevant aspect.

Unistructural : Where the response is based on a single relevant aspect.

Multistructural : Where the response is based on multiple aspects which are seen as independent.

Relational : Where the response is based on relationships between several aspects.

Extended abstract : Where the response goes beyond the thinking of the mode in question and exhibits a new way of thinking.

The specific characteristics of each SOLO level differ, depending on the mode of functioning within which the level resides. Of main interest are the three levels referred to as unistructural, multistructural, and relational as they lie within a mode. The remaining two levels are outside a given (or target) mode. For example, the pre-structural level refers to those responses that fail to address the question asked. As such, they may indicate thinking within an earlier developed mode or they may be an irrelevant response. The extended abstract level represents thinking of a more abstract nature and hence can usually be interpreted in terms of criteria consistent with a latter developed mode. This information is summarised diagrammatically in Figure 1. To reduce the complexity of the diagram, both pre-structural and extended abstract levels have been omitted.
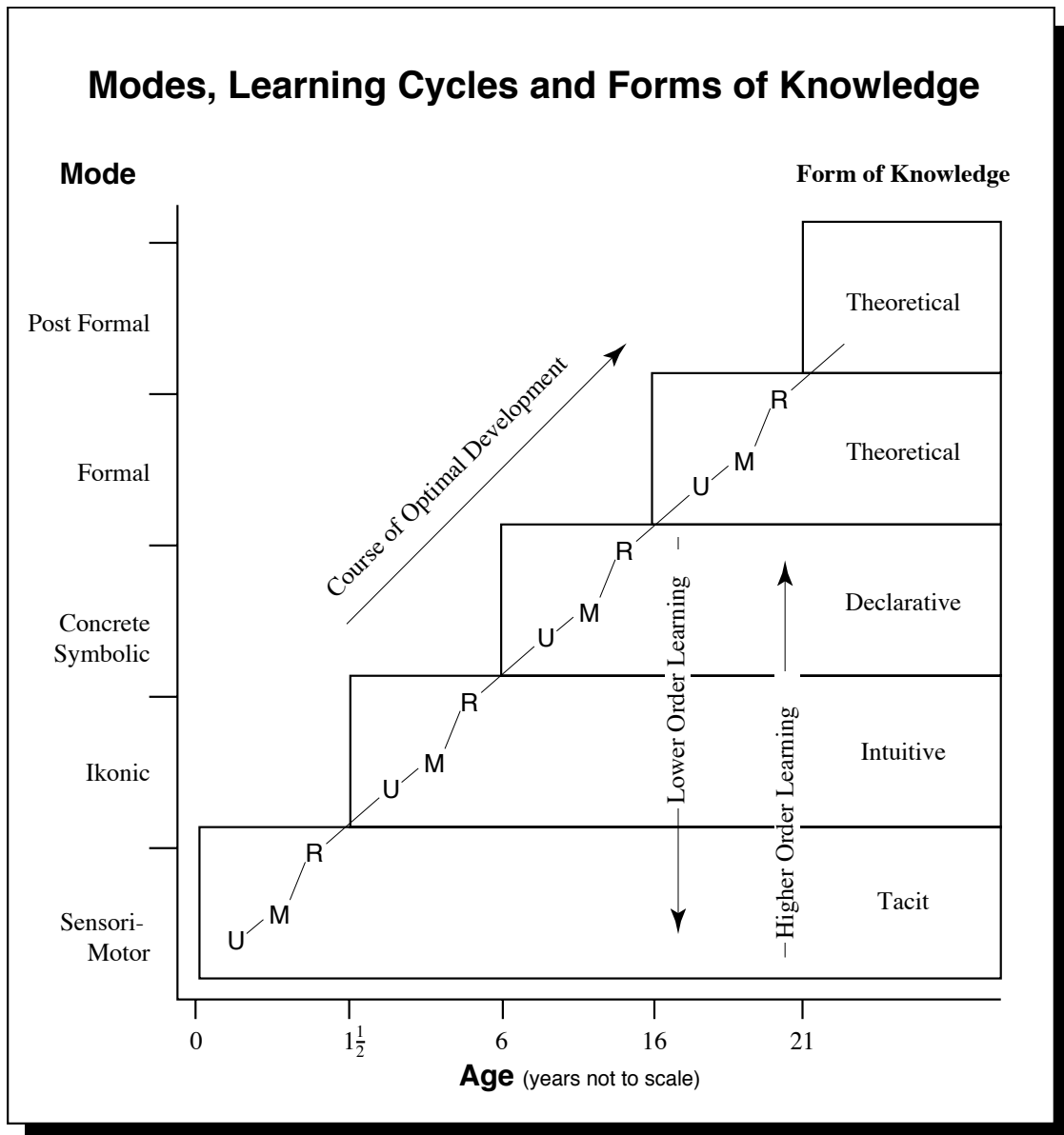
Figure 1. The SOLO model: Modes, Learning Cycles and Forms of Knowledge (Adapted from Biggs and Collis, 1991).

Recent work into the SOLO model has identified that there exists not a single cycle of unistructural, multistructural and relational levels (as shown in Figure 1) but that there are at least two cycles in each mode (Campbell, Watson & Collis 1992; Pegg 1992a). For example, a typical growth pattern for geometry within the concrete symbolic mode can be described diagrammatically as represented in Figure 2.

$$R_1 \quad R_2 \nearrow U_1 \text{ Formal mode}$$
$$M_1 = \searrow M_2 = \searrow \text{New cycle in concrete}$$
$$U_1 \quad U_2 \quad \text{symbolic mode}$$
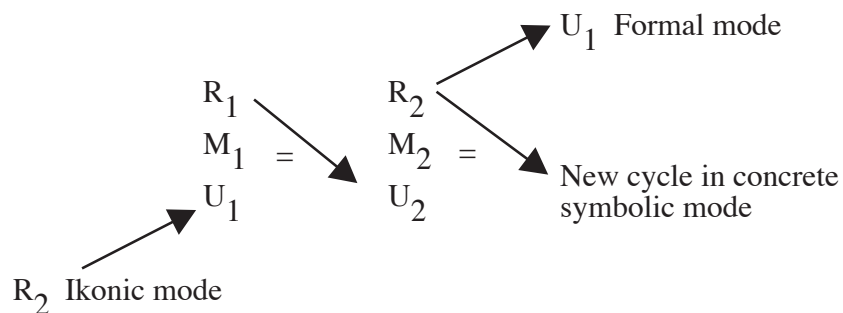$$\nearrow R_2 \text{ Ikonic mode}$$

Figure 2. Diagrammatic representation of SOLO levels associated with the concrete symbolic mode.

To explore the meaning of the levels mentioned in Figure 2, consider an example of pupils' descriptions of a certain 2-D shape. A response that describes the figure by some global visual image that may incorporate features (such as, pointy, flat, corners, lines, sides, colours, and the like) would be coded at the relational level (R_2) in the ikonic mode. An early response in the concrete symbolic mode involves building and elaborating upon some single feature associated with the figure. In practice this turns out usually to be sides but it need not. *"The shape has 4 sides"* is an example of unistructural response of the first cycle (U_1) in the concrete symbolic mode. A pupil who responded consistently at this level would provide the same basic answer for a range of quadrilaterals, such as a square, rhombus, rectangle and parallelogram. The next level of response, coded as multistructural (M_1), attempts to qualify this feature (in this case sides) by attempting to bring in the lengths of the sides into the description, i.e., a pupil might consider several like shapes or compare (say) the length and breadth dimensions of the figure and then report *"This shape* (a rectangle) *has the top and bottom sides longer than the side sides"*. Responses are coded as relational (R_1) when a single property is mentioned, e.g. *"A square has all the sides around it the same length"*. This first cycle of development in the concrete symbolic mode has many links to the ikonic mode and represents a transition from an intuitive feeling of space to the development of geometric thinking. In addition, there are usually references to images associated with the above responses, especially if the pupils are pushed to provide additional information. This cycle also charts the growing emergence of the mathematical concept of property as it pertains to shape identification.

The second cycle commences with a response that succinctly states a single property, e.g. *"A square has four equal sides"*. This response is coded as U_2 but often it is difficult to distinguish between responses at R_1 and U_2 without an interview. The essence of the difference hinges on the ability of a U_2 response to be less drawn out (less verbose) or convoluted and more precise than an R_1 response.

The example above illustrates coding in the first cycle based upon the development of the feature, sides, beginning with identification of the number and ending with the incorporation of length considerations. At the same time, pupils are developing property concepts related to other features. These are associated with features such as corners or the gap between lines. This development results eventually in properties related to angles and parallelism, although these are usually not mentioned in relation to shapes, at this level, unless pupils are specifically prompted to do so. The next level of response, coded as M_2, includes the spontaneous reference to several properties, i.e. the various properties developed individually during the first cycle become overt. This marks the ability of a pupil to link, or list in a sequential fashion, several properties. There is no indication here of a relationship between these properties. Finally, relationships between properties resulting in descriptions that indicate an overview (coded as R_2) are reported.

The difference between the two cycles identified in the concrete symbolic mode can best be articulated by comparing the response level R_1 (from the first cycle) and R_2 (from the second cycle). Both levels indicate a combining together and overview of aspects within the relevant cycle. R_1 represents an end product resulting from combining various aspects which culminate in pupils being able to provide a single property in a non-prompted situation. An R_2 response represents an end product of combining several properties so that relationships between these properties as well as between figures are made explicit.

In Figure 2 there are two paths indicated for responses whose quality is an improvement on an R_2 response. One path is to provide a U_3 response which would be an initial response in an hypothesised third cycle in the concrete symbolic mode. A second path would be to provide a U_1 level response in the formal mode. In this latter case one might expect to see the beginning emergence of the use of necessary and sufficient conditions.

The detail provided by the SOLO structure offers flexibility and versatility in categorising responses. In particular, four issues stand out which are relevant to assessment. First, in SOLO categorisations, the target of assessment is the quality of the structure of learners' responses. This notion avoids the problem of *decalage* associated with Piaget's ideas (and van Hiele's), as a respondent is not expected to respond consistently at a given level across a number of topic areas or within the same topic on different occasions. Instead, responses are sensitive to the motivation of the learner, the amount of experience with the material, the learners' interpretation of the stimulus item, and so on.

Second, the categorisations possible within the current formulation of the SOLO taxonomy are extensive. Not only are there at least six levels within the concrete symbolic mode but there are similar numbers of levels in the other modes.

Third, the taxonomy has multiple dimensions. Not only is there the opportunity to describe the quality of responses within a mode but the interaction of other modes allows for an explanation of diversity and individuality in learners' responses that has not previously been available.

Fourth, while there is no teaching scheme associated with the SOLO model the additional arrows in Figure 1 imply important general principles about learning. For example, in regards to the diagram, there is potential for bottom-up and top-down support to improve learning in some target mode for a particular skill. In a teaching situation the majority of learning experiences should be directed at the target mode. Nevertheless, it is valuable to ensure that relevant skills in a previously acquired mode are available before learning has commenced and that it provides ongoing support, where appropriate, to enhance learning. Alternatively, it may enrich the learning experience if more abstract skills, which have been previously acquired by the learner, are used to enrich the development of new skills whose target area is associated with an earlier mode.

4. Towards an integrated assessment in geometry

From the previous descriptions, links can be established between the van Hiele Theory and the SOLO Taxonomy. For example, the van Hiele levels can be matched to SOLO modes (Pegg, 1992b):

V. Hiele levels	SOLO modes
Level l	Ikonic (I)
Level 2 and 3	Concrete symbolic (CS)
Level 4	Formal (F)
Level 5	Post Formal (PF)

It is possible also to make a more elaborated matching by allocating SOLO levels to the responses that indicate van Hiele levels. The state of the current research means that this is easier with van Hiele levels 2, 3 and 4; however, there are indications that would allow us to hypothesise about the SOLO connections with van Hiele level 1.

The different degrees of acquisition of level 1, where learners recognise figures by their global appearance, can be interpreted within the levels U_2, M_2 and R_2 in the ikonic mode.

It would seem that these three SOLO levels may represent a focus on some single aspect of a shape, several unrelated aspects and an overview, respectively. The word "aspect" is used to describe words such as pointy, flat, corners and sharp and these are very different to mathematical properties that are a feature of the concrete symbolic mode. The SOLO levels in the first cycle of the ikonic mode, namely, U_1, M_1 and R_1, can be possibly interpreted as a van Hiele level 0, defined by some investigators (Clements and Battista, 1992) to describe learners who still have not started the acquisition of level 1, i.e. they show levels of response which indicate that the imagery process is quite incomplete.

Level 2 in the van Hiele theory, where learners recognise figures by their properties, can be coded at either U_2 (CS) or M_2 (CS). In the case of a U_2 (CS) response, i.e., when learners have a low acquisition of level 2, they base their observations on a single property. When the response is coded at M_2 (CS), i.e., when learners have a medium to high (or complete) acquisition of level 2, observations can be made on the basis of many properties. The earlier U_1, M_1, R_1 (CS) cycle of SOLO is a development that is in transition between seeing a shape as a global picture (van Hiele level 1) and working with a single mathematical property (van Hiele level 2).

Level 3, where learners see relationships between properties or figures and can undertake informal proofs, can be coded at R_2 (CS). Learners with complete acquisition of level 3 are able to provide simple formal proofs; however, the main differences between this and level 4 is that the learners at level 3 can only use relationships, such as congruency, in relatively straightforward prompted situations and where there is an empirical referent.

The process of acquisition of level 4, where learners understand the deductive process and can use relationships as a tool in the solution process, can be coded by at least two cycles U_1 - M_1 - R_1 and U_2 - M_2 - R_2 in the formal mode (F). The first cycle in this mode represents a development of the deductive process, where a R_1 (F) response describes a learner who can competently apply relationships, such as congruency and similarity, in addition to being able to keep track of most conditions inherent in the question. A U_2 (F) response appears to relate closely to the successful application of necessary and sufficient conditions in non-prompted situations and is possible the best response capable learners can achieve in the secondary school.

Level 5, where learners challenge old, and adopt new, axiomatic systems is hypothesised to be represented by cycles of SOLO levels in the post formal mode (PF). Supposedly, level 5 could only be achieved by professional mathematicians or bright learners, although experiences of persons reasoning at this van Hiele level have not been reported.

Traditionally, researchers have devoted considerable effort in preparing reliable tools to assess learners from the point of view of either van Hiele levels or SOLO categories. Such tests (paper and pencil, or interview) have been designed to assess learners in different areas of geometry. While an important aim of such research is to close the gap between research and practice, teachers often are unable to use such methodologies due to various constraints, the short time for assessment being one of the most important. One attempt to give an answer to teachers of geometry in how they might assess their learners' reasoning and knowledge has been carried out by designing a single test that can be depicted from the points of view of both the van Hiele model and the SOLO taxonomy (Huerta, 1997). This test was administered to a sample of 74 pupils from Primary to the University (Table 1).

Educat. level	Average age	Number
Primary	11 years	20
Secondary	16 years	19
Secondary	18 years	23
University	21 years	12
TOTAL		74

Table 1.

The instrument was designed to address both the requirements for van Hiele tests (Jaime & Gutiérrez, 1994) and the organization of SOLO tests (Collis, Romberg & Jurdak, 1986). It consists of five super-items, with four questions in each super-item. Two super-items deal with parallelism and three super-items deal with quadrilaterals. Each super-item assess the grade of acquisition of van Hiele levels 1, 2, and 3, and some questions assess also level 4. The questions in each super-item are organized so that the reasoning requirements, and structure of response, necessary for correct answers increase in complexity as learners move through the super-item. From the point of view of van Hiele, the first questions require reasoning in lower levels than the last questions. Figure 3 shows a typical super-item used in this test. From the SOLO point of view, the first question assess answers at the unistructural level, since learners have only to use one data from the stem. The second question assess answers at the multistructural level, because learners have to use two properties, although they have not to be connected one to the other. The third question assess answers at the relational level, because angle C may only be calculated after relating at least two angles and at least a property from the stem. The fourth question assess

answers at the extended abstract level, since it asks the learners to prove a property of triangles.

STEM

When a pair of parallel lines are cut by a transversal (Figure 1), we have eight angles verifying the following properties:

- The <u>alternate interior angles</u> are equal: B = F, G = C.
- The <u>alternate exterior angles</u> are equal: A = E, D = H.
- The <u>opposite angles</u> are equal: A = G, B = H, C = E, F = D.
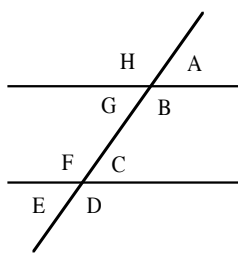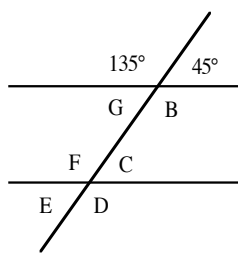- The <u>corresponding angles</u> are equal: F = H, C = A, E = G , D = B.

Figure 1   Figure 2   Figure 3
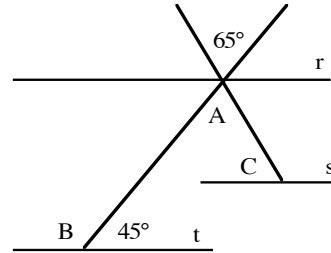
QUESTIONS

1. Determine the value of angle G in Figure 2.
2. Determine the value of the angles D and C in Figure 2.
3. In Figure 3, the lines **r** and **s** are parallel and the lines
    s and **t** are parallel. Determine angles A, B and C.
4. Prove the following property for every triangle: "The
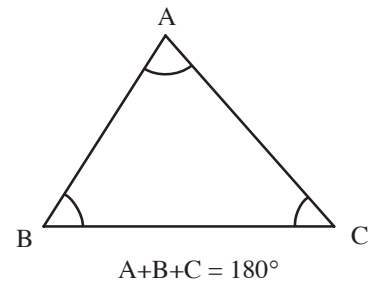sum of the three angles of a triangle is 180°".

$$A+B+C = 180°$$

Figure 3.

The learners could answer question 1 by reasoning at van Hiele levels 1 or 2. Some answers are:

G = 45° because G is beside A and so G measures 45° like A (Level 1; R-I).

G = 45° because they are opposite angles (Level 2; R-CS).

G = 45° because in opposite angles (property) is said that A = G. So, if A = 45° then G = 45° and both have the same value (Level 2; M-CS).

G = 45° because A = G and A = 45°, so G = 45° (Level 2; U-CS).

The first answer shows a possible reasoning in van Hiele level 1. The learner does not use a property to address the question, but only uses physical aspects of the figures in the answer: The value of G is justified by the position of the angles. From the point of view of SOLO, this answer is relational (R) in the ikonic mode.

The other answers show reasoning at van Hiele level 2. At the same time, they show an internal structure in the concrete symbolic (CS) mode. The second answer shows a global understanding of the property "when two lines intersect, opposite angles are equal" since the learner mentions such a property to justify that G = 45°. So, this answer is relational (R_1). In the third answer, the learner explains the property by using the appropriate equality among those exemplified in the stem. He/she uses the information as separated aspects, so this answer is multistructural (M_1). Finally, in the fourth answer, the learner only mentions the equalities to answer, showing that he/she is only able to use an aspect of the information. So, this answer is unistructural (U_1). Finally, learners who did not solve the question were considered as prestructural (PRE).

The assessment from the van Hiele point of view gave a distribution of the learners in several profiles of reasoning. Table 2 summarizes the results of the assessment from van Hiele perspective. The above mentioned profiles are:

Profile 1 : Learners with a complete grade of acquisition of van Hiele levels 1 and 2, and indications of reasoning in other levels.
Profile 2 : Learners with a complete grade of acquisition of van Hiele level 1, and indications of reasoning in other levels.
Profile 3 : Learners with no complete acquisition of van Hiele level 1, and indications of reasoning in other levels.

Profile	Number	%
Profile 1	18	24.3
Profile 2	33	44.6
Profile 3	23	31.1
TOTAL	74	100

Table 2. Distribution of the learners' van Hiele levels of reasoning.

After assessing the learners with the SOLO Taxonomy, two learning cycles in each concrete-symbolic and formal modes were found in learners' ways of use of geometrical properties. Table 3 summarizes the results of the assessment with the SOLO Taxonomy

for the learners in each van Hiele profile. Also a parallelism was found to exist among the van Hiele profiles and the SOLO levels.

	PRE	Ikn. R	Concrete-Symbolic						Formal					
			U_1	M_1	R_1	U_2	M_2	R_2	U_1	M_1	R_1	U_2	M_2	R_2
Pr. 1							1.3		4.1	1.3	2.7	1.3	9.5	4.1
Pr. 2		10.8		1.3		10.8	9.5	4.1	5.4		2.7			
Pr. 3	6.8	5.4	1.3	1.3	1.3	4.1	1.3	6.8			2.7			
TOT	6.8	16.2	1.3	2.6	1.3	14.9	12.1	10.9	9.5	1.3	8.1	1.3	9.5	4.1

Table 3. Distribution (%) of the learners' van Hiele levels of reasoning and SOLO levels of answer.

5. A Way Forward

We are entering a period when new approaches, incorporating strong philosophic positions, are emerging in the teaching and learning of mathematics. In many of these, learning mathematics is seen as much more than an acquisition of sequential succession of elements of knowledge. Such frameworks encourage the teacher to consider multiple ways for their learners to acquire knowledge. They also place an emphasis in teaching on learners being able to demonstrate that they understand the mathematics they meet by their ability to use and apply their mathematical knowledge. An aim that is very consistent with the views of van Hiele and one he would refer to as the development of insight in mathematics.

For such reforms to be successful there needs to be a close affinity between learners' potential abilities, the teaching intentions of the mathematics curriculum, its philosophy, and the range and type of assessment practices used. If teaching is to promote many ways of learning, the associated assessment should also take many forms. The models of assessment derived from SOLO and van Hiele theories, as described above, offer a chance to provide alternative forms of assessment which are a balance to more traditional methods.

Many of the features of the two proposed models of assessment are self evident but some deserve mention.

1. Assessment should not only be an end in itself. Instead, it should also inform the instruction process. This formative role is far more sympathetic to emerging views of

what it means to know and learn mathematics. An important consequence of this is that assessment procedures should contain qualitative elements. Both van Hiele and SOLO models offer teachers the opportunity to address this issue, since they integrate assessment criteria and recommendations on the organization of teaching.

2. The focus of assessment in both models is on the structural level of learners' learning, i.e., attention is on how the knowledge gained by learners is organised in coming to a decision or solution. It is not about labelling people into boxes from which it is difficult to escape.

3. Both models allow the teacher (and possibly the learner) to identify the quality of responses and hence describe where a learner is currently performing. This focus is positive, being in terms of what a learner knows and how the learner uses such knowledge.

4. The consequences of being able to ascertain a learner's understanding in terms of a picture of overall cognitive growth is a valuable feature of both models. It also means that the teacher is better informed about where the next stage of instruction might most profitably commence and where it should head.

6. Conclusion

Both the van Hiele theory and the SOLO taxonomy are means of categorising learner's achievement. They build upon the current empirical research base into learners' understanding. The identification of discrete levels in both models suggests that there are a relatively small number of identifiable categories when analysing learners' responses to a given question and, as such, these groupings provide valuable information to teachers and learners. When this information can be interpreted within established frameworks, the potential benefits are increased and the validity of the process enhanced.

Quite often frameworks, such as those offered by the two models, are seen as belonging only to the domain of researchers. They are seen as appeasing a lust for theory in academics but as largely irrelevant to teachers, who see themselves battling numerous day-to-day problems. Discussion about philosophies and theories is often seen as esoteric and out of touch with what "is really going on in schools." Even those ideas that are seen as having relevant practical value can still be very difficult for teachers to implement. This can arise because:

(i) the method of assessment does not fit the specific requirements needed by the teacher for reporting;

(ii) it may be very time consuming;

(iii) it may require an understanding of the topic or the nature of learners' responses that is not accessible to the teacher;

(iv) the teacher may need to report on a broad range of issues; and,

(v) the techniques used to assess learners may not be appropriate for a school situation.

There is no attempt in this section to devalue these concerns or to minimise difficulties inherent in the use of qualitative models of assessment. Just as assessment practices should match teaching intentions, so should researchers try to accommodate practical issues that confront teachers. We have tried here to move a step forward in the direction of linking research findings to practice, by offering a way to integrate two models of assessment in geometry which have a strong qualitative basis. Further, we have suggested that both these models can play a vital role in providing support for teaching techniques which can enhance learner learning if assessment policies reflect this philosophy as a primary purpose for assessment.

It appears that the mathematics education community has reached a critical watershed in terms of the direction mathematics instruction and assessment should take. Questions that require answers include: What should be the objectives of teaching geometry in primary and secondary grades? Should curricula move from the traditional view of geometry as the field where logic and formalism have to be learned? How can we improve how we teach and assess? Are we experiencing a shift in focus in the teaching of mathematics away from a predominantly quantitative approach that values the application of routine algorithms and the completion of a number of similar exercises using identical methods in a lesson to one that values variety in questions and encourages multiple solution paths? If this shift is happening in teaching, should assessment practices follow the same trend? An approach to teaching geometry which is consistent with the van Hiele and SOLO models of assessment seems to offer the potential for learners to have ownership of the mathematics they encounter, to show insight, to use higher-order skills (as opposed to just rote learning), and to use their geometry in creative ways.

The task ahead is not easy and requires close cooperation among many of the stakeholders, including practising teachers and researchers. However, the goal, namely, more learners knowing mathematics (in particular, geometry) better, seems very much worth the effort. The van Hiele Theory and the SOLO Taxonomy, either taken together or separately, seem an appropriate catalyst for such work. Clearly, much more research needs to be done with teachers to fully develop the potential of both models for interpreting, assessing, and improving the teaching and learning of school geometry. Nevertheless, even as they

currently stand, these models offer considerable potential to improve both teaching and learning of geometry, and, in particular as a qualitative tool, the assessment of learners' understanding in geometry.

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