

ANALYSIS OF PROOFS PRODUCED BY UNIVERSITY MATHEMATICS STUDENTS, AND THE INFLUENCE OF USING CABRI SOFTWARE

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We present a research experiment designed to analyze the ways undergraduate mathematics university students solve geometry proof problems. On the one side, we aimed to identify the types of formal proofs produced by these students. The results of this experiment inform on previous categorizations of deductive proofs. On the other side, we aimed to observe the ways these students use dynamic geometry software to solve proof problems and to determine whether using the software influenced in some way their proofs or their processes of solving the problems.

INTRODUCTION

A very active research agenda in Mathematics Education is the one focusing on mathematical proof. Some research in it described different styles of proofs produced by students (either empirical or deductive) (Balacheff, 1988; Antonini, 2003; Harel & Sowder, 1998; Zack, 1997). Other research described the mental processes followed by students when they move from producing empirical to deductive proofs or the ways students progress from producing less to more elaborated kinds of proofs (Arzarello, Micheletti, Olivero, Robutti, & Paola, 1998; Kakihana, Shimizu & Nohda, 1996; Raman, 2003). Furthermore, many of these research paid attention to the claimed advantages of teaching based on dynamic geometry software (DGS) to help students in learning deductive proof (Jones, Gutiérrez, & Mariotti, 2000); The results from research are not conclusive in confirming such claim: A majoritary conclusion is that DGS environments help students to find the way to solve geometry proof problems, but some researchers prevent from possible obstacles in making students feel the need of making deductive proofs, due to the power of conviction of dragging explorations with DGS (Chazan, 1993, and Healy, 2000).

Most research in this agenda focused on primary and, mainly, secondary school students, with only a few research projects focusing on university students (Blanton, Stylianou, & David, 2003, and Weber, 2004, are two of the very few examples), so research based on these students is insufficient (Marrades & Gutiérrez, 2000, p. 121).

A key difference among secondary and mathematics university students is that the first ones still have to learn to use deductive reasoning, while the second ones usually already have learned to do formal proofs. In this context, an unanswered research question is to identify ways the mathematics university students would use DGS to produce deductive proofs as solutions of geometry proof problems. The research presented in this paper is based on a teaching experiment with undergraduate mathematics university students who had showed expertise in writing deductive

proofs. The students were asked to solve several geometry proof problems, in a paper-and-pencil environment in some cases, and in a Cabri environment in other cases. The main objective of our research was to look for differences in the solutions of the problems solved in each environment, like producing different types of proofs or managing in different ways the difficulties found while solving the problems. More specifically, the objectives of the research were:

- 1. To identify the types of proofs produced to solve geometry proof problems in a paper-and-pencil environment and a DGS environment, and to look for differences among the types of proofs produced in each environment.
- 2. To explore the influence of a DGS (Cabri) environment, respect to a traditional paper-and-pencil environment, in the students' solutions (management of the process of solving, and proofs produced).

THEORETICAL FRAMEWORK

The theoretical framework for this research has two components: Classification of proofs produced by students, and analysis of students' use of DGS.

Bell (1976) asked secondary school students to solve combinatorial proof problems so, not surprisingly, the proof types he described are based on the completeness of checking specific examples or making deductive arguments for specific sets of cases. A category particularly relevant to our study are the *complete empirical* proofs, consisting in checking a conjecture in the whole finite set of possible cases.

Balacheff (1988), based on experiments where secondary school students had to solve several proof problems, mainly paid attention to the different ways the students *selected* the examples used to write proofs. Relevant to our study are Balacheff's categories of *naive empiricism*, *crucial experiment* and *generic example pragmatic* (empirical) proofs, and *thought experiment conceptual* (deductive) proof.

Harel and Sowder (1998) complemented Balacheff's categories, since they worked with mathematics undergraduate university students, and they obtained detailed data for types of deductive proofs. Relevant to our study are the categories of *inductive* and *perceptual empirical* proof schemes and the categories of *transformational* and *axiomatic analytical* (deductive) proof schemes. Harel and Sowder coined the term *proof scheme* to refer to "what constitutes ascertaining and persuading" for a person (p. 244). To maintain a unique terminology in this paper, in what follows we use the term "proof" instead of "proof scheme" to refer to Harel and Sowder's categories.

Recently, several researchers have applied the above mentioned sets of categories of proofs to their own data, and they have found necessary to introduce some modifications for better matching to the data. For instance, Marrades and Gutiérrez, (1998, 2000) completed Balacheff's empirical categories by considering the ways students *used* the examples in their proofs and defining several subcategories. Similarly, Ibañes (2001) introduced several pairs of subcategories in Harel and Sowder's (1988) proof schemes to classify some types of proofs that didn't mach any of their categories: *Static/dynamic* perceptual proofs; *Authentic/false*, *a case/several*

cases, and *systematic/non systematic* inductive empirical proofs; *Static/dynamic*, *particular/general*, and *complete/incomplete* transformational analytic proofs.

The framework used in our research to classify the proofs produced in the experiment (synthesized in Figure 1) is an integration of elements taken from the previously mentioned sets of categories that we considered would be useful to classify our students' outcomes, plus some original subcategories. Due to space limitations, and because the cases analyzed in this paper are formal proofs, we only explain here in detail the classification of deductive proofs in the theoretical framework.

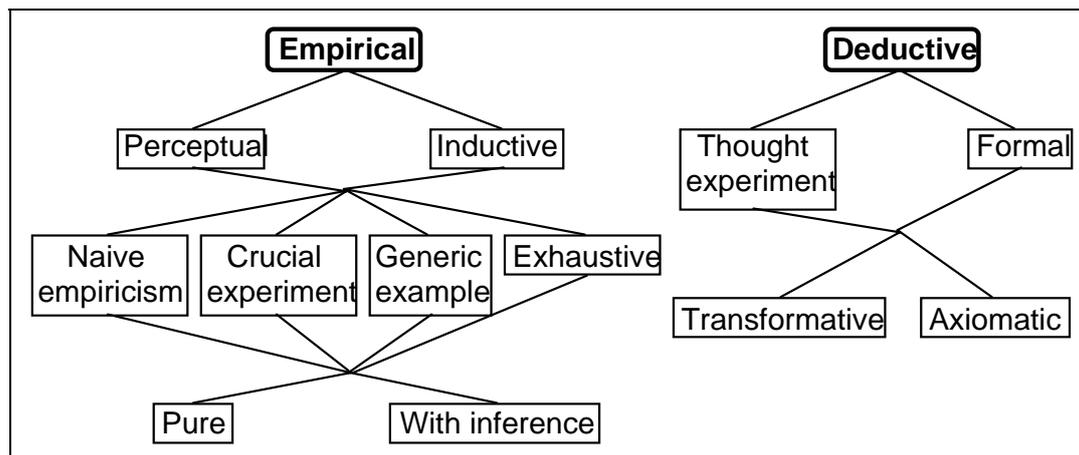


Table 1: Categories of proofs.

A empirical proof is *pure* when it only includes empirical verifications, and it is *with inference* when, apart from the empirical verifications, it includes some kind of reference to known definition, property, etc. To analyze deductive proofs, we consider two aspects of the proofs, the presence (or not) of examples in the proof, and the explicit use (or not) of elements of an axiomatic system: We differentiate, first, among *thought experiments*, when students use examples as sources of information and hints to write several steps in the proofs (Balacheff, 1988), and *formal* proofs, when students write the proofs without any support from the examples apart from, maybe, using a figure to visualize the elements involved in the problem; in this case, an example might provide the students with an initial idea of how to solve the problem, but then the example is not used any more to write the proof.

We differentiate two subclasses of deductive proofs: *Transformative* proofs, when they are based on mental operations involving goal oriented operations on objects and anticipation of the operations' results (Harel & Sowder, 1998, p. 258), and *axiomatic* proofs, when the proofs are based on elements of an axiomatic system (p. 273).

Respect to the use of Cabri by our students, the literature offers several elements that are pertinent to this research: The *ascending and descending phases* (Arzarello et al., 1998) and the *cognitive unity of theorems* (Boero, Garuti, Lemut, & Mariotti, 1996) may help to explain the relationships among empirical experimentations with Cabri and the production of a formal proof. The *modalities of dragging* (Arzarello, Olivero, Paola, & Robutti, 2002) may help to identify the aims of the students when they observe or transform a drawing on the screen.

THE EXPERIMENT

The sample was a class group of undergraduate students in their 4th or 5th year at the Faculty of Mathematics of the Univ. de les Illes Balears (Spain) studying a course on Euclidean Geometry. The 8 students in this class participated in the experiment. The students worked in 4 pairs, and they were asked to present only a joint answer to each problem. The experiment took place during the ordinary classes (October to January); There were two classes per week, about 100 minutes per class. The first classes were devoted to remind students' previous knowledge on Euclidean Geometry, to teach them some new concepts necessary for next classes, and to teach them to use Cabri II+. The rest of the course was organized as a problem solving setting jointly conducted by the teacher of the subject and the first author of this paper.

During the teaching experiment, the students solved 16 geometry proof problems. The statements of these problems didn't include any drawing. First the students solved 9 problems in their usual paper-and-pencil environment. Then they solved 7 problems in the Cabri environment. Each pair of students used a computer.

During all the experiment, both the teacher and the first author were present in the classes. Their role was to state the problems, to help students or answer their questions, and to manage the time of the classes. For each problem, there was a time for the pairs to work on the solution followed by a time to discuss the solutions obtained by the students and to institutionalize the new knowledge.

METHODOLOGY

The research was organized as a quasi-experiment, with one of the researchers acting as a participant observer. Different sources of data were used: In both environments we collected i) Researcher's field notes; ii) Students' written solutions; iii) Students' *self-protocol* – this is an innovative tool where, emulating the “think aloud” technique for oral problem solving, students were asked to write, during the process of solving each problem, notes commenting their way of solving the problem, the ideas discussed either accepted or rejected, their decisions, etc. – Furthermore, in the Cabri environment, we collected: iv) The files saved by students with the figures constructed; and v) The *record of session* files.

To analyze the information gathered, we have put together the written solutions, the self-protocols, and the record of session (for the Cabri problems); The other data (researcher's notes and Cabri files) were used when convenient. This gave us a detailed picture of the way each pair of students had worked to solve every problem.

DATA AND ANALYSIS OF RESULTS

We are presenting here, as representatives of the 16 problems solved by the 4 pairs of students, abridged versions of a pair of students' self-protocols and solutions to the paper-and-pencil problem 7 and the Cabri problem 13. Note that these students consistently use the verb “see” to mean “prove”.

Problem 7.

Let H be the orthocenter of triangle ABC . Let A' be the intersection of height AH and side BC . Let A'' be the intersection of height AH and the circle circumscribed to ABC , with centre O . Let r be the straight line parallel to BC through O . Prove that H is the image of A by the product of symmetries with axes r and BC , respectively.

1. We begin to draw Figure 1.
2. We see that $r \perp AA''$ because r is parallel to BC which is $\perp AA''$ (the height).
3. We want to see [prove] that $HA' = A'A''$.

The students draw point M' as intersection of r and AA'' .

5. We also have to see that $AM' = M'A''$.
6. To see it [conjecture 5] we draw Figure 2. We can see that B_1M_1 and M_1A_1 are congruent because [in triangles B_1OM_1 and A_1OM_1] two sides and the angle opposite to the longest side are congruent.
7. Therefore $AM' = M'A''$ perpendicular to r .
8. Now let's see that $HA' = A'A''$ in Figure 1.
10. $A'B$ is a side common to both triangles [$A'BA''$ and $A'BH$] and $\angle HA'B = \angle A''A'B = 90^\circ$.

Now we have to find another equal [congruent pair of] angle[s] to prove that the triangles are congruent and that $HA' = A'A''$.

The students drew another figure similar to Figure 1, and they labelled as B' the intersection of height BH and side AC .

11. We see that $\triangle B'HA \approx \triangle ACA'$ because both have a right angle and a common angle.
 Also $\triangle CA'A \approx \triangle CB'B$ because both have a right angle and $\angle HA'B = \angle A''A'B$ the common angle [C].
 $CBB' = CAA'' = \alpha$. Now, $A''BC = CAA'' = \alpha$ because both angles contain the arch CA'' .
12. Then, $\triangle A'A''B = \triangle A'HB$ because they have two equal angles and an equal side. $\Rightarrow A'A'' = HA'$.

The students have produced a correct transformative though experiment proof, since several drawings have guided them to write the proof in different key moments.

Problem 13.

Let ABC be a triangle. Let r and s be two non-parallel straight lines. For each side of ABC , draw a parallelogram having its sides parallel to r and s and having the given side of the triangle as a diagonal. Prove that the other diagonals of the three parallelograms are concurrent.

The students draw Figure 3 and drag the vertices of the triangle to check the truth of the statement. They also use the command *member?* to verify that the three diagonals

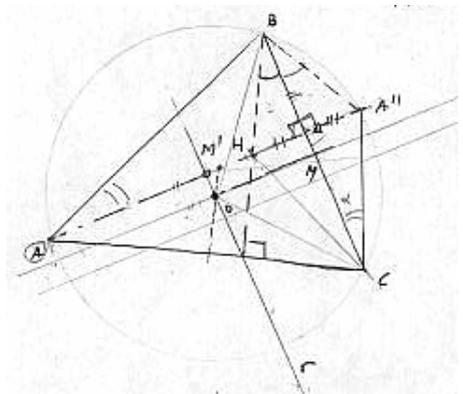


Figure 1.

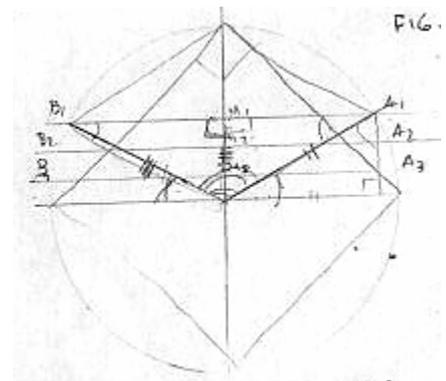


Figure 2.

meet at a single point. Now they try to prove the conjecture ad absurdum:

2. We draw another straight line. Let's suppose that this line is the diagonal and it intersects the two other diagonals in different points A, B. [Figure 4]
5. Let M_1 be the midpoint of the diagonals of parallelogram PQRS. Then it is the midpoint of side PR of the given triangle [ABC].
7. John suggests to change to the dual, three concurrent straight lines are three points of the same straight line in the dual. But we don't follow this way.

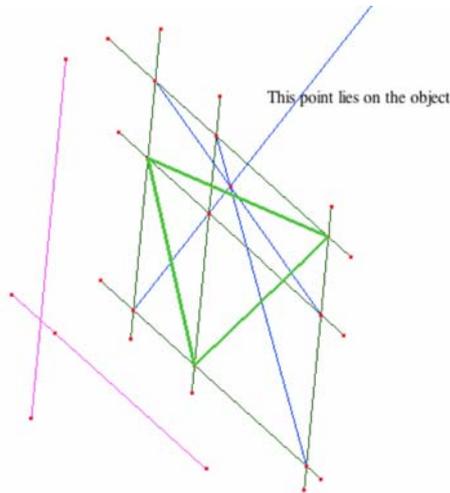


Figure 3.

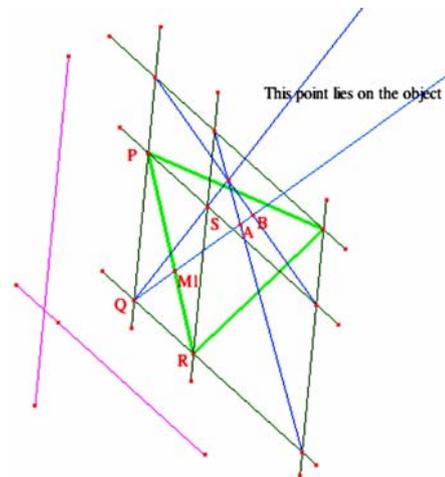


Figure 4.

8. We made a drawing on paper trying to do it wrongly to see the problem [Figure 5].
10. John suggests that we can see that the area of triangle ABE is zero, but it seems difficult, and we don't follow this way.

The students used the *Trace* in Cabri to see that point E moves along the diagonal when they dragged vertex R.

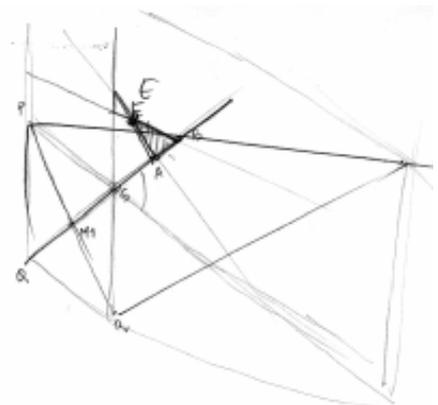


Figure 5.

12. We are looking at A and B, but we don't see any property characterizing them.
15. We look for similarities. (we don't pursue)
16. We should see that the diagonals are known cevians of some triangle.

A *cevian* of a triangle is a segment from a vertex to any point of the opposite side.

17. We create the parallel to a side through the opposite vertex [they do it for the three vertices of ABC] We check on the drawing that the diagonals don't have any relationship to these lines. [they delete the parallels]
18. We try a triangle whose vertices are intersections of the diagonals with the sides of ABC.
19. We check if they [the diagonals] are bisectors [they measure several angles], but they aren't.

The students remind the Ceva's theorem, they write the theorem's statement, and look for a way to prove it, but they don't know how to do it. Finally, they make another unsuccessful trial on the Cabri figure, and they stop working.

As a summary, the students made a sequence of transformative thought experiment trials, since they have permanently handled figures looking for valid conjectures, that they were not able to prove.

CONCLUSIONS

The comparison of the answers to the two problems by this pair of students lets us get some conclusions related to different aspects of the experiment:

- Classifying the proofs according to the categories mentioned in the Theoretical Framework section gives little information about high level mathematics university students' behaviour, since all the proofs produced by them were deductive, and most proofs will be transformative thought experiments, since the geometry problems are prone to induce such kind of proofs. Therefore, other directions of analysis are necessary to have a deeper picture of the students.
- The relationship among drawings (either in paper or DGS) and the production of proofs, that is the role of the figures/examples when the students are writing a proof, is quite subtle, and has to be observed carefully:
 - In a thought experiment proof, the examples guide the students' steps to write the proof. This has been evident in the protocols of the two problems analyzed here.
 - In a formal proof, the steps in the proof guide the drawing of examples. Their role is not to suggest ideas to the students, but to help the reader understand the proof.
 - In any deductive proof, an example may be the a source of ideas for students but, in a formal proof, the example is, at most, the source of the initial idea, and the subsequent process of writing the proof doesn't rest on the example any more.
- The DGS helps students to empirically identify and check conjectures (by dragging) but, when students are reasoning deductively, some times the DGS doesn't help them to find the way to a deductive proof. In these cases, using DGS doesn't mean any advantage over the traditional paper-and-pencil environment.
- The self-protocol has proved to be a useful methodological tool to get information on students' activity, since it has let us to track their actions, both successful and unsuccessful, and decisions.

References

- Antonini, S. (2003). Non-examples and proof by contradiction. In N. A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th PME International Conference* (Vol. 2, pp. 49-56).
- Arzarello, F., Micheletti, C., Olivero, F., Robutti, O., & Paola, D. (1998). A model for analysing the transition to formal proofs in geometry. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd PME International Conference* (Vol. 2, pp. 24-31).
- Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *Zentralblatt für Didaktik der Mathematik*, 34(3), 66-72.
- Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216-235). London: Hodder & St.

- Bell, A. W. (1976). A study of pupil's proof-explanations in mathematical situations. *Educational Studies in Mathematics*, 7(1), 23-40.
- Blanton, M., Stylianou, D. A., & David, M. (2003). The nature of scaffolding in undergraduate students' transition to mathematical proof. In N. A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th PME International Conference* (Vol. 2, pp. 113-120).
- Boero, P., Garuti, R., Lemut, E., & Mariotti, M. A. (1996). Challenging the traditional school approach to theorems: A hypothesis about the cognitive unity of theorems. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th PME International Conference* (Vol. 2, pp. 113-120).
- Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, 24(4), 359-387.
- Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education* (Vol. III, pp. 234-283). Providence, USA: American Mathematical Society.
- Healy, L. (2000). Identifying and explaining geometrical relationships: Students interactions with robust and soft Cabri constructions. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th PME International Conference* (Vol. 1, pp. 103-117).
- Ibáñez, M. (2001). *Aspectos cognitivos del aprendizaje de la demostración matemática en alumnos de primer curso de bachillerato*. PhD dissertation. Valladolid, Spain: Universidad de Valladolid.
- Jones, K., Gutiérrez, A., & Mariotti, M. A. (Eds.) (2000). Proof in dynamic geometry environments. *Educational Studies in Mathematics*, 44(1/2).
- Kakihana, K., Shimizu, K., & Nohda, N. (1996). From measurement to conjecture and proof in geometry problems. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th PME International Conference* (Vol. 3, pp. 161-168).
- Marrades, R., & Gutiérrez, A. (1998). Organizing the learning in a Cabri environment for a journey into the world of proofs. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22th PME International Conference* (Vol. 4, p. 276).
- Marrades, R., & Gutiérrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics*, 44(1/2), 87-125.
- Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, 52(3), 319-325.
- Weber, K. (2004). A framework for describing the processes that undergraduates use to construct proofs. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th PME International Conference* (Vol. 4, pp. 425-432).
- Zack, V. (1997). "You have to prove as wrong": Proof at the elementary school level. In E. Pehkonen (Ed.), *Proceedings of the 21st PME International Conference* (Vol. 4, pp. 291-298).