Design criteria of proof problems for mathematically gifted students

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Mathematically gifted students have needs that deserve to be considered by the mathematics education research. One of these is learning proof and the incidence of dynamic geometry software in such a process. In this document we present some considerations for the design of 3-dimensional geometry problems that contribute to achieve this goal. We rely on hypothetical learning trajectories and a characterization of the types of proofs to show how dynamic geometry software can favour the learning of proof while solving problems based on 3-dimensional geometric concepts and properties.

Keywords: mathematically gifted students, proof learning, dynamic geometry software, design of problems, 3-dimensional geometry.

INTRODUCTION

Nowadays it's common to find students with different mathematical abilities in any classroom, including some mathematically gifted students (MGS) (Benedicto, Acosta, Gutiérrez, Hoyos, & Jaime, 2015). MGS have a mathematical ability higher than average students with the same age, grade or learning experiences. Despite this, many teachers do not recognize that MGS require special attention, they believe that MGS learn easily on their own (Jaime, Gutiérrez, & Benedicto, 2018). Research has shown that mathematical talent, like any other skill, must be fostered through experiences, appropriate teaching, and challenges. This leads to investigate MGS's mathematical thinking processes and the way these students process and assimilate new mathematical ideas (Dimitriadis, 2010).

Research on the understanding of mathematical proof and the development of proving skills is a lively field of mathematics education. Research has shown that, in a proving process, there is an epistemological discontinuity between the phases of identification of a conjecture and elaboration of its proof. However, the ways in which dynamic geometry software (DGS) may influence aspects of proving, such as exploration, conjecture, and explanation, make this resource a strong mediator between those phases (Sinclair & Robutti, 2013). Considerable research efforts have been made on designing DGS environments based on plane geometry to teach proof and deductive reasoning (Sinclair & Robutti, 2013; Marrades & Gutiérrez, 2000). The recent availability of 3-dimensional (3D) DGS offers a new context for teaching and learning proof, with differential characteristics, that need to be explored. Research on 3D DGS environments is just starting; particularly, the design and analysis of 3D DGS environments based on space geometry and focusing on the learning of proof require specific attention, since research on this topic is scarce.

The design of a teaching sequence based on problem solving requires anticipation of possible students' behaviour and outcomes. The hypothetical learning trajectories (Simon & Tzur, 2004) are an efficient tool for the achievement of this goal. This is particularly true with MGS.

The objective of this document is to present a 3D DGS environment based on a sequence of space geometry proof problems aimed to promote the learning of proof by MGS. Such an experimental environment can provide useful information about MG students' processes of reasoning and can serve as an inspiring example to prepare sequences of space geometry problems based on 3D DGS to teach proof. We present an ongoing research, since, at the time of writing this document, we are completing the design of the experimental setting, to start the experiments later. We first present the theoretical framework supporting the environment, consisting of a classification of the types of proofs produced by students and the construct of hypothetical learning trajectories, as the organizer of the sequence of problems. Then, we present and analyse some of the designed activities.

THEORETICAL FRAMEWORK

Hypothetical learning trajectories and digital technologies

Hypothetical learning trajectories (HLT) are a construct for the design of mathematical instruction and conceptual learning (Clements & Sarama, 2004; Simon, 2014). A HLT involves three components: i) a goal about students' learning, ii) a set of mathematical problems that are expected to lead to the stated goal, and iii) a hypothetical learning process, i.e., an expectation on the way students' thoughts and understanding shall evolve when they engage with the designed problems.

Simon (2014), echoing other authors, mentioned that this framework allows to describe students' thinking and learning in specific mathematical domains. It contemplates projection of routes, through mathematical problems, that promote mental processes and higher level of mathematical thinking. Adopting a HLT requires a clear learning goal and awareness that common aspects are recognized in ways of learning by students. It is also necessary to recognize that HLTs are permeated by opportunities emerging throughout the designed instruction, since their hypothetical character gives rise to the possibility that teachers modify aspects of the intervention when they consider it necessary (Simon & Tzur, 2004). In a HLT, learning goals are seen as a guide and the point to reach, hence they provide elements for the selection of problems and contribute to build the hypothetical learning process (Simon & Tzur, 2004). According to these authors, this reveals a relationship between the last two components of the model, since problems are selected considering a hypothesis about the learning process, while the learning process is conceived through the problems selected.

Sacristan et al. (2010) studied nuclear ideas of HLT in the light of digital technologies and their impact on learning. For them, student's learning varies and takes different forms according to the situations proposed to them and the tools involved, DGS in our case. This recognition leads to deep conceptual levels not commonly reached in the school context, because interaction between students and digital technologies promotes transitions from particular to general, concrete to abstract, intuition to formalization, among others. Despite this, mathematics education research on HLT with digital technologies, in particular with 3D DGS, is in a primary state of development.

Characterizing the types of proof

Some researchers have attempted to recognize students' conceptions of mathematical proof and what is convincing for them, although they focused on particular aspects and left others aside. Marrades & Gutiérrez (2000), based on Balacheff (1988) and Harel and Sowder (1998), propose an analytical framework allowing a broad understanding of students' actions and productions when they solve proof problems. For Marrades and Gutiérrez, the term *proof* encompasses reasons given to convince someone about the truth of a mathematical fact. This model allows analysing all activity performed by students when they generate a conjecture and establish a way to prove it. The model contemplates two categories, empirical and deductive proof.

Empirical proofs take examples as the main element of conviction. Observing regularity in different cases leads students to establish a conjecture and prove it based on those examples. Examples can be used to prove a conjecture in different ways: in a perceptual or intuitive way, by choosing examples without any specific planning (*naïve empiricism*). A carefully chosen special case can be used to verify a property and consider it true in general terms (*crucial experiment*). A specific example can be selected as a representative of the family it belongs to and used to identify abstract properties after its observation and handling (*generic example*). In *deductive proofs*, a decontextualization of the arguments involved takes place. Generic aspects of the problem, mental operations, and logical deductions are used to organize proofs, so conjectures are deductively validated. Examples may be used as a help to organize arguments, but specific characteristics of the examples are not part of the proof. Deductive proofs can be organized and supported by specific examples (*thought experiment*) or based on abstract mental operations, without specific examples, and pertinent mathematical definitions and properties (*formal proof*). Marrades and Gutiérrez (2000) consider that these categories allow evaluating the improvement or changes of students' proof skills across a learning process.

CONSTRUCTION OF A HYPOTHETICAL LEARNING TRAJECTORY

The goal of this HLT is to favour the learning of proof by Spanish MGS studying lower secondary education (11-14 years old). In addition to the ordinary schooling, these students have participated in programs of attention to giftedness (AVAST) and mathematical talent (ESTALMAT). As students are not in the same grade or school, the teaching experiments are organized as individual clinical interviews.

GeoGebra is the DGS environment to solve problems. We have created 22 geometric problems, asking to make a construction and prove its correctness, based on objects and properties related to spheres, lines, planes, parallelism, equidistance, perpendicular bisectors, mediator planes, tangency, etc. Students, through dragging of points and mobilizing perceptual clues, should discover mathematical ideas that will be used later in other problems. To benefit from the integration of 2D and 3D representations in GeoGebra, some problems ask to explore a property of 2D figures and then work on the corresponding property of 3D figures. For instance, when studying the equidistance between points in space, it is possible to start working in 2D with perpendicular bisectors and circumferences and then ask students to extend their findings to bisector planes and spheres in 3D. This organization of problems should allow students reaching a deeper level of understanding by integrating and articulating different objects of 2D and 3D geometry.

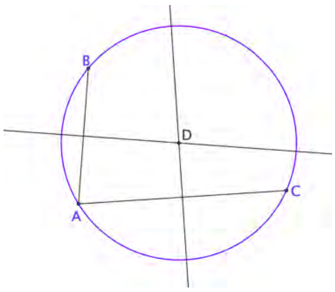
Regarding learning to prove, construction problems offer an opportunity in which students must use the tools provided by the software, as well as the geometric relationships learned from previous problems, in order to construct objects with some properties associated to equidistance and prove that the constructions are correct. In this sense, transit through various equidistance relationships, knowledge and gradual mastery of various software tools, as well as geometric relationships, configure a scenario in which students have availability of more theoretical and instrumental elements to solve new problems. Hence, we expect that the teaching sequence will induce an increment in MGS's deductive abilities. Teacher's role becomes relevant, since he has the possibility of talking to the students and asking them questions based on their productions, to induce them to express ideas in a higher level of proving.

As can be seen, this instrumental and conceptual integration combines objects of 2D and 3D geometry, as well as a non-basic knowledge of GeoGebra tools, in situations that require making particular geometric objects and proving that the construction works. Such an orchestration is not usual in conventional curricular configurations at this school level, so we consider that these problems are challenging for MGS. We present two problems to illustrate two moments of the HLT designed.

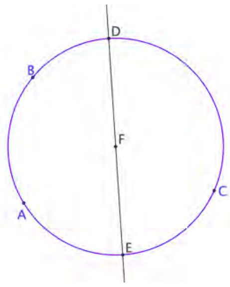
Two problems in the sequence

> Problem 6: Open GeoGebra and activate the 2D view. Construct three points, A, B, and C. Use the *Circumference by three points* tool to construct the circumference that contains the three points.
>
> Determine the centre of the circumference shown on the screen. How do you guarantee that the construction is valid? Do you think that this point is unique? How could you justify your answer?
>
> Open the 3D view and close the 2D view. Construct a point M which is at the same distance from A, B and C, but is not located in the same plane as these points. How could you guarantee that M is at the same distance from the other three points? Do you think that M is unique? If so, why do you think so? If not, what property do points M have?

Problem 6 has two parts. First, the centre of the circumference determined by three points has to be found and the construction has to be justified. Before solving this problem, students will have solved another problem and learned that perpendicular bisector is the locus of points equidistant from two fixed points, so now students will be able to find the intersection of the perpendicular bisectors of two pairs of given points and use that property to prove that this point is the centre (Fig. 1a): the intersection of the perpendicular bisectors is the centre of the circumference because each perpendicular bisector contains the points that are equidistant from the two points that determine it, and their intersection is a point simultaneously equidistant from the three points. Another way to determine the centre is by using the perpendicular bisector of only one pair of the given points. This line determines a diameter of the circumference, so its midpoint is the centre (Fig. 1b).



a)



b)

Figure 1: Finding a point equidistant from three given points in 2D

Those two solutions illustrate deductive proofs. However, MGS may also produce empirical types of proof. For example, students may create a point and place it on the centre of the circumference with the help of the distances from this to points A, B and C (*naive empiricism*). Next, the problem asks about the uniqueness of this point. We expect an affirmative answer to this question. One way to support it is to construct any point and determine the distances from it to points A, B and C, to show that the only point equidistant from them is the centre (*experiment crucial*). If the centre of the circumference is the intersection of two perpendicular bisectors, it is possible to guarantee its uniqueness because the intersection of the two lines is unique (*thought experiment*).

The second part of the problem is based on the construction made in 2D, now seen as part of the 3D space. Students are asked to create a point M at the same distance from points A, B, and C, but not located in the plane of these three points. We believe that students may produce quite diverse approaches, but having in common the combined use of the dragging of point M and the distances from it to points A, B, and C (Fig. 2). When point M is located fitting the condition, dragging it vertically will allow students to find other solutions. A proof of this result may be based on showing that distances are equal when the point is dragged vertically (*naive empiricism*) or involve geometric objects such as the sphere with centre M that contains any of the other points and note that this kind of dragging does not affect that points A, B and C always belong to the sphere, so the equidistance is conserved (*generic example*). We do not expect a deductive justification because it needs an element that is not known by students, namely perpendicularity between lines and planes, which is the learning goal of the last question in this problem.

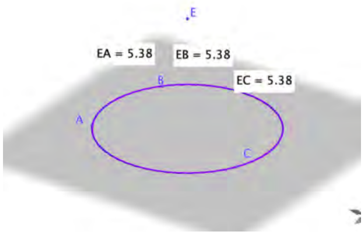
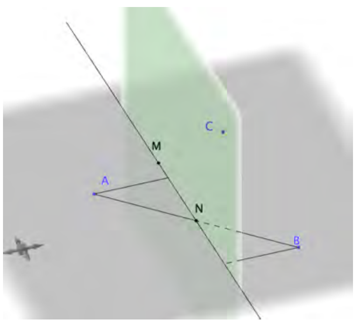
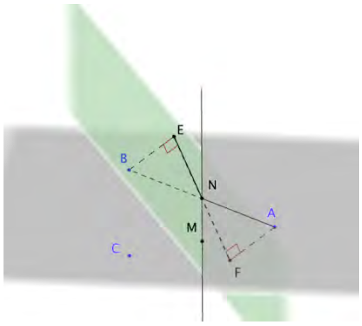


Figure 2: Finding a point equidistant from three given points in 3D

> Problem 11: Open GeoGebra and activate the 3D view. Construct three non-collinear points, A, B, and C on the base plane (grey). Construct a plane equidistant from the three points. Is this plane unique? What property does this plane satisfy, in addition to being equidistant from the three points? Explain why the property is true.

a) 

b) 

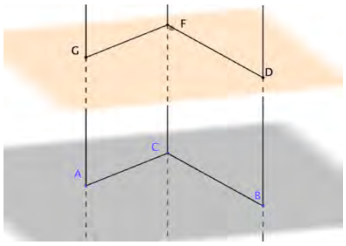
c) 

Figure 3: Finding an equidistant plane

In a previous problem, students will have constructed in 2D a line equidistant from three non-collinear points and they will have learned that the line is determined by two of the midpoints between A, B, and C. Problem 11 has two solutions. A solution can be formulated with the help of the mentioned property, that is, by constructing the line MN, where M and N are midpoints of A - B, and A - C respectively. Then, a perpendicular plane to the plane containing A, B, and C is constructed, which also contains the line MN (Fig. 3a). In this case, the proof of such a result will require constructing perpendicular lines to the new plane through A, B and C, as well as segments AB and AC. As these segments determine congruent triangles, compliance with the requested property in the problem can be proved. Students will be expected to recognize that any plane containing the line MN satisfies the stated property (Fig. 3b). In this case, the proof is similar to the previous one.

Another solution is any plane parallel to the plane determined by A, B and C. In this case, the proof can be based on constructing the perpendicular lines to the planes through A, B, and C (Fig. 3c). Since these lines are parallel, and their points of intersection with the two planes (A, B, and C; G, D, and F) determine pairs of parallel segments, three parallelograms are formed, so their opposite sides are congruent, in particular, AG, BD, and CF are congruent, thereby reaching the desired result. These solutions are deductive proof where some examples are used to help to find the properties necessary to build the proof (*thought experiment*). However, it is also possible to develop different types of empirical proofs based on specific examples and numerical values of distances.

CONCLUSIONS AND CONSIDERATIONS

We have created a HLT which has served as an organizer for designing a sequence of problems of spatial geometry in a 3D DGS environment, whose objective is to support learning of proof by lower secondary school MGS. On the other side, the Marrades and Gutiérrez (2000) framework allows us to characterize proofs produced by the students and recognize their progress from empirical to deductive proofs along the sequence. We consider that this is an advance in the research on the use of HLT to design teaching sequences of space geometry problems in 3D DGS environments to promote MGS's learning of mathematical proof.

Nature of learning is related to the means through which it occurs (Sacristan et al., 2010). In our case, GeoGebra provides the possibility of exploring simultaneously related concepts and properties in 2D and 3D configurations (e.g., perpendicular bisector and mediator plane), as well as allowing MGS explore geometric contexts usually not studied at school level. This allows students to integrate different geometrical objects and get experience to move from empirical and perceptual proofs to deductive and formal ones (Sinclair & Robutti, 2013).

The HLT we have created offers the possibility of integrating concepts and properties from school geometry and others that are not studied in secondary school. This integration, framed in situations that demand the construction of specific geometric objects, as well as the proof of the validity of such constructions, produces contexts that are challenging and interesting for MGS, due to their non-routine nature. Furthermore, GeoGebra allows MGS to explore the proposed situations, thus favouring their creativity and problem-solving strategies. These are aspects that, in line with Dimitriadis (2010) and Jaime and Gutiérrez (2017), make this proposal a valid option to promote the talent of MGS, with 3D DGS being a relevant part of it.

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