LEARNING OF PROOF BY MATHEMATICALLY GIFTED STUDENTS: AN EXPERIMENT WITH 3-DIMENSIONAL GEOMETRY

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Abstract. *One of the main mathematical activities is proving. We present a teaching experiment aimed to improve mathematically gifted students' abilities of proving. It is based on the solution in a dynamic geometry environment of construction problems, in which equidistance plays a central role. The problems ask for transitions between 2-dimensional and 3-dimensional geometries, where the elaboration of analogies between properties in plane and space supports the construction of meanings in 3-dimensional geometry and provides students with elements to elaborate deductive proofs. As an example of mathematically gifted students' outcomes, we present the solution to a problem by a student.*

Key words: 3-dimensional geometry, analogy, construction problems, dynamic geometry environment, learning of proof, lower secondary school, mathematically gifted students.

INTRODUCTION

Mathematically gifted students (MGS hereafter) have a mathematical ability higher than average students with the same age, grade or learning experiences. Despite this, many teachers do not recognize that MGS require special attention, since MGS do not face the same learning difficulties as their peers (Jaime & Gutiérrez, 2017). Research has shown that mathematical giftedness, like any other skill, must be fostered through challenging experiences and appropriate teaching. This leads to investigate MGS's mathematical thinking processes and how these students process and assimilate new mathematical ideas (Dimitriadis, 2010). The non-exhaustive review of specialized literature on MGS carried out by Jaime and Gutiérrez (2017) showed a progress in some research areas, like problem solving, identification, characterization, and analysis of learning processes, but there is a lack of research in the topic of MGS's learning to prove.

The learning of proof in geometry has been greatly favored by the arising of dynamic geometry environments (DGE hereafter). It has been shown that DGEs positively influence aspects of proving like exploration, conjecture, and justification (Sinclair & Robutti, 2013). Experiments in this direction have been made mostly in contexts of 2D geometry (Sinclair & Robutti, 2013), but the development of research in contexts of 3D geometry is not similar (Gutiérrez & Jaime, 2015). We argue that 3D DGEs offer a new context for teaching and learning proof, having differential characteristics that need to be explored. Research on 3D DGEs is just starting; particularly, the analysis of 3D DGEs focusing on the learning of proof requires specific attention, since research on this topic is scarce.

We present an exploratory research based on an experiment aimed to promote the learning of proof by lower secondary MGS. It is centered on 3D geometry and based on a DGE. The research objectives of this paper are i) to analyze the role of analogy in solving 3D geometrical construction problems and producing the corresponding proofs by MGS in a DGE and ii) to show the role of the DGE in students' solutions and proofs.

THEORETICAL BACKGROUND

Construction problems and learning to prove within a DGE

Mariotti (2012) argue that DGEs support the learning of proof. In our study, we placed emphasis on the learning of proof in the context of construction problems. Solutions to *construction problems* consist of (i) creating in the DGE a figure having some properties that remain constant under dragging and (ii) explaining the procedure used to construct the figure and validating it (Mariotti, 2019). We consider a *proof* as a mathematical argument, both empirical and deductive, aiming to convince of the validity of a mathematical statement. In our case, students produced proofs to convince of the validity of a construction they had made in the DGE. The tools used by students to construct geometric objects on the screen provoke informal meanings by suggesting dependency relationships, that students may confirm by dragging objects. Furthermore, the tools are related to theoretical elements of Euclidean geometry that could help students to create proofs of the validity of their constructions (Mariotti, 2012). Since DGEs embody systems of theoretical relationships, solving construction problems leads students to use the possibilities that the software offers them and the underlying logical system. Therefore, geometric constructions also have a purely theoretical nature, so the solutions may involve proving a theorem to validate them, so solving construction problems in a DGE can make students evoke the theoretical meaning of the constructions embodied in the tools (Mariotti, 2019).

Mathematically gifted students: characterizing their behavior

Research on MGS, besides informing about attention to these students, as we showed in the introduction, has also provided elements to identify their behavioral traits, mainly through the observation of their activity when solving problems, since there is an established connection between problem solving and mathematical creativity. Among these traits, researchers emphasize mathematical creativity and its components fluency, flexibility, and originality (Leikin & Lev, 2013). *Fluency* is observed through the generation of different mathematical ideas that lead to the exploration of a situation, the formulation of approaches, and multiple answers to a mathematical problem. *Flexibility* is present in the generation of new approaches when solving a problem. *Originality* refers to the smaller frequency of an answer given to a problem, compared to that of other answers given by different individuals to the same problem (Leikin, Koichu & Berman, 2009).

Analogy: a way to extend mathematical ideas

Two objects or systems are analogue if, based on partial similarity, the respective entities are similar in other respects as well (Fischbein, 2002). According to Richland and Simms (2015), analogical reasoning is a thinking "process of representing information and objects … as systems of relationships and drawing connections across these systems" (p. 179). When two systems are analogous, it is possible to establish a relational statement or structure in one of them and apply it to the other. In this context, the analogical reasoning leads to the organization of available information in sets of relationships and the recognition of communal aspects between different systems. Then, establishing analogies is a way of learning. In our case, the students developed meanings in a 2D geometry (source system) and they extended those meanings to 3D geometry (target system). Furthermore, students identified similarities/differences and created new 3D objects or relationships.

METHOD

3D geometry is not frequently taught in the school levels due to the complexity of geometric relationships or graphic representations in the 3D configuration (Mammana et al., 2012). According to those authors, some proposals to address this problem have used analogy and DGE to establish links between 2D and 3D geometries. In our experiments, we have used analogy between 2D and 3D geometries relying particularly on the equidistance relationship. The study of equidistance through construction problems provides an opportunity to exhibit solution strategies guided by perceptual and theoretical aspects, as well as the possibility of recognizing similarities and differences between the 2D and 3D configurations handled and the geometric relationships behind them.

We have designed a research experiment where several MGS solved a sequence of 18 problems in a DGE based on GeoGebra. The first author acted as teacher. Due to the pandemic restrictions, the experiment consisted of several virtual individual clinical interviews with each student. The interviews were video recorded. We consider the students participating in our experiments as MGS because they had participated for several years in workshops to support mathematical giftedness (AVAST and ESTALMAT).

In the problems we posed, students had to base their constructions and the posterior proofs on properties related to equidistance. Most objects and properties involved (circumference, sphere, perpendicular bisector, bisector plane, parallelism, perpendicularity, and congruent triangles) have to do with equidistance.

When students finished the solution of a problem, the teacher made questions to know their solution strategies, their reasons for considering that the construction made solved the problem, to clarify ideas expressed by them, and to provide help if it was necessary. For the analysis of the data, we selected those episodes in which students' activity showed evidence of creativity, relying on analogical reasoning and facilities from the DGE.

The first problems we posed were aimed to let students have a first contact with geometric objects in 2D or 3D, or remind them. In the 2D configuration, students dragged some geometric objects and discovered some properties or other geometric objects. Subsequently, a related 3D problem was posed, aimed to bring students closer to the corresponding 3D geometric objects and properties. In the 3D constructions, students had also to drag points to satisfy some equidistance relationship.

After those problems, we posed *construction problems*. They required the construction of objects having specific properties in 2D or 3D, in which equidistance was present. These problems could ask for the same object to be constructed in both 2D and 3D spaces (e.g., an equilateral or isosceles triangle) or the construction of a particular geometric object in 2D (e.g., the center of a circle) and then its corresponding one in 3D (e.g., the center of a sphere). In this type of problems, it was necessary to use geometric objects as tools for the construction of other objects or certain configurations. This required knowing which properties of the objects would be useful in solving the problems.

AN EXAMPLE: EQUIDISTANCE IN 2D AND 3D GEOMETRIES

We present the analysis of a case study, drawn from the broader research experiment described, in which we trace the activity of a student named John (pseudonymous) to solve one of the construction problems. John was a 14-year-old student in grade 4 of secondary school (he had been advanced one grade due to his giftedness). He knew and used confidently the main construction tools of GeoGebra 2D, but his experience with GeoGebra 3D and the 3D geometrical objects involved in the problems was scarce.

The construction problem that we present here was the eleventh problem in the sequence. The objects and geometric relationships necessary to solve the problem were known to John at this point. The statement of the problem is:

- Open GeoGebra and activate the Graphics view. Construct three non-collinear points, A, B, and C. Construct a line at the same distance from these points.
- Open the 3D Graphics view and close the Graphics view. Construct a line, not contained in the gray plane, equidistant from the three points.

In the 2D configuration, John created the non-collinear points A, B and C. He constructed the line AB, the midpoint between B and C, named D, and the line parallel to line AB containing D (Figure 1a). He justified the correctness of the construction by stating that *a line that is at the same distance from A and B must be parallel to line AB… otherwise, the distance from A to the line would be greater or lesser than the distance from B to the line… by the definition of parallel lines the distance is constant*. John had in mind the infinity of parallel lines to line AB, as well as the need for the parallel line to be also equidistant from C. In this regard, John explained that *what I have done has been to draw the midpoint between B and C, and the parallel line to AB containing that midpoint, which will be equidistant from B, C, and A*.

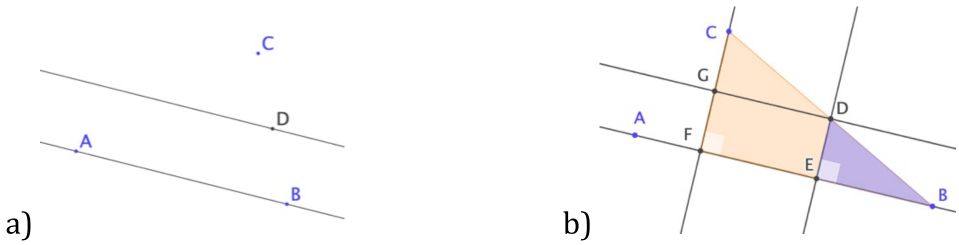

Figure 1. Construction of equidistant line from three non-collinear points in 2D configuration.

John deductively proved that his construction solved the problem: *To explain this, what I am going to do is draw… I mean, there is a simpler explanation, but I see it like this. I am going to draw two perpendicular lines from D and C to AB. Then two right triangles are formed, BDE and BCF*. Next, he justified the similarity between triangles

CFB and DEB (Figure 1b). Therefore, the distance CG is equal to the distance GF, each corresponding to the distance from the line GD to points C, B and A.

In the 3D view, all points and lines created in 2D were visible. John's first attempt involved the bisector planes between A and B, and B and C, and the line intersection of these planes (figure 2a). John proved his construction starting from the equidistance of each point of the bisector plane with respect to the two points determining it, to conclude that the intersection between the two planes must be equidistant from A, B, and C. In his explanation, he stated that *the bisector plane between A and B is perpendicular to the gray plane and the bisector plane between B and C is also perpendicular... so the intersection* [of the two bisector planes] *is also perpendicular to the gray plane. Any point on the line* [intersection between bisector planes] *that is not the intersection* [between gray plane and the line]*, the intersection, let's call it K, and a point of the blues* [A, B or C] *form 90°* [a right angle]. He added that point K is equidistant from points A, B, and C, so the line is equidistant from these points.

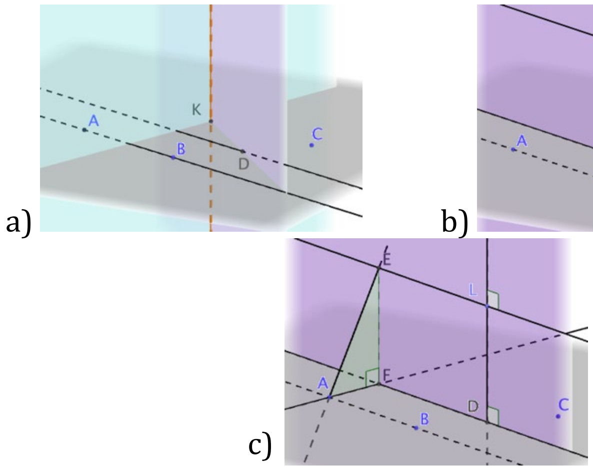

Figure 2. Construction of the line that is equidistant in the 3D configuration

The teacher asked about the possible existence of other solutions. John answered affirmatively, anticipating their characteristics: *the line that is the same distance from A, B and C... if I draw a parallel line to this line* [the solution in 2D] *through the plane perpendicular to the gray plane* [which contains the solution in 2D]*, then any of these lines is equidistant from A, B and C.* The teacher asked to construct these lines, so John created the configuration (Figure 2b). John explained the validity of his construction by making an auxiliary construction based on perpendicular to A, with respect to the two lines in the purple plane, which allowed him to obtain a triangle with vertex in A (Figure 2c). He explained that the same thing could be done with B and C, so he would obtain three triangles. In these triangles, there would be a right angle, because the perpendicularity between the planes, and two corresponding congruent sides, due to the equidistance between each parallel line and A, B, and C. Thanks to this and the congruence of triangles (SAS criterion), the line not contained in the gray plane is equidistant from A, B and C. Finally, John mentioned that this set of lines provided infinite solutions to the problem.

DISCUSSION AND CONCLUSION

We have presented results from a research experiment aimed to induce MGS's learning of 3D geometry and mathematical proof, in which the DGE and the analogy between 2D and 3D geometries are important components. An original contribution of our research is to demonstrate how the link between 2D and 3D geometries is established thanks to the central role of equidistance, construction problems, and the diversity of proofs that can emerge from students. However, further efforts in this direction have to be made.

As our research is based on a case study, it does not allow the results obtained to be generalized. However, these results provide evidence about how analogy-based problems induce students to display traits of mathematical giftedness. The activity carried out during the problem-solving exhibited traits of creativity and the use of analogical reasoning, supported by DGE in some cases.

Regarding creativity, as we have seen in this experiment, the diversity of strategies that students may offer to build the requested 3D object and the chain of theoretical properties with which each strategy was supported (including the 2D solution) is evidence of *fluency* and *flexibility* in those actions. Likewise, the reasoning employed to determinate the solution in the 2D configuration, combining geometrical properties, shows traits of *originality* in the solutions to the problem.

The DGE was used to represent anticipated ideas at the beginning of the solution. The construction of the lines or the proof of the results are examples of how GeoGebra allowed to materialize initial ideas. Although there was not exploration or fundamental use of the DGE tools to discover geometric properties, the 2D-3D integration provided by GeoGebra allowed the objects built in 2D to be used in 3D, which favored the extension by analogy of the ideas and properties used in 2D to construct the solution in the 3D space.

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