Construction of an equilateral triangle in the plane and space: an analysis from the theory of semiotic mediation

Camilo Sua, Angel Gutiérrez and Adela Jaime

Departamento de Didáctica de la Matemática, Universidad de Valencia, Spain; jeison.sua@uv.es, angel.gutierrez@uv.es, adela.jaime@uv.es

Dynamic geometry environments provide users with several tools to construct geometric objects. In the use of these tools, convergence of personal to mathematical meanings has been recognized, because the tools provided by the environment embody theoretical relationships of Euclidean geometry. In this document we analyze the productions of two mathematically gifted students when solving a construction problem in a dual 2-dimensional and 3-dimensional setting. Supported by the theory of semiotic mediation, we show the nature of the signs exhibited by the students and some difficulties that can occur when they move from a 2-D representation to a 3-D one.

Keywords: Theory of semiotic mediation, 3-Dimensional geometry, Geometrical construction, Dynamic geometry environments.

Introduction

There is wide recognition of dynamic geometry environments (DGE), their characteristics and the challenges to incorporate this resource in ordinary teaching and learning of mathematics (Sinclair et al., 2016). These authors inform on efforts by mathematics education researchers made in this regard. Despite this, many research has been carried out focusing on DGE for 2D geometry, but it is much scarcer the research on the use of DGE for 3D geometry (Gutiérrez & Jaime, 2015).

The relationships established among mathematics and DGE is of an epistemological nature, since it happens through a process of mediation (Bartolini-Bussi & Mariotti, 2008). In this process, mathematical objects are shared and thought through their representations. Technology offers possibilities for such representations, as well as a space for communication between student and teacher in which a shared language circulates (Noss & Hoyles, 1996).

The construction on DGE of geometric objects and the justification of its validity can be conceived as a route towards the identification of geometric properties in the representations offered by the software, since the available construction tools allow establish a semiotic relationship between (i) personal meanings caused by the activities of construction and verification by dragging, and (ii) theoretical aspects of geometry that underlie these constructions through the tools used, which support the validity of the construction (Mariotti, 2012). To analyze this relationship, a semiotic perspective about the students' mathematical activity while solving geometric construction problems is pertinent. We have adopted the *theory of semiotic mediation* (TSM) (Mariotti, 2019) to study the production, nature, and evolution of signs present in students' activity in a DGE. In this paper we present solutions to a problem asking to make a construction in 2-D and next to make the same construction in 3-D and justify that the constructions are correct. The research objective is to analyze the students' solutions to that construction problem to identify the signs and signs' meanings exhibited by the students while solving the problem, and the influence of the 2-D construction on the 3-D construction.

The Theory of Semiotic Mediation

The TSM is supported by the Vygotskian notion of *semiotic mediation*, as well as the role of technology (Mariotti, 2019). This theory explains the process of learning mathematical contents when an artifact is used to solve a task (Drijvers et al., 2010). When experts (teachers) use an artifact to solve a problem, they recognize mathematical notions through its use. However, when novices (students) use the artifact, they do not recognize immediately the mathematical meanings that emerge, but students see them linked to the artifactual context in which the mathematical contents are used (Mariotti, 2019).

A *sign*, from Pierce's perspective, is "anything" that represents "something" to "someone" in some aspect or degree (Bartolini-Bussi et al., 2012). The TSM interprets students' production of signs and the evolution of such signs, whose meanings move from personal to mathematical, close to the teaching objective (Mariotti, 2019). In this process, the use of specific tools by the teacher, together with her actions, to promote students' learning is recognized and has relevance (Mariotti, 2012).

The signs can be characterized according to the nature of students' productions (verbalizations, writings, gestures…) and the ways the artifact is used. Some signs are close to the activity done with the artifact and others are close to the aimed mathematical meanings. When student's explanations move from references to the use of the artifact towards references to the mathematical context, different signs are used, which mirror the state of students' learning process. We consider three types of signs (Bartolini-Bussi & Mariotti, 2008): *Artifacts signs* point the use of the artifact or to actions linked to it. *Mathematics signs* refer to mathematical elements of the context; they are related to mathematical meanings and are expressed by propositions that satisfy standards of the mathematical community. They constitute the achievement of the semiotic mediation process orchestrated by the teacher. *Pivot signs* are present in actions, carried out with the artifact, where specific instrumented actions and natural language referring to mathematical contents are involved; their polysemic nature implies that these signs are used to advance from artifactual to mathematical context.

Construction problems and learning to prove within a DGE

DGE support the learning of proof (Mariotti, 2012). In our study, special emphasis is placed on learning of proof in the context of construction problems; we consider a proof as a mathematical argument, both empirical and deductive, aiming to convince of the validity of a mathematical statement, in our case, the validity of a construction in the DGE. Construction problems consists of (i) creating in the DGE a figure having some properties that remain constant under dragging and (ii) explaining the procedure used to construct the figure and validating it (Mariotti, 2019). DGE tools allow the construction of geometric objects on the screen, which provoke personal meanings by suggesting dependency relationships, that may be confirmed by dragging some objects. Furthermore, the tools are related to theoretical elements of Euclidean geometry that could help students to create a proof of the validity of constructions (Mariotti, 2012). Since DGE embody systems of theoretical relationships, solving construction problems leads students to accept the possibilities that the software offers them and the underlying logical system. Therefore, geometric constructions also have a purely theoretical nature, so the solutions may involve proving a theorem to validate them (Mariotti, 2019).

According to Mariotti (2019), in some DGE tools we can recognize presence of (i) graphical representations of geometric objects having some known properties, which are useful for the construction of the objects thanks to the verification by dragging; and (ii) geometric properties that can be evoked, useful to support constructions and that are framed in a mathematical theoretical domain. Therefore, solving geometrical construction problems in a DGE can make students evoke the theoretical meaning of the constructions embodied in the artifact (Mariotti, 2019).

Methodology

The content of this paper is part of a doctoral research, based on case study methodology. We analyze mathematically gifted students' learning of mathematical proof in 3-D geometry with GeoGebra mediating their activity. We designed and implemented a sequence of 18 construction problems in 60-minute sessions, conducted by the first author. Students initially solved the problem and then discussed their results with him, who conducted the dialogue to justify of the results. The problems involved objects and properties of equidistance in the plane and the space, in a way that each problem provided instrumental and conceptual elements useful to solve subsequent problems. The students had to construct the geometric objects with GeoGebra and justify that the constructions were correct.

Four Spanish mathematically gifted students (11 to 14 years old) in grades 1 to 4 of secondary school participated in the experiments. Besides the ordinary schooling, the students had participated in programs of attention to general giftedness (AVAST) and mathematical giftedness (ESTALMAT). As students were in different school grades and had different previous knowledge, the teaching sessions were organized as individual clinical interviews.

We present a fragment of the solutions to the 7th problem by two students, Juan (14 years old in grade 4) and Jorge (11 years old in grade 1); the names are pseudonymous. This problem asked to create an isosceles and an equilateral triangle in GeoGebra 2D and next in GeoGebra 3D. We analyze the two students' solutions to show the similarities and differences. In their approaches to the solution of the problem, we identified whether their expressions alluded exclusively to actions carried out with GeoGebra (artifact sign), if they also included mathematical properties close to those expected (pivot sign) or if they were completely in the domain of mathematics (mathematic sign).

An example: creating an equilateral triangle in the plane and the space

Students had to create an isosceles triangle and then an equilateral triangle, both having a given segment as a side. The triangles had to be created first in GeoGebra 2D and then in GeoGebra 3D, where the given segment was not contained in the XY plane. In previous problems the students had learned to create and use in GeoGebra spheres, circles, perpendicular bisectors, and bisector planes. Here we analyze the construction of the equilateral triangles.

Juan's solution

Juan had constructed 2D isosceles triangles by using the given segment, AB, its perpendicular bisector, and the circle with center in B and AB as radius (Figure 1a). The teacher asked Juan to delete the constructed triangles and find the position of the third vertex of the requested triangle. Juan pointed at point E, one of the intersection points of the circle and the perpendicular bisector, as a solution (Figure 1b). He explained that *the distances BE and BA will be the same… Then, AB and AE* *will be the same… Then, the three* [segments] *will be equal and it will be equilateral*. In this answer, Juan did not evoke actions with GeoGebra, but only geometrical properties used in the construction and other properties logically derived from them, which shows the emergence of a mathematical sign.

Figure 1: Juan's construction of an equilateral triangle in 2-D

As it was not clear how Juan had deduced the congruence of the three segments, the teacher asked him for a more detailed justification. Juan justified it in a different way: *By symmetry, if I draw the circle with center A and radius AB, the intersection would be… E… Then, AE would be equal to AB. AB would be equal to BE, all sides would be equal*. In this new explanation, Juan evoked an auxiliary construction and properties of it allowing him to prove that the triangle is equilateral. This answer reveals the (mental) use of GeoGebra and mathematical properties, so now Juan showed a pivot sign.

The last answer led the teacher to make Juan note that he had not used the perpendicular bisector of AB, previously used by Juan to create an isosceles triangle (Figure 1a). Juan changed again his justification: *If I draw the perpendicular bisector of AE, it passes through B. As BA is a radius, it is equal to BE. And since E is in the perpendicular bisector of AB, then all 3 sides are equal, and the triangle is equilateral*. This new answer had the potential to support the validity of the construction and, like the previous justification, has the characteristics of a pivot sign.

In GeoGebra 3D, Juan first constructed the bisector plane of AB and a point C on this plane, to create an isosceles triangle ABC (Figure 2a). Then, he built the sphere with center B and point A (Figure 2b). Initially, Juan wanted point C to be at the intersection of the sphere and the plane, although later he expressed that he wanted point C to be *on the perpendicular bisector of AB and on the bisector plane*, a property that he re-stated as C *being at the intersection ... between the plane and the sphere, but at the midpoint*. These last ideas were not clear, so Juan continued exploring the construction, and he created the circle intersection between the sphere and the bisector plane (Figure 2c).

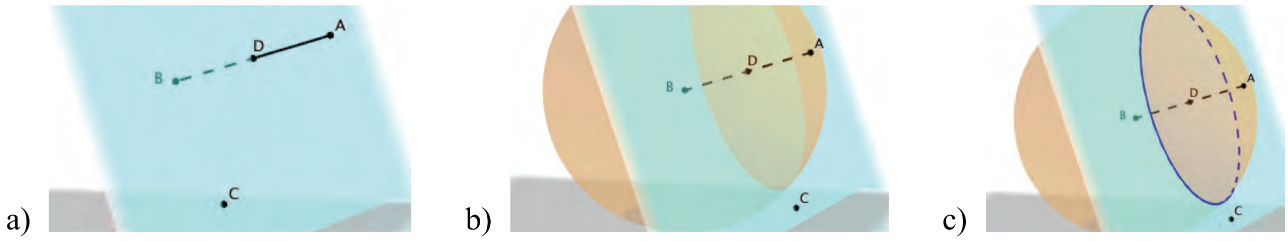

Figure 2: Juan's construction of an equilateral triangle in 3-D (first part)

Juan needed point C to belong to this circle, so he dragged C to the circle. Juan explained that point C *would have to be like here ... Or here* [while dragging C to two specific positions in the circle (Figure 3). Juan's specificity in his answer led the teacher to question these positions for point C. Juan considered it necessary because *there, C would be in the perpendicular bisector of A and B*. The teacher asked him about the possibility of placing C in another position, and Juan replied that *any point of the intersection serves*. To finish, Juan explained that C must be in the circle because *the distance from A to C will be the same as from C to B, because it* [C] *is in the bisector plane*. He completed this justification by saying that *since the sphere has the center in B, and radius BA and also BC, because C is at the intersection of the sphere and the bisector plane, then the three segments* [AB, AC, BC] *are equal*.

Figure 3: Juan's construction of an equilateral triangle in 3-D (second part)

By using the bisector plane of segment AB as reference and placing point C in the positions presented in Figure 3 indicate that, although using GeoGebra 3D, Juan used 3-D geometric objects as if they were in GeoGebra 2D. Juan's idea of using the intersection between the sphere and the plane suggests the recognition, albeit implicit, that the characteristic properties of the points of the 2-D circle used in the first part of the problem are also properties related to equidistance between the constructed points in the 3-D circle; hence this discourse shows a pivot sign, since it combines mathematical and artifactual elements. Additionally, the actions declared by Juan at the beginning of the second construction show that he knew the way to obtain the third vertex of the equilateral triangle given the 3-D configuration that he had on the screen. This leads to his actions serving a declared objective, which is clear from a mathematical point of view, as well as permeated by a deductive language. Therefore, this part of the solution evidences a mathematical sign.

Jorge's solution

Given segment GH as side, Jorge's construction of the 2-D equilateral triangle began with the construction of the circle with center in G and point H (Figure 4a). Jorge then created point I on the circle, segment GI and the circle with center at H and point G (Figure 4b). Jorge dragged point I to an intersection of the circles (Figure 4c), but he changed his mind: eliminated point I, constructed a new point I as one of the intersection points of the circles, and created the triangle GHI (Figure 4c).

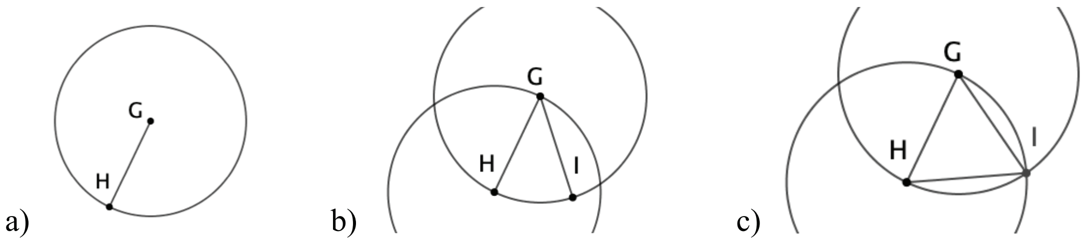

Figure 4: Jorge's construction of an equilateral triangle in 2-D

When the teacher asked Jorge about the validity of his construction, he replied: *the circles… were to copy segments, so I have used them so that all sides are the same length*. Jorge was asked about the choice of point I as the intersection of the circles, and he mentioned that any other point would be closer to one point (G or H) than the other.

To create the 2-D triangle, from the very beginning Jorge tried to use two circles having segment GH as their radii. So, the third vertex would then be one of the intersection points of these circles. Jorge first wanted to guarantee the congruence of segments GH and GI and then, with the help of the second circle, determine the place to which point I should be dragged so the triangle GHI would be equilateral. This strategy was modified, and the justification offered by Jorge was supported by the representation on the screen. His actions had been mobilized as reaction to what he observed on the GeoGebra 2D screen, for example the location of point I. Jorge did not use mathematical elements to justify the congruence of the segments, so these actions determinate a pivot sign.

In GeoGebra 3D, given the segment AB, Jorge created the sphere with center A and point B, and the sphere with center B and point A (Figure 5a). He constructed point C very close to the intersection of the two spheres and the triangle ABC (Figure 5b). Jorge believed that this triangle was equilateral because *each point is at the same distance from the two others... A from B and C, B from C and A, and C from A and B*. He continued explaining *we already have an intersection point. Well, we have two that are the same distance from A and B... we already have an equilateral triangle*. The teacher asked Jorge about the number of solutions, and he replied that he believed that there were more.

The construction that Jorge proposed, although correct, did not have any theoretical support. It is interesting to note that the solution presented by Jorge only considers two points at the intersection between the spheres as potential vertices of the equilateral triangle. This is evidence that Jorge had in mind the 2-D construction of the equilateral triangle he had just made, where the intersection of two congruent circles with the same radii leaded to this solution. We then see a potentially useful idea worked in GeoGebra but not fully developed, so presenting a pivot sign.

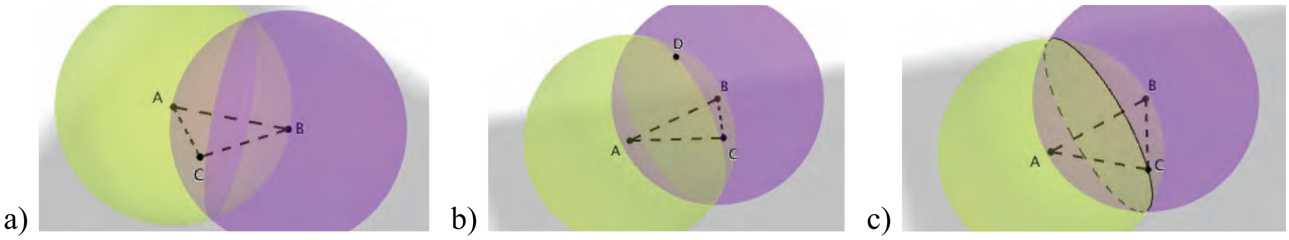

Figure 5: Jorge's construction of an equilateral triangle in 3-D

With some teacher's help, Jorge stated that the intersection of the spheres contained the solutions to the problem, so he created it on the screen (Figure 5c). Jorge tried to validate his answer by constructing some points in the intersection circle and calculating the distances from them to the centers of the spheres, A and B, but the teacher did not allow him to develop this idea and, instead, he asked Jorge to justify why the equidistance that he had stated was true. Jorge replied that *it is a circle, and all the points in a circle are at the same distance from the center*. Although the teacher showed Jorge that the centers of the spheres were not the center of the intersection circle, Jorge explained that *they are like a kind of center that has moved ... it is not in the center, but it is like a line that crosses the center*. Jorge tried to validate his construction based on some cases with the help of the artifact (an artifactual sign). He then made a progress by incorporating theoretical elements, such as the definition of circle, albeit in a wrong way, and equidistance to support the equidistance of each point of the intersection circle to the centers of the spheres. It seems that Jorge considered the centers of the spheres as points belonging to a line perpendicular to the circle and passing through its center; this constitutes a pivot sign that could have been used by the teacher to delve in his ideas. Even so, the language used is not precise and does not confirm this hypothesis.

The teacher did not develop Jorge's last idea but, instead, he asked him about the nature and properties of the circle in which the solutions to the problem were contained. Jorge mentioned that the circle was the intersection of the spheres, *which measure the same*, referring to the congruence of their radii. Additionally, the conversation with the teacher about a new point E on the circle led Jorge to mention that *they are at the same distance… Because E is at the intersection of A and B. So, like point C… it could be made equilateral triangles at all positions of the intersection*. This explanation was not entirely convincing, so the teacher insisted on asking why the points on the circle were equidistant from the centers of the spheres. Initially Jorge assured that this was true because *it is where… the spheres A and B intersect. So, it is like when the perpendicular bisector intersects the segment, they will always be at the same distance, any point on the circle*. The teacher modified his question and asked Jorge to explain why the distances AB and AE were the same. Jorge quickly replied: *Because they are in the… the spheres, any point is the same distance, right? from the center ... So, if there is a point where two spheres intersect, that point will be the same distance from the two centers*. We see that, in this conversation, Jorge managed, with the help of the 2-D construction, to connect the necessary elements of the 3-D construction to deductively justify the validity of his conjecture. This reveals at the end of Jorge's explanations a mathematical sign.

Discussion and conclusions

We have illustrated the mediating role of DGE and some semiotic signs produced by two mathematically gifted students while solving a construction problem, as well as differences and similarities between them. An interesting aspect which arises is the dynamic nature of the semiotic signs: in the excerpts presented, we can see that, at different moments in the solution of a problem, the students have exhibited signs of different types, not hierarchically sequenced; it happened that, even when mathematical signs were identified, setbacks with the presence of other types of signs were observed later. This could be because the problem selected was in the middle of the experimental sequence and students felt the need to produce deductive justifications, although they also showed traces of empirical, artifactual reasoning or parts of informal reasoning and language.

A second interesting aspect has to do with the content of students' outcomes when solving the problems in GeoGebra 3D. When constructing the 2-D equilateral triangle, differences could be observed in the students' performances, both in the geometric objects used and the degree of abstraction with which they justified their constructions. However, when working on the 3-D construction, both students agreed on the number of points that could serve as the vertices of the equilateral triangle. The solutions to this problem illustrate the ease with which the students modified the objects involved in the 2-D construction to adapt them to the 3-D construction. However, the reasoning that accompanied the constructions carried out was not adapted to the 3-D space; for instance, both students, guided by their 2-D experience, only identified two 3-D points as solutions.

Although this error was easily fixed, it is interesting to note the sustained influence of the representations and results got in GeoGebra 2D when moving to GeoGebra 3D.

We have presented only one short episode. More far-reaching studies are necessary to be able to advance in the characterization of the semiotic signs present in the justifications of solutions to geometric construction problems, the evolution in the meanings of the signs used, the relationship between the signs' meanings and the types of proofs produced, and the influence of the DGE.

Acknowledgment

The results reported in this paper are part of the research project PID2020-117395RB-I00, AEI, and the predoctoral grant EDU2017-84377-R.

References

Bartolini-Bussi, M., Corni, F., Mariani, C., & Falcade, R. (2012). Semiotic mediation in mathematics and physics classrooms: artifacts and signs after a Vygotskian approach. *The Electronic Journal of Science Education, 16*, 1-28.

Bartolini-Bussi, M. & Mariotti, M. (2008). Semiotic mediation in the mathematics classroom: artefacts and signs after a Vygotskian perspective. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (pp. 746-783). Routledge.

Drijvers, P., Kieran, C., Mariotti, M. A., Ainley, J., Andresen, M., Chan, Y. C., Dana-Picard, T., Gueudet, G., Kidron, I., Leung, A., & Meagher, M. (2010). Integrating technology into mathematics education: theoretical perspectives. In C. Hoyles & J. Lagrange (Eds.), *Mathematics education and technology - Rethinking the terrain* (pp. 89-132). Springer. https://doi.org/10.1007/978-1-4419-0146-0_7

Gutiérrez, A. & Jaime, A. (2015). Analysis of the learning of space geometry in a 3-dimensional geometry environment. *PNA*, *9*(2), 53-83. https://doi.org/https://doi.org/10.30827/pna.v9i2.6106

Mariotti, M. A. (2012). Proof and proving in the classroom: dynamic geometry systems as tools of semiotic mediation. *Research in Mathematics Education*, *14*(2), 163-185. https://doi.org/10.1080/14794802.2012.694282

Mariotti, M. A. (2019). The contribution of information and communication technology to the teaching of proof. In G. Hanna, D. A. Reid, & M. de Villiers (Eds.), *Proof technology in mathematics research and teaching* (pp. 173-195). Springer. https://doi.org/10.1007/978-3-030-28483-1

Noss, R. & Hoyles, C. (1996). *Windows on mathematical meanings: learning cultures and computers*. Springer. https://doi.org/110.1007/978-94-009-1696-8

Sinclair, N., Bartolini Bussi, M. G., de Villiers, M., Jones, K., Kortenkamp, U., Leung, A., & Owens, K. (2016). Recent research on geometry education: an ICME-13 survey team report. *ZDM - Mathematics Education, 48*(5), 691-719. https://doi.org/10.1007/s11858-016-0796-6