FROM 2D TO 3D: SUPPORT OF A 3-DIMENSIONAL DYNAMIC GEOMETRY ENVIRONMENT IN LEARNING PROOF

Camilo Sua, Angel Gutiérrez and Adela Jaime

Universidad de Valencia, Spain

Dynamic geometry environments support the learning of proof in plane geometry. Researchers have studied this process by using theoretical frameworks that allow us to understand how these environments provide such support. However, there is scarce research carried out on 3-dimensional dynamic geometry environments, which rise questions about the ways in which the learning of proof occurs in this scenario and how those environments intervene in the process. Based on the case of a mathematically gifted student, we analyze the development of the student's proving skills while solving a sequence of construction-and-proof problems in a 3-dimensional dynamic geometry environment and the way in which the environment stimulated those skills through utilization schemes put to work by the student to use some tools.

INTRODUCTION

Research on the influence of dynamic geometry environments (hereafter, DGE) is notable and has a long history. One of the several aspects of teaching and learning geometry with DGE on which research has been developed in recent years (Sinclair et al., 2016) is learning of proof (Sinclair & Robutti, 2013). However, this research has been carried out mainly in two-dimensional DGE (2D-DGE) and there is a lack of related research on three-dimensional DGE (3D-DGE) (Gutiérrez & Jaime, 2015). Particularly, it's necessary to carry out research informing on the influence that 3D-DGE can have in the learning, by ordinary and mathematically gifted students, of spatial geometry and, in this mathematical context, the learning of mathematical proof.

Research on the influence of DGE in the learning of mathematics has used different theoretical frameworks to interpret this process through the actions that individuals perform when using a technological artifact (mainly software in a computer, tablet, etc.). Some of these frameworks are based on the premise that, through the actions of a person with an artifact, the cognitive activity that occurs in that person's mind can be understood, so it is possible to provide observable evidence about the mental processes performed by the students (Drijvers et al., 2009).

We have carried out a case study research where four mathematically gifted students solved a sequence of construction-and-proof problems in a 3D-DGE based on GeoGebra (i.e., problems asking to create a geometrical figure and then prove that the figure fulfils the conditions required by the problem). The objective of that research was to analyze the students' reasoning processes and their progress in learning to do deductive proofs, to get information about mathematically gifted students' learning trajectories and different styles of mathematical reasoning. This paper focuses on one of those students, with the research objectives of i) analyzing the improvement of his

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proving skills when he solved the mentioned sequence of problems and ii) showing the way the 3D-DGE helped promote this change, through the utilization schemes put into action by the student when he used some GeoGebra tools.

THORETICAL BACKGROUND

Construction-and-proof problems and the learning of proof

DGEs support the learning of proof (Mariotti, 2012). We consider a *proof* as a mathematical argument, either empirical or deductive, aimed to convince someone of the truth of a mathematical statement (Fiallo & Gutiérrez, 2017). In our study, we emphasize the learning of proof through *construction-and-proof problems*. These problems ask i) to create on the DGE a geometric figure having some properties required by the problem, that must be preserved under dragging, and ii) to prove that the procedure used to create the figure is correct, by explaining and validating the way of construction (Mariotti, 2019). The statement to be proved is that the sequence of actions of the construction fits the conditions of the problem.

By using DGE tools in the construction of geometric objects, personal meanings are produced thanks to the dependency interrelationships that are discovered and verified through dragging. These tools are also related to theoretical elements of the Euclidean geometry which can support students when they develop proofs for the constructions (Mariotti, 2012). Solving construction-and-proof problems allows students to take advantage of the DGE possibilities and the logical system that underlies it. Therefore, geometric constructions have also a purely theoretical nature, where their validity is linked to prove that a set of constructions steps provide a specific result, so solving this kind of problems can make students evoke theoretical meanings of the tools they have used in the solutions (Mariotti, 2019).

From artifacts to instruments: instrumentalization and instrumentation

An *artifact* is any object used as a tool to perform a task (Rabardel, 1995). When a subject establishes a relationship with an artifact to do a specific task, in which the artifact is used in a particular way for a specific purpose, the artifact becomes an instrument. An *instrument* is a theoretical notion, the combination of an artifact and some mental schemes developed by the user, to which they refer when using the artifact to perform a task (Rabardel, 1995). A *scheme* is an invariant organization of mental habits for a group of situations, a stable way of dealing with specific tasks (Vergnaud, 1996). For Rabardel, the transition from an artifact to an instrument requires two intertwined processes that come from the individual's relationship with the artifact: *instrumentalization*, seen as the recognition of the components of an artifact, its limitations, and possibilities to solve a task; and *instrumentation*, seen as the emergence and development of *utilization schemes* on the artifact when solving tasks.

Due to their nature, the schemes are not directly observable, so it is necessary to have an observable counterpart to be able to refer to them. We describe the schemes in terms of students' behavior while are using the artifacts provided by a DGE, like dragging, construction tools, etc.

METHODOLOGY

We present a case study drawn from a broader research project where we analyze the learning of proof, in the context of a GeoGebra 3D-DGE, by four Spanish mathematically gifted students (11 to 14 years old) in grades 1 to 4 of secondary school. The identification of the students as mathematically gifted is because, for several years before our experiments, the students had participated in special out-of-school programs of attention to generally gifted students (AVAST) and mathematically gifted students (ESTALMAT) where they attended mathematics workshops.

We designed a sequence of 18 construction-and-proof problems which involved the equidistance relationship between points and between points and lines. Some problems requested the construction of a geometric object satisfying certain properties associated with equidistance (e.g., construct an equilateral triangle), first in 2D and then in 3D. Other problems requested first the construction of a 2D object and then the construction of an analogous object in 3D (e.g., construct the center of a circle in 2D and a sphere in 3D). These problems were implemented in several 60-minute sessions. The students' solutions of each problem provided instrumental and conceptual elements useful to solve subsequent problems. Students solved each problem and then discussed their solution with the first author of the paper, who led the conversation to justify the results. These sessions were audio and video recorded. As students were in different school grades and had different previous knowledge, the experimental sessions were organized as individual clinical interviews.

In this paper we present episodes of the solutions of three problems by a student named Hector (pseudonym). We chose those episodes because they show the development of Hector's skills for the elaboration of proofs and how he benefited from utilization schemes of some GeoGebra tools he created.

Table 1: Indicators of instrumentalization

Code	Indicator	Description
Tsa1	Discover possibilities of a tool	Previously unknown possibilities and functions of a tool (or set of tools) that allow solve the task are discovered
Tsa2	Identify limitations of a tool	An inadequate result is identified when using a tool with a defined purpose and such a purpose is ruled out for the tool.
Tsa3	Customize and adjust a tool to personal interests	Various uses of a tool are identified, according to specific interests. The tool is used in different schemes according to the requirements of the task.
Tsa4	Appropriate the artefact	The user becomes aware that the artifact is useful to obtain a specific result in a particular context.

Code	Indicator	Description
Sas1	Identify a scheme associated with one or more tools	A set of steps with one or more tools is recognized as effective to obtain a particular result. The set of steps was not known before.
Sas2	Elaborate a scheme to obtain a particular result	A scheme is elaborated/defined when using the tool that leads to obtain the same result.
Sas3	Adapt a scheme when solving a problem	A scheme is improved by including or removing some steps, to give it more scope or refine it.
Sas4	Use the same scheme in different tasks	The scheme that has been elaborated/modified is routinely used when solving different problems.

Table 2: Indicators of instrumentation

To analyze Hector's instrumental activity, we use an adaptation of the indicators of instrumentation and instrumentalization proposed by Sua and Camargo (2019) (Tables 1 and 2), which offer an analytical device to characterize both processes and come from the interpretation of their corresponding definitions in the specialized literature. The indicators of each process suggest a possible trajectory followed by a subject who uses artifacts to solve different tasks, leading them to become progressively instruments.

THE CASE OF HECTOR: THE CONSTRUCTION OF A BISECTOR PLANE

The episodes presented below report the evolution of the drawing of the perpendicular bisector of a segment with ruler and compass, which was transformed into a procedure with 3D-DGE that allowed the construction of the bisector plane of a segment. Solving previous problems had allowed introducing the definition of circle, sphere, and bisector plane as locus, as well as the definition of perpendicular bisector as a perpendicular line to the segment through its midpoint, the equidistance property of this line and that the bisector plane of a segment contained its perpendicular bisectors and therefore each point in this plane was equidistant from the ends of the segment. Problems that we present below used these properties and definitions. Before solving the problems, Hector already knew the definitions of sphere, circle, and perpendicular bisector. The other properties and definitions were obtained by solving previous problems.

Problem 7: Construction of an equilateral triangle in 2D and 3D

The first part of this problem asked to construct in GeoGebra 2D an equilateral triangle having the given segment GH as a side. To construct this triangle, Hector created the circles with centers on G and H and radius GH (Figure 1a).

To prove the validity of the construction, Hector expressed that he did not know why it worked, but that was the way he had been taught at school. However, the conversation with the teacher led him to look for support in the congruence of the radii of the circles, to state that the sides of the triangle were congruent because: ... *as these* *two circles are equal... as we have created the circles, so that the radius is this* [pointing at GH]*, these two sides* [IG and IH] *are the same… so, they are all the same.*

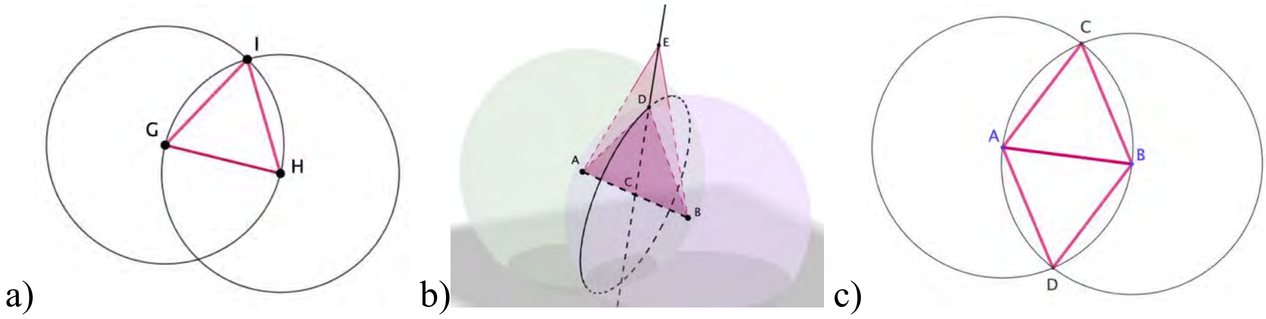

Figure 1. Construction of an equilateral triangle in 2D and 3D

The second part of the problem asked to construct an equilateral triangle in GeoGebra 3D having the given segment AB as a side. Hector created the midpoint C of AB, the spheres with centers on A and B and radius AB, and a point D at the intersection of these and the line CD (Figure 1b). Hector first described the construction: *... I have done the same as in the 2D version, only that instead of circles I have used spheres ... I learned the process that I used before for the equilateral triangle to make the perpendicular bisector with a compass on the subject of arts ... I have used that same technique to make the perpendicular bisector*.

Hector proved the result of his construction considering that *the point* [D] *is on the perpendicular bisector... that is, these two sides* [AD and BD] *are equal. I have used spheres of radius AB, so these two segments here* [pointing to AD and BD] *are radius of circles* [he means spheres] *with radius AB. So, they're all the same*.

Although Hector had built the requested triangles, at the end the professor talked to him about the mechanism with ruler and compass that had been used by Hector, but it was not known why it worked. The objective of this conversation was to provide an explanation about that mechanism, given its relevant role in the actions carried out by the student. Hector made the construction showed in Figure 1c and, although it was not easy to him at first, he proved that line CD is perpendicular to segment AB, by using the fact that ADBC is a rhombus, because its sides are congruent, so its diagonals bisect each other and are perpendicular.

Problem 10: Construction of the bisector plane of segment AB

The objective of this problem was to discover that, given three points A, B and C, the intersection of the bisector planes of A, B and A, C was contained in the bisector plane of B and C. Therefore, it was necessary to construct the first two mentioned planes and drag point A to different places in space. This was the first time Hector had built this plane in a robust way. Previously the construction that had been made was soft.

He first tried to construct several perpendicular bisectors of segment AB by using the corresponding tool or with a perpendicular line to AB through its midpoint D, but it was not possible to follow this procedure in GeoGebra 3D. For this reason, he decided to build two spheres with centers in A and B and radius AB (Figure 2a). Then he created three points at the intersection of those spheres, as well as the plane determined by these points (figure 2b). In the conversation with the teacher, Hector tried to validate his construction of this plane arguing that *it is a bisector plane... because this plane is perpendicular bisector of AB, so it is perpendicular to AB.*

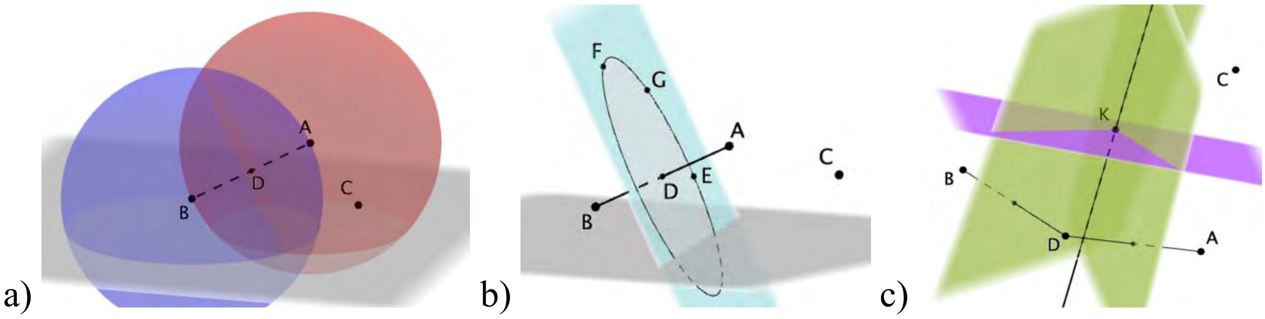

Figure 2. Construction of the bisector plane

Problem 17: Constructing a point equidistant from four non-coplanar points

This problem gave four non-coplanar points A, B, C, and D and asked to construct a point equidistant from all of them. Hector built the bisector planes of points A, D and B, D (green planes, Figure 2c), by using the scheme he created when solved problem 10, and the line intersection of these planes. He then built the bisector plane of points A, C in the same way (purple plane, Figure 2c), determined the point K of intersection between the last bisector plane and the line previously created. Hector stated that point K was the solution to the problem.

To prove the correctness of the construction, Hector assured that any point in the intersection line of the green bisector planes is equidistant from A, B, and D: *I knew that, on this line, these three points* [A, B, D] *would be at the same distance from K ... Because it was the intersection of the bisector planes of A, D and D, B*). Then, he mentioned that his reason to build the third (purple) bisector plane was to determine a set of points equidistant from A and C: *... I have made the perpendicular bisector* [bisector plane] *so that C and A were* [at] *equal* [distance from any point in the bisector plane]. Hector stated that point K was also equidistant from C, since *If C* [distance CK] *is equal to A* [distance AK]*, A is equal to B* [distance BK] *and A is equal to D* [distance DK]*, then C is equal to B, C is equal to D…*

ANALYSIS AND DISCUSSION

We have presented a trajectory that began with the procedure that Hector elaborated to build the perpendicular bisector of a 2D segment. Solving problem 07, he recreated on GeoGebra a well-known construction with ruler and compass with the help of the Circumference (Center-Radius) artifact [Sas1]. This procedure was also used for the construction of an equilateral triangle, which led Hector to discover new possibilities for this artifact [Tsa1]. Although this procedure came from his school experience and he did not know why it was valid, the conversation with the teacher and his geometric knowledge allowed him to elaborate a proof of the construction of the perpendicular bisector with circles, as well as that of the equilateral triangle.

When Hector moved to GeoGebra 3D and tried to build an equilateral triangle given one of its sides, he discovered that the scheme that had been useful in 2D was now not available, due to the limitations of the tools in GeoGebra 3D [Tsa2]. To overcome this difficulty, Hector modified the 2D scheme by replacing the circles by spheres, and he obtained a way to construct perpendicular bisectors in 3D and the requested triangle [Sas3]. Hector justified that that construction was correct, using an explicit correspondence between circles and spheres properties, as support for his deductions.

In problem 10, Hector had to modify the scheme again when he built the bisector plane because he did not have the tools used in 2D [Sas3]. This construction, however, was based on the scheme that Hector had elaborated to obtain a perpendicular bisector in 3D and the characterization of this plane as a set of all perpendicular bisectors of the segment. The latter is evidenced in the argumentation he made for the validity of the construction. Hector was modifying the utilization scheme to adapt it to the requirements of each new problem [Tsa3].

Hector's solution to last problem (17) showed a joint use of the modified schemes and the properties of the geometric objects represented through them: on the one hand, the geometric properties of the constructed planes allowed him to guarantee the equidistance of a set of points with respect to three other points; on the other hand, it is the equidistance provided by this scheme what mobilizes the construction of another bisector plane with which the problem was solved.

Hector's solutions to the problems presented, and others that we cannot mention due to the limited length of this paper, show the uses he made of utilization schemes to build bisector planes and perpendicular bisectors, according to the proposed problems [Sas2]. The recurrent use of those schemes to solve different problems [Sas4] and the confidence with which Hector referred to the results obtained through the schemes [Tsa4], revealed in a global way the relationship between him and the circle-sphere artifact. It provides elements to ensure the emergence of an instrument.

The case that we have analyzed provides evidence showing that solving construction-and-proof problems in a 3D-DGE induced the development of proving skills in this mathematically gifted student (we have also obtained similar results with the other students participating in the research experiment). We have showed some glimpses of Hector's advance in his instrumental activity with GeoGebra 3D and the ways it supported the development of his proving skills, since it allowed him to evoke mathematical meanings of represented objects on screen to prove the validity of his constructions.

The nature of this study does not allow generalize its results, but it opens a direction of research to better understand the differentiating characteristics of the processes of learning to prove by mathematically gifted students. A natural continuation of this study is to make similar experiments with average students of the same ages or grades as the mathematically gifted students participating in this experiment. On the other side, it is necessary to experiment with other 3D geometric relations as well, to provide a broader view of the influence of 3D-DGE on the learning of proof.

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