Plane representations in a 3-dimensional dynamic geometry environment: An analysis of soft constructions

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In the use of dynamic geometry environments, notable attention has been paid to robust constructions. Learning to prove approaches based on construction problem-solving have recognized a potential in robust constructions, derived from the nature of software construction tools. Although it seems to leave soft constructions aside, some aspects in favour of their relevance have also been mentioned. As part of an ongoing research that analyses and characterizes the learning of proof in three-dimensional geometry by mathematically gifted students, with the support of three-dimensional dynamic geometry environments, we analyse the potential that soft constructions can have in construction problem-solving and how it allows us to understand the way students interpret three-dimensional geometric objects when these are represented on the computer screen.

Keywords: Three-dimensional dynamic geometry environments, two-dimensional representations, soft constructions, construction problems, learning to prove.

Introduction

Dynamic geometry environments (hereafter, DGE) support the learning of proof (Sinclair & Robutti, 2013), being *construction problems* one of the the strategies that promote this learning (Mariotti, 2012). Although this type of problems shows the relevance of doing robust constructions in DGE to solve them, the benefits that soft constructions offer in problem solving cannot be neglected (Laborde, 2005). Characterizing the learning of proof requires a broad view of the students' mathematical activity, in which each solving strategy exhibited and different uses of the DGE are recognized.

Considering construction problems in three-dimensional (3D) geometry offers an additional variable to consider. Since the treatment and study of 3D geometric objects use two-dimensional (2D) representations of these, generally, the elaboration or interpretation of these representations may be problematic (Parzysz, 1988). This problem is also present in representations by 3D-DGE on computer screen or mobile device screen because, despite their realistic 3D appearance, they are plane representations (Mithalal & Balacheff, 2019). Although the perspective dragging function of 3D-DGE can help to solve this problem, it is necessary to use this function properly.

We present here a part of a research project in which we analyse the processes of reasoning and the progress in learning to do deductive proofs by several mathematically gifted students when solving construction problems in a combined 2D and 3D-DGE. We are not aware of studies with similar characteristics in 2D or 3D involving mathematically gifted students. The research objective of this document is to show and analyse different strategies exhibited by the students to solve a problem, in which soft constructions and different uses of the 2D screen view of 3D objects are proposed. We show how the use of plane representations, although supported by soft constructions, can support the solving process, and promote deductive reasoning, as well as become an obstacle in this process.

Theoretical background

Construction problems: robust and soft constructions

DGE support the learning of proof (Mariotti, 2012). We consider a *proof* as a mathematical argument, either empirical or deductive, aimed to convince someone of the truth of a mathematical statement. We emphasize the learning of proof through *construction problems*. These problems ask i) to create on the DGE a geometric figure having some properties required by the problem, that must be preserved under dragging, and ii) to prove that the procedure used to create the figure is correct, by explaining and validating the way of construction (Mariotti, 2019). The statement to be proved is that the sequence of actions of the construction produced a figure that fits the conditions of the problem.

When creating in a DGE a construction having to fit some given mathematical requirements and preserve them under dragging, two kinds of figures can be produced. *Robust constructions*: figures fitting all the requirements (i.e., based on tools bearing necessary mathematical properties). *Soft constructions*: figures not fitting some requirement, because some step in the construction has been done by eye and the figure does not preserve under dragging that requirement (Healy, 2000).

Although the relevance and usefulness of robust constructions is argued in the construction problems approach (Mariotti, 2019), our interest in analysing the learning of proof also gives a place to soft constructions (Laborde, 2005). The use of empirical and perceptual evidences in soft constructions leads to the fact that students' behaviour in the recognition of geometric properties is different from that which would be evidenced when using robust constructions. In addition, soft constructions can be a starting point for the development of robust constructions, through a continuous integration of geometric properties that allow moving from mainly visual approaches (visual control) to theoretical approaches (theoretical control) (Laborde, 2005).

Iconic and non-iconic visualization: two ways to see 2D representations

Learning to do geometric proofs is a complex process. Some difficulties lie in the interpretation of the graphical representations, either because false properties are attributed to the objects or because true properties are used in the proof without having been previously proved. Laborde (1998) refers to this phenomenon as a conflict between theoretical and graphic-spatial dimensions. Duval (2005), who also considers this as an obstacle in learning deductive geometry, mentions two ways of visualizing a drawing that guide its exploration and interpretation: iconic and non-iconic visualization.

Iconic visualization makes it possible to recognize an object through the similarity of its global shape with that of another known object. It is a usual and basic form in students' visualization, in which relationships between the parts of the represented object are not recognized. Under this approach, students do not modify the graphic representation, since, if they do it, the nature of the represented object will change. *Non-iconic visualization* allows students to see a drawing as a representation of a geometric object; the shape and appearance of the drawing are not fundamental characteristics of the represented object, so its modification does not generate conflicts. The object is seen by the students as the composition of other objects of the same or lower dimension, through certain relationships.

When 3D-DGE is used, the treatment and interpretation of 2D screen views accentuates the conflict between the theoretical and the graphic-spatial dimensions (Laborde, 2008). It is not possible to trust a single plane representation of a 3D object, in terms of the visual information it provides, so it is necessary to make an analysis of the 3D configuration based on the lower order objects that determine it (points, lines, planes, ...) and the relationships between them (congruence, parallelism, ...) (Mithalal & Balacheff, 2019). Since iconic visualization is insufficient in this situation, non-iconic visualization becomes a way to get reliable information. In 3D-DGE, dragging points and using the perspective function may help students to move towards a non-iconic visualization, since these resources provide them multiple perspectives of the object or the transformations that they can undergo by manipulating the points that determine the 3D object. Construction problems can also promote a non-iconic visualization, since building a 3D object requires decomposing it to recognize objects and relationships that allow obtaining the object to be constructed (Laborde, 2008).

Methods

Drawn from a larger ongoing research project whose objective is to analyse the learning of proof in the context of 3D geometry with the mediation of GeoGebra by four Spanish mathematically gifted students, we present here a case study of three of them, aged 11 to 14 years and studying in grades 1 to 4 of secondary school. We recognize them as mathematically gifted because, besides the ordinary schooling, the students had participated in programs of attention to general giftedness (AVAST) and mathematical giftedness (ESTALMAT). Students' knowledge of 3D geometry was scarce, it came from his school experience and was limited to recognizing the sphere, some polyhedra and simple solids. Their experience with GeoGebra was limited to the use of some 2D tools.

We designed and implemented a sequence of 18 construction problems, solved in sessions of 60 minutes per problem. Some problems requested the construction, first in 2D and then in 3D, of a geometric object satisfying properties associated with equidistance (e.g., construct an equilateral triangle given its side). Other problems requested the construction of a 2D object and an analogous 3D one (e.g., find the centre of a circle in 2D and then the centre of a sphere in 3D). The process of solution of each problem provided students with useful instrumental and conceptual elements to solve subsequent problems. Because learning to do deductive proofs is not immediate or simple, we prepared a long sequence of problems. For each problem, the students first had to solve it and then discuss the solution and justify its correctness with the teacher (the first author of this paper), who led the conversation. The sessions were audio and video recorded after informed approval by the students and their parents. As students were in different school grades and had different previous knowledge, the teaching sessions were organized as individual clinical interviews.

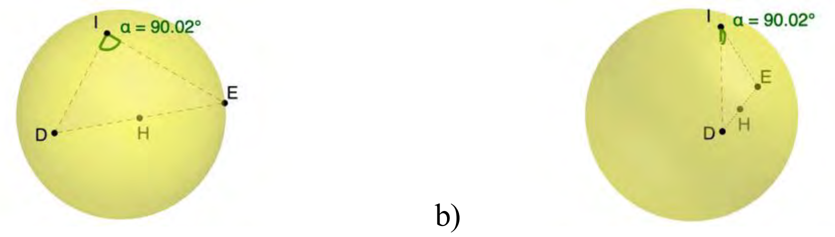
We present episodes of the solutions of one of the last problems in the sequence by three of the four students, Hector, Mario, and Rafael (pseudonyms). These students were chosen for the characteristics of their activity when they solved this problem, which were based on soft constructions as a first approximation, although they knew they had to make robust constructions. Additionally, the students' solutions show differences in the interpretation and use of the plane representations provided by the 3D-DGE, the elaborations of soft constructions to solve it, the explanations of the actions carried out, and the results derived from them. The visualization modes proposed by Duval (2005) allow us to analyse the actions of the students and help us to recognize differences in the use of soft constructions and their relationship with the interpretations of 3D objects from plane representations.

Constructing the centre of a sphere

Problem 16 was posed in GeoGebra 3D, showing the XY plane and a sphere without its centre over it. The problem requested to build the centre of the sphere, without limiting the use of tools or theoretical elements. In GeoGebra 3D there is no tool or command allowing to build the centre of the sphere. The solutions of previous problems had offered a context for the study of the circle, sphere, perpendicular bisector, and bisector plane, all of them characterized as loci and the solutions based on the property of perpendicularity in the case of the two last objects. We present below the three students' outcomes, and, in next section, we analyse, interpret, and discuss them.

Hector's construction, based on a property of the right triangle

The actions carried out by Hector had the objective of building a diameter of the sphere (Hector: *with a diameter it is basically done...*). One of the proposals to achieve his goal took advantage of a property of right triangles (in a right triangle, the length of the hypotenuse is twice the length of the segment determined by the vertex in the right angle and the midpoint of the hypotenuse). Hector constructed the triangle EID, with E, I, and D points on the sphere, and the midpoint H of segment ED. Then, he measured the angle EID and dragged point I until angle EID was very close to 90º (Hector: *...if I get angle I to measure 90 degrees, H will be the exact centre of the circle* [sphere]*...*) (Figure 1a). He claimed to have found the centre of the sphere (*H is very close to the centre*).



a) b)

Figure 1: Hector's construction

After explaining his construction, and reacting to a teacher's request, Hector made a perspective dragging to obtain a new view of the sphere and the constructed objects (Figure 1b). He recognized the mistake in his construction, and he commented what he hoped to get:

> I thought that if we managed to make that triangle a rectangle, that property would be fulfilled. When this happens, the distance from E to D would be twice that of I to H. The distance from I to H would be the radius of the sphere.

Mario's construction, based on a diameter

Mario's first strategy used the top view of the setup given by the problem. Using the sphere as a "circle", Mario constructed a point C on the sphere, carefully locating it by eye as the centre of the "circle" (Figure 2a). Then, he selected a new view of the sphere and constructed the line perpendicular to plane XY through C, the second point of intersection F of this line and the sphere, and the midpoint G of points C and F (Figure 2b). Then Mario explained the process he had followed and its validity: *C and F is* [determine] *the diameter*. So, Mario thought that point G was the centre of the sphere.

Mario expressed that his construction was not robust, in response to a question from the teacher, so he decided to look for another solution. Mario expressed his desire for two points on the sphere that would determine a diameter (if I *make a point and then the opposite point... we would have a diameter and the midpoint* [of these points] *would be the centre*). He created a point C on the sphere, placing it on “an edge” of the “circle” that represented the sphere on screen. Next, Mario built a point D in a position in which it perceptually determined with C a diameter of the “circle” (Figure 2c).

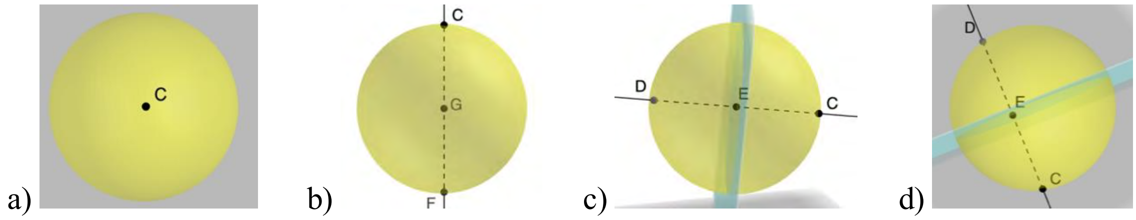


Figure 2: Mario’s constructions

He also constructed the line CD, the midpoint E of C and D, and the perpendicular plane to line CD containing point E (Figure 2c). Mario changed the perspective and noticed that, under a new perspective, the appearance of the “diameter” he had built was different and it was not a solution to the problem (Figure 2d), so he rejected this solution and deleted the objects he had built.

Rafael's construction, based on a combination of 2D perspectives and dragging

To determine the centre of the sphere, Rafael tried different auxiliary constructions (i.e., polyhedrons, chords of the sphere, and circles in the surface of the sphere). One of the strategies was based on the construction of two points on the sphere and the line they determine. Using the “circle” as a plane representation of the sphere, both points were dragged to positions in which they perceptually determined a diameter of the “circle”. The procedure was repeated with another pair of points (Figure 3a), so both lines simulated intersecting at the “centre of the circle”. Rafael observed the lines constructed from different perspectives, each of these showing different positions of the lines, where their intersection was close to the centre of the sphere. Rafael chose a perspective, from which he dragged the points along the “edge of the circle”, while the two lines did not intersect close to the centre of the “circle” (Figure 3b). Rafael decided to remove the points and lines he had constructed.

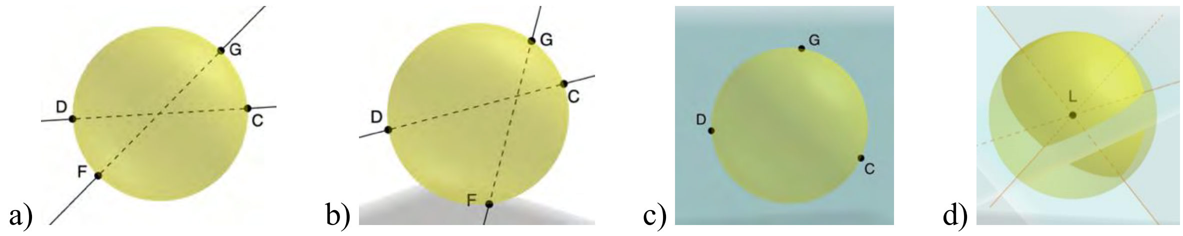


Figure 3: Rafael's constructions

In his second attempt, Rafael constructed three points on the sphere and the plane that they determined. He dragged the points to the “edge of the circle” that embodied the sphere on the screen, with which the constructed plane was seen from above, in a position parallel to the computer screen (Figure 3c). Rafael dragged to different positions of the “edge of the circle” the points that determine the plane and noticed that the plane did not change its inclination but remained static. Changing the 2D screen view, Rafael constructed two other planes using the same procedure. In the end, the intersection point L of the three planes was determined (Figure 3d).

Rafael argued his construction stating that:

> I think this [point L] would be the centre of the sphere, but... I have not used mathematical properties.

When the teacher asked him to explain the procedure he had carried out, Rafael replied:

> Taking advantage of the fact that from a single point of view it is as if it were a plane, as if you could only see in two dimensions, I have made a plane and I have moved the points as much as possible [to the "edge of the circle"]. That gives a bisector plane of the sphere. That's not because of math, but because of the limitations the program has.

Discussion

We have presented excerpts of students' solutions to a problem, where they produced soft constructions. Actions carried out with 3D-DGE to construct the centre of the sphere showed differences in the use of the 2D representation of 3D objects produced by perspective dragging.

The actions carried out by Hector were mobilized by the interest of incorporating a 2D property to a 3D object by using the 2D screen view of the sphere. He built and manipulated a triangle, caring only that one of its angles were about a right angle, ignoring that all triangle vertex belonged to the "edge of the circle". Hector used a single 2D screen view of the sphere when he developed the construction, without using perspective dragging to verify the result obtained. These actions and the verbalizations that accompanied them illustrate a solution of the problem based on iconic visualization. This characterization is confirmed by changing the plane representation after the teacher's suggestion. Hector's rejection of his construction was evident when he saw perceptual differences between the new flat representation and that he had in his mind, also revealing visual control in decision-making.

The actions carried out by Mario aimed to build a diameter of the sphere. He took advantage of the plane representation of the sphere and made two different constructions. In the first one, he used the top view of the sphere and the plane XY under a scheme not available if another 2D screen view were taken. The second construction combined soft and robust elements on a plane representation of the sphere, which was removed when the perspective was changed. In both constructions, Mario used perspective dragging and different plane representations of the same 3D configuration to build the diameter of the sphere and verify or reject his constructions. These actions show that Mario didn't trust in the information provided by a single 2D representation, thus recognizing the 3D nature of the objects represented on the screen. Mario exhibited behaviours linked to a non-iconic visualization, which also served a visual control, given the use of 2D representations in the validation or rejection of strategies, and theoretical elements that would allow him to validate the results obtained.

Rafael's actions in preparing his first construction were similar to Mario's, in terms of the use of flat representations to verify compliance with the desired properties. However, he went beyond the evidence provided by multiple perspectives, which was no longer sufficient, and he discarded his proposal, noting that the property held by objects built in a given position did not remain true when points were dragged. The second construction proposal confirmed that Rafael was not relying solely on various 2D representations. Using a 2D screen view of the 3D configuration, Rafael dragged the points that determined the plane to specific positions where the appearance of the plane was not changed. This gave him evidence of the validity of his construction, which allowed him to determine the centre of the sphere. Echoing the explanation he offered at the end, Rafael was taking advantage of the limitations of the plane representation offered by the DGE, not only to use the sphere as if it were a circle, but also to guarantee that his construction, although soft, had some invariance through dragging. Rafael's words and his actions on 3D-DGE showed how he related to 3D objects from their 2D representations, an aspect that reflects a visual and theoretical control of his behaviour, as well as a non-iconic visualization in the work done.

Characterizing the students' actions in terms of the use made of plane representations, perspective, or object dragging, and the geometric properties involved, reveals different levels of quality in students' elaborations and uses of soft constructions when a construction problem is solved. At the lowest level we find solutions, like Hector's, where the 2D representation of the 3D objects is not used properly and the dragging of the points that determine the objects on the screen is not carried out correctly. At this level, guided by an iconic visualization and visual control, 3D objects are used from a single plane representation, unaware that the properties declared on this representation may not be generally true. An intermediate level is characterized by solutions like Mario's, where different 2D representations are used to analyse or validate constructions (non-iconic visualization). This implies making a correct dragging of points when elaborating the construction, through which the objects are endowed with specific properties that are perceptually verified (visual control). The highest level includes solutions like Rafael's; they are elaborated constructions which are validated not only by perceptual aspects (visual control), since they also include theoretical elements. This makes the construction partially soft and partially robust (theoretical control), since the points that determine the objects constructed are placed in positions that allow a free dragging that does not affect the properties of the objects that they determine. Perspective dragging can be used as a verification tool or to build other geometric objects (non-iconic visualization).

Conclusions

Researchers have focused on solutions to construction problems based on robust constructions (Laborde, 2005), but solutions based on soft constructions are also frequent and interesting. Our interest in analysing and understanding the learning of proof has led us to analyse soft constructions, recognizing the opportunities that this kind of approaches can provide to solve a problem.

The analysis of the strategies exhibited by the mathematically gifted students in our case study that included soft constructions in their solutions showed differences in the use of perspective and point dragging, the 2D representations on the screen, and the theoretical elements involved. Supported by the analysis of the use of the tools provided by a 3D-DGE and the way in which a graphic representation can be analysed (Duval, 2005), we were able to characterize the students' actions and recognize different levels in the use of soft constructions. These results give relevance to this kind of approach when solving construction problems in a DGE, since the use of soft constructions allows us to understand the interpretations given by students to 3D objects when they are presented on the screen as 2D representations. Results shown in this document cannot be generalized, so additional studies analysing students with different characteristics and geometric relationships are necessary.

Acknowledgment

This paper is part of the R+D+I project PID2020-117395RB-I00, and the predoctoral grant EDU2017-84377-R, both funded by MCIN/AEI/10.13039/501100011033.

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