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ANALOGIES: A WAY TO PROMOTE THE LEARNING OF PROOF IN 3D GEOMETRY USING DYNAMIC GEOMETRY ENVIRONMENTS

Camilo Sua, Angel Gutiérrez, Adela Jaime
Universidad de Valencia, Departamento de Didáctica de la Matemática

Research on the influence of dynamic geometry environments on the teaching and learning of 3D geometry is embryonic. Some proposals that use these dynamic environments have involved analogy as a methodological approach to studying geometric objects in space through comparison with objects on the plane so that some properties of plane geometry are taken as the basis of the exploration and formulation of properties in space. Adhering to this line of work and as part of ongoing doctoral research, we present in this document some ideas about the role that analogy plays in students' mathematical activity when they solve construction-and-proof problems with the help of a 2D and 3D dynamic geometry environment. We emphasize the relationships between analogy, elaboration of proofs, and the use of a digital environment when students' productions are analyzed.

INTRODUCTION

The impact and benefits of dynamic geometry environments (hereafter, DGE) in the development of different processes of mathematical activity, including learning to prove, are significant (Sinclair & Robutti, 2013). However, the focus and development of the research has been mainly on 2D configurations, with the attention paid to 3D geometry or the use of 3D-DGE not being similar (Gutiérrez & Jaime, 2015). And, in the schools, 3D geometry is scarcely studied in depth. Some arguments that support this decision allude to the complexity of the study of 3D objects when plane representations of them are involved, since it is difficult to correctly visualize or represent the properties of these objects in these representations (Parzysz, 1988). However, mathematically gifted students can benefit of the challenges provided by 3D geometry problems and, in particular, proof problems. Then, in our research, we analyze the learning by mathematically gifted students because it is an aspect on which there is scarce literature (Jaime & Gutiérrez, 2017).

Some proposals for the study of 3D geometry have involved 3D-DGE and analogies between 2D and 3D objects as a method to recognize similarities and differences between analogous objects in both domains (Echeverry et al., 2018; Ferrarello et al., 2020). We are determined to provide results along this same path. We are conducting a doctoral research to analyze the learning of proof by mathematically gifted students in 3D geometry with the support of 2D and 3D GeoGebra. This research focuses on the establishment and use of analogies between 2D and 3D objects when students solve construction-and-proof problems.

This document aims to show how the use of analogy, as a driver of students' mathematical activity when entering the world of 3D geometry, with the support of a 3D-DGE, promotes the learning of mathematical proof. We analyze the sequence of problems we posed to a sample of mathematically gifted students and the results of its implementation. We focus on the relationships built between analogy, the use of 3D-DGE, and the elaboration of proofs.

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THEORETICAL BACKGROUND

Analogy: A way to extend ideas

An *analogy* is a relationship of similarity between two domains through specific objects and relationships. A domain is a representation of certain aspects of a situation, model, problem, conceptual structure, etc. (Schlimm, 2008). By establishing a set of relationships in one domain and taking them to another domain, in which these relationships are valid, analogy allows the formulation of hypotheses and the simplification of complex mental operations in the second domain, when these are performed in the first domain that is already known (Fishbein, 2002).

Making analogies is also a product of human activity, with several benefits (Richland & Simms, 2015). This process demands recognizing corresponding conceptual structures between different domains and their similarities or differences, thus advancing from a comparison based on superficial aspects or characteristics to one based on relationships between those domains. It means that this process has implications for the learning of mathematics (English, 1997) since new objects and relationships in an unknown domain can be discovered by extending ideas from a known domain.

Learning of proof and support of dynamic geometry environments

To analyze the learning of proof, we consider different strategies of the students when they validate mathematical statements. We consider that students, as learners of this practice, use with understanding theoretical elements, modes of reasoning, and forms of communication that are valid within the frame of this activity. We find support for this need in the approaches of Stylianides et al. (2016), making explicit the different nature of the ways of validating. In this sense, we understand the proof as an empirical or deductive mathematical argument, a sequence of connected assertions for or against a mathematical statement.

In our study, we emphasize the learning of proof through construction-and-proof problems. These problems ask i) to create on the DGE a geometric figure having some properties required by the problem that must be preserved under dragging, and ii) to prove that the procedure used to create the figure is correct by explaining and validating the way of construction (Mariotti, 2019). The statement to be provided is that the sequence of actions of the construction produced a figure that fits the conditions of the problem.

The tools provided by a DGE are related to theoretical elements of Euclidean geometry. When students use these tools to construct geometric objects, personal meanings are produced, thanks to the dependency relationships they discover and verify through dragging. Using DGE tools to solve different problems, personal meanings can progressively become mathematical meanings incorporating theoretical elements. Bartolini-Bussi and Mariotti (2008) call this relationship the semiotic potential of an artifact. Therefore, solving construction-and-proof problems allows students to take advantage of the possibilities of DGE, the logical system that underlies it, and evoke theoretical meanings of the tools they have used in the solutions (Mariotti, 2019).

METHODOLOGICAL CONSIDERATIONS

The content of this paper is part of a doctoral research adopting case study methodology. The research design considered a hypothetical learning trajectory (Simon & Tzur, 2004), whose objective was the learning of proof by mathematically gifted students. To achieve this objective, we developed a sequence of 18 construction-and-proof problems that involved the equidistance relationship and the analogy between 2D and 3D domains. Equidistance offers an opportunity to observe solution strategies guided by theoretical and perceptual aspects, as well as similarities and differences between the geometric objects of the compared domains. To solve the problems, students only used GeoGebra. This software was used because it provided simultaneity between the 2D and 3D configurations, an aspect that we consider useful to support the establishment and use of analogies.

The sequence brought students closer to the perpendicular bisector and circle and their 3D analogs (sphere and bisector plane) to construct these objects, define each object as a locus, and provide valuable properties to solve subsequent problems. Some problems required the construction, first in 2D and then in 3D, of geometric objects that satisfied equidistance properties (e.g., construct an equilateral triangle given its side). Other problems required constructing a 2D object and an analogous 3D object (e.g., finding the center of a circle in 2D and then the center of a sphere in 3D).

Four Spanish mathematically gifted students participated, aged between 11 and 14 years, belonging to different school grades and with different academic experiences. We recognize the students as mathematically gifted because they had participated in workshops on general and mathematical giftedness. The students had used GeoGebra 2D previously but not to carry out robust constructions in a context like the ones described. His knowledge of GeoGebra 3D and spatial geometry was scarce. The characteristics of the students were suitable to the process of solution of each problem to provide them with useful conceptual and instrumental elements to solve subsequent problems and for them to mobilize their knowledge of 2D geometry to the 3D domain, where it would be amplified and deepened by converting it into support for the mathematical activity deployed there.

For each problem, the students had to solve it and then discuss the solution and justify its correctness to the teacher (the first author of this paper), who led the conversation. The teacher's participation was relevant; his questioning of the students' productions promoted the ideas and arguments to advance from empirical and perceptual approaches to deductive and theoretical ones. The sessions were held by video conference and recorded in audio and video after informed approval by the students and their parents. As students were in different school grades and had different previous knowledge, the teaching sessions were organized as individual clinical interviews.

We present three moments in the sequence and what we expected the students to do in each one regarding the analogies developed, the use of GeoGebra tools, and the elaboration of proofs. Student names have been changed by pseudonyms. We contrast these considerations with the students' productions when implementing the sequence. Supported by this information, we discussed the articulation between the three elements that intervened in the sequence design.

THREE MOMENTS OF THE SEQUENCE

First moment: discovery of new geometric objects

The first problems were solved in 2D and 3D GeoGebra. These problems aimed to introduce the circle, perpendicular bisector, and their 3D analogs (sphere and bisector plane) as loci. Drag and trace functions were relevant. This characterization would be the basis of the analogies established between the circle, sphere, perpendicular bisector, and bisector plane. In instrumental terms, the recognition of tools and schemes that would allow each object to be constructed robustly was anticipated. As in this phase, geometric objects and properties were discovered, deductive proofs were not expected.

Instead, the use of empirical and perceptual evidence was expected, as well as the description of each object or set of points in terms of its appearance.

The students quickly incorporated the circle and perpendicular bisector into their activity since they knew them in advance. His school experiences also allowed the sphere to be introduced in 3D as a substitute for the circle and support equidistance concerning a point. This did not happen with the bisector plane. The students were unaware of this object, which provoked different exploration strategies when solving the problem that involved it.

Mario relied on the trace and drag function to move points of the perpendicular bisector out of the XY plane and drag them on it, controlling this action with the dotted line that drag function provides in GeoGebra 3D (Figure 1a). He used the labels of the distances of the displaced point concerning the two fixed points to validate his strategy inductively and said: *Point C continues going through the perpendicular bisector... by a perpendicular, a vertical line, then it will always be at the same distance*. Hector realized the sufficiency of changing the height of the points of the perpendicular bisector (z component of its coordinates) to preserve equidistance in 3D, so he saw in vertical dragging a way to obtain points that solved the problem, as he explained: *As high as point D is, up or down the perpendicular bisector, there will always be the same distance*. To prove equidistance, he referred to the line determined by the displaced point and the midpoint of fixed points as a perpendicular bisector, without being aware that this lacked sufficient theoretical support (Figure 1b). Both students characterized the set of points obtained as a perpendicular plane to the XY plane.

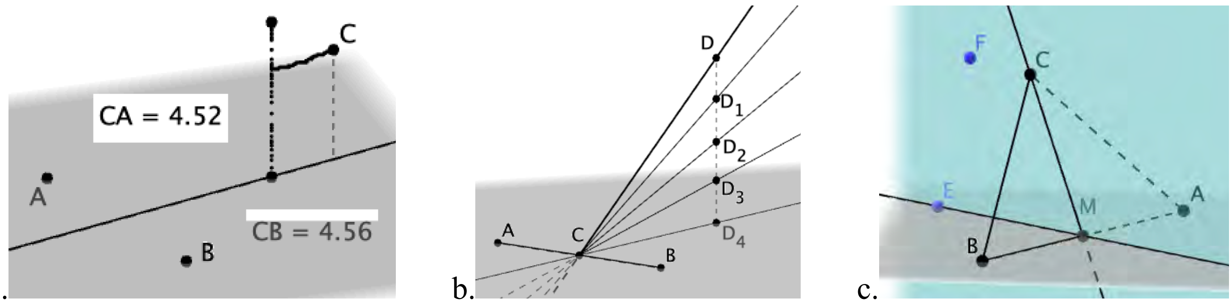

Figure 1. Discovering the bisector plane

Rafa anticipated a plane as a locus and the solution to the problem (Figure 1c). Like Hector, Rafa obtained this plane by dragging the points of the perpendicular bisector vertically, and he justified his actions: *We have moved point C on the Z axis, A and B are at the same height, so it does not vary (the distances between A, C and B, C remain the same)*. However, Rafa also explained that it was possible if the configuration taken as a base (perpendicular bisector and points that determine it) were contained in a horizontal plane. Like Hector, he proved the validity of his construction by referring to the line determined by each dragged point and the midpoint of the fixed points as a perpendicular bisector. Juan developed a different strategy. From the beginning, he recognized the existence of infinite perpendicular bisectors of a segment in 3D, so he constructed one of those lines and rotated it to determine all the solutions to the problem: *There are infinitely many perpendiculars to segment AB. I could draw any one and move point C on it*. The proof of his construction took advantage of the fact that the line constructed was the perpendicular bisector of the segment. The set of bisectors was characterized as a plane.

Second moment: substitution of 2D objects with 3D objects

The second block of problems was intended to use the geometric objects introduced. Students had to build objects with specific characteristics, first on the plane and then in space. Replacing the plane with space was intended to exchange the circle and perpendicular bisector for the sphere and bisector plane when necessary. Working in 3D would take advantage of strategies for constructing objects and proving properties worked out on the plane.

The substitution of 2D objects for 3D objects was seen, for example, in the strategies of some students to construct points in space that are equidistant from two fixed points. Hector and Rafa, aware of the lack of a tool that would allow constructing the perpendicular bisector or the bisector plane, adapted the procedure to construct the perpendicular bisector on paper with a ruler and compass, replacing the circles with spheres and using one of the points of their intersection and the midpoint of the given points to determine a perpendicular bisector (Figure 2a), or three points of the intersection to determine the bisector plane (Figure 2b). For them, replacing circles with spheres, as long as they preserve the equidistance to a fixed point, supports the validity of their construction. Hector argued: *I have done the same as in 2D, instead of circles I used spheres... I have created the perpendicular bisector*.

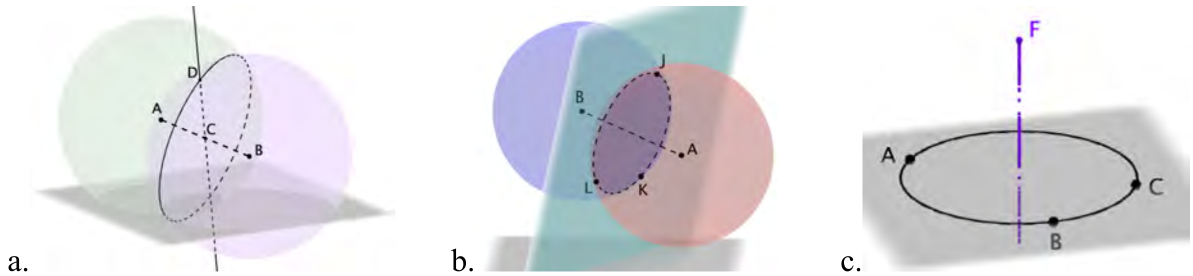

Figure 2. Construction of perpendicular bisector and bisector plane in 3D

When the problem conditions did not suggest the replacement of 2D objects with 3D objects, the students used the configuration represented in 2D as the basis of their exploration in 3D. An example of this was obtaining the set of points in 3D that are equidistant from the points of a circle. Previously, the center of the circle had been constructed in 2D. Mario combined vertical dragging and the trace of the center of the circle to determine a perpendicular line to the XY plane as a solution to the problem (Figure 2c): *It is a circle where the center is at a different height*. The other students anticipated this locus, ignoring the vertical dragging and constructing the perpendicular line. Mario proved his construction based on the conservation of equidistance under vertical dragging. Hector combined empirical evidence and theoretical elements to prove it. Juan and Rafa defined the perpendicularity relationship between a line and a plane, which allowed them to prove the equidistance.

Third moment: sophisticate use of objects and properties

The final problems requested constructions only in 3D, leaving behind the simultaneous work with the plane and the use of analogies. At this sequence stage, robust constructions of geometric objects and fluidity in using their properties to solve problems were expected. The proofs developed should be deductive or have features very close to these.

The students' actions when solving the problems reflected a purposeful use of 3D objects and robust constructions of them. This was the case of Rafa, who used 3D objects in the construction process and proved it based on their properties, leaving behind the reference to vertical drag or the analogy between these objects and their corresponding ones on the plane. Juan and Hector, in addition to Rafa's action, explained some constructions in similar ways, both based on an analogy between the plane and space. An example of this is what was made when they constructed the center of a given sphere. They constructed chords of the sphere and their bisector planes (Figure 3a), stating that the point of intersection of the planes was the center of the sphere by simultaneously equidistant from the ends of the chords. Juan explained that the choice of the bisector planes was based on an experience in the plane, in which they used a perpendicular bisector of chords in the circle to construct its center (Figure 3b): *In 2D, we used the intersection of the perpendicular bisectors of points in the circle. In this case* (3D)*, instead of perpendicular bisectors, we used the bisector plane*.

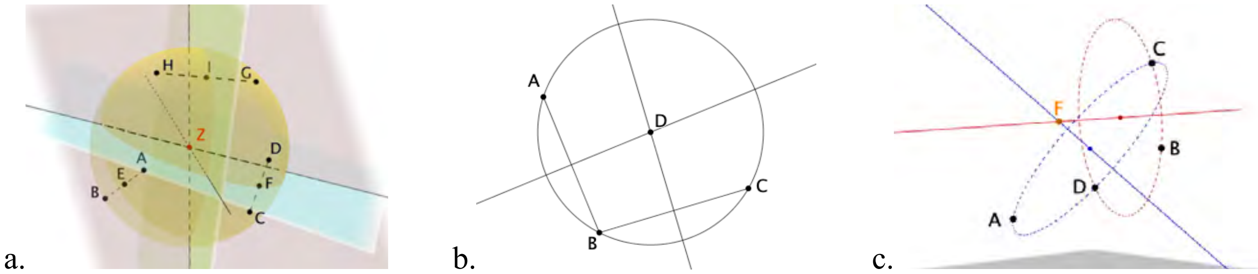

Figure 3. Construction of perpendicular bisector and bisector plane in 3D

Mario's actions do not allow us to generalize the ideas presented above. Although Mario knew procedures for constructing geometric objects that solved problems and he was sure about the result obtained, he could not develop a deductive proof of its construction. As the vertical dragging of points marked his experience moving objects from plane to space, he inductively established these results as properties that, although true, did not have theoretical support. He determined the center of the sphere with the help of bisector planes, as did Juan and Hector. When he solved another problem that asked to construct an equidistant point from four non-coplanar points, he used lines containing points that were equidistant from the points of the circles determined by three of the given points and their intersection (Figure 3c). In both cases, the robustness of his constructions was accompanied by gaps in the elaborate proofs.

DISCUSSION AND CONCLUSIONS

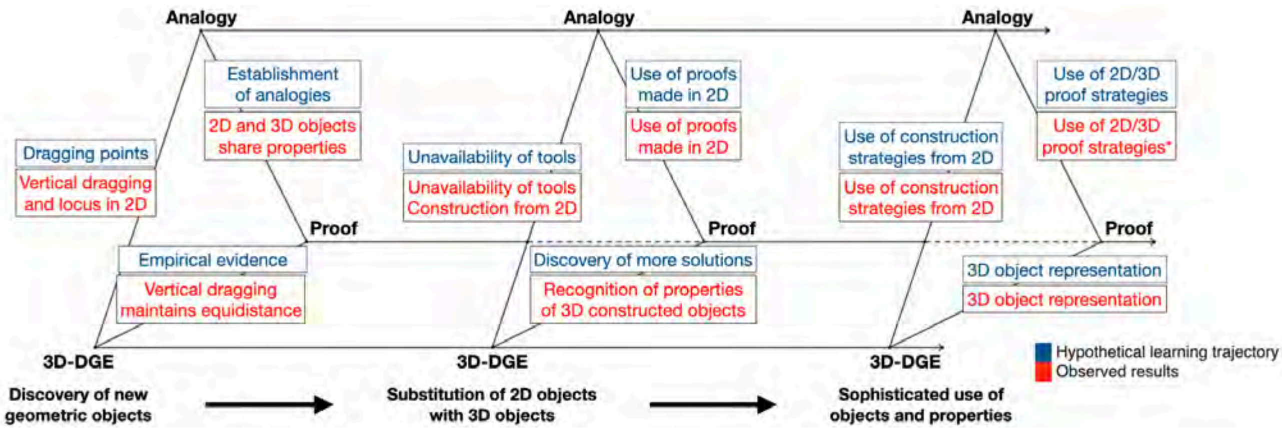

Figure 4. Moments of construction of perpendicular bisector and bisector plane in 3D

Figure 4 shows a diagram with the trajectory described previously. Each moment is represented through a triad in which the analogy, the use of the DGE, and the elaboration of proofs are articulated. The articulation derived from the hypothetical learning trajectory is presented in blue. In red is the articulation of the elements at each moment, according to the results obtained.

In the first moment, the combined use of vertical dragging and a locus in the plane became the basis for exploring and discovering analogous objects in space. Although we expected that the dragging of points would be an expected action, we did not expect the ways some students used it to guarantee the conservation of equidistance. This relationship led them to formulate proofs for equidistance in space, characterizing the discovered object in 3D through the 2D object taken as a base.

The relationship between vertical dragging and equidistance is a personal meaning promoted by using DGE functions. This meaning evolved in the second moment of the sequence when the characteristics of one of the problems led the students to replace dragging points with the construction of the perpendicular line to the plane. In this way, vertical dragging became a vehicle for establishing the relationship between perpendicularity and equidistance in 3D (mathematical meaning). Determining the meaning of the perpendicularity between a line and a plane allowed some students to deductively prove compliance of equidistance and not offer empirical proof as an explanation.

The absence of some construction tools favored the analogy between the circle and the sphere, an aspect considered when the sequence was designed. However, the procedures of some students to construct the perpendicular bisector and the bisector plane in 3D were not part of our assumption. These students simplified 3D mental operations to build these objects by recovering and adapting to the 3D-DGE construction protocols with a ruler and compass. Proving these constructions led to evoking the corresponding proofs in 2D, a valid strategy for them since the circle and sphere shared the property of equidistance.

The third moment shows a more mature state of transition in which the analogy begins to disappear. The use of geometric objects is observed, endowed with useful properties for constructing and elaborating deductive proofs. The analogy, if used, allows ideas to be extended from one domain to another by recovering construction strategies in the plane that are then adapted to the characteristics of the 3D configuration, thereby advancing the resolution of the problem in the latter domain.

Mario's actions offer a different picture. The use of vertical dragging in the exploration and characterization of sets of points had a greater presence when compared with the actions of his colleagues. Mario transformed vertical dragging into a tool for construction and proof when he worked in 3D, but the inductive nature with which this tool was developed did not allow the level of his proofs to be similar to that of his colleagues. Although the analogy allowed him to solve problems in 3D, his journey to get to this point was slower. This was evident in the episodes in which other students anticipated locus while he resorted to vertical dragging in search of solutions or used the discovered geometric objects without being aware of why they provided a solution to the problem.

Construction-and-proof problems provide a scenario in which the need to construct objects with specific properties leads to the need to articulate the available tools and knowledge. In the context of 3D geometry, there is also a search and connection with what is known about 2D geometry when the tools provided by 3D-DGE do not allow replicating the actions executed in the plane. Analogy can support these processes but requires the design of appropriate problems that promote it. As we have presented, using a 3D-DGE in a scenario like this produces personal meanings that progressively, and at unequal rates among students, will become mathematical meanings or approximate them.

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