
Wally Pacholka

Numerical Relativistic Magnetohydrodynamics in Dynamical Spacetimes

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Numerical Methods for Hyperbolic Equations: Theory and Applications

Santiago de Compostela, July 4-8, 2011

Scheme

1. Introduction: What a wonderful *(relativistic)* world
2. The Hydro-solver: General Relativistic Hydrodynamics (GRHD)
3. The Magnetohydro-solver (II): GRMHD
4. The Einstein-solver
5. The current virtual *(numerical)* relativistic world: Simulations
6. Conclusions and Perspectives

Outline

1 Introduction: What a wonderful *(relativistic)* world

2 The Hydro-solver: General Relativistic Hydrodynamics (GRHD)

3 The Magnetohydro-solver (II): GRMHD

4 The Einstein-solver

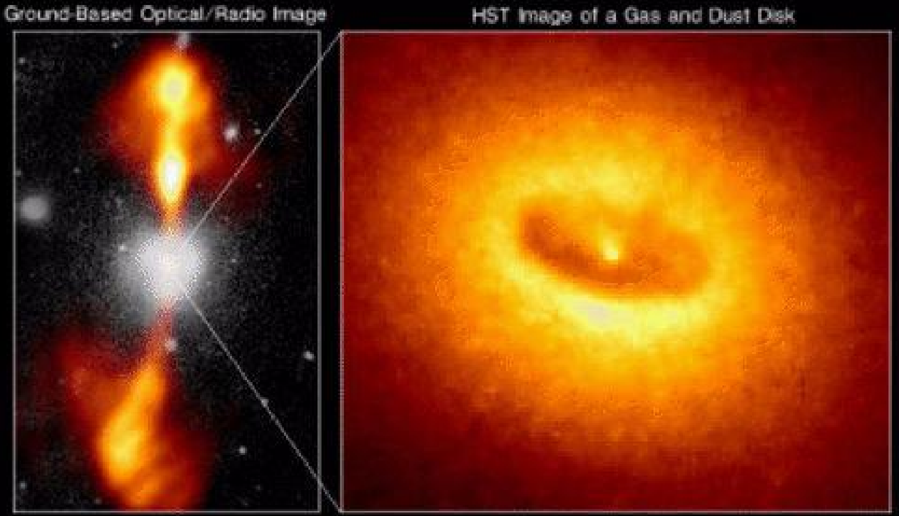
5 The current virtual *(numerical)* relativistic world: Simulations

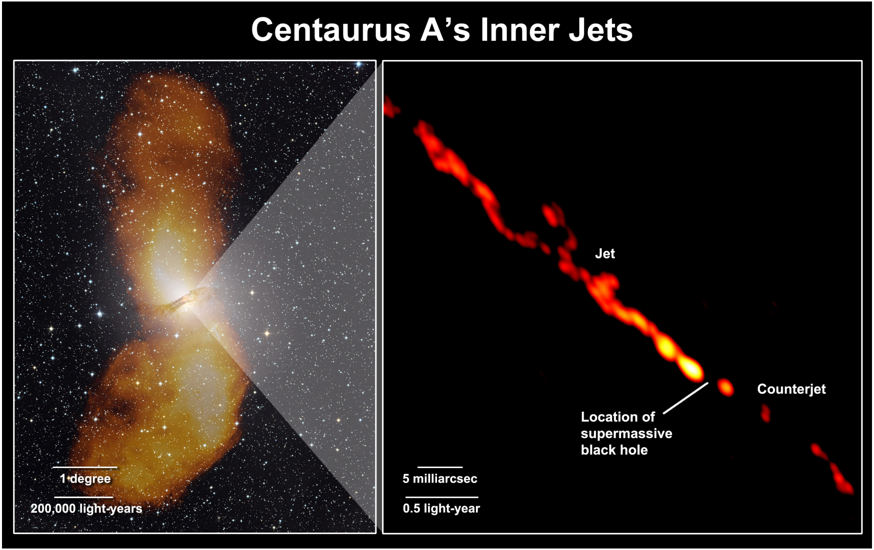
6 Conclusions and Perspectives

Core of Galaxy NGC 4261

Hubble Space Telescope

Wide Field / Planetary Camera



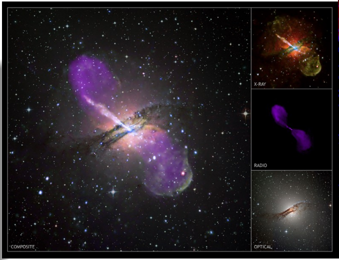


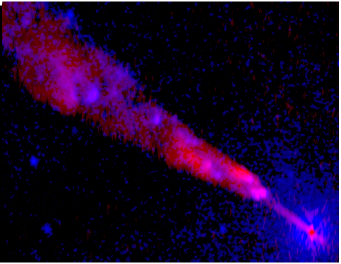
Left: Vast radio-emitting lobes (shown as orange in this optical/radio composite) extend nearly a million light-years from the galaxy. *(Credit: Capella Observatory (optical), with radio data from Feain, Cornwell, and Ekers (CSIRO/ATNF), Morganti (ASTRON), and Junkes (MPIfR)). Right:* Radio image from the TANAMI project. This view reveals the inner 4.16 light-years of the jet and counterjet. The image resolves details as small as 15 light-days across. Undetected between the jets is the galaxy's 55-million-solar-mass black hole. *Credit: NASA/TANAMI/Müller et al.*

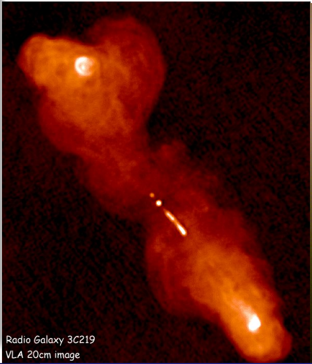
Astrophysical scenarios governed by relativistic (magneto-)hydrodynamical processes

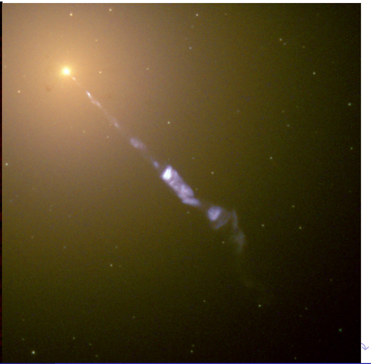
Relativistic Jets (in AGNs)

- Statistics:
  $\approx$ 10% radio-loud
- $v_{\rm jet}$
  $\approx 0.995c$
- $L_{\rm jet}/L_{\odot}$
  $\approx 10^{10} - 10^{15}$
- Size:
  $\approx 0.1 - 1$ Mpc
- Collimation:
  few degrees
- Central engine:
  SMBH + disc









Astrophysical scenarios governed by relativistic (magneto-)hydrodynamical processes

Gamma-ray Bursts

- $v_{\rm jet/wind} \approx 0.99995c$
- $L_{\rm GRB} \approx 10^{52}$ erg/s ($T \approx 1\ s$)
- Size: $\approx 1\ pc$
- Collimation: few tens of degrees
- Central engine: stellar BH + torus

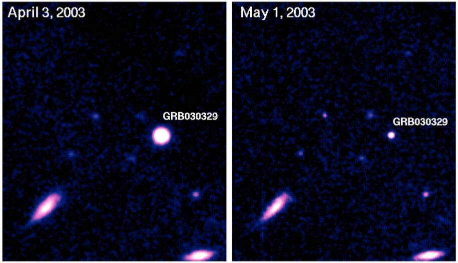
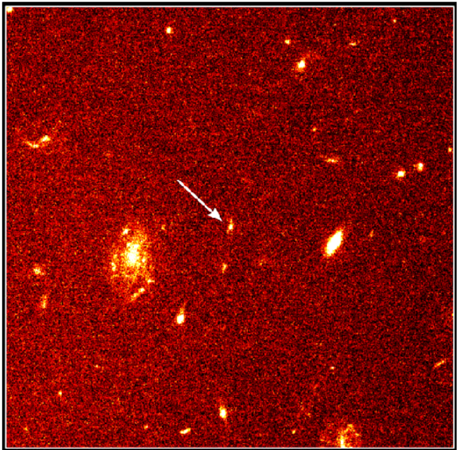


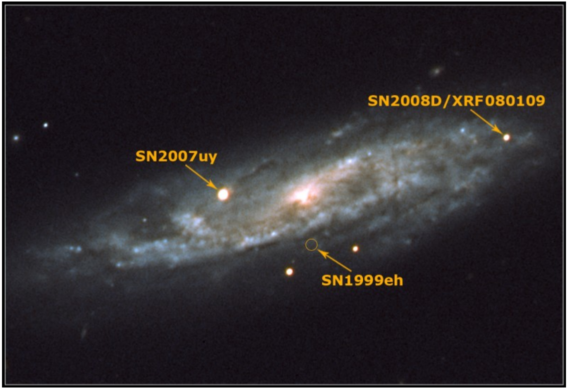
Image of Afterglow of GRB 030329
(VLT + FORS)

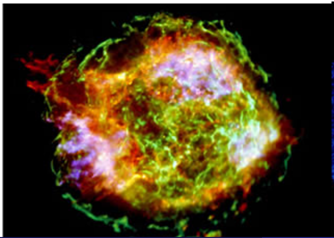
ESO PR Photo 17a/03 (18 June 2003)

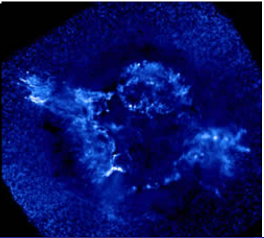
©European Southern Observatory



Astrophysical scenarios governed by relativistic (magneto-)hydrodynamical processes







Hydrodynamical Supernovae: SNII, SNIb/c

Top. The spiral galaxy NGC 2770. The three supernovae, indicated in this image, are now thought to be hydrodynamical (core-collapse), but the most recent of the trio, SN2008D, was first detected by the Swift satellite at more extreme energies as an X-ray flash (XRF) or possibly a low-energy version of a gamma-ray burst on January 9th. Located a mere 90 million light-years away in the northern constellation Lynx. (A. de Ugarte Postigo et al., 2007)

Bottom. False-colour image highlighting the jet and counterjet of silicon atoms around the SNR CasA (Hwang et al., 2004)

Einstein's world: Special Relativity vs General Relativity

Special-Relativistic Hydrodynamics *(SRHD)*: $\mathbf{v \rightarrow c \quad (c = 1)}$

- **Relativistic Jets**: *AGNs, GRBs, microQSOs*

General-Relativistic Hydrodynamics *(GRHD)*: $\mathbf{R \rightarrow \frac{2GM}{c^2} \quad (G = c = 1)}$

- **Stellar Core Collapse**: *Hydrodynamical SNe (SNIb/c, SNII), NS/BH formation*
- **Tidal Disruption by a SMBH**: *SNIa-like (WD)*
- **Coalescing Compact Binaries**: *NS-NS, WD-NS, WD-WD, NS-BH*
- **Progenitors of lGRBs**: *Collapsars*
- **Progenitors of sGRBs**: *Mergers of (magnetized-) NS-NS*
- **(Relativistic-) Jet Formation**: *Accretion torus around BHs, ...*

SRHD $\subset$ GRHD

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General Relativistic Hydrodynamics (GRHD): Characteristic structure

3+1 Formalism: *Darmois (1927), Lichnerowicz (1939), Choquet-Bruhat (1948), Arnowitt, Deser & Misner (1962), York (1979)*

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

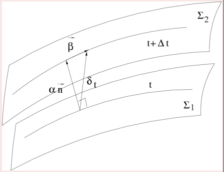
Geometrical quantities:

- ∗ lapse function: α
- ∗ shift vector: β^i
- ∗ three-metric: γ_{ij}

Minkowski space-time (Cartesian coordinates):

$\alpha = 1, \beta^i = 0$

$\gamma_{ij} = \delta_{ij}$.



GRHD: The equations (perfect fluid)

- $\mathbf{T^{\mu\nu} = (\rho(1+\varepsilon)+p)\, u^\mu u^\nu + g^{\mu\nu} p}$

u^μ is the four-velocity, ρ the proper rest mass density, ε the specific internal energy, p the pressure

- Local Conservation Laws:

Mass: $\nabla_\alpha(\rho \mathbf{u}^\alpha) = \mathbf{0}$

Energy-Momentum: $\nabla_\alpha \mathbf{T}^{\alpha\beta} = \mathbf{0}$

EOS: $p = p(\rho, \varepsilon)$

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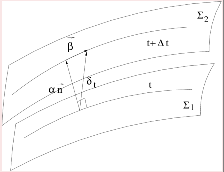

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Mass:	Energy-Momentum:	EOS:
$\nabla_\alpha(\rho \mathbf{u}^\alpha) = \mathbf{0}$	$\nabla_\alpha \mathbf{T}^{\alpha\beta} = \mathbf{0}$	$p = p(\rho, \varepsilon)$

GRHD: The Conserved Variables

Martí, Ibáñez, Miralles (1991), Banyuls, Font, Ibáñez, Martí, Miralles (1997)

$$D = \rho W$$
$$S^i = \rho h W^2 v^i$$
$$\tau = \rho h W^2 - p - D$$

$v^i = u^i/(\alpha u^0) + \beta^i/\alpha$ three-velocity

$W = 1/\sqrt{1 - \gamma_{ij} v^i v^j}$ Lorentz factor

$h = 1 + \epsilon + p/\rho$ specific enthalpy

GRHD equations as a HSCL

Conserved quantities: $\mathbf{F^0} = (\mathbf{D}, \mathbf{S^i}, \tau)$

$$\frac{1}{\sqrt{-g}}\left[\frac{\partial\sqrt{\gamma}\mathbf{F^0}}{\partial x^0} + \frac{\partial\sqrt{-g}\mathbf{F^i}}{\partial x^i}\right] = \mathbf{S}$$

$\mathbf{F^i}$ are the fluxes, and $\mathbf{S}$ the sources ($g = \det g_{\mu\nu}$, $\gamma = \det \gamma_{ij}$).

General Relativistic Hydrodynamic (GRHD) Equations

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\rho W}{\partial t}+\frac{\partial\sqrt{-g}\rho W v^i}{\partial x^i}\right) = 0$$

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\rho h W^2 v^j}{\partial t}+\frac{\partial\sqrt{-g}(\rho h W^2 v^i v^j + P\delta^{ij})}{\partial x^i}\right) = \mathcal{S}^j_M$$

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}(\rho h W^2 - P - \rho W)}{\partial t}+\frac{\partial\sqrt{-g}(\rho h W^2 - \rho W)v^i}{\partial x^i}\right) = \mathcal{S}_E$$

$$\mathcal{S}^j_M = T^{\mu\nu}\gamma^{jk}\left(\frac{\partial g_{\nu k}}{\partial x^\mu} - \Gamma^\eta_{\mu\nu} g_{\eta k}\right)$$

$$\mathcal{S}_E = \alpha\left(T^{\mu 0}\frac{\partial \ln \alpha}{\partial x^\mu} - T^{\mu\nu}\Gamma^0_{\mu\nu}\right)$$

$\Gamma^\lambda_{\mu\nu}$ are the **4-Christoffel symbols** .

GRHD: Characteristic velocities, *Banyuls, Font, Ibáñez, Martí, Miralles (1997)*

Eigenvalues: $(\mathbf{A}^x - \lambda \mathbf{A}^0)\mathbf{r}^* = 0 \;,\; \mathbf{A}^\mu = \frac{\partial \mathbf{F}^\mu(\mathbf{U})}{\partial \mathbf{U}}$

$$\lambda_0 = \alpha v^x - \beta^x \quad , \text{ (triple)}$$

$$\lambda_\pm = \frac{\alpha}{1-v^2 c_s^2}\left\{ v^x(1-c_s^2) \pm c_s \sqrt{(1-v^2)[\gamma^{xx}(1-v^2 c_s^2) - v^x v^x (1-c_s^2)]} \right\} - \beta^x$$

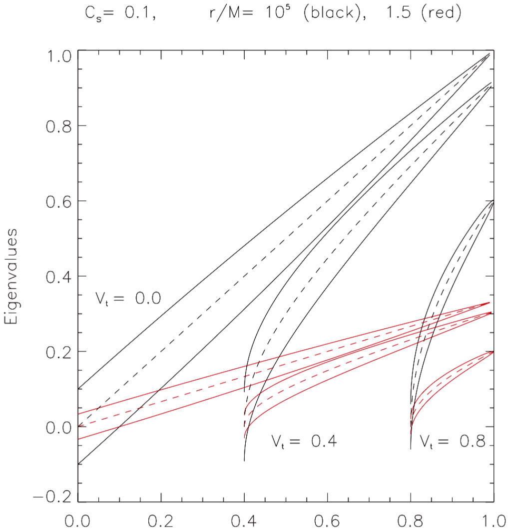
Local sound velocity, $c_s \implies h c_s^2 = \chi + \frac{p}{\rho^2}\kappa \;,\; \chi = \frac{\partial p}{\partial \rho} \;,\; \kappa = \frac{\partial p}{\partial \varepsilon}$

Minkowski space-time (Cartesian coordinates) $\implies \alpha = 1 \;,\; \beta^i = 0 \;,\; \gamma_{ij} = \delta_{ij}$

1D $\implies$ $\mathbf{v} = (v, 0, 0) \implies \lambda_0 = v \;,\; \lambda_\pm = \frac{v \pm c_s}{1 \pm v c_s}$

GRHD: Characteristic Velocities vs Total velocity, *Banyuls, Font, Ibáñez, Martí, Miralles (1997)*

$$\lambda_0 = v \quad , \quad \lambda_\pm = \frac{\alpha}{1-v^2c_s^2}\left\{v^x(1-c_s^2) \pm c_s\sqrt{(1-v^2)[\gamma^{xx}(1-v^2c_s^2) - v^xv^x(1-c_s^2)]}\right\} - \beta^x$$



Black lines $\Longrightarrow$ **SRHD**

Red lines $\Longrightarrow$ **GRHD**

RELATIVISTIC EFFECTS:

- Tangential component of the flow velocity: $v_t = 0, 0.4, 0.8$,
- Gravitational field: $r = 1.5 r_s$ *(r_s: Schwarzschild radius)*,

$\Updownarrow$

A very rich physics

GRHD: Right and Left eigenvectors , *Ibáñez, Aloy, Font, Martí, Miralles, Pons (2001)*

Right-Eigenvectors

$$\mathbf{r}_{\pm} = \begin{bmatrix} 1 \\ hW\left(v_x - \dfrac{v^x - \Lambda^x_{\pm}}{\gamma^{xx} - v^x\Lambda^x_{\pm}}\right) \\ hWv_y \\ hWv_z \\ \dfrac{hW(\gamma^{xx} - v^x v^x)}{\gamma xx - v^x\Lambda^x_{\pm}} - 1 \end{bmatrix}, \quad \mathbf{r}_{0,1} = \begin{bmatrix} \dfrac{\mathcal{K}}{hW} \\ v_x \\ v_y \\ v_z \\ 1 - \dfrac{\mathcal{K}}{hW} \end{bmatrix}$$

$$\mathbf{r}_{0,2} = \begin{bmatrix} Wv_y \\ h\left(\gamma_{xy} + 2W^2 v_x v_y\right) \\ h\left(\gamma_{yy} + 2W^2 v_y v_y\right) \\ h\left(\gamma_{zy} + 2W^2 v_z v_y\right) \\ Wv_y(2hW - 1) \end{bmatrix}, \quad \mathbf{r}_{0,3} = \begin{bmatrix} Wv_z \\ h\left(\gamma_{xz} + 2W^2 v_x v_z\right) \\ h\left(\gamma_{yz} + 2W^2 v_y v_z\right) \\ h\left(\gamma_{zz} + 2W^2 v_z v_z\right) \\ Wv_z(2hW - 1) \end{bmatrix}$$

Left-Eigenvectors

$$\mathbf{l}_{0,1} = \frac{W}{\mathcal{K} - 1}(h - W, Wv^x, Wv^y, Wv^z, -W)$$

$$\mathbf{l}_{0,2} = \frac{1}{h\xi}\begin{bmatrix} -\gamma_{zz}v_y + \gamma_{yz}v_z \\ v^x(\gamma_{zz}v_y - \gamma_{yz}v_z) \\ \gamma_{zz}\mathcal{W}^x + \gamma_{xz}v_z v^x \\ -\gamma_{yz}\mathcal{W}^x - \gamma_{xz}v_y v^x \\ -\gamma_{zz}v_y + \gamma_{yz}v_z \end{bmatrix}, \quad \mathbf{l}_{0,3} = \frac{1}{h\xi}\begin{bmatrix} -\gamma_{yy}v_z + \gamma_{zy}v_y \\ v^x(\gamma_{yy}v_z - \gamma_{zy}v_y) \\ -\gamma_{zy}\mathcal{W}^x - \gamma_{xy}v_z v^x \\ \gamma_{yy}\mathcal{W}^x + \gamma_{xy}v_y v^x \\ -\gamma_{yy}v_z + \gamma_{zy}v_y \end{bmatrix}$$

$$\mathbf{l}_{\mp} = (\pm 1)\frac{h^2}{\Delta}\begin{bmatrix} hW\mathcal{V}^x_{\pm}\xi + l^{(5)}_{\mp} \\ \Gamma_{xx}(1 - \mathcal{K}\tilde{\mathcal{A}}^x_{\pm}) + (2\mathcal{K} - 1)\mathcal{V}^x_{\pm}(W^2 v^x\xi - \Gamma_{xx}v^x) \\ \Gamma_{xy}(1 - \mathcal{K}\tilde{\mathcal{A}}^x_{\pm}) + (2\mathcal{K} - 1)\mathcal{V}^x_{\pm}(W^2 v^y\xi - \Gamma_{xy}v^x) \\ \Gamma_{xz}(1 - \mathcal{K}\tilde{\mathcal{A}}^x_{\pm}) + (2\mathcal{K} - 1)\mathcal{V}^x_{\pm}(W^2 v^z\xi - \Gamma_{xz}v^x) \\ l^{(5)}_{\mp} \end{bmatrix}$$

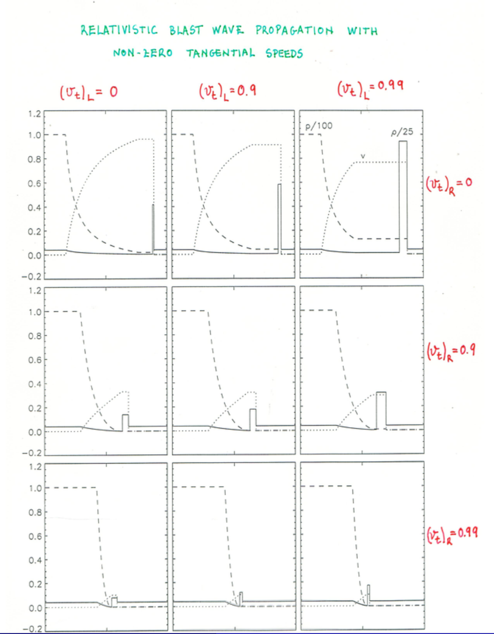
$$\Lambda^i_{\pm} \equiv \tilde{\lambda}_{\pm} + \tilde{\beta}^i, \; \tilde{\lambda} \equiv \lambda/\alpha, \; \tilde{\beta}^i \equiv \beta^i/\alpha, \; \mathcal{K} \equiv \frac{\tilde{\kappa}}{\tilde{\kappa} - c_s^2}, \; \tilde{\kappa} \equiv \kappa/\rho, \; \mathcal{C}^x_{\pm} \equiv v_x - \mathcal{V}^x_{\pm}, \; \mathcal{V}^x_{\pm} \equiv \frac{v^x - \Lambda^x_{\pm}}{\gamma^{xx} - v^x\Lambda^x_{\pm}}, \; \mathcal{W}^x \equiv 1 - v_x v^x, \; \tilde{\mathcal{A}}^x_{\pm} \equiv \frac{\gamma^{xx} - v^x v^x}{\gamma^{xx} - v^x\Lambda^x_{\pm}}$$

$$\Delta \equiv h^3 W(\mathcal{K} - 1)(\mathcal{C}^x_{+} - \mathcal{C}^x_{-})\xi, \; l^{(5)}_{\mp} \equiv (1 - \mathcal{K})[-\gamma v^x + \mathcal{V}^x_{\pm}(W^2\xi - \Gamma_{xx})] - \mathcal{K}W^2\mathcal{V}^x_{\pm}\xi, \; \xi \equiv \Gamma_{xx} - \gamma v^x v^x, \; \Gamma_{xx} = \gamma_{yy}\gamma_{zz} - \gamma^2_{yz}, \; \ldots$$

Solution of the Riemann Problem in RHD: tangential velocities *(Pons, Martí & Müller, 2000)*

Intrinsic relativistic effects
⇕
Coupling of tangential speeds

- In RHD, all the components of the flow velocity are coupled, through the Lorentz factor, in the solution of the Riemann problem.
- In adition, the specific enthalpy also couples with the tangential velocities ⟹ Important in the thermodynamically ultrarelativistic regime
- Two FORTRAN programs (**RIEMANN, RIEMANN-VT**) are provided by Martí, & Müller, **www.livingreviews.org/lrr-2003-7** ⟹ To compute the *exact solution of an arbitrary special relativistic Riemann problem* for an ideal gas (constant adiabatic index), both with zero and non-zero tangential speeds.



General Relativistic Hydrodynamics: Convexity

Ibáñez, Aloy, Mimica, Antón, Miralles, Martí, Proc. ASTRONUM-2010, San Diego, USA (2011)

$$\mathcal{P}_{\pm} \ := \ \left(\vec{\nabla}_{\mathbf{w}} \ \lambda_{\pm}(\mathbf{w}) \ \right) \ . \ \mathbf{r}_{\pm}(\mathbf{w}) \ = \ \pm \ \mathcal{K}_{\pm} \ (\gamma_{ij}, v_m) \ \ \mathcal{T}(\rho, \epsilon)$$

$$\mathcal{K}_{\pm} \ (\gamma_{ij}, v_m) \ = \ (v^x \ c_s \ \pm \ \Delta^{1/2})^{-2} \ (\gamma^{xx} - v^x v^x)^2 \ W^2 \ \Delta^{-1/2}$$

$$\Delta \ := \ W^2 \ \gamma^{xx} \ (1 \ - \ v^2 \ c_s^2) \ - \ v^x \ v^x \ (1 \ - \ c_s^2)$$

$$\mathcal{T} \ = \ \frac{\partial \ c_s}{\partial \ \rho} + \frac{p}{\rho^2} \ \frac{\partial \ c_s}{\partial \ \epsilon} \ + \ \frac{c_s}{\rho} \ (1 - c_s^2)$$

$$\mathcal{T} \ = \ \frac{c_s}{\rho} \ \left(\mathcal{G} - \frac{3}{2} \ c_s^2\right)$$

where G (the fundamental derivative) is defined by (Menikoff & Plohr, 1987)

$$\mathcal{G} \ := \ - \ \frac{1}{2} \ V \ \frac{\left(\frac{\partial^2 \ p}{\partial \ V^2}\right)_s}{\left(\frac{\partial \ p}{\partial \ V}\right)_s} \ = \ 1 \ + \ \left(\frac{\partial \ ln \ c_s}{\partial \ ln \ \rho}\right)_s$$

being $V \ := 1/\rho$ the specific volume.

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$$\mathcal{P}_\pm \ := \ \left(\vec{\nabla}_{\mathbf{w}} \ \lambda_\pm(\mathbf{w})\right) \ . \ \mathbf{r}_\pm(\mathbf{w}) \ = \ \pm \, \mathcal{K}_\pm \ (\gamma_{ij}, v_m) \ \ \mathcal{T}(\rho, \epsilon)$$

$$\mathcal{K}_\pm \ (\gamma_{ij}, v_m) \ = \ (v^x \ c_s \ \pm \ \Delta^{1/2})^{-2} \ (\gamma^{xx} - v^x v^x)^2 \ W^2 \ \Delta^{-1/2}$$

$$\Delta \ := \ W^2 \ \gamma^{xx} \ (1 \ - \ v^2 \ c_s^2) \ - \ v^x \ v^x \ (1 \ - \ c_s^2)$$

$$\mathcal{T} \ = \ \frac{\partial \ c_s}{\partial \ \rho} + \frac{p}{\rho^2} \ \frac{\partial \ c_s}{\partial \ \epsilon} \ + \ \frac{c_s}{\rho} \ (1 - c_s^2)$$

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being $V \ := \ 1/\rho$ the specific volume.

$$\mathcal{G} - \frac{3}{2} \ c_s^2 > \ 0 \ \Longrightarrow$$ **the HSCL of relativistic HD is convex.**

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General Relativistic Magnetohydrodynamics (GRMHD): Equations *Anile A.M. (1989)*

Definitions

$$T^{\mu\nu} = T^{\mu\nu}_{\text{pf}} + \frac{1}{2} b^\alpha b_\alpha g^{\mu\nu} - b^\mu b^\nu$$

$$T^{\mu\nu}_{pf} = (\rho(1+\varepsilon) + p)\, u^\mu u^\nu + g^{\mu\nu} p$$

ρ: proper rest mass density, ε: specific internal energy, p: pressure

$$h^* = 1 + \varepsilon + \frac{p^*}{\rho} \;,\; p^* = p + \frac{|b|^2}{2}$$

$$H^{\mu\nu} = u^\mu b^\nu - u^\nu b^\mu \;,\; u^\mu = W(1, v^x, v^y, v^z)$$

$b^\mu = (b^0, b^x, b^y, b^z)\;,\; b_\mu b^\mu = | b |^2 \geq 0$, where

$$b^\mu = \left(\frac{W B_k v^k}{\alpha}, \frac{B^i}{W} + W B_k v^k (v^i - \frac{\beta^i}{\alpha}) \right)$$

GRMHD: The Equations

Conservation of mass: $\nabla_\alpha (\rho u^\alpha) = 0$

Conservation of energy and momentum:

$$\nabla_\alpha T^{\alpha\beta} = 0$$

Maxwell Equations: $\nabla_\mu H^{\mu\nu} = 0$

Induction equation: $\nabla_\alpha (u^\alpha b^i - u^i b^\alpha) = 0$

$\left(\Rightarrow \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0\right)$

Differential constraint on the initial magnetic field configuration:

$\partial_\alpha (u^\alpha b^0 - u^0 b^\alpha) = 0 \; (\Rightarrow \nabla \cdot \mathbf{B} = 0)$

Algebraic constraints: $u^\alpha u_\alpha = -1\;,\; u^\alpha b_\alpha = 0$

Equation of state: $p = p(\rho, \varepsilon)$ $[p = (\gamma - 1)\rho\varepsilon]$

GRMHD equations as a HSCL: Conserved variables (U), fluxes (F) and sources (S)

Antón, Zanotti, Miralles, Martí, Ibáñez, Font, Pons (2006)

$$\frac{1}{\sqrt{-g}} \frac{\partial \sqrt{\gamma}\, \mathbf{U}}{\partial t} + \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{\gamma}\, \mathbf{F^i}}{\partial x^i} = \mathbf{S} \quad \text{and} \quad \frac{\partial \sqrt{\gamma}\, B^i}{\partial x^i} = 0$$

$$\mathbf{U} = \begin{pmatrix} \rho W \\ (\rho h + b^2) W^2 v_j - \alpha b^0 b_j \\ (\rho h + b^2) W^2 - \left(p + \frac{1}{2} b^2\right) - (\alpha b^0)^2 \\ B^k \end{pmatrix}$$

$$\mathbf{F^i} = \begin{pmatrix} \rho W (v^i - \frac{\beta^i}{\alpha}) \\ (\rho h + b^2) W^2 v_j (v^i - \frac{\beta^i}{\alpha}) + \left(p + \frac{1}{2} b^2\right) \delta^i_j - b^i b_j \\ (\rho h + b^2) W^2 (v^i - \frac{\beta^i}{\alpha}) - \alpha b^0 b^i - \left(p + \frac{1}{2} b^2\right) \frac{\beta^i}{\alpha} \\ B^i (\alpha v^k - \beta^k) - B^k (\alpha v^i - \beta^i) \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} 0 \\ T^{\mu\nu} \left(\partial_\mu g_{\nu j} - \Gamma^\delta_{\mu\nu} g_{\delta j}\right) \\ \alpha \left(T^{\mu 0}\, \partial_\mu \log \alpha - T^{\mu\nu}\, \Gamma^0_{\mu\nu}\right) \\ 0^k \end{pmatrix}$$

Classical Ideal MHD: Characteristic Structure   Brio & Wu (1988)

There are SEVEN physical waves:

Two ALFVEN WAVES: $\lambda_{a_\pm} \Longrightarrow \lambda_a = v_x \pm v_a$

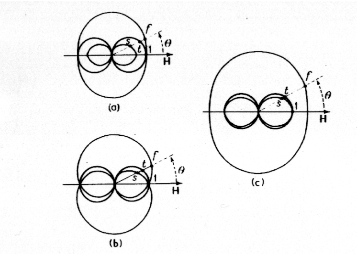
Two FAST MAGNETOSONIC WAVES: $\lambda_{f_\pm} \Longrightarrow \lambda_{f_\pm} = v_x \pm v_f$

Two SLOW MAGNETOSONIC WAVES: $\lambda_{s_\pm} \Longrightarrow \lambda_{s_\pm} = v_x \pm v_s$

One ENTROPY WAVE: $\lambda_e \Longrightarrow \lambda_e = v_x$

$$v_{f,s}^2 = \frac{1}{2}\left\{c_s^2 + \frac{B_x^2+B_y^2+B_z^2}{\rho} \pm \sqrt{\left(c_s^2 + \frac{B_x^2+B_y^2+B_z^2}{\rho}\right)^2 - 4\left(\frac{B_x^2}{\rho}\right)c_s^2}\right\}, \quad v_a = \sqrt{\frac{B_x^2}{\rho}}$$

$$\mathcal{M}_{f,s}^2 = \frac{1}{2}\left\{(1+s) \pm \sqrt{(1+s)^2 - 4s\cos^2\theta}\right\}, \quad \mathcal{M}_{f,s}^2 := \frac{v_{f,s}^2}{\mathbf{B}^2/\rho}, \quad s := \frac{c_s^2}{\mathbf{B}^2/\rho}$$



Wave front diagrams: Surfaces of normal speeds for s=0.5 (a), 1 (b), and 2 (c)
By rotating these curves around the axis taken in the direction of the magnetic field, we obtain the surfaces in the three dimensional space (surfaces of normal velocity, Jeffrey & Taniuti, 1964)

DEGENERACIES: States for which two or more wavespeeds become equal

Degeneracy I: $B_n = 0$

Degeneracy II: $B_t = 0$

Renormalization (Brio & Wu, 1988)

Ideal RMHD (I). Characteristic equation: $\mathbf{Ea^2A^2N_4 = 0}$ *(Anile, 1989)*

Definitions: $E, a, \mathcal{A}, N_4$

$$E \;:=\; \eta + |b|^2 \;,\; \eta \;:=\; \rho h = \rho + \rho\varepsilon + p = e + p$$

$$\mathcal{A} = Ea^2 - \mathcal{B}^2 \;,\; a = u^\alpha \Phi_\alpha \;,\; \mathcal{B} \;:=\; b^\alpha \Phi_\alpha$$

$$\mathcal{G} \;:=\; \Phi^\alpha \Phi_\alpha \;,\; \Phi_\alpha = (-\lambda, 1, 0, 0)$$

$$N_4 := \eta(e'_p - 1)a^4 - (\eta + e'_p\,|b|^2)a^2\mathcal{G} + \mathcal{B}^2\mathcal{G}$$

$$e'_p := \left(\frac{\partial e}{\partial p}\right)_s =: c_s^{-2} \;,\; \omega^2 \;:=\; c_s^2 + \mathbf{c_a}^2 - c_s^2\mathbf{c_a}^2$$

$$c_a \;(relativistic\;\; Alfv\acute{e}n\;\; velocity) : c_a^2 = \frac{|b|^2}{E}$$

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$$c_a \; (relativistic \;\; Alfv\acute{e}n \;\; velocity) : c_a^2 = \frac{|b|^2}{E}$$

Entropy waves: $a^2 = 0$

$$a := W(v^x - \lambda) \rightarrow \lambda = v^x$$

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$$a \; := \; W(v^x - \lambda) \rightarrow \lambda = v^x$$

Alfvén waves: $\mathcal{A} \; = \; 0$

$$\mathcal{A} \; := \; E\,W^2\,(v^x - \lambda)^2 - (b^x - \lambda b^0)^2 \rightarrow$$

$$\lambda = \frac{b^x \pm u^x\,\sqrt{E}}{b^0 \pm u^0\,\sqrt{E}}$$

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Definitions: $E, a, \mathcal{A}, N_4$

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Entropy waves: $a^2 = 0$

$$a \; := \; W(v^x - \lambda) \rightarrow \lambda = v^x$$

Alfvén waves: $\mathcal{A} \; = \; 0$

$$\mathcal{A} \; := \; E \; W^2 \; (v^x - \lambda)^2 - (b^x - \lambda b^0)^2 \rightarrow$$

$$\lambda = \frac{b^x \pm u^x \; \sqrt{E}}{b^0 \pm u^0 \; \sqrt{E}}$$

Magnetosonic waves: $N_4 \; = \; 0$

$$\sum_{i=0}^{4} C_i \; \lambda^i \; = \; 0$$

$$C_4 = 1 - \omega^2 v^2 - \frac{(b^0)^2 c_s^2}{EW^4}$$

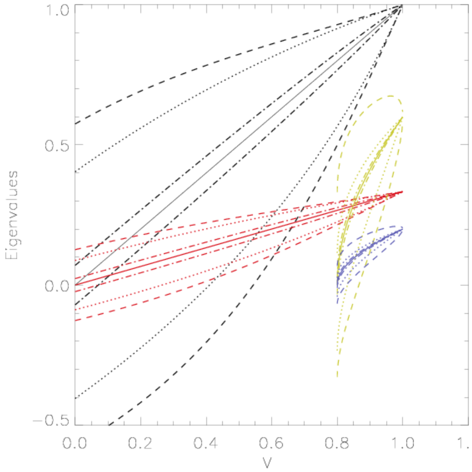
$$C_3 = -4v^x(1-\omega^2) - 2v^x\omega^2(1-v^2) + \frac{2b^0 b^x c_s^2}{EW^4}$$

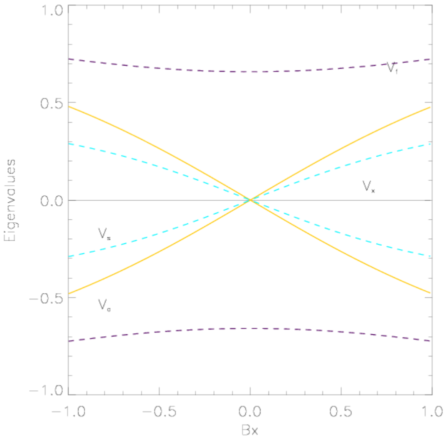
$$C_2 = 6(v^x)^2(1-\omega^2) - (1-(v^x)^2)\omega^2(1-v^2) + \frac{((b^0)^2 - (b^x)^2)c_s^2}{EW^4}$$

$$C_1 = -4(v^x)^3(1-\omega^2) + 2v^x\omega^2(1-v^2) - \frac{2b^0 b^x c_s^2}{EW^4}$$

$$C_0 = (v^x)^4(1-\omega^2) - (v^x)^2\omega^2(1-v^2) + \frac{(b^x)^2 c_s^2}{EW^4}$$

Ideal RMHD: Characteristic Structure (II)



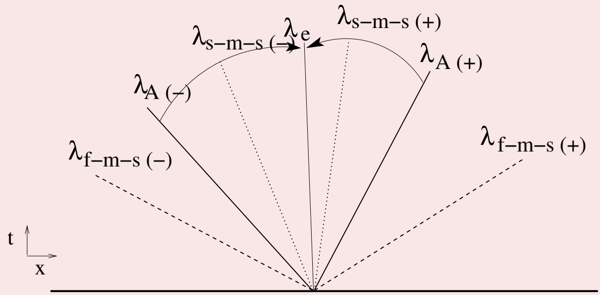


Black and yellow lines $\Longrightarrow$ **SRHD**
Red and blue lines $\Longrightarrow$ **GRHD**

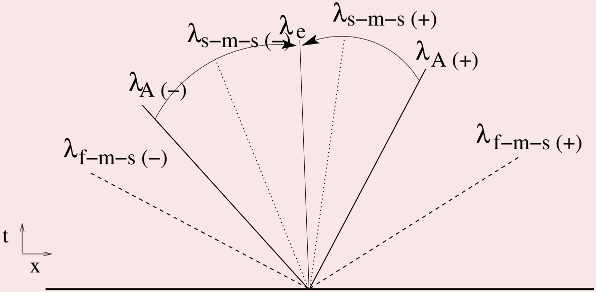
- $v_t = 0$ (black and red), 0.8 (yellow and blue)
- Gravitational field: $r = 1.5 r_s$ *(r_s: Schwarzschild radius)*,

Black lines $\Longrightarrow$ $\lambda_{f\pm}$
Yellow lines $\Longrightarrow$ $\lambda_{a\pm}$
Blue lines $\Longrightarrow$ $\lambda_{s\pm}$
Degeneracy arises at $B_x = 0$:

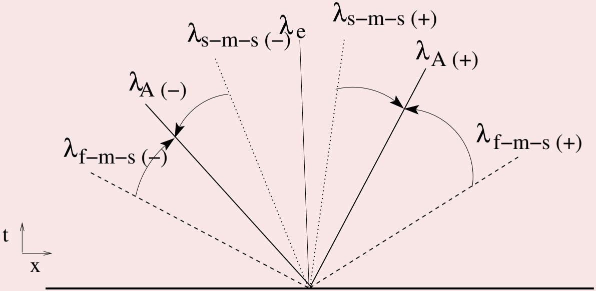
Type I degeneracy $\Longrightarrow$ *component normal to the Alfvén wave front:* $b_n^\mu = 0^\mu \rightarrow B^x = 0$



Type I degeneracy $\Longrightarrow$ *component normal to the Alfvén wave front:* $b^\mu_n = 0^\mu \;\; \rightarrow B^x = 0$



Type II degeneracy $\Longrightarrow$ *component tangential to the Alfvén wave front:* $b^\mu_\tau = 0^\mu \;\; \Longrightarrow b^2_n = b^2$



RMHD: Numerical issues

Some Numerical Problems

- *Fast and slow magnetosonic velocities*

  No practical ANALYTICAL expressions exist

  (but... *Dongsu Ryu's talk given at ASTRONUM-2011*, Valencia)

- *Recovering the primitive variables*:

  $$\mathbf{U} = (D, S^i, \tau, H^i) \rightarrow (\rho, v^i, \varepsilon, B^i)$$

- *Constraint to be preserved:*

  $$\nabla \cdot \mathbf{B} = 0$$

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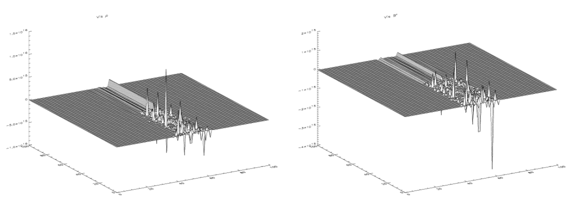
  $\rightarrow \;\; (\rho, v^i, \varepsilon, B^i)$

- *Constraint to be preserved:*

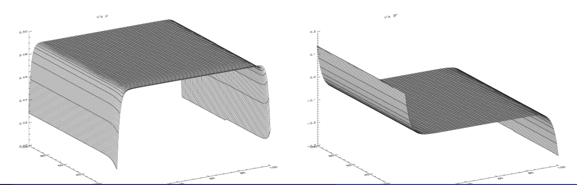
  $\nabla \cdot \mathbf{B} = 0$

Renormalization (L. Antón, Ph.D. Thesis, Univ. of Valencia, 2008)

$$\hat{\mathbf{f}}^{\mathrm{ROE}} = \frac{1}{2}\left[\mathbf{f}(\mathbf{u}_{\mathrm{l}}) + \mathbf{f}(\mathbf{u}_{\mathrm{r}}) - \sum_p |\widetilde{\lambda}^{(p)}| \widetilde{\alpha}^{(p)} \widetilde{\mathbf{r}}^{(p)}\right] \;,\; \widetilde{\alpha}^{(p)} = \widetilde{\mathbf{l}}^{(p)}(\mathbf{u}_{\mathrm{r}} - \mathbf{u}_{\mathrm{l}})$$



FIFTEEN orders of magnitude around the degenerate state



Characteristic wavespeeds (1): *Ideal gas with* $\gamma = 4/3$

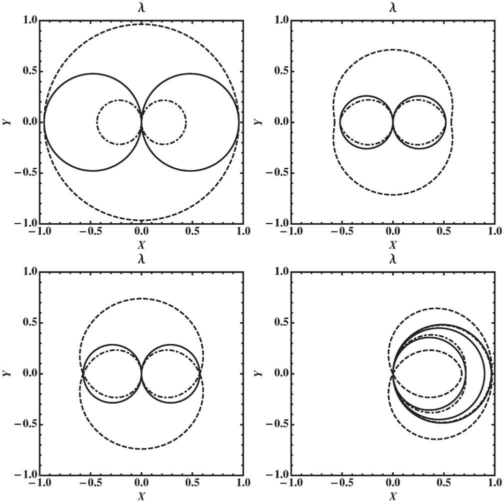


Figure 15.- **Dashed lines correspond to** *fast magnetosonic waves*, **continuous lines to** *Alfvén wavespeeds*, **and dash-dotted lines to** *slow magnetosonic waves*. *Entropic wave speed* **corresponds to dash-triple-dotted line.** *Top-left panel*: **Fluid at rest with** $\rho = 1.0$, $\varepsilon = 1.0$, $B^x = 5.0$, $B^y = 0.0$, $B^z = 0.0$. *Top-right panel*: **Fluid at rest with** $\rho = 1.0$, $\varepsilon = 50.0$, $B^x = 5.0$, $B^y = 0.0$, $B^z = 0.0$. *Bottom-left panel*: **Fluid at rest with** $\rho = 1.0$, $\varepsilon = 37.864$, $B^x = 5.0$, $B^y = 0.0$, $B^z = 0.0$. **In these three panels, the curve associated with the entropy wave degenerates in a point located at the origin, and Type I degeneracy is along the** y**-axis, whereas the three subcases of Type II degeneracy are along the** x**-axis.** *Bottom-right panel*: **Fluid state as in the top-right panel but moving along the** x**-axis at a speed** $v^x = 0.9$. **Type I and II degeneracies are again along** y **and** x**-axes, respectively.**

Characteristic wavespeeds (2): *Ideal gas with* $\gamma = 4/3$

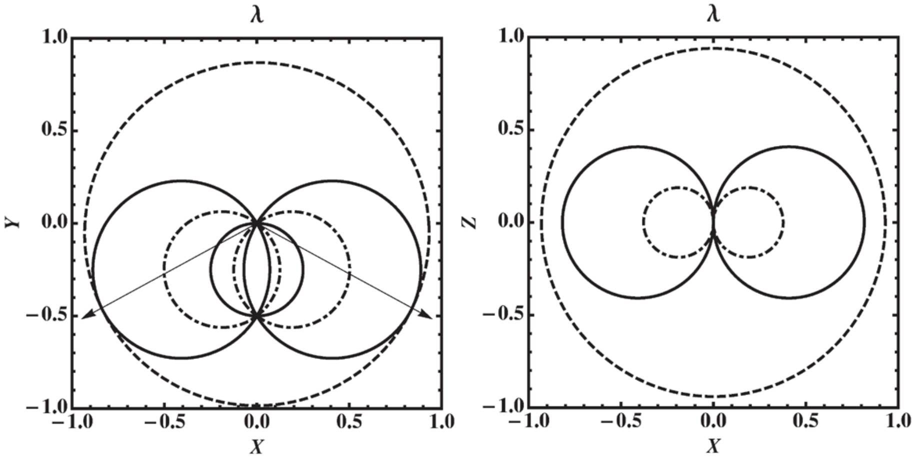


Figure 16.- **Line types as in Figure 15.** $\rho = 1.0$, $\varepsilon = 1.0$, $v^x = 0.0$, $v^y = -0.50$, $v^z = 0.0$, $B^x = 5.0$, $B^y = 0.0$, $B^z = 0.0$. *Left panel*: xy**-plane. Type I degeneracy is along the** y**-axis. Type II degeneracy is along the directions pointed by the arrows.** *Right panel*: xz**-plane. Type I degeneracy is along** z**-axis. There is no Type II degeneracy on this plane.**

Characteristic wavespeeds (3): *Ideal gas with* $\gamma = 4/3$

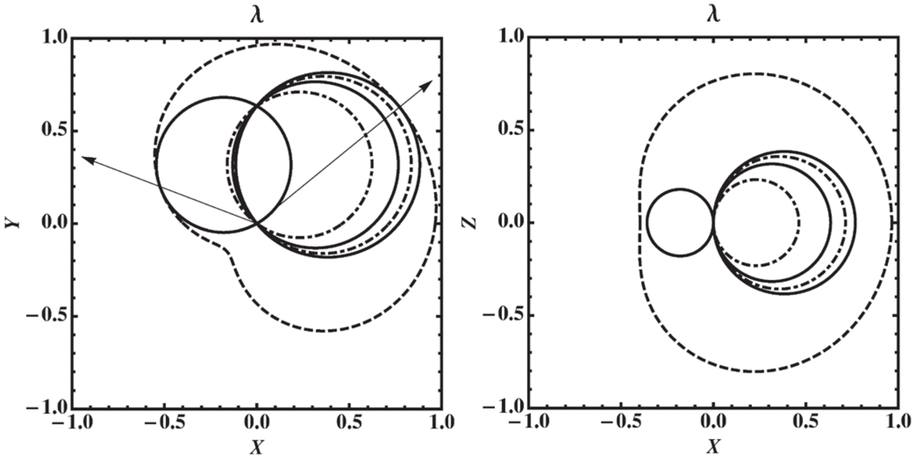
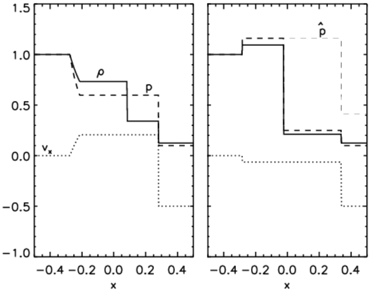


Figure 17.- **Line types as in Figure 15.** $\rho = 1.0$, $\varepsilon = 1.0$, $v^x = 0.634$, $v^y = 0.634$, $v^z = 0.0$, $B^x = 5.0$, $B^y = 0.0$, $B^z = 0.0$. *Left panel*: **xy-plane. Type I degeneracy is along y-axis. Type II degeneracy is along the directions pointed by the arrows.** *Right panel*: **xz-plane. Type I degeneracy is along z-axis. There is no Type II degeneracy on this plane**

RMHD: Exact Riemann Solver (Transversal Field) *Romero, Martí, Pons, Ibáñez, Miralles (2005)*

An exact (magneto-)shock-tube test



Solution of the Riemann problem $(t = 0.4)$

$p_L = 1.0$, $\rho_L = 1.0$, $v^x_L = 0.0$;
$p_R = 0.1$, $\rho_R = 0.125$, $v^x_R = -0.5$

Ideal gas EOS $(\gamma = 5/3)$

(Left) **A purely hydrodynamical problem**

(Right) **Tangential magnetic field:** $b_R = 2.0$

Analytical Magnetosonic velocities $(\mathbf{u}^\mu \mathbf{b}_\mu = \mathbf{0})$

$$u^\mu = W(1, v^x, 0, v^z) \; , \; b^\mu = (0, 0, b, 0)$$

$$\lambda_0 = v^x$$

$$\lambda_\pm = \frac{v^x(1-\omega^2) \pm \omega\sqrt{(1-v^2)[1-(v^x)^2-(v^2-(v^x)^2)\omega^2]}}{1-v^2\omega^2}$$

$$\omega^2 = c_s^2 + \mathbf{c_a}^2 - c_s^2 \mathbf{c_a}^2 \quad , \quad c_a^2 = \frac{|b|^2}{\rho h^*}$$

1D $\Longrightarrow$ $v^z = 0 \rightarrow v = v^x$:

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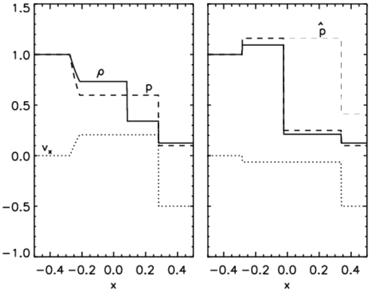

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$$\omega^2 = c_s^2 + \mathbf{c_a}^2 - c_s^2 \mathbf{c_a}^2 \quad , \quad c_a^2 = \frac{|b|^2}{\rho h^*}$$

1D $\Longrightarrow$ $v^z = 0 \rightarrow v = v^x$: $\lambda_0 = v$, $\lambda_\pm = \dfrac{v \pm \omega}{1 \pm v\omega}$

¶ $|\lambda_\pm| \geq |\lambda_\pm^{(fms)}|$ *(Leismann, Antón, Aloy, Müller, Martí, Miralles, Ibáñez, 2005)*

The exact solution of the Riemann problem in RMHD, in the general case, has been derived by *Giacomazzo & Rezzolla, 2006* **(Rezzolla's talk, NMA-MinySymp)**

FWD: A Full Wave Decomposition Riemann Solver in RMHD

Antón, Miralles, Martí, Ibáñez, Aloy, Mimica, ApJS, **187** *, 1-31 (2010)*

Steps towards a FWD RS in RMHD *(46 pages in the manuscript version. 13 pages in the printed version !!!)*

- Covariant variables (Anile, 1989) : $\widetilde{\mathbf{U}} = (u^\mu, b^\mu, p, s)^T$
- Primitive variables : $\mathbf{V} = (\rho, v_i, p, B^y, B^z)^T$
- Conserved variables : $\mathbf{U} = (D, S_j, \tau, B^y, B^z)^T$
- Eigenvectors : $\mathbf{R} := r_{\mathbf{U}} = \left(\frac{\partial \mathbf{U}}{\partial \mathbf{V}}\right) r_{\mathbf{V}}$, $r_{\mathbf{V}} = \left(\frac{\partial \mathbf{V}}{\partial \widetilde{\mathbf{U}}}\right)^{\star} r_{\widetilde{\mathbf{U}}} \Longleftarrow$ $r_{\widetilde{\mathbf{U}}}$ renormalized !!!

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$$\times u^y u^z + (b^0 u^z - 2(u^0)^2 B^z)\frac{u^y u^x B^x}{u^0} - (B^y + (B^y u^z - B^z u^y)u^z) \times \left(\frac{u^z b^0}{u^0} + 2B^z \frac{1-(u^0)^2}{u^0}\right)\Bigg\}. \quad (181)$$

In the previous expressions,

$$C_{a,1,\pm} = -\mathcal{E}u^0 u^z(1 - au^x) - (\mathcal{E}u^0 \pm \sqrt{\mathcal{E}}b^0) \times \frac{b^0}{Z}(b^z(1+au^x) + u^y(b^z u^y - b^y u^z) - 2b^x a u^z) + (Z+b^2)u^z - b^y(b^z u^y - b^y u^z)((u^0)^2 - 1) \pm \sqrt{\mathcal{E}}\left\{u^z\left((u^0)^2 u^x \mathcal{B} + \frac{b^0 b^x B^x}{Z}\right) - ((u^0)^2 - 1)(b^z + u^y(b^z u^y - b^y u^z))\right\}, \quad (182)$$

$$C_{a,2,\pm} = -\mathcal{E}u^0 u^y(1 - au^x) - (\mathcal{E}u^0 \pm \sqrt{\mathcal{E}}b^0) \times \frac{b^0}{Z}(b^y(1+au^x) + u^z(b^y u^z - b^z u^y) - 2b^x a u^y) + (Z+b^2)u^y - b^z(b^y u^z - b^z u^y)((u^0)^2 - 1) \pm \sqrt{\mathcal{E}}\left\{u^y\left((u^0)^2 u^x \mathcal{B} + \frac{b^0 b^x B^x}{Z}\right) - ((u^0)^2 - 1)(b^y + u^z(b^y u^z - b^z u^y))\right\}. \quad (183)$$

Vectors $\mathbf{W}_{a,1,\pm}$ and $\mathbf{W}_{a,2,\pm}$ are well defined in Type I degeneracy. In the case of the Type II degeneracy, coefficients f_1 and f_2 in (167) must be taken equal to $1/\sqrt{2}$.

6.3.3. Magnetosonic Eigenvectors

We present now the magnetosonic eigenvectors in conserved variables from the transformation of the corresponding eigenvectors in the reduced system of covariant variables before the renormalization process for Type II degeneracy, those defined through expressions (139)–(145). We start by defining the quantities

$$H_{m,\pm} = (G + 2a^2)\left(b^0 - \left(\frac{\mathcal{B}}{a}\right)_{m,\pm} u^0\right) - (b^x \lambda_{m,\pm} - b^0), \quad (184)$$

$$C_{m,\pm} = \frac{b_t^2}{a^2 - (G+a^2)c_s^2}\left\{(c_s^2 - 1)(G+a^2)au^0\left(u^x + \frac{b^0 B^x}{Z}\right) + \left((u^0)^2 + \frac{b^2}{\rho h}\right)(\lambda_{m,\pm}u^x - u^0)G\right\} - b_t^0\left\{\frac{2(G+a^2)B^x u^0}{u^0 - \lambda_{m,\pm}u^x}\left(u^0 a + \lambda_{m,\pm} + \frac{b^0 \mathcal{B}}{Z}\right) + \left((u^0)^2 + \frac{b^2}{\rho h}\right) G \left(\frac{\mathcal{B}}{a}\right)_{m,\pm}\right\} - g_2\left\{u^0 H_{m,\pm}\left(u^0 u^y + \frac{b^0 b^y}{Z}\right) + \left(b^y((u^0)^2 - 1) - u^0 u^y b^0 \frac{2Z + b^2}{Z}\right)(u^0 - \lambda_{m,\pm}u^x)\right\} - g_1\left\{u^0 H_{m,\pm}\left(u^0 u^z + \frac{b^0 b^z}{Z}\right) + \left(b^z((u^0)^2 - 1) - u^0 u^z b^0 \frac{2Z + b^2}{Z}\right)(u^0 - \lambda_{m,\pm}u^x)\right\}. \quad (185)$$

Now, the components of the magnetosonic eigenvectors read

$$L_{m,\pm,D} = C_{m,\pm}\frac{\partial Z}{\partial D}, \quad (186)$$

$$L_{m,\pm,S^x} = C_{m,\pm}\frac{\partial Z}{\partial S^x} + \frac{b_t^2}{a^2 - (G+a^2)c_s^2}\left\{(1 - c_s^2)a(G+a^2) \times \left((u^0)^2 + \frac{(B^x)^2}{\rho h}\right) + \frac{B^2 u^x - b^0 B^x}{\rho h u^0} \times (\lambda_{m,\pm}u^x - u^0)G\right\} + b_t^0\left\{\frac{2(G+a^2)B^x}{u^0 - \lambda_{m,\pm}u^x}\left((1 + au^x)u^0 + \frac{B^x \mathcal{B}}{\rho h}\right) - \frac{B^2 u^x - b^0 B^x}{\rho h u^0}\left(\frac{\mathcal{B}}{a}\right)_{m,\pm} G\right\} + g_1\left\{u^0 H_{m,\pm}\left(u^x u^y + \frac{B^x b^y}{\rho h u^0}\right) - (u^0 - \lambda_{m,\pm}u^x) \times \left(u^y\left(b^x u^0 + B^x \frac{b^2}{\rho h}\right) - B^y u^x\right)\right\} + g_2\left\{u^0 H_{m,\pm}\left(u^x u^z + \frac{B^x b^z}{\rho h u^0}\right) - (u^0 - \lambda_{m,\pm}u^x) \times \left(u^z\left(b^x u^0 + B^x \frac{b^2}{\rho h}\right) - B^z u^x\right)\right\}, \quad (187)$$

$$L_{m,\pm,S^y} = C_{m,\pm}\frac{\partial Z}{\partial S^y} + \frac{b_t^2}{a^2 - (G+a^2)c_s^2}\left\{(1 - c_s^2)a(G+a^2) \times \frac{B^x B^y}{\rho h} + \frac{B^2 u^y - b^0 B^y}{\rho h u^0}(\lambda_{m,\pm}u^x - u^0)G\right\} + b_t^0\left\{\frac{2(G+a^2)B^x}{u^0 - \lambda_{m,\pm}u^x}\left(au^y u^0 + \frac{B^y \mathcal{B}}{\rho h}\right) - \frac{B^2 u^y - b^0 B^y}{\rho h u^0}\left(\frac{\mathcal{B}}{a}\right)_{m,\pm} G\right\} + g_1\left\{u^0 H_{m,\pm}\left(1 + (u^y)^2 + \frac{B^y b^y}{\rho h u^0}\right) - (u^0 - \lambda_{m,\pm}u^x)\left(u^y B^y \frac{b^2}{\rho h} + b^0(1 + (u^y)^2)\right)\right\} + g_2\left\{u^0 H_{m,\pm}\left(u^y u^z + \frac{B^y b^z}{\rho h u^0}\right) - (u^0 - \lambda_{m,\pm}u^x) \times \left(u^z\left(b^y u^0 + B^y \frac{b^2}{\rho h}\right) - B^z u^y\right)\right\}, \quad (188)$$

$$L_{m,\pm,S^z} = C_{m,\pm}\frac{\partial Z}{\partial S^z} + \frac{b_t^2}{a^2 - (G+a^2)c_s^2}\left\{(1 - c_s^2)a(G+a^2) \times \frac{B^x B^z}{\rho h} + \frac{B^2 u^z - b^0 B^z}{\rho h u^0}(\lambda_{m,\pm}u^x - u^0)G\right\}$$

RMHD and Convexity *(L. Antón, Ph.D. Thesis, University of Valencia, 2008)*

In Anile's covariant variables: $\tilde{U} = (u^\alpha, b^\alpha, p, s)$.

$$\mathcal{P}_A \; := \; \left(\vec{\nabla}_{\tilde{\mathbf{U}}} \; \lambda_A(\tilde{\mathbf{U}}) \right) \; . \; \mathbf{r}_A(\tilde{\mathbf{U}}) \;\; , \;\; (A = e, a, m := entropic, Alfvén, magnetosonic)$$

Entropic characteristic fields $\Longrightarrow \;\; \mathcal{P}_e \; = \; 0 \Longrightarrow$ *Linearly degenerate fields*
Alfvén characteristic fields $\Longrightarrow \;\; \mathcal{P}_a \; = \; 0 \Longrightarrow$ *Linearly degenerate fields*
Magnetosonic characteristic fields

1. *Characteristic equation* $\Longrightarrow \;\; N_4 = 0 \;\; \Longrightarrow \;\; C_4 \; \lambda^4 \; + \; C_3 \; \lambda^3 \; + \; C_2 \; \lambda^2 \; + \; C_1 \; \lambda \; + \; C_0 \; = \; 0$

$$\Longrightarrow \;\; \partial_{\tilde{U}} \lambda = - \frac{\lambda^4 \partial_{\tilde{U}} C_4 + \lambda^3 \partial_{\tilde{U}} C_3 + \lambda^2 \partial_{\tilde{U}} C_2 + \lambda \partial_{\tilde{U}} C_1 + \partial_{\tilde{U}} C_0}{4C_4\lambda^3 + 3C_3\lambda^2 + 2C_2\lambda + C_1} \;\; := \; - \frac{Q_{\tilde{U}}}{P_\lambda}.$$

2.

$$\mathcal{P}_m \; = \; \left\{ a^2 \left(\frac{4a^2}{c_s^2} + \frac{\partial \rho h}{\partial p} \right) \left(a^2 - c_s^2 (a^2 + G) \right) + 2(1 - c_s^2) a^6 \right.$$
$$\left. - \frac{2b^2 a^4}{\rho h c_s^2} + \frac{2\mathcal{B}^2 G a^4}{\rho h} - \left(\rho h a^2 (a^2 + G) + \mathcal{B}^2 G \right) \frac{\partial c_s^2}{\partial p} \right\} \cdot \left\{ \frac{-a\mathcal{A}}{4C_4\lambda^3 + 3C_3\lambda^2 + 2C_2\lambda + C_1} \right\}$$

$\mathcal{P}_m \; \neq \; 0 \Longrightarrow$ *Genuinely nonlinear fields*

$a\mathcal{A} \; = \; 0 \;\; \Longrightarrow \;\; \mathcal{P}_m \; = \; 0 \Longrightarrow$ **RMHD is non-convex** $\Longrightarrow$ **Type I and Type II degeneracies**

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4. The Einstein-solver
5. The current virtual *(numerical)* relativistic world: Simulations
6. Conclusions and Perspectives

Einstein Equations (3+1 Formalism): $\mathbf{G}^{\mu\nu} = 8\pi\mathbf{T}^{\mu\nu}$

Evolution of geometrical (dynamical) quantities :

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i$$

$$\partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \alpha(R_{ij} + K K_{ij} - 2 K_{im} K^m_j) + \beta^m \nabla_m K_{ij} + K_{im} \nabla_j \beta^m + K_{jm} \nabla_i \beta^m - 8\pi T_{ij}$$

$$\bigoplus$$

Constraints:

$$0 = R + K^2 - K_{ij}K^{ij} - 16\pi\alpha^2 T^{00}$$
$$0 = \nabla_i (K^{ij} - \gamma^{ij} K) - 8\pi S^j$$

$R_{\mu\nu}$ the Ricci tensor, R the Riemann scalar, K_{ij} the extrinsic curvature, $K = \gamma^{ij} K_{ij}$

¶ **E. Gourgoulhon, J.L. Jaramillo, A 3+1 perspective on null hypersurfaces and isolated horizons, Phys.Rept. 423 (2006) 159–383,**

3+1 Formalism: Formulations of Einstein Equations (*BSSN, FCF, ...*)

Free evolution scheme: Solving the constraint equations only to get the initial data.

- **BSSN**: **Shibata & Nakamura, 1995; Baumgarte & Shapiro, 1999.**
  - **Much improved stability with respect to the standard ADM formulation.**
  - **Rigurous mathematical analysis (well-posedness, symmetric hyperbolic,...)** *Gundlach & Martín-Garcia (2004,2005,2006)*
  - **The most successful computations in numerical relativity to date**
    - *Binary neutron stars* **(Shibata and Uryu, 2000, 2001, 2002; Shibata et al., 2003).**
    - *Long-term evolution of neutron stars* **(Font et al. 2002).**
    - *Gravitational collapse of neutron stars to black holes* **(Shibata 2003; Baiotti et al., 2004).**

- **Other First Order Hyperbolic formulations**: **Reula,** *Hyperbolic Methods for Einstein's Equations,* Living Reviews in Relativity (www.livingreviews.org), 1998. **Bona & Palenzuela-Luque,** *Elements of Numerical Relativity,* Lecture Notes in Physics, v. 673 (2005).

FCF (Fully Constrained Formulation): Solving the four constraint equations at each time step.

Bonazzolla, Gourgoulhon, Grandclément & Novak, 2004

- Elliptic equations are much **more stable** than hyperbolic ones.
- **The constraint-violating modes do not exist** by construction.
- The equations describing stationary spacetimes are usually elliptic and are **naturally recovered**
- **Very efficient numerical techniques, based on spectral methods** .
- **CFC** is recovered in a simple way.

BSSN: *Conformal rescaling, conformal connection functions and evolution*

- **Conformal metric:** $\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij} \implies$ *Algebraic constraint:* $\tilde{\gamma} := det(\tilde{\gamma}_{ij}) = 1$
  $\rightarrow \phi = (\ln\gamma)/12, \quad \gamma := det(\gamma_{ij})$.
- **Conformal rescaling of the traceless part of the extrinsic curvature:**

$$A_{ij} := K_{ij}^{TF} \implies \tilde{A}_{ij} = e^{-4\phi}A_{ij}$$

- **Conformal connection functions:** $\tilde{\Gamma}^i := \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk} = -\tilde{\gamma}^{ij}{}_{,j}$
- **Evolution equations:**

$$\partial_t\phi - \beta^k\partial_k\phi = s^{(1)}(\alpha, \partial_i\beta^i, K)$$

$$\partial_t K = s^{(2)}(\alpha, \beta^i, \gamma^{ij}, \tilde{A}_{ij}, K, \rho, S)$$

$$\partial_t\tilde{\gamma}_{ij} - \beta^k\partial_k\tilde{\gamma}_{ij} = s^{(3)}_{ij}(\alpha, \beta^i, \gamma^{ij}, \tilde{A}_{ij})$$

$$\partial_t\tilde{A}_{ij} - \beta^k\partial_k\tilde{A}_{ij} = s^{(4)}_{ij}(\alpha, \beta^i, \gamma^{ij}, \phi, K, \tilde{A}_{ij}, R_{ij}, S_{ij})$$

$$\partial_t\tilde{\Gamma}^i - \beta^k\partial_k\tilde{\Gamma}^i = s^{(5)i}(\alpha, \beta^i, \gamma^{ij}, \phi, \partial_j K, \tilde{A}_{ij}, \tilde{\Gamma}^i_{jk}, S_j)$$

- **17 PDEs: Hyperbolic system of balance laws**

FCF

Bonazzola, Gourgoulhon, Grandclément & Novak, 2004; Cordero-Carrión, Ibáñez, Jaramillo, Gourgoulhon, Novak, 2008; Cordero-Carrión, Cerdá-Durán, Dimmelmeier, Jaramillo, Novak Gourgoulhon, 2009 (**Cordero-Carrión's talk, NMA-MiniSymp**)

¶ **Time independent fiducial flat metric f_{ij}, and conformal decomposition of the 3+1 fields:**

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij} \quad , K^{ij} = \Psi^4 \tilde{A}^{ij} + \frac{1}{3} K \gamma^{ij} \quad , \det(\tilde{\gamma}_{ij}) = \det(f_{ij}),$$

¶ **Deviation of the conformal metric from the flat fiducial metric:** $h^{ij} := \tilde{\gamma}^{ij} - f^{ij}$.

¶ **Gauge conditions: Maximal slicing and the generalized Dirac gauge** $\Longrightarrow$ $K = 0, \quad H^i := \mathcal{D}_k \tilde{\gamma}^{ki} = 0,$

¶ **The Elliptic Sector**

$$\Delta \psi = f^{(1)} \left(\alpha, \beta^i, \tilde{\gamma}_{ij}, \psi, E, h^{ij} \right)$$

$$\Delta \left(\alpha \psi \right) = f^{(2)} \left(\alpha, \beta^i, \tilde{\gamma}_{ij}, \psi, E, S^{ij}, h^{ij} \right)$$

$$\Delta \beta^i + \frac{1}{3} \mathcal{D}^i \mathcal{D}_j \beta^j = f^{(3)i} \left(\alpha, \beta^i, \tilde{\gamma}_{ij}, \psi, E, S^i, h^{ij} \right)$$

¶ **The Hyperbolic Sector**

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{\alpha^2}{\psi^4} \tilde{\gamma}^{kl} \mathcal{D}_k \mathcal{D}_l h^{ij} - 2 \mathcal{L}_{\boldsymbol{\beta}} \frac{\partial h^{ij}}{\partial t} + \mathcal{L}_{\boldsymbol{\beta}} \mathcal{L}_{\boldsymbol{\beta}} h^{ij} = \ldots$$

1. **The characteristic velocities:** $\lambda_0 = 0 \quad (multiplicity \; = \; 8),$
   $\lambda_{\pm}^{(\zeta)} = -\beta^\mu \zeta_\mu \pm \frac{\alpha}{\psi^2} \left(\tilde{\gamma}^{\mu\nu} \zeta_\mu \zeta_\nu \right)^{1/2} = -\beta^\mu \zeta_\mu \pm \alpha \left(\zeta^\mu \zeta_\mu \right)^{1/2} \quad (multiplicity \; = \; 6)$
2. **Dirac's gauge is a sufficient condition for the hyperbolicity of the PDEs governing the evolution of h^{ij}**
3. **The (right-)eigenvectors define a complete system iff: i) the lapse α does not vanish, and ii) $\left(\beta^{\|}\right)^2 \neq \alpha^2$.**
4. **Strong hyperbolicity if t^μ is timelike, i.e. if $\alpha \neq 0$ and $\alpha^2 - \beta^i \beta_i > 0$.**
5. **All the characteristic fields are linearly degenerate**
6. **For a coordinate system adapted to a spacelike inner worldtube $\mathcal{H}$, where $\beta^\perp > \alpha$, no ingoing radiative modes flow into the integration domain Σ_t at the excision surface.**

Special Relativistic Riemann Solvers: Extension to General Relativity

Pons, Font, Ibáñez, Martí, Miralles (1998)

- According to the **equivalence principle**, physical laws in a **local inertial frame** of a curved spacetime have the same form as in special relativity.
- **Locally flat (or geodesic) systems of coordinates** , in which the metric is brought into the Minkowskian form up to second order terms, are the realization of these **local inertial frames**. However, whereas the coordinate transformation leading to locally flat coordinates involves second order terms, **locally Minkowskian coordinates** are obtained by a linear transformation.
- **Our proposal** $\Longrightarrow$ *To perform, at each numerical interface, a coordinate transformation to locally Minkowskian coordinates assuming that the solution of the Riemann problem is the one in special relativity and planar symmetry*:

$$x^{\hat{\alpha}} = M^{\hat{\alpha}}_{\alpha}\,(x^{\alpha} - x^{\alpha}_{0})$$

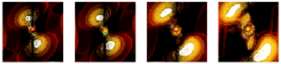
where x^{α}_0 are the coordinates of the center of the interface (at time t) and the matrix $M^{\hat{\alpha}}_{\alpha}$ is given by $\partial_\alpha = M^{\hat{\alpha}}_{\alpha}\mathbf{e}_{\hat{\alpha}}$, calculated at x^{α}_0. The 4 four-vectors $\mathbf{e}_{\hat{\alpha}}$ with $\mathbf{e}_{\hat{0}} := \mathbf{n}$ and $\mathbf{e}_{\hat{\imath}}$ form an orthonormal basis in the plane orthogonal to $\mathbf{n}$, with $\mathbf{e}_{\hat{1}}$ normal to the surface Σ_{x^1} and $\mathbf{e}_{\hat{2}}$ and $\mathbf{e}_{\hat{3}}$ tangent to that surface. The vectors of this basis verify $\mathbf{e}_{\hat{\alpha}} \cdot \mathbf{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$ with $\eta_{\hat{\alpha}\hat{\beta}}$ (the Minkowski metric)

- **All the theoretical work developed in the last years, concerning SRRS, can be exploited** $\Longrightarrow$ Easy extension of HRSC methods to (test) GRHD: A **local change of coordinates** allows one to use **any of the special relativistic Riemann solvers** in numerical calculations of **general relativistic** flows

EU-Training Network (2000-2003): Whisky

Welcome to the EU Training Network "Sources of Gravitational Waves"!

Theoretical Foundations of Sources for Gravitational Wave Astronomy of the Next Century: Synergy between Supercomputer Simulations and Approximation Techniques

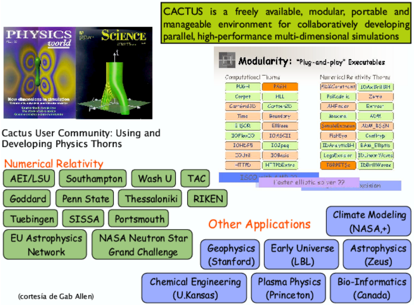


The European Commission has funded a Research Training Network (contract HPRN-CT-2000-00137) of ten institutions to study sources of gravitational waves in preparation for the upcoming generation of gravitational wave detectors. The project started in August 2000 and will run for three years. The project aims to use a combination of large scale numerical simulation techniques and approximation techniques. For details, please see the Project Description.

This timely and exciting Network links the following ten European institutions:

- Max-Planck Institut fuer Gravitationsphysik, Golm bei Potsdam, Germany. (coordinating partner).
- Theoretisch-Physikalisches Institut, University of Jena, Germany.
- Observatoire de Paris, Section de Meudon, France.
- Departamento de Astronomia y Astrofisica, University of Valencia, Spain.
- Departament de Fisica, University of Palma, Spain.
- Department of Physics, University of Thessaloniki, Greece.
- Dipartimento di Fisica, Universita di Roma "La Sapienza", Italy.
- Scuola Internazionale Superiore di Studi Avanzati, Trieste, Italy.
- Department of Mathematics, Southampton Univesity, UK.
- School of Computer Science and Maths, University of Portsmouth, UK.

Third EU-Network meeting (Southampton)



Members of the Valencia group have contributed to the *development of fully general relativistic three-dimensional codes such as the CACTUS code for Numerical Relativity (www.cactuscode.org). and the WHISKY code for Numerical Relativistic Hydrodynamics (www.whiskycode.org).*

Two RMHD codes: MRGenesis (SR), CoCoNuT (GR)

MRGenesis: A common framework for RHD, RMHD and RRMHD
Aloy et al.(1999), Leismann et al. (2005), Mimica et al. (2009)

- A multidimensional (1D, 2D or 3D) parallel (MPI) code which allows one to study astrophysical scenarios governed by Special Relativistic magnetohydrodynamical processes.
- Finite Volume approach
- Method of lines: separate semi-discretization of space and time
- Time advance: TVD Runge Kutta methods of 2nd and 3rd order
- High-Resolution Shock Capturing methods (HRSC)
- Inter-cell reconstruction: Up to 3rd order using PPM algorithm
- RMHD: constrained transport
- RRMHD: Munz's method (Lagrange multipliers) to conserve $\vec{\nabla} \cdot \vec{B}$ and charge
- Several orthogonal coordinate systems: Cartesian, Cylindrical, Spherical
- SPEV for problems where the non-thermal emission of R(M)HD

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CoCoNuT *(Core Collapse Code with Nu Technologies)*

A general-relativistic (magneto-)hydro code to evolve matter fields in a FCF dynamical spacetime

CoCoA = HRSC (Hydro Solver) $\bigoplus$ Standard Methods for elliptic PDEs (Einstein Solver: CFC)

Dimmelmeier, Font, Müller (2002)

CoCoA+ = HRSC (Hydro Solver) $\bigoplus$ Standard Methods for elliptic PDEs (Einstein Solver:CFC+)

Cerdá-Durán, Dimmelmeier, Font, Ibáñez, Müller (2005)

CoCoNuT = HRSC (Hydro Solver) $\bigoplus$ Spectral Methods (Einstein Solver: CFC)

Dimmelmeier, Font, Ibáñez, Müller, Novak (2005)

CoCoNuT-M = HRSC (Magneto-Hydro Solver) $\bigoplus$ Spectral Methods (Einstein Solver: FCF)

Cerdá-Durán, Cordero-Carrión, Dimmelmeier, Font, Gourgoulhon, Ibáñez, Jaramillo, Müller, Novak

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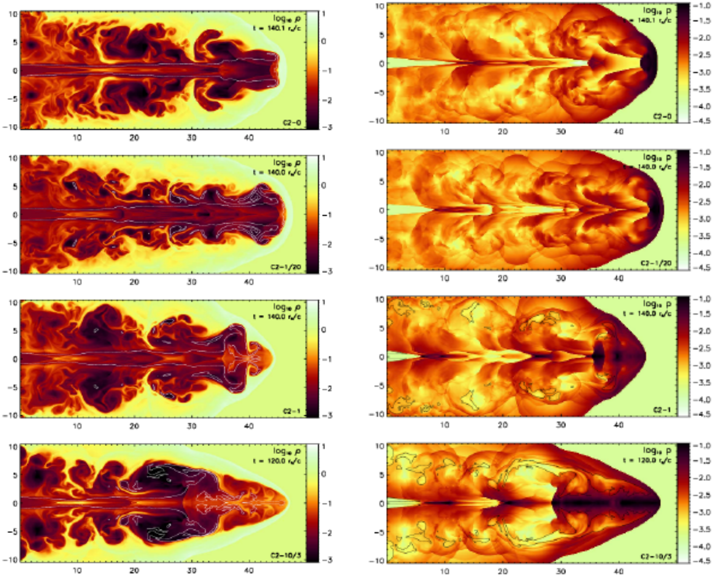
Magnetized Relativistic Jets *Leismann, Antón, Aloy, Müller, Martí, Miralles, Ibáñez (2005)*



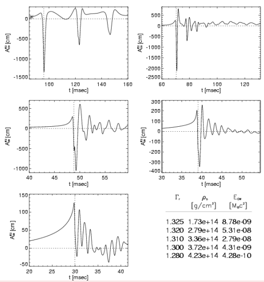
The intensity of the toroidal field increases from top to bottom. Density (*Left*). Pressure (*Right*)

Gravitational Waves from Core Collapse Supernovae (simplified EOS)

Historical achievements

- **Müller, 1982** ⟹ **2D Newtonian. First numerical evidence of the low gravitational efficiency ($E < 10^{-6} Mc^2$) of the core-collapse scenario.**
- **Finn & Evans, 1990** ⟹ **Confirmed Müller's results. Improved quadrupole formula.**
- **Bonazzola & Marck, 1993** ⟹ **First 3D simulations using pseudospectral methods, very accurate and free of numerical or intrinsic viscosity. They found that, still, low amount of energy is radiated in gravitational waves, regardless of whether the initial conditions of the collapse are axisymmetric, rotating or tidally deformed**
- **Zwerger & Müller, 1997** ⟹ **2D Newtonian. Rotating stellar cores.**
- **Rampp, Müller & Ruffert, 1998** ⟹ **3D Newtonian. Rapidly-rotating core-collapse, focusing on non-axisymmetric instabilities**

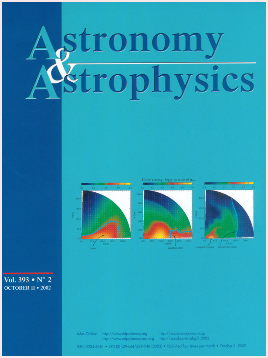
Zwerger & Müller's catalogue of wave-forms (1997)



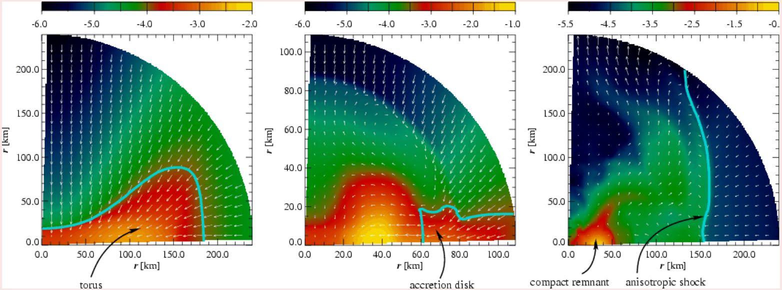
Γ_r	ρ_b [g/cm³]	E_{gw} [$M_\odot c^2$]
1.325	1.73e+14	8.78e-09
1.320	2.79e+14	5.31e-08
1.310	3.36e+14	2.79e-08
1.300	3.72e+14	4.31e-09
1.280	4.23e+14	4.28e-10

Dimmelmeier, Font & Müller, 2001,2002

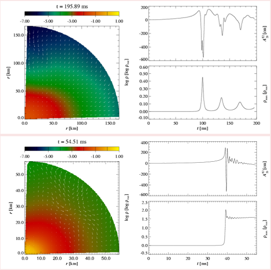
- *First relativistic attempt. 2D axisymmetric simulations with CFC metric (Isenberg, Wilson & Mathews).* ⟹ **CoCoA code**
- **To extend Newtonian simulations** ⟹ **To determine whether GR effects make a difference in overcoming the angular momentum threshold.**
- **To extract gravitational radiation from core collapse and more realistic waveforms .**
- **To develop a versatile and extensible code for simulating highly relativistic rotating stars .**



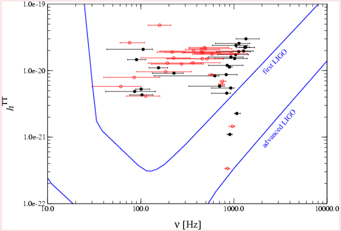
Rotational Core Collapse: DFM's catalogue of wave-forms *(Dimmelmeier, Font & Müller, 2002)*



Formation of a torus and shock-propagation in the very rapidly and highly differentially rotating model A4B5G5. The three snapshots show color coded contour plots of the density, $(\log\ \rho,$ scaled to nuclear matter density), together with the meridional flow field during the infall phase at $t\ = 25.0$ ms (left panel), shortly before the centrifugal bounce at $t\ = 31.2$ ms (middle panel), and at $t\ = 35.0$ ms (right panel).



(Top) Model A2B4G1 *(Bottom)* Model A3B2G4

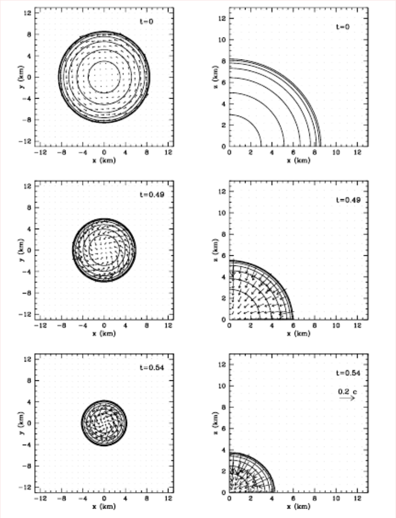


Prospects of detection of the gravitational wave signal from axisymmetric rotational supernova core collapse in relativistic (black filled circles) and Newtonian (red unfilled circles) gravity (D = 10 kpc). 26 models.

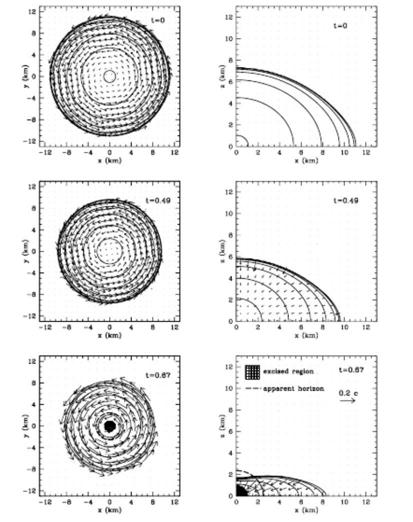
Three-dimensional relativistic simulations of rotating neutron star collapse to a Kerr BH

Baiotti, L., Hawke, I., Montero, P.J., Loeffler, F., Rezzolla, L., Stergioulas, N., Font, J.A., Seidel, E., (2005)

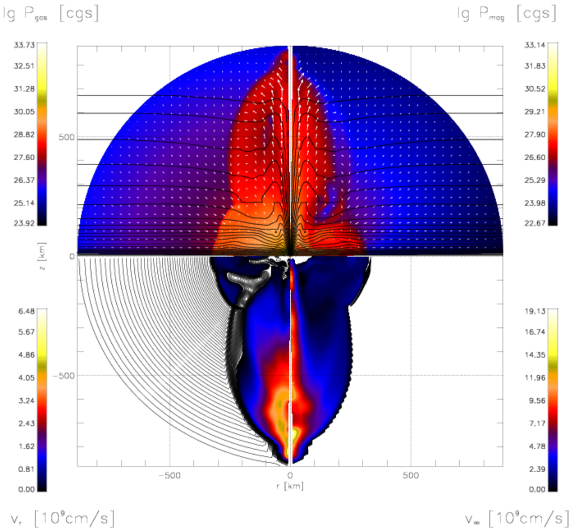
Collapse sequence for the slowly rotating model.

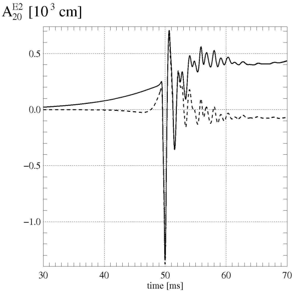


Collapse sequence for the rapidly rotating model.



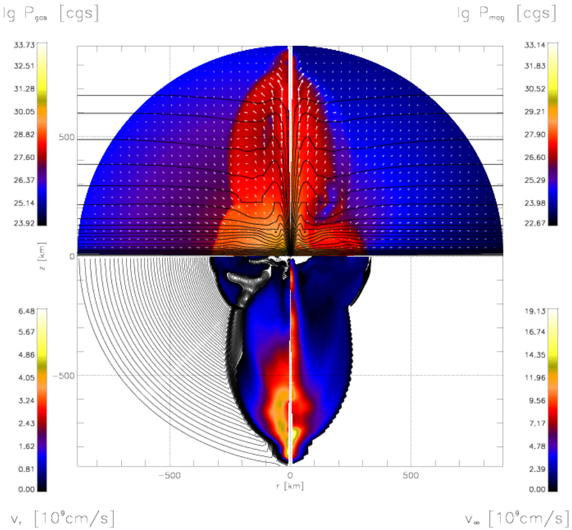
Magnetorotational core collapse *Obergaulinger, Aloy, Müller, A&A*, **450**, *1107 (2006); Obergaulinger, Aloy, Dimmelmeier, Müller, A&A*, **457**, *209 (2006)*

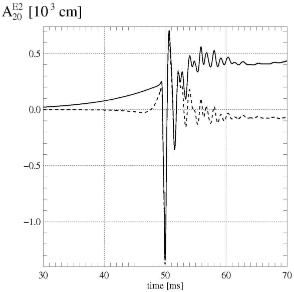




- A complete parameter set up ($>$ 50 models).
- Dynamical development of strong magnetic fields (MRI).
- New types of GW signals related with the presence of the magnetic field $\Longrightarrow$ signatures of the production of jets.

Magnetorotational core collapse *Obergaulinger, Aloy, Müller, A&A,* **450***, 1107 (2006); Obergaulinger, Aloy, Dimmelmeier, Müller, A&A,* **457***, 209 (2006)*





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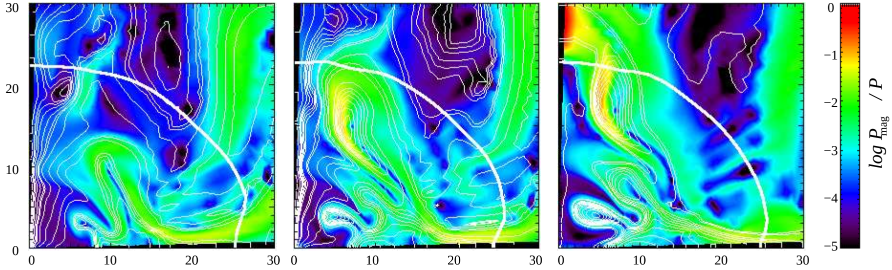
Current works

- Cerdá-Durán and coll. (2010) *CFC, FCF (CoCoNuT)*
- Sato's school (Yamada, Kotake, ...)

Magneto-rotational Instability in Core-Collapse Supernovae

M. Obergaulinger, P. Cerdá-Durán, E. Müller, M.A. Aloy, (2009, 2010)

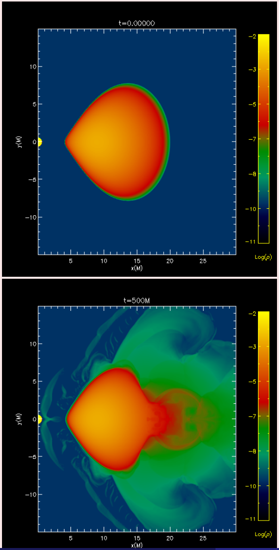


Three snapshots of the innermost 30 km in the post-bounce evolution of a magnetorotational stellar core collapse simulation. Color coded in the logarithm of the ratio of magnetic pressure to thermal pressure. Thin lines indicate magnetic field lines, and thick white line the position of the neutrino-sphere. The elongated structures inside the Proto-Neutron star are channel flows, a distinctive feature of the growth of the magneto-rotational instability.

Self-gravitating accretion tori around a black hole

P. Montero, J.A. Font, M. Shibata (2010)

Three snapshots: $t = 0, 2, 7$ (units of t_{orb})



Self-gravitating accretion tori around a black hole

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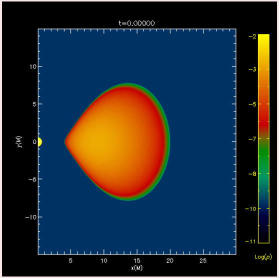
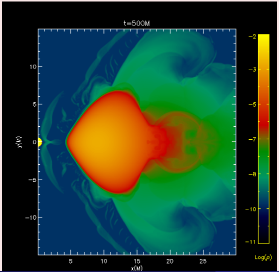
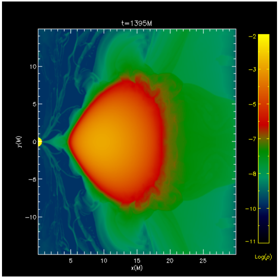






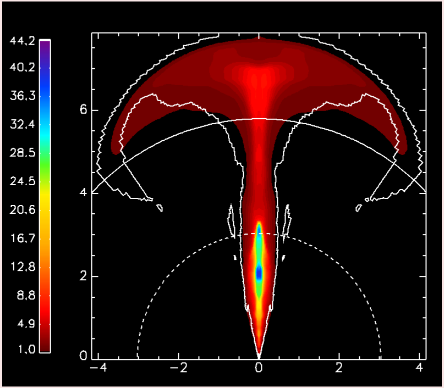
Captions

Figures show three different snapshots of the evolution of a self-gravitating torus, up to a final time of $t = 7\ t_{orb}$. The initial perturbation triggers the accretion of mass and angular momentum through the cusp and on to the BH . Our simulations show that such a quasi-periodic oscillatory behaviour, which had already been found in the test-fluid simulations of non-self-gravitating disks, is also present when the self-gravity and fully dynamical spacetime and hydrodynamical evolutions are incorporated in the numerical modelling. The colour coded iso density contours displayed also show the interesting dynamics at the boundary region separating the disk from the external medium. In particular, Kelvin-Helmhotz-driven eddies are seen being shed downwind from the edge of the disk during each oscillation Parameters: $M_{torus}/M_{bh} = 1.0, r_{in} = 4.02, r_{out} = 19.97, t_{orbit} = 199.54$ (units of $G = c = M_{\odot} = M_{bh} = 1$)

GRBs of long duration (lGRB): Relativistic Jets from Collapsars

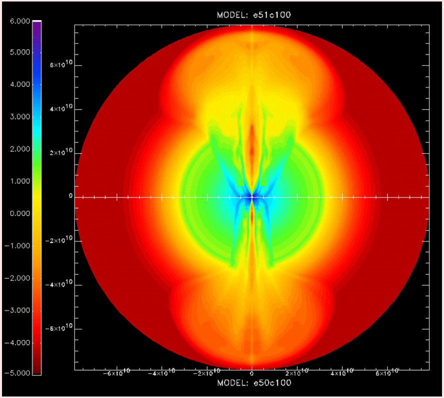
(Aloy, Müller, Ibáñez, Martí, MacFadyen, 2000)

lGRBs: *Lorentz factor*



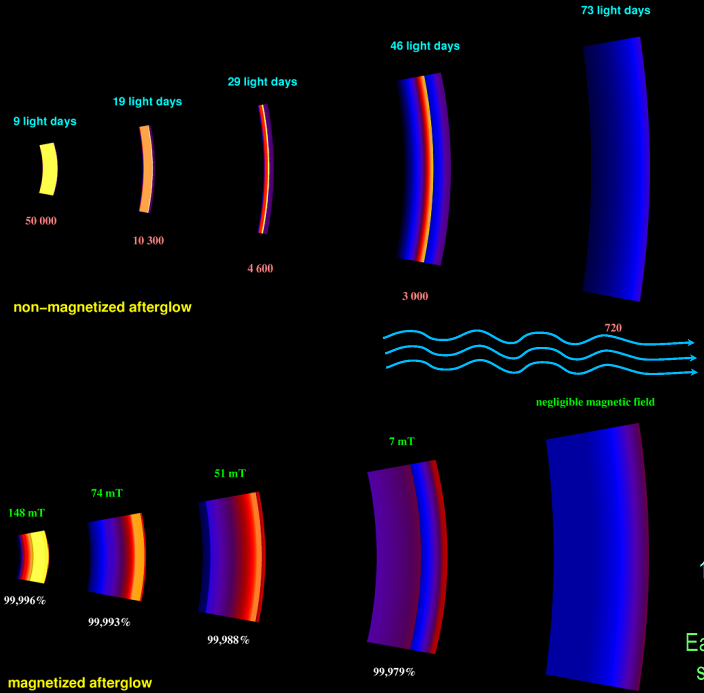
Lorentz factor about 1.8 s after shock breakout. The numbers on the axes give the length in units of 100,000 km. Dashed and solid arcs mark the stellar surface and the outer edge of the exponential stellar atmosphere, respectively. The other solid line encloses matter with a radial velocity larger than 0.3c, and an internal energy density larger than 5% of the rest-mass energy.

lGRBs: *Rest-mass density*



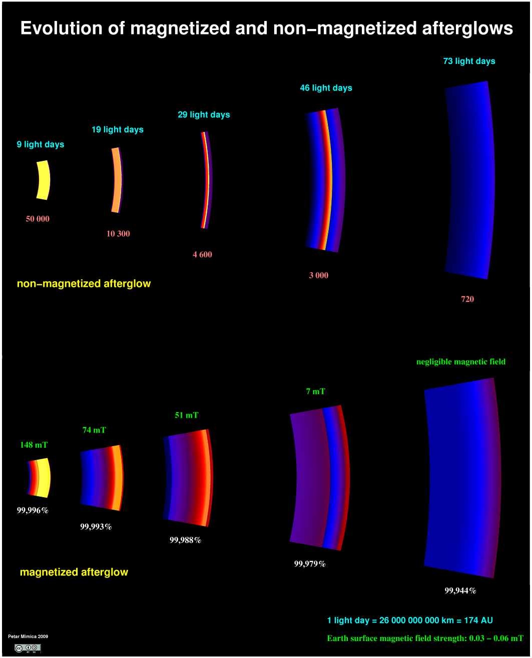
Rest-mass density for the models with a constant energy deposition rate of 10^{51} erg/s (top) and 10^{50} erg/s (bottom) respectively, about 1.8 s after shock breakout. The numbers on the axes give the length in units of centimeters.

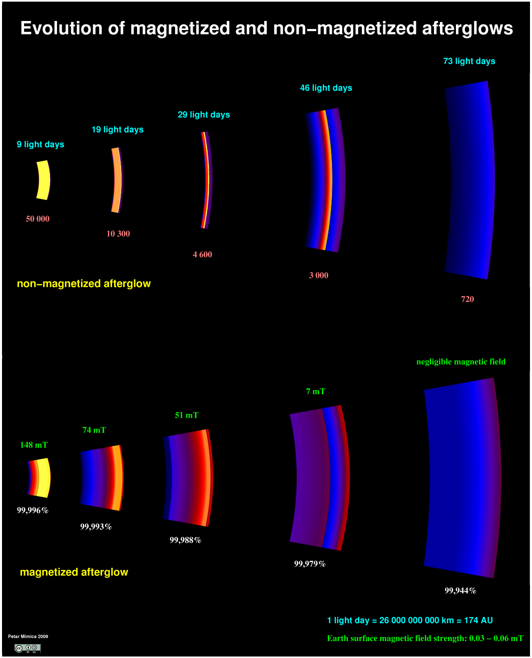
Evolution of magnetized and non-magnetized afterglows



1 light-day = 2.6×10^{15} cm
= 174 AU

Earth surface magnetic field strength = 0.03 - 0.06 mT

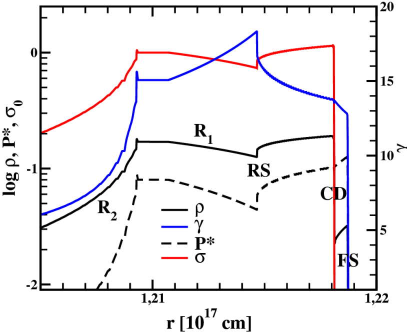

Evolution of magnetized and non–magnetized afterglows
9 light days
19 light days
29 light days
46 light days
73 light days
50 000
10 300
4 600
3 000
720
non–magnetized afterglow
negligible magnetic field
148 mT
74 mT
51 mT
7 mT
99,996%
99,993%
99,988%
99,979%
99,944%
magnetized afterglow
1 light day = 26 000 000 000 km = 174 AU
Earth surface magnetic field strength: 0.03 – 0.06 mT
Petar Mimica 2009



GRB: Magnetized afterglows

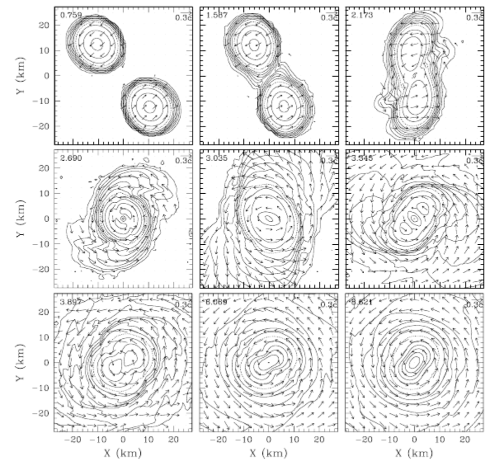
Mimica, Giannios & Aloy (2009)

- Ejecta/ISM interaction $\Longrightarrow$ a *forward shock* forms in both magnetized and unmagnetized ejecta, paired with either a *"reverse" shock* (yellow arc in the third snapshot of the upper panel) or a *"reverse" rarefaction* (red-to-blue shades in the third snapshot of the lower panel), depending on the degree of magnetization.
- *The work has allowed to put quantitative limits on the typical strength of the magnetic field in the ejecta, above which we do not expect to observe an optical flash*
- A new set of scaling laws $\Longrightarrow$ to extrapolate the results of numerical models with moderate values of the initial bulk Lorentz factor (~ 15) of the ejecta to *equivalent* models with much larger Lorentz factors 100.



Merger of binary neutron stars *Shibata, Taniguchi & Uryu (2005)*

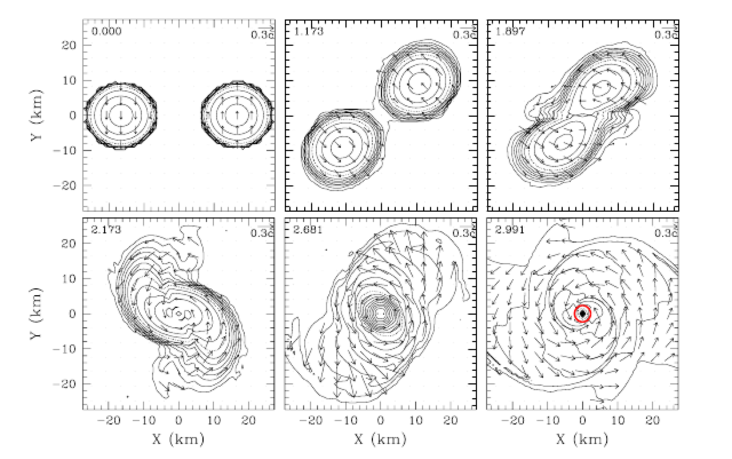
Model SLy1313a ($NS + NS \rightarrow SMNS$): $\Gamma_{\rm th} = 2$, $Grid = (633, 633, 317)$, $\lambda_0 = 316km$, $\lambda_{\rm merger} = 94km$



Snapshots of the density contour curves for ρ in the equatorial plane for model SLy1313a. The solid contour curves are drawn for $\rho = 2 \times 10^{14} \times i \ {\rm g/cm^3}$ $(i = 2 \sim 10)$ and for $2 \times 10^{14} \times 10^{-0.5i} \ {\rm g/cm^3}$ $(i = 1 \sim 7)$. The dotted curves denote $2 \times 10^{14} \ {\rm g/cm^3}$. The number in the upper left-hand side denotes the elapsed time from the beginning of the simulation in units of ms. The initial orbital period in this case is 2.110 ms. Vectors indicate the local velocity field (v^x, v^y), and the scale is shown in the upper right-hand corner.

Merger of binary neutron stars *Shibata, Taniguchi & Uryu (2005)*

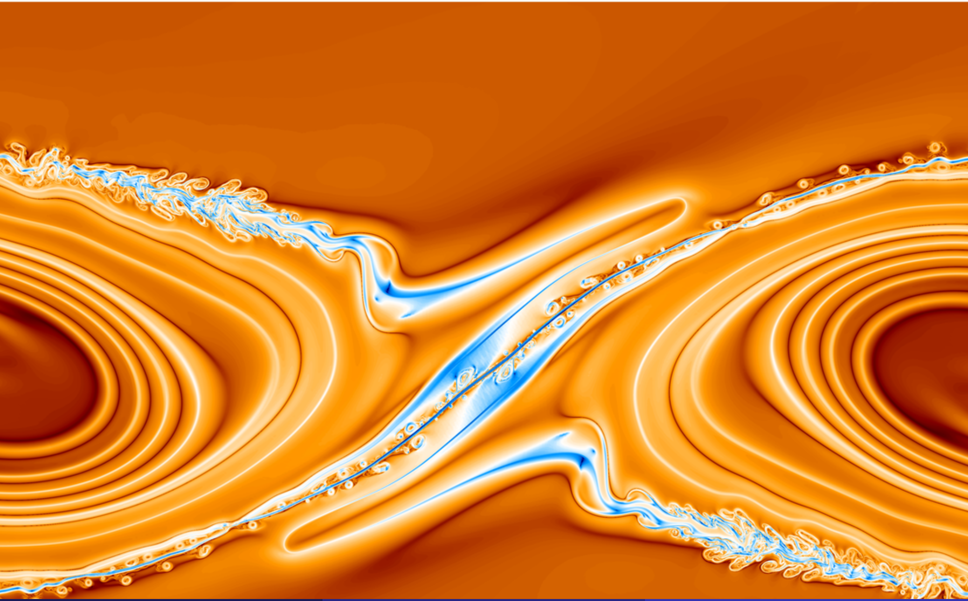
Model SLy1414a $(NS + NS \rightarrow BH)$: $\Gamma_{\rm th} = 2$, $Grid = (633, 633, 317)$, $\lambda_0 = 302 km$



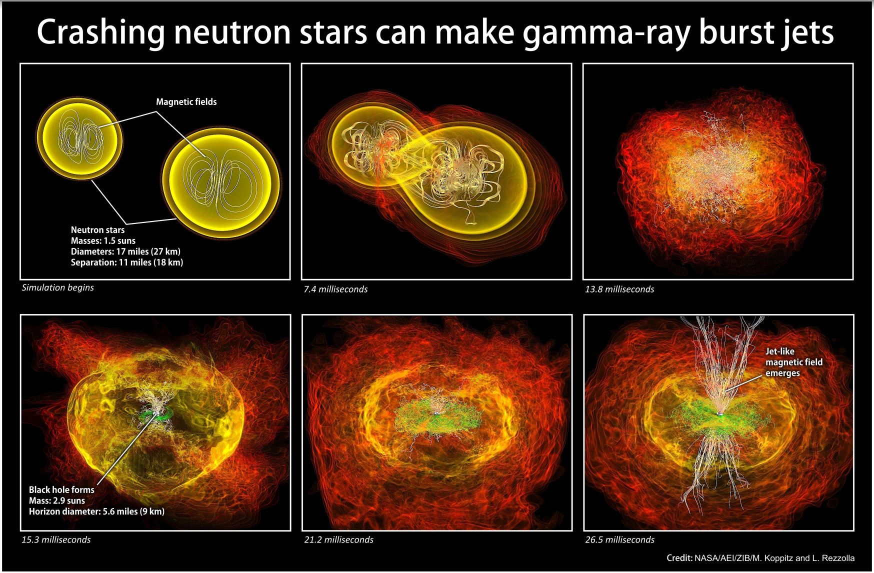
Snapshots of the density contour curves for ρ in the equatorial plane for model SLy1414a. The solid contour curves are drawn for $\rho = 2 \times 10^{14} \times i \ {\rm g/cm^3} \ (i = 2 \sim 10)$ and for $2 \times 10^{14} \times 10^{-0.5i} \ {\rm g/cm^3} \ (i = 1 \sim 7)$. The dotted curves denote $2 \times 10^{14} \ {\rm g/cm^3}$. The number in the upper left-hand side denotes the elapsed time from the beginning of the simulation in units of ms. The initial orbital period in this case is 2.012 ms. Vectors indicate the local velocity field (v^x, v^y), and the scale is shown in the upper right-hand corner. The thick circle in the last panel of radius $r \sim 2$ km denotes the location of the apparent horizon.

Simulations of the magnetized Kelvin-Helmholtz instability in neutron-star mergers

Obergaulinger, Aloy, Müller (2010)



Astrophysical scenarios governed by relativistic (magneto-)hydrodynamical processes



Merging neutron stars produce jet-like structures and can power short Gamma-Ray Bursts

(Rezzolla, Giacomazzo, Baiotti, Granot, Kouveliotou, Aloy, 2011)

Outline

1 Introduction: What a wonderful *(relativistic)* world

2 The Hydro-solver: General Relativistic Hydrodynamics (GRHD)

3 The Magnetohydro-solver (II): GRMHD

4 The Einstein-solver

5 The current virtual *(numerical)* relativistic world: Simulations

6 Conclusions and Perspectives

Summary and conclusions

1. The theoretical foundations on the characteristic structure of the GRHD & GRMHD equations are at hand.
2. The exact solution of the Riemann problem (both in RHD, and RMHD) are known.
3. The full spectral decomposition of the flux vector Jacobians of the RHD and RMHD equations is known. A renormalized sets of right and left eigenvectors of the flux vector Jacobians of the RMHD equations, which are regular and span a complete basis in any physical state including degenerate ones have achieved.
4. As a consequence of the above conclusions, it is possible to develop robust and efficient linearized Riemann solvers based on the renormalized spectral decomposition of the RHD and RMHD equations.
5. Choosing the adequate restrictions on characteristic variables serves optimally the purpose of setting boundary conditions. The computation of characteristic variables needs of the knowledge of the complete set of left and right eigenvectors.
6. Stable formulations of Einstein Equations $\bigoplus$ *combined with accurate numerical techniques.*

Perspectives

- **Hydro-solver: Beyond the ideal case**
  1. **Perfect fluid $\Longrightarrow$ Radiation (photons, neutrinos,...) transport, ...**
  2. **Ideal MHD $\Longrightarrow$ Finite conductivity, reconnection,...**
- **Einstein-solver: Beyond "BSSN"**
  1. **Characteristic formulation $\Longrightarrow$ Winicour and coll.**
  2. **Hyperbolic formulations $\Longrightarrow$ Bona and coll. (Z_i , $i = ..., 3, 4, ...$)**
  3. **FCF $\Longrightarrow$ Meudon-Valencia formulation.** *Numerical implementation (into* `CoCoNuT` *) is underway.*
- **The golden age of (General) Relativistic Astrophysics/Astrophysical Relativity.**