Some Recent Results on Finite Volume, Gradient Discretization methods and Ill Posed Problems

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Based on joint works with Benkhaldoun (USPN-Paris) and Lesnic (Leeds University)

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Plan of this presentation

Main References on FVMs

1. R. Eymard, T. Gallouët, and R. Herbin: Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. SUSHI: A scheme using stabilization and hybrid interfaces. IMAJNA, 2010.

2. R. Eymard, T. Gallouët, and R. Herbin: Finite volume methods. Ciarlet, P. G. (ed.) et al., Solution of equations in $\mathbb{R}^n$ (Part 3). Techniques of scientific computing (Part 3). Amsterdam: North-Holland/ Elsevier. Handbook of Numerical Analysis 7, 713–1020, 2000.

Principles of Finite Volume methods

Finite volume methods are numerical methods approximating different types of Partial Differential Equations (PDEs). They are based on three principle ideas:

- Subdivision of the spatial domain into subsets called Control Volumes.
- Integration of the equation to be solved over the Control Volumes.
- Approximation of the derivatives appearing after integration.

[1] We mean here the "pure" finite volume methods and not finite volume-element methods

Finite Volume methods on admissible meshes

Definition

Let $\mathcal{T}$ be an Admissible Mesh in the sense of Eymard et al. (Handbook, 2000).

$K \in \mathcal{T}$ are the control volumes and σ are the edges of the control volumes K.

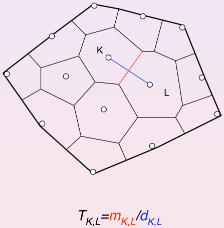

Figure: transmissivity between K and L: $\mathcal{T}_\sigma = \mathcal{T}_{K|L} = \frac{m_{K,L}}{d_{K,L}}$

Finite Volume methods on admissible meshes

Main properties of Admissible mesh:

1. Convexity of the Control Volumes.
2. The orthogonality property: the $(\boldsymbol{x}_K \boldsymbol{x}_L)$ is orthogonal to the common edge σ between the control volumes K and L.

Finite Volume methods on admissible meshes

Model to be solved:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \quad \text{and} \quad u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega. \tag{1}$$

Principles of Finite Volume scheme:

1. Integration on each control volume K : $-\int_K \Delta u(\boldsymbol{x}) d\boldsymbol{x} = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$,
2. Integration by Parts gives : $-\int_{\partial K} \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$
3. Summing on the lines of K: $-\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) dx$

Finite Volume methods on admissible meshes

Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

$$-\sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_{K|L}}(u_L - u_K) = \int_K f(\boldsymbol{x})d\boldsymbol{x}. \tag{2}$$

Matrix Form

$$\mathcal{A}^{\mathcal{T}} u_{\mathcal{T}} = f_{\mathcal{T}}.$$

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Finite Volume methods on admissible meshes

Theorem

Let $\mathcal{X}(\mathcal{T})$: functions which are constant on each control volume K. Let $e_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$ be defined by $e_K = u(\boldsymbol{x}_K) - u_K$ for any $K \in \mathcal{T}$. Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence results hold:

1. *H_0^1-error estimate*

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \leq Ch\|u\|_{2,\overline{\Omega}}, \tag{3}$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the H_1^0-norm $\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 = \sum_{\sigma=K|L\in\mathcal{E}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u_L - u_K)^2.$

2. *L^2-error estimate:*

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{4}$$

Finite Volume methods using nonconforming grids, SUSHI scheme

Definition (New mesh of Eymard et *al.*, IMAJNA 2010)

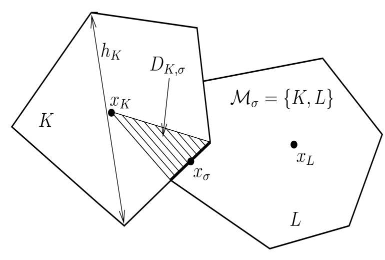

Figure: Notations for two neighbouring control volumes in $d = 2$

Finite Volume methods using nonconforming grids, SUSHI scheme

Main properties of this new mesh:

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

Finite Volume methods using nonconforming grids, SUSHI scheme

Principles of discretization for Poisson's equation:

1. Discrete unknowns: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{\left((v_K)_{K\in\mathcal{M}}\,,\,(v_\sigma)_{\sigma\in\mathcal{E}}\right),\ \ v_K, v_\sigma \in \mathbb{R},\ \ v_\sigma = 0,\ \forall\sigma \in \mathcal{E}_{\text{ext}}\}$$

2. Discretization of the gradient: the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):

   1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
   2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

Finite Volume methods using nonconforming grids, SUSHI

Weak formulation for Poisson's equation: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{5}$$

SUSHI (Scheme Using stabilized Hybrid Interfaces) for Poisson's equation: Find $u_\mathcal{D} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_\Omega \nabla_\mathcal{D} u_\mathcal{D}(\boldsymbol{x}) \cdot \nabla_\mathcal{D} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{6}$$

Finite Volume methods using nonconforming grids, SUSHI

Theorem

Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence result hold:

1. H_0^1*-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{7}$$

2. L^2*-error estimate:*

$$\|u - \Pi_{\mathcal{M}} u_{\mathcal{D}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{8}$$

Some works related to the item: FVMs

1. F. Benkhaldoun, A. Bradji: SUSHI for a non-linear time fractional diffusion equation with a time independent delay. Numerical methods and applications, 73–84, Lecture Notes in Comput. Sci., 13858, Springer, Cham, 2023.
2. F. Benkhaldoun, A. Bradji: Convergence analysis of a finite volume scheme for a distributed order diffusion equation. Numerical methods and applications, 59–72, Lecture Notes in Comput. Sci., 13858, Springer, Cham, 2023.
3. Bradji: An analysis of a second order time accurate scheme for a finite volume method for parabolic equations on general nonconforming multidimensional spatial meshes. AMC, 2013.
4. Bradji: Convergence analysis of some high–order time accurate schemes for a finite volume method for second order hyperbolic equations on general nonconforming multidimensional spatial meshes. NMPDE, 2013

Main references on GDM (Gradient Discretization Method)

Main reference on GDM

1. J. Droniou, R. Eymard, T. Gallouët, C. Guichard, R. Herbin: The Gradient Discretisation Method. Mathématiques and Applications (Berlin) [Mathematics and Applications], 82. Springer, Cham, 2018.
2. J. Droniou, R. Eymard, T. Gallouët, C. Guichard, R. Herbin: Gradient schemes: a generic framework for the discretisation of linear, nonlinear and nonlocal elliptic and parabolic equations. Math. Models Methods Appl. Sci. 23/13, 2395–2432, 2013.
3. R. Eymard, C. Guichard, R. Herbin: Small-stencil 3D schemes for diffusive flows in porous media. ESAIM Math. Model. Numer. Anal. 46/2, 265–290, 2012.

GDM: Definition of the Approximate Gradient Discretization

Definition (Definition of a generic approximate gradient discretization)

Let Ω be an open domain of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$. An approximate gradient discretization $\mathcal{D}$ is defined by $\mathcal{D} = (\mathcal{X}_{\mathcal{D},0}, h_{\mathcal{D}}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$, where

1. The set of discrete unknowns $\mathcal{X}_{\mathcal{D},0}$ is a finite dimensional vector space on $\mathbb{R}$.
2. The space step $h_{\mathcal{D}} \in (0, +\infty)$ is a positive real number.
3. The linear mapping $\Pi_{\mathcal{D}} : \mathcal{X}_{\mathcal{D},0} \to L^2(\Omega)$ is the reconstruction of the approximate function.
4. The mapping $\nabla_{\mathcal{D}} : \mathcal{X}_{\mathcal{D},0} \to L^2(\Omega)^d$ is the reconstruction of the gradient of the function; it must be chosen such that $\|\nabla_{\mathcal{D}} \cdot \|_{L^2(\Omega)^d}$ is a norm on $\mathcal{X}_{\mathcal{D},0}$.

Additional hypotheses on the approximate gradient discretization

Definition (Additional hypotheses on the approximate gradient discretization)

- The **coercivity** of the discretization is measured through the the constant $C_{\mathcal{D}}$ given by:

$$C_{\mathcal{D}} = \max_{v \in \mathcal{X}_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}} v\|_{L^2(\Omega)}}{\|\nabla_{\mathcal{D}} v\|_{L^2(\Omega)^d}}. \tag{9}$$

- The **strong consistency**: $S_{\mathcal{D}} : H_0^1(\Omega) \to [0, +\infty)$ defined by, for all $\varphi \in H_0^1(\Omega)$

$$S_{\mathcal{D}}(\varphi) = \min_{v \in \mathcal{X}_{\mathcal{D},0}} \left(\|\Pi_{\mathcal{D}} v - \varphi\|^2_{L^2(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla \varphi\|^2_{L^2(\Omega)^d} \right)^{\frac{1}{2}}. \tag{10}$$

- The **dual consistency**: For all $\varphi \in H_{\text{div}}(\Omega)$, $W_{\mathcal{D}}(\varphi)$ is given by

$$\max_{u \in \mathcal{X}_{\mathcal{D},0} \setminus \{0\}} \frac{1}{\|\nabla_{\mathcal{D}} u\|_{L^2(\Omega)^d}} \left| \int_{\Omega} \left(\nabla_{\mathcal{D}} u(\boldsymbol{x}) \cdot \varphi(\boldsymbol{x}) + \Pi_{\mathcal{D}} u(\boldsymbol{x}) \text{div} \varphi(\boldsymbol{x}) \right) d\boldsymbol{x} \right|.$$

First example on the approximate gradient discretization: Conforming finite element method

Let $\{\mathcal{T}_h; h > 0\}$ be a family of shape regular and quasi-uniform triangulations of the domain Ω. Let $\mathcal{V}^h$ be the standard finite element space of continuous, piecewise polynomial functions of degree less or equal $l \in \mathbb{N} \setminus \{0\}$ and we denote by $\mathcal{V}_0^h = \mathcal{V}^h \cap H_0^1(\Omega)$. Assume that $\mathcal{V}_0^h$ is spanned by the usual basis functions $\varphi_1, \ldots, \varphi_M$. The space $\mathcal{X}_{\mathcal{D},0}$ can be $\mathbb{R}^M$ and for any $(u_1, \ldots, u_M) \in \mathcal{X}_{\mathcal{D},0}$, we define $\Pi_{\mathcal{D}} u = \sum_{i=1}^M u_i \varphi \in \mathcal{V}_0^h \subset H_0^1(\Omega)$ and $\nabla_{\mathcal{D}} u = \sum_{i=1}^M u_i \nabla \varphi = \nabla \Pi_{\mathcal{D}} u$. Using the Poincaré inequality, we have for all $u \in \mathcal{X}_{\mathcal{D},0}$, $\|\Pi_{\mathcal{D}} u\|_{\mathbb{L}^2(\Omega)} \leq C(\Omega) \|\nabla_{\mathcal{D}} u\|_{\mathbb{L}^2(\Omega)}$.

Therefore, the assumption (9) of Definition 5 holds with constant $C_{\mathcal{D}}$ only depending on Ω. In addition to this, we have $W_{\mathcal{D}}(\varphi) = 0$, for all $\varphi \in H_{\rm div}(\Omega)$, and $S_{\mathcal{D}}(\varphi)$ is bounded above by (up to a multiplicative constant independent of the mesh) $h^l |\varphi|_{l+1,\Omega}$, for all $\varphi \in H^{l+1}(\Omega)$.

Other example on the approximate gradient discretization: SUSHI method

Second example

SUSHI method, cf. Eymard et al. (IMAJNA, 2010).

Third example

Mimetic Finite Difference methods, cf. Brezzi et al. (Math. Models Methods Appl. Sci., 2005).

Fourth example

Mixed Finite Volume method, cf. Droniou et al. (Numer. Math., 2006).

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It is shown in Droniou et al. (Math. Models Methods Appl. Sci., 2010) that the Second example–Fourth example mentioned can be identified to each other.

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How to use GS: an example of application

Weak formulation for Poisson's equation

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{11}$$

GS for Poisson's equation

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$$\int_\Omega \nabla_\mathcal{D} u_\mathcal{D}(\boldsymbol{x}) \cdot \nabla_\mathcal{D} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in\in \mathcal{X}_{\mathcal{D},0}. \tag{12}$$

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Control of the error, cf. Eymard, Guichard, and R. Herbin (M2AN, 2012)

Theorem

Assume that $u \in H^2(\Omega)$. The following convergence results hold:

1. H_0^1*-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \leq W_{\mathcal{D}}(\nabla u) + 2S_{\mathcal{D}}(u). \tag{13}$$

2. L^2*-error estimate:*

$$\|u - \Pi_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)} \leq C_{\mathcal{D}} W_{\mathcal{D}}(\nabla u) + (C_{\mathcal{D}} + 1) S_{\mathcal{D}}(u). \tag{14}$$

Some works related to the item: GDM

1. F. Benkhaldoun, A. Bradji, Abdallah: $L^\infty(H^1)$-error estimate for gradient schemes applied to time fractional diffusion equations. Finite volumes for complex applications X. Vol. 1. Elliptic and parabolic problems, 177–185, Springer Proc. Math. Stat., 432, Springer, Cham, 2023.
2. A. Bradji: Notes on the convergence order of gradient schemes for time fractional differential equations. C. R. Math. Acad. Sci. Paris 356/4, 439–448, 2018.
3. A. Bradji: Convergence analysis of some first order and second order time accurate gradient schemes for semilinear second order hyperbolic equations. NMPDE, 2017
4. A. Bradji: Some new first order and second order time accurate gradient schemes for semilinear parabolic equations. Comput. Math. Appl. 2016.

An approach to establish high order approximations

Standard approach to get high order approximations

The classical approach to construct high order approximations for PDE $\mathcal{L}u = f$, we have either to use a high order difference method/high degree trial functions in the case of finite difference and finite element methods respectively.
The drawback of these approaches is that they lead to systems to be solved with matrices with many non-zeros entries.

Our approach to follow to establish high order approximations

We use low order schemes to generate successively high order approximations. The advantage of low order schemes is that the systems will have sparse matrices. Consequently, the computational cost is not expensive.

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Some works related to the item: High order approximations

1. A. Bradji: A theoretical analysis of a new second order finite volume approximation based on a low-order scheme using general admissible spatial meshes for the one dimensional wave equation. J. Math. Anal. Appl. 422/1, 109–147, 2015.

2. A. Bradji: A full analysis of a new second order finite volume approximation based on a low-order scheme using general admissible spatial meshes for the unsteady one dimensional heat equation. J. Math. Anal. Appl. 416/1, 258–288, 2014.

3. A. Bradji, A. -C. Chibi: Optimal defect corrections on composite nonmatching finite-element meshes. IMA J. Numer. Anal. 27/4, 765–780, 2007.

A main reference on the subject: restoring the stability of a fractional PDE

A main reference on the subject

N. H. Tuan, T. D. Xuan, N. A. Triet, D. Lesnic: On the Cauchy problem for a semilinear fractional elliptic equation. Appl. Math. Lett. 83, 80–86, 2018.

Description of the problem

We consider the Cauchy ill-posed problem:

$$\partial_t^\alpha u(\boldsymbol{x},t) + \Delta u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \Omega_T := \Omega \times (0,T), \tag{15}$$

where $1 < \alpha < 2$, Ω is an open bounded connected subset of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$), $T > 0$ and ∂_t^α is the Caputo derivative of order α given by

$$\partial_t^\alpha \varphi(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \varphi_{\tau\tau}(\tau) d\tau, \tag{16}$$

where Γ is the Gamma function. In addition

$$u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \partial\Omega \times (0,T). \tag{17}$$

$$-u_t(\boldsymbol{x},0) = G(\boldsymbol{x}) \quad \text{and} \quad u(\boldsymbol{x},0) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{18}$$

Description of the problem (Suite)

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This problem has a unique solution (can be computed for instance using the method of separation of variables) but it does not depend continuously on the data.

Some works related to the item: restoring the stability

Restoring the stability of problem (15)–(18)

A. Bradji, D. Lesnic: A note on an iterative algorithm for solving an inverse problem for a fractional-order partial differential equation. Commun. Anal. Comput. 3, 31–39, 2025.

Description of the contribution

We suggested a series of well-posed problems, by keeping the same equation (15), whose solutions converges to the solution of problem (15)–(18).

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Conclusion and a Perspective

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We sketched some recent results on FVMs, GDM, and High order numerical solutions for PDEs and on an ill-posed Cauchy problem.

A perspective

Collaboration with Prof. Dr. Baeza, Antonio and his team.

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