Chapter 5

# The Utility of Money

'James wiped his napkin all over his mouth.
"You don't know the value of money," he said, avoiding her eye.'

*The Man of Property*, Ch. 3.

## 5.1 THE USE OF ASSETS, RATHER THAN GAINS OR LOSSES

Utility is a number measuring the attractiveness of a consequence—the higher the utility, the more desirable the consequence—the measurement being made on a probability scale. It is sometimes difficult to attach a number to a consequence because the relevant features are not naturally quantifiable. The pleasures to be derived from a walk in the country, a visit to a theatre, or winning an argument with a colleague are aesthetic and psychological and not immediately expressible in numerical terms. (In passing, note that one buys a theatre ticket or a train ticket to the country, so that there is some measured element in those situations.) Therefore it will be simpler if we start our study of utility by considering cases where there is already a numerical value present. The situations to be considered are those in which the consequences are entirely monetary. Examples are bets in which one stands to win or lose prescribed sums of money, investments on the stock exchange, insurance against loss of tangible assets, or the many circumstances in industrial technology where the outcomes of the choice of any particular design can be expressed in terms of costs and rewards. Almost all consequences have some monetary element (for example, the visit to the theatre cited above) but in the present chapter our concern is with those that are *entirely* monetary and do not have aesthetic, psychological, or moral overtones.

Consider someone contemplating the investment in a stock of 500 dollars for a period of three months. At the end of that time the stock may be worth more or less than the original sum spent on purchasing it. Suppose, for simplicity, that at the end of three months the stock will either have appreciated to 600 dollars or have dropped to 400. In practice there are many more than just these two possibilities, but the simpler situation will capture the essence of the argument whilst avoiding arithmetic complexities. Hence there are two uncertain events: $\theta_1$, increase to 600; $\theta_2$, decrease to 400. With two decisions, $d_1$ to invest and $d_2$ to keep the 500 dollars in the bank, we have a simple decision problem in the standard form already discussed. It is presented in the usual tabular form in Table 5.1, except that the entries in the body of the table are the monetary outcomes now about to be described, and not the utilities.

Consider any one of the consequences, say investment followed by a fall in price $(d_1, \theta_2)$. The important feature is a loss of 100 dollars compared with the result of not investing. This is a comparison of two consequences, not a description of one. In order to obtain the latter suppose that the investor had capital of amount $C$ at the time he bought the stock, then $(d_1, \theta_2)$ can be described by saying his capital is now $C-100$, assuming no other factors have affected it in the intervening period. The correct monetary entries are those given in Table 5.1, and it is to those that utilities must be attached, not to the losses or gains of 100 dollars, for these do not describe consequences but only differences between consequences. For any capital sum, $x$, the utility of $x$ will be written $u(x)$. In the example we require $u(C-100)$, $u(C)$, and $u(C+100)$. One immediate result of these considerations is that the decision whether or not to invest may depend on $C$, the capital at the time the decision is made, and this is surely a realistic conclusion.

Throughout this chapter we shall therefore be considering monetary consequences, where the sum of money involved describes the total assets of the decision-maker if that consequence results. The capital concerned is the total realizable capital and not just the fluid surplus capital, as can be seen by recognizing that a decision whether to invest or not may depend on the former and not merely on the latter. The security that comes from the ownership of property, which can be mortgaged if necessary, affects one's investment policy. So the sums of money that we are discussing are always non-negative and it is not sensible to talk of negative assets. The law recognizes this when fines are levelled against offenders or maintenance orders imposed on defecting husbands, and no attempt is made to extract more than the litigants possess.

Our task, therefore, is to discuss the form of the utility function $u(x)$ describing the relationship between utility and total monetary assets, $x$. Before doing this we introduce an example designed to illustrate two points: first, the need for such a function; and second, but perhaps more important, the way in which the coherence principles that have been advocated work in practice.

Table 5.1. The entries are sums of money

| | $\theta_1$: Stock appreciates | $\theta_2$: Stock depreciates |
|---|---|---|
| $d_1$: Invest | $C+100$ | $C-100$ |
| $d_2$: Leave in bank | $C$ | $C$ |



## 5.2 INCOHERENCE

The example concerns four bets listed as follows:

| Bet | Lose | Win |
|---|---|---|
| I | 10 | 10 |
| II | 10 | 20 |
| III | 20 | 10 |
| IV | 20 | 20 |

Thus bet I, if accepted, will either lose you 10 dollars or win you 10. (Dependent on the reader's interpretation of our dollar, and also on his assets, he may wish to scale these values up or down—for example, by multiplying by 10—in order to increase the interest of the bets.) So far no probabilities have been mentioned. The reader, before proceeding further, is asked to state for each bet the *least* value of $p$, the probability of winning, that will lead him to accept the bet. The bet at this value of $p$ must be as attractive as declining the bet and preserving the *status quo*, since a smaller value would mean declining the bet and a larger value would make it more worthwhile.

In carrying out the requested task the reader has solved four separate decision problems, in each case the decisions being to accept or to refuse a bet. Thus, if $p_1$ is his stated value for the first bet, he has decided to accept if $p$ exceeds $p_1$ and to refuse otherwise. Let us now see how the four solutions fit together, or cohere. Experience with the example on subjects unfamiliar with utility concepts has shown a great variety of reactions, some rather ridiculous and others apparently sensible but still incoherent. To illustrate consider assessments in the latter class with $p_1 = 0.6$, $p_2 = 0.5$, $p_3 = 0.8$, and $p_4 = 0.6$. These are reasonable since II is clearly the most favourable bet and would have the least value of $p$, whereas III is the worst and would require the largest probability, the others being intermediate and differing only in the equal sums to be won or lost. Nevertheless we proceed to show that these values are incoherent.

(Some readers will have chosen $p_1 = 1/2$, $p_2 = 1/3$, $p_3 = 2/3$, and $p_4 = 1/2$; that is, those values that make the bets *monetarily* fair. For example, with bet III, $-20(1-p_3)+10p_3 = 0$ when $p_3 = 2/3$. Such decisions are, of course, coherent and the utility of $x$ dollars is equal to $x$ dollars over the range of $x$-values involved in the bets. These readers may care to change their interpretation of the dollar to make it a larger amount, or alternatively to multiply all the amounts by, say, 100. If a sufficiently large factor is chosen the above probabilities will cease to be reasonable. Generally in this chapter we are considering important gambles where the rewards and/or losses are large in comparison with the assets. With smaller gambles one can work with values that are monetarily fair and treat utility as proportional to money.)

In assigning these four values, the decision-maker has admitted that the four bets at these probabilities are equally desirable, because they are all equivalent in his mind to the *status quo*. Now consider what is called a *mixture* of bets. This is a situation in which one of the four bets is selected by chance with probabilities which we denote by $a_1$, $a_2$, $a_3$, $a_4$. For example, with $a_1 = a_2 = 1/2$, $a_3 = a_4 = 0$ a fair coin might be tossed and if it falls heads the bet I is selected whereas tails results in bet II. Then since these two bets (at the selected probabilities) are equivalent to the *status quo*, the same must be true of the mixture. In the numerical example take

$$a_1 = 0.3, a_2 = 0.4, a_3 = 0.3, a_4 = 0.0$$

and call this, mixture *A*. Then bet I will be used with chance 0.3, and if it is, 10 dollars will be won with chance 0.6. By the mutliplication law the chance of bet I being used and resulting in a win is $0.3 \times 0.6 = 0.18$. 10 dollars can also be won with bet III, the same argument giving a chance of $0.3 \times 0.8 = 0.24$. These are the only two ways a win of 10 dollars can be obtained, so by the addition law the chance of such a win is $0.18 + 0.24 = 0.42$. Proceeding similarly with the other sums, the mixture with the above probabilities leads to the probabilities given the first row of the following table

| Mixture | Lose 20 | Lose 10 | Win 10 | Win 20 |
|---|---|---|---|---|
| *A* | 0.06 | 0.32 | 0.42 | 0.20 |
| *B* | 0.06 | 0.355 | 0.42 | 0.165 |

The second line is similarly obtained from a mixture with

$$a_1 = 0.7, \ a_2 = 0.15, \ a_3 = 0.0, \ a_4 = 0.15$$

called mixture *B*.

Now both mixtures are equivalent to the *status quo* and therefore are themselves equivalent, in the sense that the decision-maker, if coherent, should have no preferences between *A* and *B*. This is plainly nonsense, because the mixtures have the same chances of winning 10 or losing 20 dollars, the only difference between them being that *A* has a higher chance of winning 20 dollars and a lower chance of losing 10, and as a result is preferred to *B*. Thus in assigning the four probabilities $p_1 = 0.6$, etc. the decision-maker has been incoherent.

The example has demonstrated our second point, namely the manner in which coherence operates. The avoidance of incoherence is achieved through a utility function, but a demonstration that is so must be delayed until the function has been studied in more detail. Notice that in its combination of bets leading to absurdity the ideas of this section are closely related to those of a Dutch book discussed in section 3.15. The treatment here takes full account of the utility structure: that in section 3.15 effectively supposed utility to be the same as money. (Some readers may wish to know how the mixtures *A* and *B* were found. One way is by trial and error. A more efficient procedure is to express the problem as one in linear programming. To discuss this would take



us too far away from the central topic of this book but the reader familiar with linear programming may like to carry out the exercise. We return to the discussion of a utility function for money.)

### 5.3 UTILITY IS INCREASING AND BOUNDED

The first obvious remark is that an increase in $x$ causes an increase in utility, so that $u(x)$ increases with $x$. Most of us would prefer the larger of two sums of money, and since utility measures the desirability of the money, the larger sum must have the larger utility. If this were not so then we should find people literally throwing money away. Experience shows that people do not do this. They may give money away but this is usually associated in their minds with a gain at least in utility for society and the consequences are not entirely monetary.

A second reasonable feature of $u(x)$ may be obtained by considering very large values of $x$. This is a little harder to think about because we are so unfamiliar with really substantial assets. However, suppose you contemplate a consequence with which is associated such a value, say a win of a million dollars; or if that is not large enough (as it may not be if the decision-maker is an industrial corporation) some really big value. Then you would find it quite hard to distinguish between an $x$ of one million and an $x$ of two millions. Admittedly the latter is better, but the former would enable you to do all those wonderful things you have wanted to do for so long, and buy all those marvellous extravagances that you never thought you would be able to have, so that a further million dollars on top would only gild an already very attractive lily. To put it differently, there would come a point where the extra capital would cease to excite you. It therefore seems natural to suppose that utility does not increase without limit as $x$ does, but that utility is bounded by some upper limit as $x$ increases. This upper limit need never be attained for any value of $x$ but values of $x$ can be found whose utility is as near to the limit as is desired. A glance at Figure 5.1, or at the two figures at the end of the book, will demonstrate the point.

Since $u(x)$ increases with $x$ and $x$ cannot fall below zero, the utility of zero, $u(0)$, must be a lower bound for utility. Similarly we have just seen there is an upper limit. It will agree with the discussion in section 4.10 if we assign a utility of zero to $x$ being zero (that is, put $u(0) = 0$), and utility of 1 to the upper limit. The worst possible consequence, $c$, will correspond to $x = 0$. The best possible consequence, $C$, is this upper limit which is just slightly beyond any attainable value of $x$ and for a large value of $x$, $u(x)$ is almost 1. The outcome of these considerations is that $u(x)$ is reasonably an increasing function of $x$, with $u(0) = 0$ and an upper limit, as $x$ increases, of 1. Such functions are illustrated in the figures in this chapter and at the end of the book. We saw in section 4.10 that it does not affect decision-making if the utility values have a constant added to them, or if they are all multiplied by a constant.

The use of 0 and 1 as the lower and upper limits enables us to employ the method of obtaining utilities described in the previous chapter, namely to replace any sum $x$ by a gamble with some chance of a utility of one and a complementary chance of zero, the chance being equated to the utility, $u(x)$. This method is unsatisfactory here if only because we find it so hard to think of these extreme utilities of 1 and 0, representing perfection and disaster, respectively. As has been emphasized before, that device was used to establish the existence of utilities, for which purpose it is perhaps the simplest. To explore the values of utilities in most situations it is preferable to employ other checks on coherence which are simpler to gauge, just as it is better to use theodolites rather than rulers to measure distances of the order of miles.

## 5.4 DIMINISHING MARGINAL UTILITY

Consider again the investor contemplating purchasing stock. Suppose that he does so and that the stock appreciates, so that he gains 100 dollars, his initial capital growing from $C$ to $C + 100$. Then his gain in utility is $u(C + 100) - u(C)$, and the difference measures the satisfaction he obtains from the additional 100 dollars. Now for most of us this satisfaction will depend on $C$, the initial capital. If $C$ is also 100, the gain represents a doubling of capital and the satisfaction is presumably rather high. On the other hand, if $C$ is a million then the gain is infinitesimal in comparison and there is little satisfaction to be had. These two extremes illustrate the phenomenon that, for many of us under typical conditions, the increase in utility derived from the increase of 100 dollars in capital is smaller the larger the initial capital. We express the idea more generally and more mathematically.

For a fixed gain $a$ in monetary capital, the increase in utility

$$u(x + a) - u(x)$$

is a diminishing function of the initial monetary capital, $x$. (Here $a$ is positive, as also is $x$ by a previous assumption.)

This is often referred to as the principle of the diminishing *marginal utility* of money, marginal utility being the term used for the increase in utility due to an increase in capital, as distinct from the utility of capital. Before discussing this any further it is necessary to make another observation.

## 5.5 A DISTINCTION BETWEEN NORMATIVE AND PRESCRIPTIVE VIEWS

In the first four chapters of this book it has been shown that if certain assumptions are made then certain results necessarily follow; there is no disputing the truth of the results granted the premises. Furthermore we have argued strongly in favour of these premises. The properties of utility that are being discussed in the present chapter do not have this inevitability and they do not follow from the assumptions. We merely put them forward as reasonable attributes of a decision-maker's utility function. As we shall see below, there are individuals who act as if they do not accept the principle of diminishing marginal utility of money: for them the marginal utility increases, at least for some values of $x$. It is perfectly possible for people to act in this way and yet be coherent. My personal opinion is that people who appear to violate the principle on some occasions are incoherent and would change some of their actions if they realized this, but certainly I would not insist that they accept the principle, at least in the same way that I would insist they should accept the basic axioms or else give me reason why they should not. The same holds for the other properties of $u(x)$ so far examined, for example its increasing with $x$. They do not follow from the initial premises. They could, of course, be added as additional premises but they would be premises which, for myself, do not have the conviction of those already defended. There is therefore a powerful distinction between the properties of utility described in the present chapter and the laws of probability derived in Chapter 3. The latter are indisputable, the former are merely reasonable. If you violate the one you are wrong: disagree with the other and you are merely unusual. In the language of section 1.4, the laws of probability and the existence of utility are part of the normative approach; whereas the properties of utility now being discussed are prescriptive. You do not have to take the prescription.



## 5.6 TWO EXAMPLES OF UTILITY FUNCTIONS FOR MONEY

We now have several reasonable properties for $u(x)$: increasing from $u(0) = 0$ to an upper limit of 1 as $x$ increases, and such that $u(x + a) - u(x)$ decreases with $x$ for every positive $a$. Such a function is tabulated in Table A.I, at the end of the book, and graphed in Figure A.I. A second such function is similarly displayed in Table A.II and Figure A.II. Detailed explanations of these are given alongside them. The reader may wish to interrupt the reading of this chapter in order to study them, since they will be referred to in many of the subsequent examples. We invent two individuals called for the moment I and II (names will be provided in a moment) with these two functions as their respective utilities. Although the two utilities look somewhat similar we shall later see that they differ in at least one important respect, and under similar circumstances the individuals I and II might behave differently owing to their possessing different utilities.

It is easy to see that the two utility functions have all the properties so far suggested as appropriate. They certainly increase from $u(0) = 0$ to an upper limit of 1. It is almost as easy to see that they have diminishing marginal utility. A reference to Figure 5.1 will make it clear. The points $A$ and $B$ are at values of $x$ and $x + a$ dollars, respectively, so that the distance $AB$ represents $a$ dollars. The distance $CD$ also represents $a$ dollars but $C$ corresponds to a much larger sum than does $A$ (or even $B$). In changing from $C$ to $D$ there is the same marginal increase in money (namely $a$) as in changing from $A$ to $B$, but the initial assets are larger. The marginal increase in utility corresponding to $AB$ is $PQ$: that corresponding to $CD$ is equal to $RS$. Clearly $RS$ is less than $PQ$,

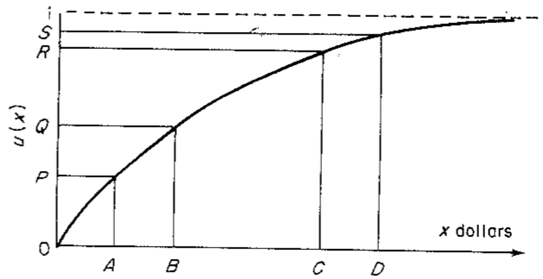


Figure 5.1. A concave, bounded, increasing utility function

that is, the marginal increase in utility at the larger capital sum is smaller than that at the lesser capital; or the marginal utility decreases with capital. An alternative way of expressing the same result is to say that the slope of the graph decreases as $x$ increases—imagine walking up a hill shaped like the graph: you would find it harder at the beginning (small $x$) than on the almost level plateau at large $x$. (Those familiar with the necessary mathematical language will appreciate that the second derivative of $u(x)$ is negative.) A function having these properties is said to be *concave*. Our two utility functions are increasing, bounded by the values 0 and 1, and are concave.

From Figure 5.1 it will also be appreciated how $u(x)$ manages always to increase with $x$ and yet never exceeds 1. The marginal utility is always positive and yet gets less and less as the assets increase. Indeed, it can be made arbitrarily small by supposing the assets large enough. To see this, imagine $C$ moved to the right, together with $D$, so that $CD$ is fixed; then $RS$ diminishes without limit. In other words, 100 dollars is chicken feed for a rich individual.

In using the utility functions given at the end of the book it is often convenient to change the scale of the assets, $x$, thereby obtaining another function. For example, as it stands, function I attaches a utility of 0.632 to 100 dollars. This is a high value for most people, and it might be more realistic to multiply the horizontal scale by 10, thereby assigning the same value of 0.632 to 1000 dollars and reducing the utility of 100 dollars to 0.095, the value originally found against $x = 10$. These two utility functions are distinct and we can use whichever is appropriate: the tabulation of one effectively provides, by simple changes of scale, a lot of others. Experience shows that many people associate a utility of around 0.5 with their current assets. A function having this value can be obtained as follows. At the moment function I has a utility of one half at $x = 69.4$ dollars, or in round figures, 70 dollars. If someone has a capital of 7000 dollars he might find I an acceptable utility function on multiplying the horizontal scale by a hundred, when, for example, a doubling of capital to 14000 would give a utility of 0.753, an increase of about 50%. Generally, with a capital of $C$ you might find I an acceptable utility for you if the scale is multiplied by $C/70$. (We shall see below that II, with factor $C/36$, is more likely to be suitable.) Remember that there is nothing mandatory in the use of these functions: they are merely suggestions, and the reader is encouraged to invent and use his own.



## 5.7 THE INVESTMENT EXAMPLE

To illustrate the use of these utility functions, consider the investment example of Table 5.1. To define the problem completely it is necessary to specify $C$ and the probability of the stock appreciating, $p(\theta_1)$. Let the latter be denoted by $p$. We study the dependence of the solution on $C$, $p$, and whichever of the two utility functions is employed. In all cases the scale of $x$ will be supposed multiplied by 10 for convenience. Suppose first, that $C = 100$, the lowest possible value. (Even here the invested sum, 500 dollars, will have had to have been borrowed. It is presumed that the associated interest charges have been taken into account.) The relevant utilities can be read off the graphs or taken from the tables and are as folows:

| | Decision-maker | |
|---|---|---|
| Dollars | I | II |
| 200 | 0.181 | 0.364 |
| 100 | 0.095 | 0.221 |
| 0 | 0.000 | 0.000 |

Thus, for the first decision-maker, at 200 dollars we enter Table A.1 at $x = 20$ (a scaling by 10) and read 0.181. For him, Table 5.1 reads

| | $\theta_1$ | $\theta_2$ |
|---|---|---|
| $d_1$ | 0.181 | 0.000 |
| $d_2$ | 0.095 | 0.095 |

and for the other decision-maker it reads:

| | $\theta_1$ | $\theta_2$ |
|---|---|---|
| $d_1$ | 0.364 | 0.000 |
| $d_2$ | 0.221 | 0.221 |

For I the expected utility of $d_1$ is $0.181 \times p$, and of $d_2$, 0.095. These are equal when $p = 0.52$. Hence if the chance of the stock appreciating exceeds this value, he should invest, otherwise not. For the other decision-maker, II, the similar value of $p$ is equal to 0.61. He requires the investment to be safer than the first decision-maker does and it is not until the chance exceeds 61% that the gamble is worth taking.

The low value of $C$ made this an extreme case, so next suppose $C = 1000$

dollars and they are contemplating investing half their fortunes. The relevant decision tables are, for decision-maker I,

| | $\theta_1$ | $\theta_2$ |
|---|---|---|
| $d_1$ | 0.667 | 0.593 |
| $d_2$ | 0.632 | 0.632 |

and for decision-maker II,

| | $\theta_1$ | $\theta_2$ |
|---|---|---|
| $d_1$ | 0.709 | 0.676 |
| $d_2$ | 0.693 | 0.693 |

For I, the expected utility of $d_1$ is $0.667 \times p + 0.593 \times (1-p)$, and of $d_2$ is 0.632. These are equal when $p = 0.52$, the same critical value* as before. A similar calculation for II shows that the two expected utilites are again equal when $p = 0.52$, so that the two decision-makers would behave similarly in this situation, both preferring the investment only if its chance of success exceeds 52%. As a result of his increase in total assets the second decision-maker is prepared to accept the investment at a lower chance than before. Thus if $p = 0.57$, he would not have accepted it with a capital of 100 dollars, but will with one of 1000 dollars. The first decision-maker is not so affected by the change in capital. We leave the reader to explore what happens when $C$ increases still further, say to 2000 dollars.

## 5.8 UTILITY AND THE AVOIDANCE OF INCOHERENCE

The example concerning four bets, introduced earlier in this chapter, can now be discussed using a utility function. Consider decision-maker II with assets of 50 dollars. He can win or lose 10 or 20 dollars so the relevant utilities are:

| Dollars | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|
| Utilities | 0.458 | 0.523 | 0.570 | 0.605 | 0.633 |

The probabilities of winning that the reader was asked to determine can be found by equating the expected utilities of the bets with the utility of the *status quo*, namely 50 dollars. Thus $p_1$, for bet I to win or lose 10 dollars satisfies

$$0.523(1-p_1) + 0.605p_1 = 0.570$$

* The reader who takes the trouble to perform the calculations described may sometimes find discrepancies of 1 or even 2 in the last place between his results and those in the text. These are 'rounding-errors' and no significance should be attached to them. Thus here he may find $p = 0.53$.



giving $p_1 = 0.57$. The other values, similarly obtained, are $p_2 = 0.43$, $p_3 = 0.76$, and $p_4 = 0.64$. These values do not differ much from those of the incoherent decision-maker cited above—indeed, except for $p_2$, they agree to the first place of decimals—but the change is adequate to avoid incoherence. To see this it is only necessary to note that each bet now has expected utility 0.570 and therefore each mixture has the same expected utility, unlike the two mixtures $A$ and $B$ considered above.

An alternative method of establishing the incoherence is to demonstrate the lack of a utility function corresponding to the quoted probabilities, $p_1 = 0.6$, $p_2 = 0.5$, $p_3 = 0.8$, and $p_4 = 0.6$. We saw, at the end of the last chapter, that a utility function may have its origin and scale changed without affecting the decisions, so let us suppose that this person making the probability assessments has utilities of 0 and 1 respectively for the worst outcome, a loss* of 20 dollars, and the best, a gain of 20. Then the assertion that $p_4 = 0.6$ for bet IV means that the utility of the *status quo* equals

$$0 \times 0.4 + 1 \times 0.6 = 0.6$$

Bet III, with $p_3 = 0.8$, has expected utility

$$0 \times 0.2 + u \times 0.8$$

where $u$ is the utility of a win of 10 dollars, which again must equal that of the *status quo*, so $u = 0.75$. Bet II, with $p_2 = 0.5$ has expected utility

$$v \times 0.5 + 1 \times 0.5$$

where $v$ is the utility of a loss of 10 dollars. Again, this must equal 0.6 and $v = 0.2$. Finally the first bet has $p_1 = 0.6$ and an expected utility of

$$v \times 0.4 + u \times 0.6$$

which, on inserting the values of $u$ and $v$ just found, gives 0.53 and not 0.6 as it should. Hence there is no utility function which explains the decision-maker's choices and incoherence results. This demonstrates the first point about the example, namely the need for a utility function in reaching decisions involving money. The general approach is in section 9.14.

In the remainder of this chapter we investigate some of the results which follow from having a bounded, increasing, concave function for the utility of money, illustrating the ideas by means of the two utility functions provided at the end of the book. As a by-product of this investigation we shall show that several of the criticisms that have been levelled against utility, and against selecting an act on the basis of expected utility, are not valid.

## 5.9 RISK-AVERSION

We show that a decision-maker with a concave utility function is, in a sense

* Strictly, we may not talk about the utility of a loss. The phrase used is a convenient abbreviation for 'utility of 0 for the outcome $C-20$, where $C$ is his current assets'.

now to be described, averse to taking a risk and does not like some apparently favourable gambles.

Consider a decision-maker with the utility function sketched in Figure 5.2 and total assets indicated by the point $A$ in that figure. The utility corresponding to $A$ is indicated by the point labelled $P$. Now suppose he considers a gamble in which he is equally likely to win or lose a certain sum, for example, the simple investment situation studied earlier in this chapter but with the added condition that the stock be equally likely to prosper or fall; in the former notation, $p = 1/2$. If he wins the gamble his assets increase to $C$ (in the figure), if he loses they fall to $B$ with $BA = AC$ and with $p = 1/2$ the two possibilities are equally likely. Assessed in monetary terms the gamble is fair, in the sense explained in the previous chapter that the expected gain in *dollars* is zero. With $p = 1/2$, this is because $A$ is the mid-point of $BC$. The situation is different if the gamble is judged in *utiles*. Referring to the figure, the utilities corresponding to $A$, $B$, and $C$ are denoted by $P$, $Q$, and $R$. $P$ is above $S$, the mid-point of $QR$, because of the property of diminishing marginal utility, $PR$ corresponding to $AC$ being less than $QP$ corresponding to $BA$. As $S$ is mid-way between $Q$ and $R$, representing the two utility outcomes of the gamble, $S$ represents the expected *utility* of the gamble and we see this is less than the utility, at $P$, of refusing it. Hence, in utiles, the gamble is subfair, and will be refused by the decision-maker. The next two paragraphs present two other ways of looking at the same phenomenon and may be omitted by someone who is convinced by the above argument.

The concavity of $u(x)$ may alternatively be expressed by the fact that the slope of $u(x)$ decreases with $x$. Referring again to Figure 5.2, where $L$, $M$, and $N$ are the points on the utility curve corresponding to monetary sums $A$, $B$, and $C$ and utilities $P$, $Q$, and $R$, respectively, this means that $ML$ is steeper than $LN$. If the straight line, or chord, $MN$ cuts $AL$ in $K$, so that $K$ is the mid-point of $MN$, this means that $L$ is above $K$: hence $P$ is above $S$ and the same argument works.* For a concave curve, a chord $MN$ always lies below the curve.

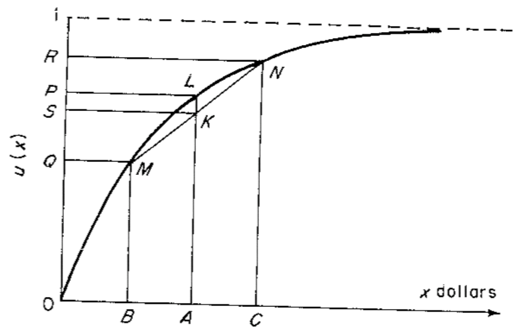


Figure 5.2. Geometrical explanation for risk-aversion



The same result may be obtained without reference to a figure. Let the assets be $x$ and let the gamble have equal chances of winning or losing $a$, so that, if accepted, it will result in assets $x + a$ or $x - a$. Then because of diminishing marginal utility

$$u(x+a) - u(x) < u(x) - u(x-a)$$

Now add $u(x) + u(x-a)$ to both sides and obtain

$$u(x+a) + u(x-a) < 2u(x)$$

and finally multiply by 1/2 to give

$$u(x+a)/2 + u(x-a)/2 < u(x)$$

The left-hand side is the expected utility of the gamble and the right-hand side the utility of not gambling, so that again we have the result that on the basis of utility it is better not to engage in a monetarily fair gamble.

The result may be generalized to a gamble which has probability $p$ of winning $a$ and complementary probability $1 - p$ of losing $b$, provided that it is monetarily fair; that is, provided

$$a \times p - b \times (1-p) = 0 \qquad (5.1)$$

(The above was the special case $p = 1/2$, $a = b$.) The argument is similar though $A$ will not be the mid-point of $BC$ but will, because of equation (5.1), divide $BC$ in the ratio $p$ to $1 - p$. We leave the details to the interested reader: the essential point is that $K$ will be below $L$ and $S$ below $P$. The result may be further generalized to a bet with several possible outcomes, and not just two. The procedure here is similar to that adopted in section 4.3 when the $n$ possible consequences of decision $d_i$ were reduced to just two, $c$ and $C$. The several outcomes can be reduced to two without destroying the monetary fairness, and we are back at the original situation. Again details are left to the interested reader.

The upshot of all this is that a decision-maker with concave utility for money will refuse a monetarily fair bet. He is said to be *risk-averse*. In using this last term remember that it refers to monetarily fair bets and he is not averse to a risk with sufficiently high monetary expectation.

## 5.10 PROBABILITY PREMIUM

We have already carried out some simple calculations to illustrate the point. With the investment that could gain or lose 100 dollars, we saw in section 5.7 that in every numerical case it would not have been made had the chances been equal. In one case the chance would have had to increase to 61% before the investment became worthwhile. In general, let $p$ be the probability that makes

* Since $S$ is the mid-point of $QR$ and $A$ the mid-point of $BC$, $K$ is the mid-point of $MN$ and $SK$ is horizontal.

a bet on two outcomes monetarily fair; and let $P$ be the probability that makes the same bet fair on a utility basis, so that the expected gain in utiles is zero. Then we have shown that $P$ exceeds $p$. The difference between them, $P-p$, is called the *probability premium* of the gamble. It describes the increase in probability needed to change a fair bet (from the monetary viewpoint) into an acceptable bet (from utility considerations). In the examples considered the probability premium varied between 2% and 11%.

The effect of risk-aversion becomes more noticeable the larger the sums involved are relative to the initial assets. Suppose decision-maker II has total assets of 500 dollars and contemplates a bet which will either lose him everything or win him 500 dollars, so doubling his assets. Then this is monetarily fair if the chance of a win is 1/2. The utilities of 500 and 1000 dollars are, respectively, 0.570 and 0.693 (from Table A.II with a factor of 10) and that of zero is zero. Hence the expected utility of the bet is $1/2 \times 0.693 = 0.347$, whereas if he does not gamble the utility is 0.570. It is obviously better to refuse the bet. For the bet to be worth considering its expected utility must be at least that of not betting, that is, $0.693 \times P = 0.570$, giving $P = 0.82$. He would need to be 82% certain of winning before the bet could be entertained. The probability premium is $0.82 - 0.50 = 0.32$, or 32%. The conclusion here is in reasonable agreement with common sense because this gamble has so much at stake that he would naturally want a high chance of winning before accepting it. Our point is that this natural desire is explained by maximising expected utility with a concave utility function.

The phenomenon of risk-aversion is a result of the assumption of diminishing marginal utility, most clearly expressed in the *concave* form of the graphs in which the slope decreases with the assets. If the curvature of the graph is in the opposite sense, so that the slope *increases* (the curve is said to be *convex*) then the contrary phenomenon exists and the decision-maker will accept risky gambles.

Consider Figure 5.3 in which a utility function is sketched having increasing steepness for small $x$ and only reverting to the familiar form for larger $x$. Then in the lower part of the curve a chord $MN$ lies *above* the utility curve, contrary to the case in Figure 5.2, where it lay below it. The effect of this is to reverse the argument given in connection with the earlier figure, consequently the expected utility of a monetarily fair bet exceeds the utility of not gambling and the probability premium is negative. In this part of the curve the decision-maker has increasing (not diminishing) marginal utility. He is said to be *risk-prone*, rather than risk-averse.

It has been suggested that many people have a utility function of the general shape illustrated in Figure 5.3, convex for small assets and concave for larger ones. Personally I doubt this because it would imply a reversal of the decision discussed above concerning a gamble which had equal chances of doubling one's assets or losing everything. If all the assets referred to the convex part of the curve such a gamble would be accepted, whereas I do not think most people would be so wild. Admittedly the point is not easily resolved if only



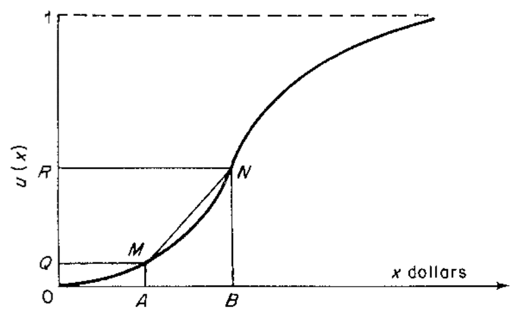


Figure 5.3. A convex utility curve

because it is difficult to see exactly what is meant by zero assets and therefore zero utility. Fortunately few of us ever have occasion to explore the extreme left of our utility curve. (Remember *monetary* consequences are under discussion: death is not being considered.)

## 5.11 TYPES OF RISK-AVERSION

Although the curvature of the utility function obviously affects the risk-aversion, it is not quite obvious how the amount of curvature is related to the amount of risk-aversion. To illustrate the relationship the two utility functions, I and II, have been provided. Decision-maker I has, in a sense now to be explained, a constant aversion to risk; whereas II has an aversion which decreases as his assets increase. This is despite the fact that the curve in Figure A.I has more curvature for small $x$ than for large $x$. The calculations already performed will illustrate the difference.

In considering the investment situation in section 5.7 we saw that I required a chance of 52% before investing, whether his assets were 100 dollars or 1000 dollars. If the reader did the calculation suggested, he will have obtained the same value at 2000 dollars. The probability premium is therefore fixed at 2%. On the other hand, for II the premium dropped from 11% at 100 dollars to 2% at 1000; for 2000 it has dropped almost to zero.*

For I, the premium, for a given bet, does not depend on the assets: nevertheless, for given assets, it does change with the bet. For example, at 1000 dollars assets, we saw that the premium for a bet to win or lose 100 dollars is 2%, whereas to win or lose 1000 dollars it is easy to calculate the value at 23%. (Remember that we were scaling the money by 10 before using Table A.I: the utility of 1000 is obtained by entering the table at $x = 100$, giving 0.632.)

Decision-maker I will be referred to as the constantly risk-averse decision-maker, meaning that the probability premium for a gamble does not vary with

* It is not possible to make very precise calculations at $C = 2000$ due to the limits of accuracy of the tables.

his assets. II will be called a decision-maker with decreasing risk-aversion, meaning that the probability premium for a gamble decreases as the assets increase. Most of us are decreasingly risk-averse, as can be seen from commonsense consideration of the investment example, where a person would typically react differently to the investment if his capital, $C$, changed and a rich person would take greater risks than a poor one. A utility function like that of Table A.II is therefore more realistic than that of Table A.I. Of course, there are several utility functions which yield decreasing risk-aversion and that in Table A.II is merely an example. It does so happen, however, that the only constantly risk-averse utility is that given in Table A.I, apart from a possible change in scale. The following paragraph explains this point in more detail but may be omitted by the reader unfamiliar with the mathematical language.

The probability premium for a bet to win or lose $h$ is given by $P-1/2$ where $P$ satisfies

$$u(x+h)P+u(x-h)(1-P)=u(x)$$

Hence

$$P=\{u(x)-u(x-h)\}/\{u(x+h)-u(x-h)\}$$

For $P-1/2$ not to depend on $x$ we must have

$$-(P-1/2)=\frac{u(x+h)-2u(x)+u(x-h)}{2\{u(x+h)-u(x-h)\}}$$

constant. In particular, the limit of this as $h$ tends to zero, namely $u''(x)/u'(x)$ must be constant, where the primes denote differentiation. The solution to this differential equation with $u(0)=0$ and upper bound one is

$$u(x)=1-e^{-cx}$$

for some positive $c$. $c$ is clearly a scale factor on $x$ and the essential uniqueness of the constantly risk-averse utility is established. Notice that

$$\log_e\{1-u(x)\}=-cx$$

so the disutility, $1-u(x)$, has its logarithm linear in $x$. There is a much wider class of decreasingly risk-averse utilitites: for example, any function of the form

$$1-we^{-ax}-(1-w)e^{-bx}$$

with $a>0$, $b>0$, $0<w<1$ has the property, and there are many others. Table A.II gives the function

$$1-e^{-x/200}/2-e^{-x/20}/2$$

## 5.12 INSURANCE

We now leave the mathematics and return to the main discussion. So far we have been discussing whether or not to accept a gamble; whether to invest the money or keep it in the bank; whether a company should try to enter a new field or stay in its own; whether a risky situation is preferred to a non-risky one. The conclusion we have come to is that, with diminishing marginal utility of money, a fair risk from a monetary point of view is unacceptable and a probability premium is required to make it worth considering. The amount of this premium depends on the risk, and usually on the assets, decreasing as the assets increase. Let us now consider the position where we have a risky situation and are considering disposing of it. The usual way of doing this is through insurance. If we own a house, that house may be destroyed by fire or other calamity, but we can take out an insurance which will compensate us in the event of the calamity at the cost of an annual premium paid to the insurance society. From a decision table viewpoint the situation is described in Table 5.2. (The table is again simplified; partial loss of the house is another possibility, but we do not wish to complicate the analysis.) If the total assets are $C$, the value of the house $h$, and the premium $m$, the monetary consequences are as in Table 5.3 (cf. Table 2.4). The inconvenience of temporarily losing the house is supposed compensated for, so that the insurance leaves the decision-maker in the same position had the calamity not occurred. Here $m$ is much less than $h$, and $p$, the probability of $\theta_1$, the chance of a calamity, is small.

Now Table 5.1 (for the investment problem) and Table 5.3 are very similar. In both cases there is one decision whose outcome is certain ($d_2$ in Table 5.1, $d_1$ in Table 5.3) and another which is risky. The difference between the two tables is that in the former inactivity results in choosing the certain situation (leave the money in the bank) whereas in the latter it results in the risky situation (possible loss of an uninsured house), but as a problem of choice between two decisions this difference is immaterial. Consequently many of our remarks about the earlier situation apply to the present one. In particular, a risky situation is avoided by a risk-averse decision-maker: therefore he will tend to opt for insurance in order to remove the risk.

Interest in the insurance situation centres on the premium quoted by the



Table 5.2

| | $\theta_1$: Calamity | $\theta_2$: No calamity |
|---|---|---|
| $d_1$: Insurance | Inconvenience, but otherwise *status quo* | Loss of premium |
| $d_2$: No insurance | Loss of house | *Status quo* |

Table 5.3

| | $\theta_1$ | $\theta_2$ |
|---|---|---|
| $d_1$: Insurance | $C-m$ | $C-m$ |
| $d_2$: No insurance | $C-h$ | $C$ |

company and only if this is low enough will the insurance be undertaken. We therefore consider a numerical illustration of Table 5.3 and see how the reasonable premium might be found. Suppose $C = 5000$ dollars and that the calamity represents total loss: that is, $h = 5000$ and $C - h = 0$. Suppose we take the decreasingly risk-averse decision-maker and scale Table A.II by a factor of 100 (not 10, as before). The utility of 5000 dollars is then found with $x = 50$, namely 0.570. If $p$ is the chance of the calamity, the expected utility without insurance ($d_2$) is $0.570 \times (1 - p)$: with insurance it is $u(C - m)$. Consequently an acceptable premium, $m$, satisfies

$$u(C - m) = 0.570 \times (1 - p)$$

with $C = 5000$. Any smaller value of $m$ will, of course, be acceptable. If $p = 0.01$, so that there is just one chance in 100 of calamity, we have

$$u(5000 - m) = 0.570 \times 0.99 = 0.564$$

Using Figure A.II or Table A.II in the reverse direction and finding the value of $x$ with a utility of 0.564, gives about 48.6, or $5000 - m = 4860$ and $m = 140$ dollars. Hence the decision-maker would be prepared to pay up to 140 dollars premium against the total loss of his assets.

This premium of 140 dollars is much more than the actuarial value of the loss based on expected monetary values. 5000 dollars with a chance of 0.01 of loss gives an expected monetary loss of only 50 dollars. Hence the premium could be about three times the actuarial loss. This suggests that insurance would not only be desirable at 140 dollars, but that a society could be found to offer this premium. The total assets, $C'$ say, of an insurance society will be large in comparison with the assets $C$ of the insured and if the society accepts insurance it will be involved with possible consequences $C' + m$, $C'$, $C' + m - h$ and, since $C'$ is much larger than $h$ or $m$, these three consequences will be similar. Hence on the society's utility curve the three possible consequences will correspond to points that are very close together. For example, 5M (M for million) might be a reasonable value for $C'$ when the other values would be 5000140 and 4995140. A scale factor of 100000 might enable Table A.II to be used with $x = 50.0014$, 50 and 49.9514. Over this small range the curvature of $u(x)$ is slight and $u(x)$ is effectively linear in $x$. We saw that a linear transformation of utility had no effect on decision-making so we can, over this range, take utility to be the same as money. Hence the company will view the situation actuarially, and a premium of over 50 dollars will be acceptable to them. In addition, they have administrative costs, but even if these equal the actuarial costs a premium of 100 dollars will cover both and, being below 140, be acceptable to the individual. Hence the concavity of the risk function explains why insurance works. The individual is operating at a curved part of his utility graph, whereas the society's graph has no appreciable curvature over the range considered and it can equate utility with money. It follows as a corollary that if the curvature for the individual is very small, he will not find it possible to obtain attractive premiums. It does not pay to insure against small losses, only against large ones.



The difference of 90 dollars between (1) the expected monetary loss associated with the calamity and (2) the amount the individual would be prepared to pay to dispose of the gamble, is called the *risk premium*. Like the probability premium defined earlier in this chapter it measures the difference between the monetary and utility assessments of the gamble. The probability premium expresses it by describing by how much the probability would have to change to make the gamble acceptable: the risk premium achieves the same end by stating what variation in the rewards (or losses) is necessary at fixed probability.

## 5.13 EXPECTED UTILITY AS THE SOLE CRITERION

An objection that is often levelled against the use of expected utility in reaching decisions is that it is unreasonable to judge an action $d_i$ on the basis of just one number: its expected utility, $\bar{u}(d_i)$. It is argued that there are many facts to be considered in connection with each action and it is not sensible to expect to be able to describe them adequately by a single value. The following two examples will illustrate the point.

A scientist in charge of an industrial research laboratory has the task of selecting the projects to be investigated under his guidance. Some projects are fairly standard and he can say with a fair degree of confidence that, if attempted, they will most likely be successful and produce a useful, but modest, return. An improvement in an existing product might come into this category. On the other hand, there are some projects which are unpredictable and hazardous, but if they come off make a fortune for the firm. A drug successful against a killing disease might be an example. The scientist will often argue that he must take into account not only what can be expected to happen with a project, but must also consider the greater hazards associated with some projects. He might argue that the two types of project just quoted have the same expectations, but that the second might land the firm in financial difficulties if it failed, and that this great variability in the possible outcomes makes it less desirable than the former. It is better to be safe than sorry.

A second example arises in advising on the selection of a portfolio of stocks. Some stocks, gilt-edged for example, are fairly sure to give a modest yield whilst others are risky and might give a handsome return, or nothing. Surely, it is argued, the variability of the latter needs to be considered in addition to its expected yield.

The counter-argument is that the expected utility does take the riskiness of the action into consideration and that the objection only arises because of a confusion between expected utility and expectations of other quantities, for example, money. It must be remembered that utility is not just a number describing the attractiveness of a consequence but is a number measured on a probability scale and obeys the laws of probability. For example, the square of utility would be a legitimate measure to associate with a consequence, in the sense that the more desirable the consequence, the larger the measure, but the square would not obey the laws of probability and therefore we could not justify using its expectation. It is this unique probability interpretation of utility that makes its expectation so important. When, in addition, the utility function is concave, the increasing riskiness of a gamble is reflected in a corresponding diminution in expected utility, without a corresponding fall in expected financial reward, in the way we have already described by risk-aversion. The previous numerical examples will be sufficient to illustrate the point.

Consider the decision-maker of constant risk-aversion (and a scale factor of 10) with assets of 100 dollars contemplating the investment which is equally likely to win or lose him 100 dollars. The expected utility of the investment is

$$1/2 \times 0.667 + 1/2 \times 0.593 = 0.630$$

whereas, left in the bank, the utility is 0.632. The riskiness of the venture has resulted in a small loss of expected utility and the safer course is preferred. Hence the effect of riskiness is included when employing expected utilities.

Now take a slight variant of this problem in which the retention of the money in the bank is certain to result in a loss of 6 dollars, all the other factors remaining the same. The utility of $1000 - 6 = 994$ dollars is 0.630, the same as the risky investment. The 6 dollars is another example of the risk premium recently referred to in connection with insurance and describes by how much the safe course of action has to be reduced to bring it to the level of the risky venture. More risky ventures have lower expected utilities and higher risk premiums. Thus, still with assets of 1000 dollars, the 'fair' bet to win or lose 1000 dollars has expected utility

$$1/2 \times 0 + 1/2 \times 0.865 = 0.432$$

much less than 0.632 obtained by leaving it in a bank. Using Table A.I (or Figure A.I) in reverse we see that 566 dollars is the sum having a utility of 0.432. Hence the risk premium is $1000 - 566 = 434$ dollars.

There is another reason for thinking that no more than one number is needed to select the best decision, namely that in general, we cannot maximize more than one thing at a time. Suppose, for example, that instead of one number, each decision, $d_i$, has associated with it two numbers, $u_i$ and $v_i$, and that the larger $u_i$ or $v_i$ the better was the decision. Thus $u_i$ might be the expected utility, formerly denoted $\bar{u}(d_i)$, and $v_i$ might be some measure of the riskiness, the larger $v$'s being associated with the smaller risks. Then how is the best decision to be selected, granted that one wants to make $u_i$ and $v_i$ as large as possible? Figure 5.4 illustrates the situation with just three decisions, each point corresponding to a decision. It is clear that $d_3$ is no good because $d_1$ and $d_2$ have values of both $u$ and $v$ larger than those of $d_3$ ($u_1$ and $u_2$ exceed $u_3$: $v_1$ and $v_2$ exceed $v_3$). Thus $d_3$ can be rejected and the choice lies between $d_1$ and $d_2$. The selection here is genuinely difficult because $d_1$ gains over $d_2$ in respect of $u$($u_1$ exceeds $u_2$) but $d_2$ is better than $d_1$ on the basis of $v$ ($v_2$ exceeds $v_1$). In



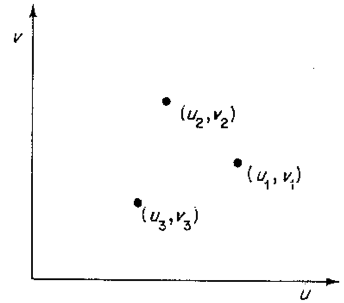


Figure 5.4. Decisions involving two considerations, $u$ and $v$

deciding between them, the benefits of $u$ increasing have to be balanced against the loss in $v$ when changing from $d_2$ to $d_1$ and until some such balance is struck, no solution is available. A decrease in riskiness has to be balanced against a loss of expectation. Expected utility does just this, and its maximization solves the decision problem.

## 5.14 PORTFOLIO MANAGEMENT

As an example of a situation where it is necessary to consider both risky and relatively secure ventures, we cited the problem of an investor choosing a portfolio of stocks, and we claimed that the expected utility of a stock is an adequate description of the stock for decision purposes. In compiling a portfolio advice is often given to diversify one's holdings by putting some money into securities and some into more speculative material. How far is this advice sound? It is sometimes argued that the principle of maximizing expected utility goes against it, because each stock has its expected utility and therefore, applying the principle, the best thing to do is to put all one's money into the stock of highest expected utility. We now show that this last argument is facile and that a more careful investigation shows that the maximizer of expected utility, the coherent decision-maker, will typically act on the advice and diversify his investments. To establish that the advice is sound in general requires some mathematics, so we will content ourselves with illustrating the phenomenon in a simple situation.

The example involves our decision-maker with constant risk-aversion (Table A.I) contemplating investing in the stock described at the beginning of the chapter, or leaving his money in the bank. We suppose he has a capital of 1000 dollars and his utility is as in Table A.I with a scale factor of 10. The stock is such that every 500 dollars invested will increase to 600 with probability $p$, or decrease to 400 with probability $1 - p$; smaller or larger amounts will increase or decrease in proportion. Money left in the bank will keep its value.

We suppose $p = 0.55$. In our earlier calculations we saw that with this value (greater than the critical value of 0.52 needed to overcome the risk-aversion) it was better to invest 500 dollars than to leave it in the bank.

Now we ask the new question: what is the optimum amount to invest in the stock? We have seen 500 dollars is better than none, but perhaps it might be best to risk all 1000 dollars. The following table lists the possible outcomes, in dollars, for five typical investments between 0 and 1000.

| Amount invested in stock | Stock appreciates | Stock depreciates |
|---|---|---|
| 0 | 1000 | 1000 |
| 250 | 1050 | 950 |
| 500 | 1100 | 900 |
| 750 | 1150 | 850 |
| 1000 | 1200 | 800 |

One way of looking at this is as a decision table with five decisions and two uncertain events. (The real decision table has 1001 decisions if investments can be made in dollar units.) From this table we can form a decision table with probabilities and utilities (from Table A.I) and hence expected utilities. This is shown in Table 5.4. The final column shows that the investment of highest expected utility amongst those considered is 500 dollars. More refined calculations, or use of the differential calculus, show that the true maximum is very close to this value. In other words, the optimum procedure for the coherent decision-maker is to invest half his capital in the stock, leaving the other half in the bank; that is, to diversify his holdings between the secure and the risky. (We leave the reader to verify that it is not until the probability of the stock appreciating increases to about 0.60 that it is worth investing all his capital in it.) The advice is therefore sound, though the increase in utility of 0.002 between the best and worst decisions is small, corresponding to a monetary change of around 5 dollars. With other examples larger differences can arise. The facile counter-argument given above fails to work, again because it ignores the diminishing marginal utility reflected in the curvature of the utility function.

Table 5.4

| Amount invested in stock | Stock appreciates | Stock depreciates | Expected utility |
|---|---|---|---|
| 0 | 0.632 | 0.632 | 0.632 |
| 250 | 0.650 | 0.613 | 0.633 |
| 500 | 0.667 | 0.593 | 0.634 |
| 750 | 0.683 | 0.573 | 0.633 |
| 1000 | 0.699 | 0.551 | 0.632 |
| Probability | 0.55 | 0.45 | |



## 5.15 REPEATED DECISION PROBLEMS

It has already been explained in section 2.4 that the approach to decision-making described here is relevant to a single occasion on which a decision is to be taken, and does not depend for its validity on the real or conceptual repetition of the same problem, as do some methods. The uniqueness of the decision-making occasion is emphasized in the probability considerations, which invite comparison of the uncertain event with the single drawing of a ball from an urn. Nevertheless there are situations in which essentially the same decision problem occurs repeatedly. A simple example is provided by the manufacturer, quoted in section 4.1, faced with the problem of whether or not to inspect material before its despatch to the customer. Presumably he is producing the material in large quantities and the same problem will present itself with each roll so that there is genuine repetition of the same decision problem.

Anything like a reasonably complete discussion of this topic is impossible because many of the problems it raises are difficult. We therefore content ourselves with illustrating one type of repetition: first, in order to show how something rather interesting and perhaps surprising can occur; and second, to dispose of another objection to the principle of maximization of expected utility. The objection says that this principle, if applied whenever a particular situation arises, will always lead to the same choice of act and therefore a stimulus will always produce the same response. It is held that this is absurd as a general rule of action. We agree with the absurdity of the conclusion but demonstrate that it does not follow from the maximization of expected utility.

The illustration is simple and the reader is warned to be cautious in extending the result to more complicated situations. Suppose a decision-maker contemplates a gamble which will either win for him a given sum with probability $p$ or lose him the same sum with probability $1 - p$: the investment situation just considered provides an example. Then by the arguments already given he can calculate the critical value of $p$ which is just enough to make the gamble attractive and counter his risk-aversion. Now suppose he is offered a second gamble which is just the previous one played over twice: will his attitude towards this gamble be the same as to the single gamble? (If the manufacturer has two items to despatch, will his attitude be the same as for a single item?) It is not hard to show that for a decision-maker with constant risk-aversion the two gambles will be treated alike; if one is accepted, so is the other and the critical values of $p$ will be the same. However, we saw that it was realistic to think of people as having decreasing risk-aversion and here the answers are more interesting.

Specifically, suppose decision-maker II has assets of 50 dollars and that he is offered a gamble with chance $p = 0.678$ of winning 25 dollars and complementary chance 0.322 of losing 25 dollars. The expected utility of the gamble is

$$0.678 \times 0.645 + 0.322 \times 0.415 = 0.571$$

which exceeds 0.570, the utility of refusing the gamble and keeping 50 dollars, and the single gamble should just be accepted. Next consider the double gamble; if he wins on both plays he will win 50 dollars, finishing up with 100, and this has probability $p^2$, by the product law. (Notice that we are assuming that the two plays are independent, both in the sense that the chance of a win on the second play is unaffected by one on the first, and also in the sense that the decision-maker does not have the option of withdrawing after one play but commits himself to both plays or neither.) Similarly, if he loses on both plays he will lose 50 dollars, and this has probability $(1-p)^2$. The final possibilities are that he wins on the first and loses on the second, with probability $p(1-p)$, or loses on the first and wins on the second, also with probability $(1-p)p$: in either case he finishes up where he started. The expected utility is therefore

$$(0.678)^2 \times 0.693 + 2 \times 0.678 \times 0.322 \times 0.570 = 0.567$$

(the utility of zero being zero) which is less than the utility, 0.570, of refusing the gamble. Consequently although the single play is worth accepting, the double play is not. It is possible to provide examples which go in the opposite direction, that is, where the double play is acceptable but the single play not. Exercise 5.8 at the end of the chapter does just this. Thus we see that decision rules for individual repetitions of the same situation may differ from those appropriate for a single occasion.

## 5.16 SOME COMMENTS ON MONETARY UTILITY

It has been explained that different persons may have different utility functions and, in particular, one may be more risk-averse than another. Equally, a person's utility function may change with time. Like a probability, to which we have seen it is intimately related, utility depends on the circumstances at the moment of its assessment and may change with those circumstances. In solving a decision problem it is the utilities at the time of choice that are relevant, not those that obtain when the consequence is realized. I might decide I have high utility for a holiday 'away from it all', only to find when I get there that it is not as attractive as I had expected. Little seems to be known about the way utilities change with time, though the manner in which probabilities change is well understood and is the topic of the next chapter.

In analysing some of the situations of the present chapter the reader may have felt himself even more risk-averse than the two people we have imagined: he might, for example, say he would not risk half his assets (500 dollars) on a venture having a 48% chance of losing him 100 dollars. Equally, the same person may find himself less risk-averse in liking to have a gamble on a horse race. This may be explained by changing the utility function for money, but there is another possibility. We have been discussing the utility of money alone and there are many situations in which money is not the only relevant factor. The person considering the possible loss of 100 dollars may feel that the mental anguish to him of the loss, and the possible criticism by his spouse, might well result in lower utility than is suggested by purely financial considerations. The



gambler at the racetrack may look upon the sacrifice of the stake as a legitimate price to pay for the added thrill that comes from watching a race in which he has a monetary interest. It is hard to generalize about the utilities of such pains and pleasures, but they should not be forgotten.

Notice that the considerations of this chapter lead to the conclusion that everything has its price. For all consequences have utility, money has utility, and therefore any consequence has, through utility as an intermediary, a monetary equivalent. This is sensible. What is not necessarily true is that the possession of the money will enable the consequence to be reached. 2000 dollars may not enable the good health to be purchased. Rather we say that were we to have the good health it would be equivalent to having an extra 2000 dollars.

Notice the use of the subjunctive 'were' in the last sentence. Its occurrence here with utility is for exactly the same reason that it was used with probability at the end of section 3.11, namely that the situation does not have to be realized, only contemplated. It is not necessary to know the defendant is guilty, we need only consider the possibility of his being guilty. Here we do not need to have the consequences, money or good health, only contemplate them. Indeed, throughout the discussion of the utility of money, the values obviously only require contemplation. For example, we have only to think about our risk-aversion were we to have 100 dollars.

### Exercises

**5.1.** The decision-maker with decreasing risk-aversion has assets of 30 dollars and has a decision problem with the following structure:

| | $\theta_1$ | $\theta_2$ |
|---|---|---|
| $d_1$ | $-10$ | $+5$ |
| $d_2$ | $+15$ | $-5$ |
| $pr$ | 0.3 | 0.7 |

The entries in the table represent gains or losses in dollars: thus with $d_1$ and $\theta_1$ he will finish up with assets of 20 dollars. Advise him on the choice of decision. Would your advice remain the same if his assets were 200 dollars?

**5.2.** The same decision-maker as in Exercise 5.1 has assets of 20 dollars and contemplates a gamble which may win him 20 or lose him 10 dollars. It is therefore actuarily fair if the chance of winning is 1/3. Determine his probability premium. Do the same for the constantly risk-averse decision-maker. Find the probability premium for the first person when his assets are 200 dollars.

**5.3.** Perform the exercise suggested in section 5.7: that is, for each decision-maker find the probability premium with assets of 2000 dollars and a chance to win or lose 100. (Remember both had a scale factor of 10.)

**5.4.** The decision-maker with decreasing risk-aversion has assets of 50 dollars and realizes that 25 dollars of it is in a risky situation which has 1 chance in 10 of collapsing and losing him his money. What is a reasonable premium for him to pay for insurance against the loss? What would the premium be if his assets were 100 dollars?

**5.5.** A decision-maker has the following utilities for money:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Money | 0 | 20 | 40 | 60 | 80 | 100 | 120 | 140 |
| Utility | 0 | 0.04 | 0.13 | 0.27 | 0.50 | 0.73 | 0.87 | 0.96 |

By sketching a graph or otherwise show that he is risk-averse for assets above about 80 dollars but not below this amount. Consider a gamble that might win or lose 20 dollars, first when his assets are 40 dollars, and then when they are 100. In each case determine the probability premium.

**5.6.** An insurance company has assets of 40M dollars and its utility is given by Table A.II with a scale factor of a million (M) dollars. It is asked to insure against an earthquake disaster of 40M dollars and assesses the chance of the earthquake at 0.05. What is a reasonable premium for it to request? (Remember you are looking at the situation from the company's side, not from that of the earthquake-sufferers. Administrative expenses can be ignored.)

The company finds a second company with the same assets and utility function and agrees to share the risk. That is, it insures against a loss of 20M dollars with probability 0.05. What is the premium now? Show that even twice this premium is less than the original premium so that the insurance is improved from everybody's point of view by the companies' sharing the risk.

**5.7.** The decision-maker with decreasing risk-aversion has assets of 50 dollars. How much would he be prepared to pay for a ticket that entitled him to a 50–50 chance of winning 20 dollars? Suppose he had the ticket already: how much would he be prepared to sell it for? Repeat the question for the constantly risk-averse decision-maker and show that in his case the buying and selling prices are the same.

**5.8.** A decision-maker has the following utilities for money:

| | | | | | |
|---|---|---|---|---|---|
| Money | 0 | 10 | 20 | 30 | 40 |
| Utilities | 0 | 0.10 | 0.17 | 0.22 | 0.26 |

With assets of 20 dollars he contemplates a gamble to win 10 dollars with probability 0.58 or otherwise lose 10 dollars. Show that the gamble should be declined. He then goes on to consider a gamble which consists of two plays of the original gamble. Show that the new gamble should be accepted.