Negative Rates in Derivatives Pricing. Theory and Practice

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Trabajo de investigación 021/017

Master en Banca y Finanzas Cuantitativas

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Negative rates in derivatives pricing. Theory and Practice

A MSc Thesis submitted to the
Complutense University of Madrid
in partial fulfilment of the requirements

for the degree

Master's degree in
Banking and Quantitative Finance

July, 2017

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Acknowledgments

To my supervisors, Gregorio and Manuel, for their constant guidance and support. Your selfless help has been one of the main pillars of this project.

To Álvaro, Alberto and Samu, who needed between 5 to 7 minutes to find and download those non-standard volatilities I had previously been searching for too many days. You are stars.

To my friends, for suffering my bad mood when things just would not work. Wherever I am, you are always with me. *Alba, Tamara, Fran, Miguel, Rober, Rubén, Carlos, Joan, Ferran*: I deserve none of you, so thank you all.

To Eli, for an unforgettable journey full of laughter and companionship. Late-night discussions about quantile regression, friends, Archimedean copulae, life, martingale representation theorem and less important topics made my day. Everyday.

To Dani, for assuming the role of being the best friend one could only imagine. You played it nicely, as every time you are on the stage. I beg you to keep playing it, just a little bit more. Your courage inspires me and everyone who is beside you.

To my parents, María and Agustín, for everything you have done. For never failing me. Do not be proud of me, be proud of you two. Because everything that is good in me was taught by you, and my many flaws are those things I just could not learn. You are my strength, and make the difference in my life.

Abstract

This MSc Thesis reviews, challenges and compares those models which have been most commonly used by the industry in pricing fixed income derivatives under the current negative rates environment. Our main aim is to analyse their relative behaviour under this new and defying context. Shifted SABR model is taken as a benchmark, since it has been the industry preferred approach among the range of suitable candidates.

Additionally, a new full-calibration method based on arbitrage-free assumptions is proposed for completing the volatility cube when negative rates are allowed. Accurate calibrations of the cube of implied volatilities for every maturity, tenor and strike outstanding are of capital importance among industry firms, since it is one of their main tools in the process of pricing any kind of interest rate derivative. Empirical behaviour of our completion methodology is tested through the Thesis by the inclusion of several illustrative examples.

Keywords: *SABR, negative rates, implied volatility, volatility cube, smile/skew, Bachelier, (shifted) Black, fixed income derivatives, interpolation methods, numerical optimization, in/out-of-sample analysis, arbitrage-free condition.*

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Introduction

Logic will get you from A to B. Imagination will take you everywhere.
Albert Einstein (1879-1955)

Negative interest rates have spread progressive and systematically all over the globe. This phenomenon, which was strongly considered to be impossible not so long ago, have elevated its category from a "curious, punctual and irrelevant fact" to a really worrying concern among every interest rates desk in the industry. The assumption that interest rates could not overstep the zero-barrier was so embedded into our knowledge that every model that admitted below-zero rates was considered to exhibit a serious drawback against alternative competitors which forced the rate above the zero-limit, by (usually) imposing lognormal specifications. Nowadays this tendency has changed its sign, and most firms are abandoning lognormal-models looking for more flexible options.

Our main contribution lies on the (ambitious) idea of developing a full-comprehensive survey comparing numerous industry-based fixed income derivatives pricing models. The research is conducted on several approaches driven by theoretical, econometric and numerical methods. As far as we know, even though some excellent papers have devoted their research to the current negative interest rates context[1], none of them have particularly coped with this issue until today.

As a by-product of the models comparison, a second essential question arises: the need for an accurate easily-comprehensive method for a full completion of the object known as *the volatility cube*: industry's fundamental tool when interest rates derivatives pricing is under concern. A new fast approach fully based on arbitrage-free relationships is thoroughly developed through the Thesis, and the results attained are summarized within several examples.

This MSc Thesis is splitted in two main parts. Part one (*Theory*) provides a fully self-contained discussion on the main theoretical topics covered through the Thesis, revisiting and updating most of them to the current negative rates environment. *Chapter one* contextualizes the appearance of negative rates in modern economies and justifies it from a credit risk perspective. *Chapter two* aims to provide the theoretical background that is strictly necessary to understand subsequent arguments and developments. *Chapter three* follows the evolution of interest rates derivatives pricing models among industry firms,

[1] See, for example, [1], [2] or [3]

focusing on the SABR approach. *Chapter four* revisits the models capable of dealing with the current negative rates environment. *Chapter five* thoroughly explains the new fast-approach proposal for the completion of the volatility cube.

Part two of the Thesis (*Practice*) analyses the topics exposed in Part one from an empirical perspective. *Chapter six* characterizes the datasets under analysis. *Chapter seven* provides an *step-by-step* methodological guide that intends to make the conducted analysis fully understandable. *Chapter eight* explains the results attained. *Conclusion* and *further research* suggestions are displayed at the end of the document, followed by several illustrative *appendices*.

Part I

Theory

Chapter 1

Context of negative rates

1.1 Chronicle

Hardly 20 years have elapsed since one of Black's most famous (and unlucky) comments was stated [4]: "the nominal short rate cannot be negative". Twenty years later this assumption looks questionable: one quarter of world GDP now comes from countries with negative central bank policy rates.

Fisher Black, a visionary whose innovative work in options [5] was considered to deserve the attainment of a Nobel Prize, was wrong. The process of assimilating and incorporating this new situation has forced practitioners to update their models accordingly, in many cases introducing greater complexity.

Until recently it was assumed that interest could not go below the "zero bound", since depositors could withdraw cash when rates became negative, averting the implied loss of money when negative rates are permitted[2]. In this argument, however, it has been omitted that cash needs to be stored and insured, which costs money. A bank account could be more convenient in use, and therefore there could be willingness to pay for having a bank account, which is equivalent to being charged negative interest rates. The question is how low interest rates can go before cash becomes more attractive.

Negative nominal interest rates are not new phenomena. As early as the 19th century, economists discussed imposing taxes on money (e.g. Gesell's tax [6]), and in the 1970s the Swiss National Bank experimented with negative rates in a bid to prevent the Swiss Franc appreciating[3]. In recent years, an unprecedented number of central banks have adopted negative policy rates. An extensive but not exhaustive list of these banks can be chronologically enumerated[4] as:

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[2] As an example, for continuously-compounded risk-free investments at a (simplified) constant rate $r(t) = r \ \forall t$, it can be easily seen that, when $r < 0$, $B(t) = B(0)\exp(rt) < B(0)$, since $\exp(rt) < 1$. Therefore, the invested amount $B(0)$ is a guaranteed money-loser.

[3] `https://snbchf.com/snb/2013-snb/reflections-on-negative-interest-rates-in-switzerland/`.

[4] See `http://www.bankofgreece.gr/Pages/en/Bank/News/Speeches/DispItem.aspx?Item_ID=347&List_ID=b2e9402e-db05-4166-9f09-e1b26a1c6f1b` for further discussion about the type of measures adopted by European policymakers during recent years.

1. **Riksbank:** Sweden's central bank was pioneer in the use of negative interest rates, by fixing the rate paid on commercial bank deposits to -0.25% in 2009. In February 2016, after not having met its 2% inflation goal for four years, this interest rate achieved a negative record of -0.5%.

2. **Danmarks Nationalbank (DNB):** Denmark's central bank followed the steps of its Swedish homologue, by imposing a below zero deposit rate of -0.20% in July 2012. In early 2015, a below zero rate of -0.75% was fixed for the deposits.

3. **Swiss National Bank (SNB):** By December 2014, the Swiss central bank adhered to the trend initiated by its Nordic equivalents, announcing that a -0.25% return would apply to sight deposit account balances. Just one month later, by January 2015, a new drop of the rate to a negative record level of -0.75% was announced.

4. **European Central Bank (ECB):** By 2014, the European policymaker introduced the below-zero return rate on the deposits by fixing a deposit facility rate of -0.20%. This rate kept on decreasing during the following years, by attaining values of -0.30% and -0.40% in 2015 and 2016 respectively.

5. **Bank of Japan (BoJ):** European central banks do not monopolize the adventure into the negative rates territory. In January 2016, Japan's central bank decided to lower the rate on new deposits to -0.1%, introducing this new paradigm in the Asian continent.

Following the trend towards negative rates among several (mainly European) regions, a significant growth in the use of financial derivatives has arisen. As stated in the BIS[5], FX, equity and interest-rate derivatives accounted for $72 trillion in 1998 in terms of notional amount. By 2015, this quantity rose sharply to $522.9 trillion. About 80% of this notional amount is covered by interest rate derivatives, which had been priced as if no negative rates were permitted until recent years. In our criteria, this fact is enough to justify industry's deep concern about models' performance on this new scenario.

1.2 Explaining negative rates

The recent financial crisis that emerged in August 2008 weakened the trustfulness among counterparties of financial transactions, jeopardizing the stability of the whole financial system. Giants of the sector collapsed, while the interconnectedness between institutions led to a quick contagion of the default risk.

Despite of not being considered until the crisis, the *credit quality* of the counterparty suddenly became a key aspect of the market risk. For (mostly) small institutions, trading became either *too risky* or *too expensive* to be afforded under the price of the credit risk. A halt into the economy was starting to be feared by monetary authorities, and the low (and even negative) interest rates policies appeared as a response to that issue.

[5] `http://www.bis.org/statistics/about_derivatives_stats.htm`

These exceptional measures were headed by the ECB, that progressively lowered interest rates from 2008 to 2011 to make borrowing cash cheaper. This policy should encourage investors to borrow money and invest into the economy, which would therefore find the funds and grow.

By June 2014, ECB's policies appeared insufficient to boost the economy, and more drastic measures were understood as necessary. ECB fixed a key interest rate to -0.20%, overstepping (for the first time) the (theoretically) unattainable zero-barrier. This is a (fairly) aggressive move, which aims to inspire investors further to bring in new money into the economy to help activity surge. The use of negative rates is an unconventional but not unprecedented tool of economic policy. As mentioned in the previous section, in recent times some central banks have also taken the decision to move some of their key interest rates into the negative territory.

By definition, a negative rate forces leaving money at rest in a bank to be a guaranteed money-loser activity. ECB would, in fact, *punish* investors and banks for holding their cash in their respective deposits. In this paradigm, banks would strongly prefer to lend money to each other, provided that EURIBOR/LIBOR[6] remains *less negative* than ECB's punishment for securing their money. In any case, financial institutions still prefer to be penalized by ECB taxes rather than lending money to the investors. Expected credit risk losses largely exceed those caused by ECB negative rates on the deposits, and therefore negative rates remain in the economy, as a natural consequence of the credit risk deep fear of the financial sector towards individuals.

Negative deposit rates are presented by monetary authorities as a tax imposed by the central bank on commercial banks to encourage them to increase lending to companies and consumers. The disjunctive is therefore assumed by commercial banks' managers, who can decide whether to transfer it (or not) to their clients. By reducing their lending rates and charging negative rates for deposits, the tax is immediately transferred to the customer. Depositors are punished, but banks' benefits do not suffer the tax. Choosing not to pass the tax to their customers might not be a better option, at least in global terms. If this decision is taken, the result is an incentive to stop lending money to the real economy, since banks are then forced to endure the punishment on their own benefits.

In fact, some iconic commercial banks have already followed the path initiated by the central Banks and are charging taxes on their depositors. The process started in August 2016[7], when the *Royal Bank of Scotland (RBS)* decided that all those corporative clients of their investment banking division that were operating with derivative products were to be charged a negative interest rate on their margin accounts. German bank *Postbank*, 100% subsidiary of the Deutsche Bank, adhered to the trend and announced that a 3.90€ commission would apply on all those clients whose monthly earnings were not higher than 3000€ per month.

[6]The precise definition of LIBOR rate, if needed, would be provided in chapter 2 of the Thesis. From now on, and in the spirit of continuity of the text, it is enough to understand that EURIBOR/LIBOR are averaged-rates at which banks among Europe/UK are willing to lend money to each other.

[7]`http://www.telegraph.co.uk/business/2016/08/19/rbs-biggest-customers-face-negative-rates/`

Spanish financial sector has not escaped from this awkward situation. By June 2016, *BBVA* was the first Spanish commercial bank to admit that they were charging these taxes on their clients, although they stated that this practice was being negotiated *case by case*[8]. Several main Spanish commercial banks are progressively following this trend, and admit that they are either starting to apply these taxes on particular customers or considering to do so in the years to come.

There are several reasons why a low (even negative) interest rates policy improves economic growth and is therefore implanted by central banks all over the globe. Firstly, it enlarges credit to the real economy, contributing to an increase in asset prices and forcing investors towards riskier instead of safer assets. In addition, the exchange rate is depreciated indirectly. Individuals would change currency to invest in Government bonds of countries where these kind of policies have not been applied, and therefore account for a higher yield. Since the exchange rate is depreciated, net exports are boosted.

There are, however, several drawbacks that might arise with the implementation of this kind of exceptional policies[9]. Some of them can be enumerated according to the following list:

- **Banks' benefits are cut back:** Banks' margins shrink, jeopardizing the profitability of the banking business.
- **Excessive risk-taking:** Although this feature might boost economic growth (as explained previously), an uncontrolled flow of funds towards risky assets in the spirit of obtaining higher yields can be considerably dangerous, especially for individuals.
- **Disincentive for Government debt reduction:** A sustained negative interest rates environment can contribute to the emergence of perverse incentives for governments, which might choose not to reduce their debt since there is no pressure for them in terms of interest payments. In fact, they are actually encouraged to borrow *even* more money.
- **Operational risks:** Since most trading systems (and industry firms) are not ready (yet) for derivatives pricing under a negative interest rates context, their inability to get adapted to this new paradigm may lead to serious concerns. This Thesis is devoted to the development of a full-comprehensive survey about the kind of models that might be useful in this context, and aims to be helpful within this new and defying environment.

[8] http://www.elperiodico.com/es/noticias/economia/banca-espanola-cobrar-depositos-grandes-clientes-

[9] See http://www.bankofgreece.gr/Pages/en/Bank/News/Speeches/DispItem.aspx?Item_ID=347&List_ID=b2e9402e-db05-4166-9f09-e1b26a1c6f1b for further discussion on this topic.

Chapter 2

Theoretical background

2.1 Interest rates framework

Some preliminary *interest rate topics* are covered trough this section, since they are strictly necessary for subsequent developments. We suggest the experienced reader to omit this section (and possibly the next one) if it seems too straightforward for them. Along the exposure, we will mainly follow the standard treatment of [7]. We will also use definitions provided in [3].

- **LIBOR** (*London Interbank Offered Rate*)**:** It is the rate of interest that a selection of major banks charge each other for short-term loans. It is an indication of the average rate at which contributor banks can borrow money in the *London interbank market* for a particular period and currency.

- **OIS:** OIS rates stand for *Overnight indexed swaps*, which are interest rate swaps in which a fixed rate of interest is exchanged for a floating rate that is the geometric mean of a daily overnight rate. The payment on the floating side replicates the aggregate interest that would be earned from rolling over a sequence of daily loans at the overnight rate. The overnight rates for the EUR, USD and GBP markets are *the Euro Overnight Index Average* (*EONIA*), the *effective Federal Funds Rate* and the *Sterling Overnight Index Average* (*SONIA*) respectively.

  The suitability of several market rates (LIBOR-OIS) as inputs for the risk-free rate and discounting curve is currently under discussion, as the market practice has progressively changed its tendency since the financial crisis of 2008. As it is carefully explained in [8], the credit crunch shocked industry's conception about the optimal candidate for the risk-free discounting curve. While the standard choice among interest rate traders used to be LIBOR and LIBOR-swap rates before 2008, they have been considered a poor proxy for the risk-free rate under stressed market conditions during recent years, and therefore have been progressively replaced by the OIS rate, especially when collateralized portfolios are under concern[10].

[10]In fact, using a *unique* discounting curve is a simplification of today's standard market practice, where *multiple curves* are combined, decoupling the process of implying forward rates (in market's lingo, *forwarding*) from the process of computing *discounting factors*. We address the interested reader to [9], where this market practice is thoroughly analysed.

Following the claim of [8], the OIS curve is used as proxy for the risk-free discount curve. This choice provides a benchmark which is not intended to be discussed through the Thesis.

- **Maturity and day count convention:** The *time to maturity* $\tau := T - t$ is understood as the amount of time (in years) remaining among dates t and T. Since there are several market conventions about *how to measure* the amount of time within discrete time intervals (whether to include holidays or not, for example), the time to maturity *does depend* on the day count convention chosen. This feature cannot be avoided; it is implicit in daily operative and traders should quote the type of convention that has been chosen in every pricing in order to replicate the valuations given. Further discussion about the problematics when dealing with different day count conventions is beyond the scope of this Thesis, and we reference the interested reader to [10] for a complete discussion on this topic.

- **Tenor:** We define the tenor of an interest rates derivative as the time to maturity for the underlying fixed income product. In this sense, and in a slight abuse of notation, "maturity time" is usually understood as maturity time *for the derivative*, and "tenor" is therefore reserved for the time to maturity *of the underlying fixed income product*. This convention is followed here, unless otherwise stated.

- **Instantaneous and compounded rates. FRA contract. The forward rate:** Interest rates can be divided among *compounded* and *instantaneous* rates. Continuously, simply and annual compounding are particular cases of the compounded rates category, and they are fully discussed in [7]. They are quoted for investments on finite discrete time intervals, and differ among them in the kind of reinvestment guaranteed for the interest earned periodically.

  Instantaneous rates cover investments over infinitesimal time intervals, and therefore are hardly conciliated with real world rates. They *do not exist* in the markets. Despite of that, the literature has usually taken this approach, extracting analytical formulae for discrete time to maturity plain vanilla derivatives (such as caplets, floorlets and swaptions) from the behaviour deducted for the instantaneous rate.

  Concretely, our main concern is the *instantaneous forward rate*, since floorlets and caplets under consideration have the forward rate for a given tenor as their underlying instrument. Basically, forward rates are characterized by three time instants, namely 0 (today's date), t (investment's start date) and T (investment's end date). In fact, a forward rate can be defined from a prototypical *forward rate agreement* (FRA), which basically locks in today the forward rate to be applied for a future investment horizon $[t, T]$ on a pre-specified notional amount N.

  The process of fixing a forward rate for a future investment accounts for the *fixed leg* of the contract. The *floating leg* is therefore indexed to the behaviour of a reference index, which is specified at the beginning of the contract. It is assumed that the standard reader is used to specific details about the behaviour of a FRA contract (such as methods of payment, simply-compounding rates' specification and so on), and therefore we omit them in our development. Further details, if needed, can be found, for example, in [11].

The previous approach to the FRA contract is motivated by the concept known as the *fair value of a contract*, which leads to the first definition of a (compounded) forward rate. We say that a FRA contract has a *fair value* at settlement if $V(0,t,T)=0$ at time 0. The simply-compounded interest forward rate (since reference rates on FRA's should be quoted in simply-compounding form) is then defined by the strike that guarantees a fair value of the FRA at settlement. From now on, today's date is fixed at 0 unless otherwise stated, and is omitted in the sake of shorthand notation. Therefore, the (simply-compounded) forward rate interest rate prevailing at today's date 0 for the future investment period $[t,T], F(t,T)$, is defined by

$$F(t,T) := \frac{1}{\delta(t,T)}\left(\frac{P(0,t)}{P(0,T)}-1\right), \tag{2.1}$$

where $\delta(t,T)$ accounts for the year fraction (amount of time in years) between dates t and T[11], and $P(0,S)$ is today's price of a zero-coupon bond, which pays a monetary unit at date S.

The instantaneous forward rate, $F(t)$, is then defined as the simply-compounded forward rate (2.1) when the future investment period becomes *infinitesimal.* Only a future date is therefore needed to characterize the future investment period, and the notation becomes even shorter:

$$\begin{aligned} F(t) :&= \lim_{t\to T^-} F(t,T) = -\lim_{t\to T^-} \frac{1}{P(0,T)}\frac{P(0,T)-P(0,t)}{T-t} = \\ &= -\frac{1}{P(0,t)}\frac{\partial P(0,t)}{\partial t} = -\frac{\partial \ln P(0,t)}{\partial t}. \end{aligned} \tag{2.2}$$

(2.2) explicitly states that there exists a relationship between zero-coupon bond prices and instantaneous forward rates. This relationship is often used to extract the implied (market) forward rates from zero-coupon bond prices.[12]

- **Interest rate swaps (IRS) and forward swap rates:** A (forward start) interest rate swap (IRS) is an agreement between two parties that accord to exchange several cash flows indexed to the behaviour of two reference forward rates (*floating* and *fixed* leg) during a period of time specified by the tenor of the swap, starting from a future time instant.

  Given the set of n pre-specified payment dates $T_1, T_2, ..., T_n$, on every instant of the set T_i, the fixed leg party pays the amount $N\delta_i K$, while the floating leg pays

[11] Consequently, $\delta(t,T)$ *does depend* on the day count convention.

[12] To be precise, coupon bond prices are not used as input when models are calibrated to market data, since zero-coupon bond prices are not quoted in the markets. It is a common market practice to extract a *risk-free discounting curve* from hypothetical risk-free coupon bonds prices, and then use the bootstrapped risk-free curve as an input from where implied instantaneous (and continuously-compounded) forward rates as well as discounting factors are computed. This practice is followed through the Thesis.

$N\delta_i L(T_{i-1}, T_i)$[13]. N accounts for the total notional outstanding the contract, δ_i is the year count fraction between dates T_{i-1} and T_i, K is the strike rate designed by the contract and $L(T_{i-1}, T_i)$ corresponds to the floating reference rate resetting at the previous instant T_{i-1} for the maturity given by the current instant T_i.

A full description of the behaviour of IRS contracts might be consulted, again, in [7]. In a clear analogy with the FRA contract, requiring a fair value of the IRS at time $t = 0$[14] leads to a particular value of the strike K faced by the fixed leg of the contract, known as the *forward swap rate*. Consequently, the forward swap rate $S(t, T_{start}, T_{mat})$ observed at time t for the n sets of times specified in the interval $[T_{start}, T_{mat}]$ and year fractions δ_i can easily be obtained as:

$$S(t, T_{start}, T_{mat}) := \frac{P(t, T_{start}) - P(t, T_{mat})}{\sum_{i=T_{start}+1}^{T_{mat}} \delta_i P(t, T_i)}, \tag{2.3}$$

where the denominator is usually called the *forward level function*. As it is proved in [7], it is straightforward to rewrite expression (2.3) in terms of forward rates, which shows the equivalence between forward rates and forward swap rates.

- **Caplets, floorlets, caps and floors:** Caps and floors are usually understood as the "positive parts" of a payer/receiver IRS respectively, since their payoffs can be computed as the sum of those exchange payments which are above zero for every date of the set $[T_{start}, T_{mat}]$. Therefore, a cap/floor consists on a basket of n options, each one of them referred to the behaviour of the reference floating rate of an IRS for each one of the dates among the set $[T_{start}, T_{mat}]$. Following previous notation, it is easily deduced that the cap discounted payoff at time t is given by

$$\sum_{i=T_{start}+1}^{T_{mat}} D(t, T_i) N\delta_i (L(T_{i-1}, T_i) - K)^+, \tag{2.4}$$

where $D(t, T_i)$ accounts for the discount factor to be applied for the time interval $[t, T_i]$. Similarly, the floor discounted payoff reads

$$\sum_{i=T_{start}+1}^{T_{mat}} D(t, T_i) N\delta_i (K - L(T_{i-1}, T_i))^+. \tag{2.5}$$

Each one of the terms in both sums is called *caplet/floorlet* respectively. These options account for the most basic plain vanilla fixed income derivatives under consideration in the Thesis, since their prices are directly computed by the models outstanding.

[13] A subtle simplification has been done here, in the spirit of a simpler notation. In general, IRS payment dates do not have to be identical for both parties. Indeed, a prototypical American IRS has a fixed leg with annual payments and a floating leg with quarterly or semiannual payments.

[14] I.e., imposing that the contract has zero value for both parties at settlement time.

2.2 Mathematical framework

This section provides some mathematical insights which might be well-known by the experienced reader and can be immediately skipped if that was the case. We will mainly follow [12]. In any case, the standard reader is supposed to be familiarized with basic stochastic calculus concepts such as *Wiener processes*, *filtrations* and *martingales*, and is strongly recommended to resort to [12] if a refreshment was needed.

- **Money market account:** The value at time t of a *money market account*, $B(t)$, represents a zero-risk investment, continuously compounded at the $r(t)$ rate. Since the money held in the money market account continuously evolves at a rate $r(t)$ for every instant t, the money market account obeys to the following differential equation[15]:

$$dB(t) = r(t)B(t)dt. \tag{2.6}$$

  Solving (2.6) by ordinary differential calculus gives

$$B(t) = B(0)\exp\left(\int_0^t r(u)\,du\right), \tag{2.7}$$

  where $B(0)$ is the amount invested at time $t = 0$.

- **Zero-coupon bond price:** A T-maturity zero-coupon bond is a contract that guarantees the holder the payment of one unit of currency at time T, with no intermediate payments. $P(t,T)$ represents the value of the contract at time $t < T$. Obviously, $P(T,T) = 1\ \forall T$.

- **No-arbitrage pricing. Change of numeraire:** A milestone in the development of financial derivatives pricing is found in [13]. The authors prove that the existence of an *equivalent martingale measure*, Q,[16] is equivalent to the absence of arbitrage opportunities in a contingent-claims market[17]. If Q belongs to the set of equivalent martingales measures, the (fair and unique) price of any contingent claim $V(t)$ can be found as the (conditional) expected value under the measure Q of the product of the (in general, stochastic) discounting factor by the value of the claim at maturity, $V(T)$:

[15] Every process under study *adapts to the natural filtration* $\{\mathcal{F}\}_{t=0}^{t}$ considered through the text. This fact solves any possible uncertainty among these processes at time t (and previous instants) when time t comes. More details on this technical issue can be found in [12].

[16] Again, some technicalities are omitted for the sake of brevity. An equivalent martingale measure is a probability measure defined on the measure space which accomplishes for certain properties, such as *equivalency* with the pre-defined probability measure of the probability space, Q_0, *existence* of the Radon-Nikodym derivative and some others. The whole set of technical requirements can be found, for example, in [7] or [12].

[17] This is not their only contribution. They also prove that a financial market is *arbitrage-free* and *complete* if and only if there exists a *unique* equivalent martingale measure.

$$V(t) = E^Q\bigg(D(t,T)V(T)|\mathcal{F}(t)\bigg). \tag{2.8}$$

Not every contingent claim $V(t)$ can be priced by the expectation under the equivalent martingale measure Q (due, fundamentally, to the presence of the stochastic discount factor *inside of* the conditional expectation), and it may be convenient to *change* the original equivalent martingale measure to another one which eases the valuation. This process is known as the *change of numeraire technique.*

A numeraire $U(t)$ normalizes the value of any asset in the market $S(t)$ by referring it to the numeraire units, i.e., $S(t)/U(t)$. The only necessary conditions to be imposed to the numeraire is to be *positive* and to pay *no dividends.*

Not every numeraire choice is *useful* in terms of easing the pricing process. In fact, only two particular numeraires (defined in the previous section) are considered in this Thesis: the **money market account** and the **zero-coupon bond price**.

As stated in [7], equation (2.8) can be generalized to *any particular choice* of numeraire. Assume that there exists a particular numeraire U and a probability measure Q^U equivalent to the initial Q^0 such that the value of any asset X *in numeraire units* $(X(t)/U(t))$ is a martingale under Q^U:

$$\frac{X(t)}{U(t)} = E^U\bigg\{\frac{X(T)}{U(T)}|\mathcal{F}(t)\bigg\}, 0 \leq t \leq T. \tag{2.9}$$

Then, the *change of numeraire technique* states that, for any other numeraire W, *there exists a probability measure* Q^W, equivalent to Q^0, such that the value of any asset X in the new numeraire units is a martingale under Q^W:

$$\frac{X(t)}{W(t)} = E^W\bigg\{\frac{X(T)}{W(T)}|\mathcal{F}(t)\bigg\}, 0 \leq t \leq T. \tag{2.10}$$

- **Risk-neutral measure and T-forward measure. Change of numeraire in practice:** Choosing the money market account as a numeraire leads to the probability measure known as *the risk neutral measure*, which has been deeply discussed over several classic derivatives pricing texts, such as [11]. According to (2.10), this choice guarantees that the discounted value of any asset, $\exp(-\int_0^t r(u)du)X(t)$, follows a driftless process, and is therefore extensively used in pricing equity derivatives.

  The most interesting choice of numeraire for the objectives of the Thesis is the zero-coupon bond price, $P(t,T)$, which defines the so-called *T-forward measure.*[18] This probability measure is particularly interesting in the interest-rates world since there are several important results associated to it:

[18] It should be noted that this choice *depends explicitly* on the maturity T of the selected bond. This is the reason why the name is quoted as *T-forward* measure.

1. **Under the T-forward measure, any simply-compounded forward rate accounting for a future investment period which ends at T is a martingale**. This statement is proved in [7], and the interested reader is referred there for further details. Therefore, we have that:

$$E^T\{F(t,S,T)|\mathcal{F}(u)\} = F(u,S,T) \tag{2.11}$$

for every $0 \leq u \leq t \leq S \leq T$.

2. **The instantaneous forward rate $F(t,T)$ equals the expected value of the future instantaneous spot rate $r(T)$ under the T-forward measure.** Indeed:

$$E^T\{r(T)|\mathcal{F}(t)\} = F(t,T). \tag{2.12}$$

3. **Under the T-forward measure, the volatility of the instantaneous forward rate, $\sigma(t)$, is driftless.** This feature is explicitly mentioned in [14], and makes the T-forward measure a really convenient tool when dealing with *stochastic volatility models*, which permit the volatility to follow its own stochastic process.

The process of changing the numeraire between these particular choices (risk-neutral and T-forward measure) via Radon-Nikodym derivative is fully reviewed in [15]. The interested reader is redirected there for further details.

- **Fundamental Theorems of Derivatives Pricing:**

From (2.10), and within the particular choices of numeraires already stated, it is straightforward to obtain the two fundamental theorems of derivatives pricing under consideration through this Thesis for the price of any kind of fixed income plain vanilla derivative $V(t)$ under study. Under the risk-neutral measure Q, we have:

$$V(t) = E^Q\bigg\{\exp\bigg(-\int_t^T r(u)\ du\bigg)V(T)|\mathcal{F}(t)\bigg\}. \tag{2.13}$$

Equivalently, under the T-forward measure Q^T:

$$V(t) = P(t,T)E^{Q^T}\{V(T)|\mathcal{F}(t)\}. \tag{2.14}$$

Further details about the particular pricing process of several fixed income derivatives using (2.14), such as caplets or floorlets, can be found, for instance, in [7]. (2.14) is the pricing formula that will be mainly used through the development of the Thesis.

Chapter 3

From Black to SABR. Models history in the industry

Several interest rates models have been traditionally used by the industry during non-negative rates recent decades. This section follows their performance and states their main characteristics, aiming to provide a common reference framework for subsequent comparisons. Through this chapter, we mainly follow [16].

3.1 Black (1976)

The standard way of quoting prices of caps/floors[19] is in terms of Black's model [17], which is a version of the Black-Scholes (1973) model adapted to deal with forward underlying assets. We assume that any instantaneous forward rate $F(t)$[20] follows a driftless lognormal process reminiscent of the basic Black-Scholes model under the T-forward measure:

$$dF(t) = \sigma \cdot F(t) \cdot dW(t), \tag{3.1}$$

where $W(t)$ is a Wiener process and σ is the parameter accounting for the instantaneous forward rate (constant) volatility under lognormal specification. The solution to this stochastic differential equation (3.1) reads

$$F(t) = F(0)\, e^{\sigma W(t) - \frac{1}{2}\sigma^2 t}. \tag{3.2}$$

Therefore, as it can be seen in [16], the value at time t of a caplet/floorlet on any forward rate over the future investment period $T = [T_{start}, T_{mat}]$[21] , $F(t, T_{start}, T_{mat})$, with strike

[19] It is extended market practice to actually quote these prices in terms of *implied volatilities*, due to the *one-to-one* correspondence between both quantities. While the price of any financial product shall be unique when quoting in the market, there are *different* volatility specifications (Black and Bachelier) and, therefore, when quoting volatilities traders should also quote which one of these two models has been selected for the quotation.

[20] Such as a LIBOR forward or a forward swap rate, for instance.

[21] The following comment is made in the sake of clarifying notation. When caps(floors) and caplets(floorlets) were introduced in section 2.1, $T = [T_{start}, T_{mat}]$ denoted the set of n payment dates embedded into the cap(floor) structure. Since a caplet(floorlet) can be understood as a *unique-payment* cap(floor), $[T_{start}, T_{mat}]$ does not account for a set of dates now, but for the length of the (future)

(rate) price K, today's value of $F(t, T_{start}, T_{mat})$ equal to $F(t, 0)$, total notional amount N and constant volatility of the forward rate σ is[22]

$$Caplet(t, T, N, K) = N\delta P(t, T_{mat}) B_{call}(T_{start}, K, F(t, 0), \sigma), \tag{3.3a}$$

$$Floorlet(t, T, N, K) = N\delta P(t, T_{mat}) B_{put}(T_{start}, K, F(t, 0), \sigma), \tag{3.3b}$$

where

$$\begin{aligned} &B_{call}(T_{start}, K, F(t, 0), \sigma) = F(t, 0)\phi(d_+) - K\phi(d_-), \\ &B_{put}(T_{start}, K, F(t, 0), \sigma) = -F(t, 0)\phi(-d_+) + K\phi(-d_-), \\ &d_\pm = \frac{log(\frac{F(t,0)}{K}) \pm \frac{1}{2}\sigma^2 T_{start}}{\sigma\sqrt{T_{start}}}. \end{aligned} \tag{3.4}$$

δ is the day count fraction from time T_{start} to T_{mat} and $P(t, T_{mat})$ is the price today of a zero-coupon bond which pays a monetary unit at time T_{mat}. Immediately, since a cap/floor can be understood as a finite sum of caplets/floorlets, we have from (3.3):

$$Cap(t, T, N, K) = N \sum_{i=T_{start}+1}^{T_{mat}} \delta_i B_{call}(T_{i-1}, K, F(t, i), \sigma_i) P(t, T_i), \tag{3.5a}$$

$$Floor(t, T, N, K) = N \sum_{i=T_{start}+1}^{T_{mat}} \delta_i B_{put}(T_{i-1}, K, F(t, i), \sigma_i) P(t, T_i), \tag{3.5b}$$

where δ_i is the day count fraction applying to the period starting at T_{i-1} and ending at T_i, and $F(t, i)$ is today's underlying forward rate for that period.

Although Black's model permits to price plain-vanilla interest rates derivatives analytically, its many flaws make it unacceptable for the industry. Firstly, the lognormal solution *does not allow* the underlying rate to go below zero. As we will see later, this problem can be solved by adding a *shift* to the rate behaviour, so analytical solutions can be maintained within the so-called *shifted Black* framework.

An unsolvable problem within Black's model is that one of its founding hypothesis strongly violates the empirical behaviour of interest rates. The basic premise of Black's model, that σ is independent of K and $F(t, 0)$, is clearly rejected by the markets. In particular, for a given maturity, options implied volatilities exhibit a *pronounced dependence on their strikes*. This phenomenon is called the **skew or the volatility smile**. In order to accurately value and risk manage options portfolios, refinements to Black's model are necessary.

investment period (i.e., *the tenor* of the derivative) the forward refers to.

[22] The fundamental theorem of derivatives pricing under the T-forward measure (2.14) has been applied.

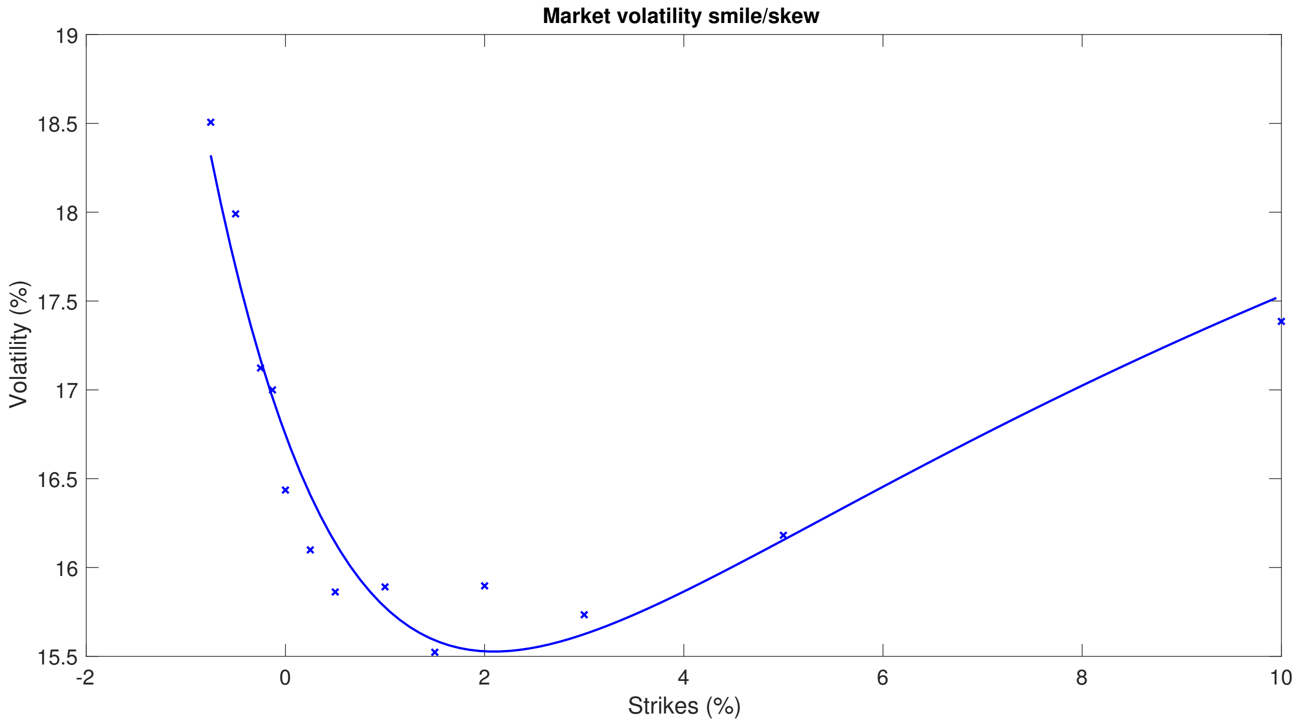

Figure 3.1: *Market usual smile/skew. σ depends on the srike K being considered, for both maturity ($T = 9Y$) and tenor (6 months) given.*

3.2 Local volatility (1994)

The first widely-embraced industry's proposal to deal with smiles and skews within the interest rates markets came by the so-called *local volatility models*, firstly introduced by Dupire, Derman and Kani (see [18], [19] and [20]).

This celebrated approach meant an improvement over Black's model by using the market prices of options to find an effective ("local") specification of the underlying process, so that the theoretical implied volatilities match the market implied volatilities. The stochastic differential equation that describes the dynamics of the forward rate under the local volatility model in the T-forward measure is given by

$$dF(t) = C(F(t), t) \cdot dW(t), \tag{3.6}$$

where $C(F(t), t)$[23] is a certain (deterministic) volatility coefficient. Although local volatil-

[23]Any mathematical-advanced reader might have realized previously that the instantaneous forward rate $F(t)$ *has to* obey equation (3.6) necessarily, due to the **Martingale representation theorem**. This theorem states that any martingale under the probability measure in which $W(t)$ is a Wiener process can be written as the Ito process shown in (3.6), where $C(F(t), t)$ denotes an (unknown) *adapted process*. Local volatility formulation, therefore, accounts for a particular case of the Martingale representation

ity models have been widely used over the industry, since the smile calibration is remarkable for any given maturity, they lead to *unstable* (and *incorrect*) hedges, since they predict a dynamic evolution for the smile that *opposes* the one observed within the markets, and therefore have been discarded by the firms during recent years. This problem is brilliantly explained in the original paper of Hagan *et al.* [21] and in the introduction of the reference book by Rebonato *et al.* [14].

The idea is as follows. When the price of the underlying increases, one expects that the smile shifts to higher levels as well. In contrast, the local volatility model predicts that the smile will shift to lower prices after an increase of the underlying. The opposite counterintuitive movement can be seen for a decrease of the underlying. Due to this contradiction, *delta* and *vega* risk metrics under the local volatility model may perform worse than the risk metrics of naive's Black.

The original paper of Hagan [21] is a highly-recommended reference at this point, since they clearly exemplify this fact. Their argument is replicated in Appendix A for interested readers. In conclusion, the local volatility model is *suited for pricing purposes*, but *not for proper risk management*.

In this context, Hagan *et al.* (2002) define the **SABR**[24] [21], a stochastic-volatility model that will be explained next. As it is mentioned in the introduction of [14], the SABR model is not as accurate for fitting today's observed smile (even though it is precise enough to do it *reasonably well*) as local volatility models do but, instead, predicts a dynamic evolution of the smile which is *completely consistent* with the one observed in the markets.

This characteristic produces stable hedges that, combined with the availability of a closed-formula for implied volatilities under lognormal or normal specifications (Black's or Bachelier's), have made the SABR industry's preferred candidate to work with interest rates derivatives, despite of some remarkable drawbacks. In any case, it is important to note that the preference for the SABR model against local volatility competitors *comes from an empirical perspective* (see [14] for further discussion on this topic).

theorem, where this process is restricted to be a deterministic function of time, to be calibrated with market data. Models to come, such as SABR, permit $C(F(t), t)$ to evolve according to its own stochastic process.

[24] The name stands for "*Stochastic alpha beta rho model*".

3.3 SABR (2002)

3.3.1 The model

The SABR model describes a single forward, such as a LIBOR forward rate, a forward swap rate or a forward stock price[25]. The volatility of the forward $F(t)$ is usually described by a parameter σ. SABR is a two-factor dynamic model in which both $F(t)$ and $\sigma(t)$ are represented by stochastic state variables whose time evolution is given by the following system of stochastic differential equations:[26]

$$dF(t) = \sigma(t) \cdot F(t)^{\beta} \cdot dW(t), \tag{3.7a}$$

$$d\sigma(t) = \alpha \cdot \sigma(t) \cdot dZ(t), \tag{3.7b}$$

with the prescribed time zero (currently observed) value $F(t,0) := f$ for the forward rate. In this representation, we have explicitly chosen the probability measure that makes the forward and its volatility *driftless* (known as the T-forward measure), and therefore the Brownian motions are referred to that measure which, obviously, depends on the maturity of each forward (this aspect is explicitly stated in [14], where a superscript T is included in both formulae). To avoid notational tediousness we do not adhere to that notation, but it should be noted that (3.7) is a different model for each maturity under consideration. $W(t)$ and $Z(t)$ are two correlated Wiener processes with correlation coefficient ρ:

$$E^{Q^T}(dW(t) \cdot dZ(t)) = \rho\, dt. \tag{3.8}$$

It should be satisfied that $\alpha \geq 0$, $0 \leq \beta \leq 1$ and $-1 < \rho < 1$. The above dynamics (3.7) is a stochastic version of the CEV model with the *skewness* parameter β. In fact, it reduces to the CEV model if $\alpha = 0$.[27]

As each forward rate is described in its own T-forward measure, the forwards that comprehend the yield curve have *no tools to interact with each other*. SABR model is not suitable, then, to provide insights about the dynamics of a yield curve, but it fits the implied volatility curves given by the markets for any single exercise date reasonably well. This is one of the reasons why so many traders choose the SABR model to price and hedge their fixed income plain-vanilla (single exercise date) derivatives, such as caplets, floorlets and swaptions. In order to price path-dependent derivatives, a proper calibration of the volatility cube becomes a must.

[25] Again, it is important to note that in any case we are considering *a single maturity*, and therefore each calibration procedure is limited to *that maturity*. Several options for including inter-maturities dependencies during the calibration process for the SABR model have been investigated in previous literature (see, for instance, [14] or [22]), but they lie beyond the scope of this Thesis.

[26] Discussion of the previous section can be recalled here, just to guarantee that the Martingale representation still applies. As both $\sigma(t)$ and $F(t)$ are adapted processes, the product given by $\sigma(t)F(t)^{\beta}$ satisfies this condition, and the Martingale representation theorem is respected under the SABR formulation.

[27] Since the CEV model is not suitable for coping with negative rates (unless $\beta = 0$, in which case we recover Bachelier's model (4.1), to be explained in the next chapter), it has not been explained during the text. We address the interested reader to Appendix B, where the model's main features are highlighted.

3.3.2 The parameters. Sensitivity analysis

Since f is currently observed in the markets, the set of parameters constituting the SABR model for any fixed maturity is given by $\{\sigma(0)\,\alpha, \beta, \rho\}$[28]. Each parameter implies different effects over the smile/skew for a given maturity:

- β: It stands for the *power parameter*. Due to parameters' degeneracy when fitting a smile for any maturity (especially, the degeneracy among ρ and β was firstly acknowledged by the original authors in [21]), calibrating the whole set of parameters is usually equivalent to "fitting the noise"[29], and therefore it is common market practice to fix the value of β according to aesthetic considerations. Mainly, it is fixed in the values of 0, 0.5 or 1, resulting in the *stochastic normal, stochastic-CIR* or *stochastic-lognormal* models respectively[30]. The option $\beta = 0.5$ seems to have gained strength among industry firms.

- $\sigma(0)$: It basically influences the *level* of the smile/skew.

- α (*volatility of the volatility*): Its effects are mainly acknowledged in the curvature of the smile/skew. In a second order of approximation, it also affects the level of the smile/skew. It does not affect the slope.

- ρ: It basically accounts for the slope of the smile/skew, expanding its influence over the curvature as well. It does not affect the level of the smile/skew.

Figure (3.2) below plots one of the fitted market smiles ($T = 8Y$) in the *empirical results* chapter via shifted SABR (see chapter 4), modifying *ceteris paribus* each one of the involved parameters $\{\sigma(0)\,\alpha, \rho\}$ to several values close to the calibrated's. From left to right, $\sigma(0)$, ρ and α are respectively modified, resulting in the previously exposed effects in the smile/skew.

[28]Notice that today's forward volatility, $\sigma(0)$, is not observed in the markets and therefore should be calibrated within the other parameters of the model.

[29]Certain analogy can be established within *SABR calibration procedure* for a given maturity and the statistical technique known as *PCA*, widely-used in the process of identifying those factors which mainly drive the term structure of interest rates (*TSIR*). Existent previous literature fix in *three* the number of necessary factors to explain among 95-99% of the variability of the TSIR for every maturity under consideration, respectively acknowledging for the *level*, the *slope* and the *curvature* of the TSIR. In this case, the *three* parameters that play this role and calibrate the smiles/skews without overparameterization are $\sigma(0)$, ρ and α respectively.

[30]This classification was originally proposed in [21], and we follow their convention here.

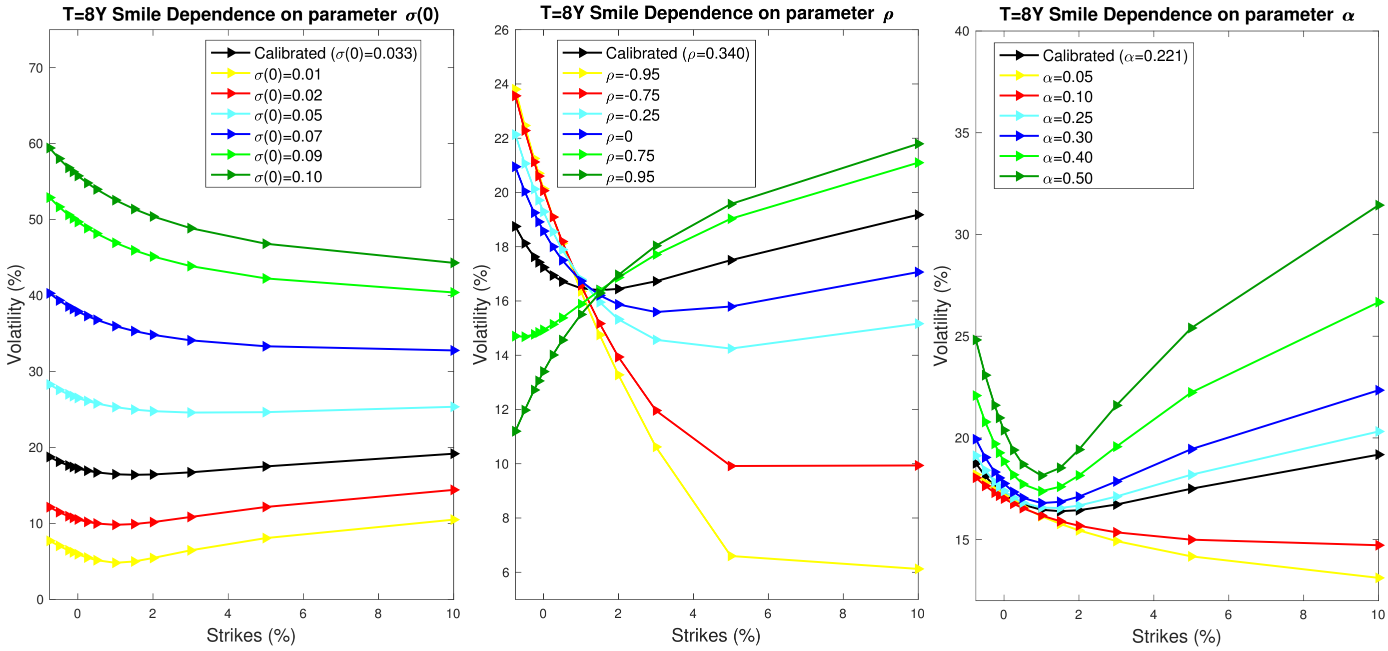

Figure 3.2: *Shifted SABR parametric sensitivity.*

3.3.3 Implied volatilities within SABR context

By means of an asymptotic expansion valid for *short-enough* maturities, the authors of [21] obtain an approximated analytical solution for the implied volatility that should be introduced in Black's formula (3.4) in order to price a caplet/floorlet for the future investment period $T = [T_{start}, T_{mat}]$, strike rate K, notional amount $N = 1$ and currently observed forward rate f, usually called *Hagan's formula*:[31]

$$\sigma(T_{start}, K, f) = \sigma(0)\left[(Kf)^{\frac{1-\beta}{2}}\left(1 + \frac{(1-\beta)^2}{24}\log^2\frac{f}{K} + \frac{(1-\beta)^4}{1920}\log^4\frac{f}{K} + \cdots\right)\right]^{-1} \cdot \frac{c}{g(c)} \cdot$$
$$\cdot\left\{1 + \left(\frac{\sigma(0)^2(1-\beta)^2}{24\cdot(Kf)^{1-\beta}} + \frac{\alpha\cdot\beta\cdot\rho\cdot\sigma(0)}{4\cdot(Kf)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24}\sigma(0)^2\right)T_{start} + \cdots\right\} \tag{3.9}$$

with:

[31]To be precise, what we understand today as Hagan's formula is not *actually* Hagan's original derivation. Hagan *et al.* committed a small mistake when deriving their formula, corrected by Obłój in [23]. From now on, Hagan's formula is presented by incorporating Obłój's correction.

$$
\begin{aligned}
c &:= \frac{\alpha}{\sigma(0)}(Kf)^{(1-\beta)/2} \cdot \log\frac{f}{K}, \\
g(c) &:= \log\left(\frac{\sqrt{c^2 - 2\rho c + 1} + c - \rho}{1-\rho}\right),
\end{aligned}
\tag{3.10}
$$

where the dots stand for higher-order negligible terms. For at-the-money options ($f = K$), Hagan's formula reduces to

$$
\sigma^{ATM}(T_{start}, f, f) = \frac{\sigma(0)}{f^{1-\beta}}\left[1+\left(\frac{\sigma(0)^2(1-\beta)^2}{24\cdot f^{2-2\beta}}+\frac{\alpha\cdot\beta\cdot\rho\cdot\sigma(0)}{4\cdot f^{1-\beta}}+\frac{2-3\rho^2}{24}\sigma(0)^2\right)T_{start}+\cdots\right].
\tag{3.11}
$$

As f changes during the day, the curve traced by the implied ATM volatility $\sigma^{ATM}(T_{start}, f, f)$ for a given maturity T_{start} is known as the *backbone* (see [24]), while the smile/skew is referred to the dependence of the implied volatility $\sigma(T_{start}, K, f)$ as a function of K for both given f and T_{start}.

As shown in [24], an alternative to the aesthetic consideration of fixing $\beta = 0.5$ lies on using (3.11) to estimate β from an auxiliary regression over a time series of both at-the-money volatilities and forward rates for a given maturity:

$$
\log(\sigma^{ATM}(T_{start}, f, f)) \approx \log(\sigma(0)) - (1-\beta)\log(f). \tag{3.12}
$$

Hagan's formula (3.9) is frequently used among traders to calibrate an implied Black volatility smile. Similarly, there exists a formula for Bachelier model (to be explained in the next chapter) to calibrate an *implied Bachelier volatility smile.* As shown in [21], the implied volatility that should be introduced in Bachelier's formula (4.4) in order to price a caplet/floorlet for the future investment period $T = [T_{start}, T_{mat}]$, strike rate K, notional amount $N = 1$ and currently observed forward rate f reads:

$$
\begin{aligned}
\sigma^n(T_{start}, K, f) = \sigma(0)(fK)^{\beta/2}&\frac{1+\frac{1}{24}\log^2\frac{f}{K}+\frac{1}{1920}\log^4\frac{f}{K}+\cdots}{1+\frac{(1-\beta)^2}{24}\log^2\frac{f}{K}+\frac{(1-\beta)^4}{1920}\log^4\frac{f}{K}+\cdots}\cdot\frac{c}{g(c)}\cdot \\
\cdot&\left\{1+\left(\frac{-\beta\cdot\sigma(0)^2(2-\beta)}{24\cdot(Kf)^{1-\beta}}+\frac{\alpha\cdot\beta\cdot\rho\cdot\sigma(0)}{4\cdot(Kf)^{\frac{1-\beta}{2}}}+\frac{2-3\rho^2}{24}\alpha^2\right)T_{start}+\cdots\right\}
\end{aligned}
\tag{3.13}
$$

with:

$$
\begin{aligned}
c &:= \frac{\alpha}{\sigma(0)}(Kf)^{(1-\beta)/2} \cdot \log \frac{f}{K}, \\
g(c) &:= \log\left(\frac{\sqrt{c^2 - 2\rho c + 1} + c - \rho}{1-\rho}\right),
\end{aligned}
\tag{3.14}
$$

where the dots stand for higher-order negligible terms.

3.3.4 Calibrating the SABR

Once β has been fitted (either using equation (3.12) with historical data or fixing it to a predetermined value attending to aesthetical reasons), $\{\sigma(0), \rho, \alpha\}$ should be calibrated for every given maturity. Two parameterizations have been explored in previous literature (see, for instance, [24]):

- **First parameterization. Estimating α, ρ and $\sigma(0)$ directly:** Given a set of implied volatilities (either Black's or Bachelier's) for some caplets with *the same maturity* and different strikes, they are compared with theoretical (Black/Bachelier) implied volatilities provided by an arbitrary choice of the parameters $\{\sigma(0), \rho, \alpha\}$ in formulae (3.9) or (3.13) respectively. The parameters for that maturity are chosen with any standard non-linear optimizer so that the sum of the quadratic errors is minimized:

$$
(\alpha, \rho, \sigma(0)) = \underset{\alpha,\rho,\sigma(0)}{\arg\min} \sum_i \left(\sigma_i^{market,(n)} - \sigma^{(n)}(T_{start}, K_i, f)\right)^2. \tag{3.15}
$$

  Obviously, different weights ω_i can be allocated to the set of market implied volatilities according to the analyst criteria (if there was a special interest in fitting some volatilities better than others, for example).

- **Second parameterization. Two-steps calibration:** This method was firstly proposed in [25], and focuses on decreasing the number of parameters to be calibrated. If market data for ATM implied volatilities is available, we can use equation (3.11) (or its Bachelier equivalent) to obtain $\sigma(0)$ by inverting the formula. Re-writing it in a suitable form, we find that a cubic polynomial equation must be numerically solved:[32]

[32] As noted in [25], this equation may have more than a single real root. In this case, it is claimed that the smallest positive root shall be selected. We adhere to this claim through the Thesis.

$$\left(\frac{(1-\beta)^2}{24\cdot f^{2-2\beta}}\,T_{start}\right)\sigma(0)^3+\left(\frac{\beta\cdot\rho\cdot\alpha}{4\cdot f^{1-\beta}}\,T_{start}\right)\sigma(0)^2+\left(1+\frac{2-3\rho^2}{24}\,\alpha^2 T_{start}\right)\sigma(0)-\sigma^{ATM}f^{1-\beta}=0. \tag{3.16}$$

Therefore, the optimization algorithm consists in two sequential iterative steps. Firstly, $\sigma(0)$ is found from the previously-step calibrated pair $\{\rho,\alpha\}$ using equation (3.16). Then, the calibration is performed among the two free parameters remaining:

$$(\alpha,\rho)=\arg\min_{\alpha,\rho}\sum_i\left(\sigma_i^{market,(n)}-\sigma^{(n)}(T_{start},K_i,f)\right)^2. \tag{3.17}$$

Since a root-finding algorithm must be used in every-step to obtain $\sigma(0)$ from equation (3.16), it has been claimed[33] that this estimation procedure might need more time to converge.

3.3.5 Further considerations

Through the Thesis, we have made an explicit differentiation between *implied Black volatilities* and *implied Bachelier volatilities.* Despite the fact that the industry has traditionally chosen the lognormal specification of implied volatilities, and therefore they have been quoted directly by using Black's formula (3.9), in a negative rates context this specification might be reformulated, as it is done in [26] or [27]. If the lognormal specification is to be maintained, both the implied volatility and the *shift* that has been used within the *shifted Black formula* (see next chapter) shall be quoted, since the caplet/floorlet/swaption price *does depend* on both unobservable parameters. If Bachelier's specification is used for quoting volatilities, normal volatility can be quoted directly.

Several drawbacks within SABR's using have been indicated by both practitioners and academics. A nice summary of them can be found in [3].

In its standard formulation (3.7), the SABR model does not admit negative rates. Another obvious drawback is that its implied volatility expression (both in Black and Bachelier form) is based on an approximated asymptotic expansion, which tends to fail when time to maturity becomes long enough. Not only that, but the dependence of its probability density function on the forward rate at maturity $F(T_{start},T_{start},T_{mat}):=F(T)$ (which, basically, plays the same role that $S(T)$ in Black-Scholes model) can be hugely problematic. The probability density function is zero for rates less or equal to zero (and even negative!), and therefore negatives rates are not permitted without the undesirable introduction of **arbitrage opportunities** (see [3] for further discussion on this topic). To extend the SABR beyond the negative rates frontier, some theoretical derivations (shifted SABR, free-boundary SABR) have been developed. Their structure and calibration procedure will be explained next.

[33] See, for instance, [24].

Chapter 4

Derivatives pricing under negative rates

The appearance of negative rates involved a full review of pre-existing pricing methodologies to cope with this new environment:

- **Lognormal models:** *Shifted models* arose as a natural response to this situation. They basically add a shift to the underlying forward rate to displace its zero-boundary into the negative domain. Since rates cannot become (theoretically) *arbitrarily negative*[34], this sounds as a rather good solution. Adding a shift to Black's model (see (3.1)) generates the *shifted Black model*, while doing it with the SABR (see (3.7)) results in the *shifted SABR model*. As original's SABR was the most usual approach until negative rates appeared, it seemed sound to adjust the pricing methodology to the shifted SABR model, where former SABR analytical solutions still apply (indeed, this is what most industry firms *have already done*).

- **Normal models:** Another possible solution is changing our focus of interest into *normal models*, which had been completely neglected until then due to their "main disadvantage": *they allowed negative interest rates from the beginning*. In fact, their domain comprehends the whole real line, and therefore no constraints are imposed to the values that the forward rate might take. This solution, rather simple and allowing for analytical formulation, seems a little unrealistic, since forward rates are not supposed to go far below the zero-barrier.

A list of suited candidates to cope with negative interest rates environment is stated next, and their basic properties are thoroughly analysed.

4.1 Bachelier (1900)

The normal model, introduced in 1900 by L. Bachelier [28], is the simplest approach to model negative interest rates. In the normal model, under the *T-forward measure* the instantaneous forward rate $F(t)$ follows the process

$$dF(t) = \sigma^n \cdot dW(t), \tag{4.1}$$

[34]See discussion on Chapter 1 of the Thesis.

where σ^n is the parameter accounting for the instantaneous forward rate (constant) volatility under normal (Bachelier) specification. The solution to (4.1) reads

$$F(t) = F(0) + \sigma^n W(t), \tag{4.2}$$

which means that the instantaneous forward rate follows a *Gaussian distribution*, with mean $F(0)$ and variance $\sigma_n^2 t$. Negative rates are therefore modelled in a natural way. Unfortunately, the solution (4.2) exhibits one of the main drawbacks of the normal model: with non-zero probability, $F(t)$ may become *arbitrarily negative in finite time*. Under typical circumstances this is, however, a relatively unlikely event.

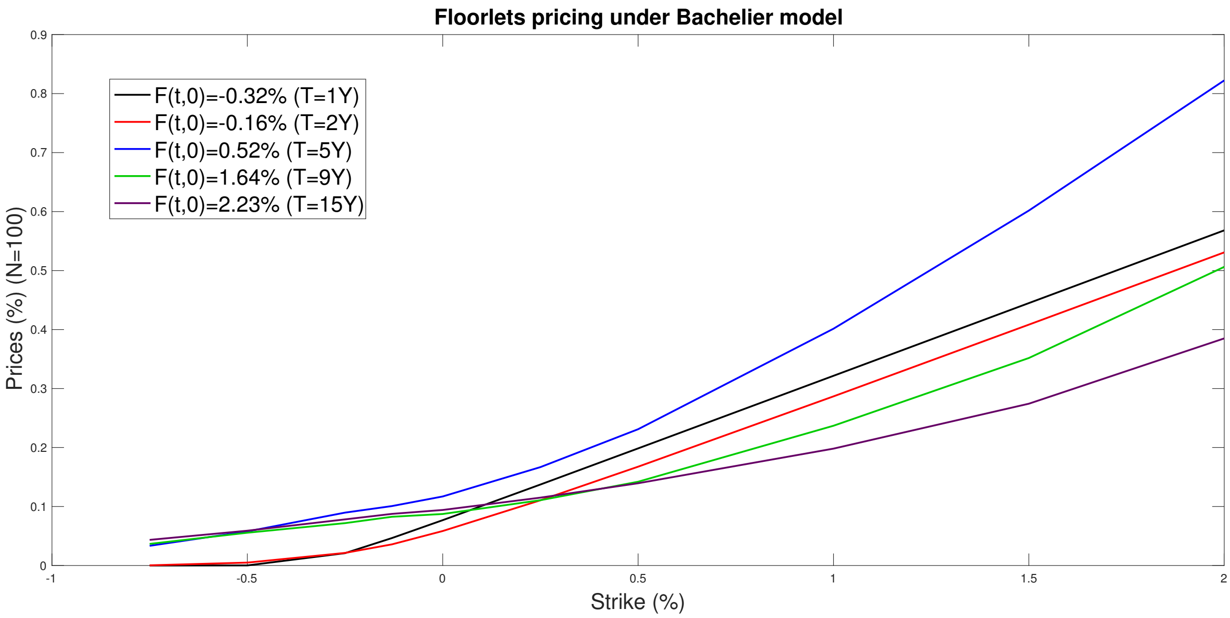

Figure 4.1: *Different floorlets valuations (formula (4.3b)) under Bachelier model. The prices are strictly possitive, even for strike rates below the zero-barrier.*

Under Bachelier model (4.1), closed-formulae for pricing caplets and floorlets can be immediately obtained by applying the fundamental theorem of derivatives pricing under the *T-forward measure* (2.14) (see [16]). The value at time t of a caplet/floorlet on any forward rate over the future investment period $T = [T_{start}, T_{mat}]$, $F(t, T_{start}, T_{mat})$, with strike (rate) price K, today's value of $F(t, T_{start}, T_{mat})$ equal to $F(t, 0)$, total notional amount N and constant (normal) volatility of the forward rate σ^n is

$$Caplet(t, T, N, K) = N\delta P(t, T_{mat}) B^n_{call}(T_{start}, K, F(t, 0), \sigma^n), \tag{4.3a}$$
$$Floorlet(t, T, N, K) = N\delta P(t, T_{mat}) B^n_{put}(T_{start}, K, F(t, 0), \sigma^n), \tag{4.3b}$$

with

$$
\begin{aligned}
&B^n_{call}(T_{start}, K, F(t,0), \sigma^n) = \sigma^n \sqrt{T_{start}}(d_+\phi(d_+) + \phi'(d_-), \\
&B_{put}(T_{start}, K, F(t,0), \sigma^n) = \sigma^n \sqrt{T_{start}}(d_-\phi(d_-) + \phi'(d_-), \\
&d_\pm = \pm \frac{F(t,0) - K}{\sigma^n \sqrt{T_{start}}},
\end{aligned}
\tag{4.4}
$$

δ is the day count fraction from time T_{start} to T_{mat}, $\phi'(x)$ refers to the pdf of the standard normal distribution evaluated at point x and $P(t, T_{mat})$ is the price today of a zero-coupon bond which pays a monetary unit at time T_{mat}.

Aggregating the caplets/floorlets underlying any given cap/floor results in:

$$
Cap(t, T, N, K) = N \sum_{i=T_{start}+1}^{T_{mat}} \delta_i B^n_{call}(T_{i-1}, K, F(t,i), \sigma^n_i) P(t, T_i), \tag{4.5a}
$$

$$
Floor(t, T, N, K) = N \sum_{i=T_{start}+1}^{T_{mat}} \delta_i B^n_{put}(T_{i-1}, K, F(t,i), \sigma^n_i) P(t, T_i). \tag{4.5b}
$$

The normal model is (in addition to the lognormal model) an important benchmark in terms of which implied volatilities are quoted (remember equation (3.13)). In fact, many traders are in the habit of thinking in terms of normal implied volatilities. The normal model allows valuation of options with negative strikes and negative current forward rates, in contrast to the lognormal model. Figure (4.1) (above) shows the value of several floorlets with different underlying forward rates (bootstrapped from *OIS EONIA* curve at valuation date *24th May, 2017.*) As shown in the figure, the value of a floorlet with any strike under the normal model is strictly positive, since any (positive or negative) forward rate has a non-zero probability of being attained.

However, in the lognormal model a floorlet with strike zero has zero value *by definition*. Since floorlet market prices are not zero even for small strikes, a large (Black) implied volatility is needed to provide a positive price. In fact, letting the strike go to zero while maintaining a positive value for the floorlet (as it actually *happens* within the markets) results in an *unbounded growth* of Black's implied volatility (3.9), which goes to infinity even for strictly non-zero strikes (see figure (4.2) below). This idea is thoroughly discussed in [3], where the existence of a vertical asymptote at a given strike K is proved.

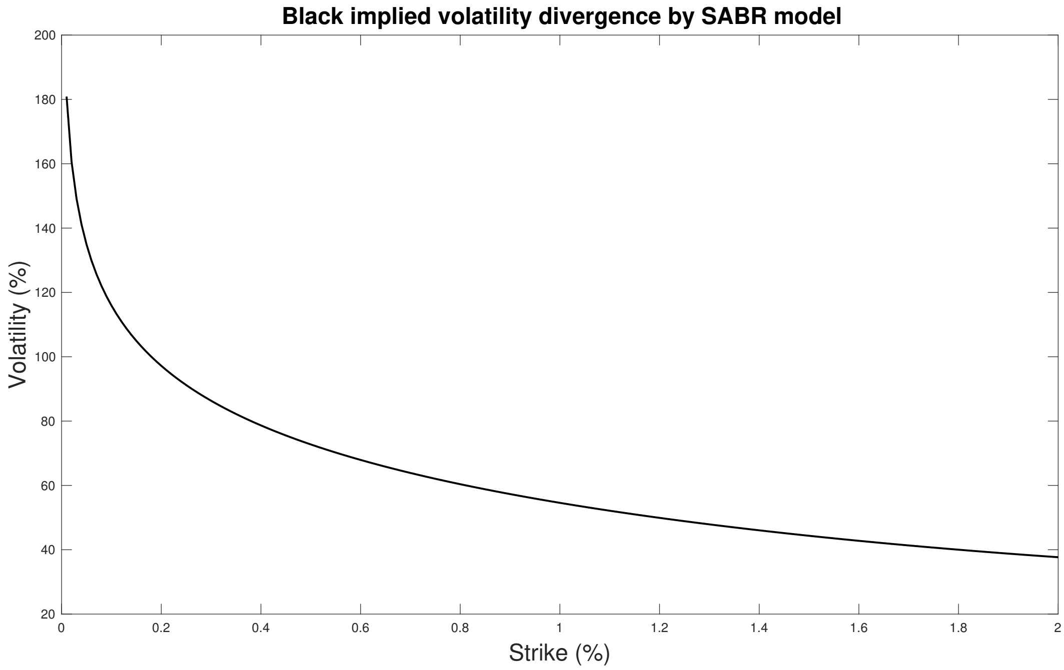

Figure 4.2: *Black's implied volatility divergence when pricing floorlets by SABR model. $T = 4Y$, tenor=3 months and $F(t, 0) = 2.10\%$.*

4.2 Normal SABR (2002)

Fixing $\beta = 0$ in the original SABR model (see (3.7)) restricts it to the so-called **normal SABR model**. This is the only version of the SABR that can model negative forward rates directly, without adding any shift or free boundary condition.

Normal SABR model can be understood as a direct generalization of Bachelier's model, since it basically maintains the same evolution for the forward rate while postulating a lognormal diffusion-process for the forward's rate instantaneous normal volatility for any given maturity:

$$\begin{aligned} dF(t) &= \sigma(t) \cdot dW(t), &\text{(4.6a)}\\ d\sigma(t) &= \alpha \cdot \sigma(t) \cdot dZ(t), &\text{(4.6b)}\\ E^{Q^T}(dW(t) \cdot dZ(t)) &= \rho\, dt. &\text{(4.6c)} \end{aligned}$$

Given a set of market Bachelier's implied volatilities[35] for any given maturity, the parameters are usually calibrated by using equations (3.15) or (3.17).

[35]Note that the normal SABR model permits the instantaneous forward rate to be *arbitrarily* negative and, therefore, *it cannot follow a lognormal distribution*. Consequently, the calibration process is exclusively conducted via Bachelier implied volatilities.

4.3 Shifted Black (2012)

Shifted Black model accepts negative forward rates while maintaining a lognormal specification, since it postulates that the instantaneous forward rate obeys the following process:

$$dF(t) = \sigma \cdot (F(t) + s) \cdot dW(t), \tag{4.7}$$

where s is a constant displacement parameter, which should be chosen *a priori* by the analyst, being *high enough* to avoid the magnitudes $F(t) + s, K + s$ going below zero for any given time.

In fact, this is the main criticism to the shifted model (see, for instance, [27]): the analyst *is supposed to know* which is the most negative value the forward rate may attain. In practice, s is chosen so every observed value of the underlying forward rate can be modelled in this context, and should be redefined if the forward rate escapes from this given constraint. It should be noted that the process of fixing s should be done careful and precisely, since choosing an extremely high value leads to the problems already explained in Bachelier's model (arbitrarily low values for the forward rate may be attained).

As it can be seen (for example) in [3], shifted Black formulation is completely equivalent to Black's (formulas (3.1) to (3.5)), by changing $K \to K + s, F(t, 0) \to F(t, 0) + s$ and $F(t, i) \to F(t, i) + s$ respectively. Analytical formulae and calibration procedure are, therefore, obtained and performed in a similar way.

4.4 Shifted SABR (2014)

Since shifted Black model (4.7) inherits unrealistic constant-volatility hypothesis from Black formulation (3.1), shifting the SABR model (3.7) seems a good choice for both *calibrating observed smile* precisely enough and *including negative forward rates* into our framework.

Additionally, closed-approximated formula for implied Bachelier and Black volatilities would still be available[36] and our problem of choosing an appropriate shift parameter would (unfortunately) reappear[37]. Displaced SABR model was originally proposed in [29]. It postulates that both instantaneous forward rate and its instantaneous volatility should obey the following system of equations:

$$\begin{aligned} dF(t) &= \sigma(t) \cdot (F(t) + s)^{\beta} \cdot dW(t), && (4.8a) \\ d\sigma(t) &= \alpha \cdot \sigma(t) \cdot dZ(t), && (4.8b) \\ E^{Q^T}(dW(t) \cdot dZ(t)) &= \rho\, dt. && (4.8c) \end{aligned}$$

[36] Although they *should include* the shift parameter s, and therefore calibrating the shifted SABR model requires its own process. It cannot be recovered from a previous SABR calibration, since the shift parameter explicitly appears in implied volatilities' formulae.

[37] In fact, it could be of *high interest* to perform an empirical study on the influence of the shift parameter s in the process of calibrating, pricing and hedging within a shifted SABR context. This analysis is left for further research.

The whole development and formulae deducted in section 3.3 of the Thesis (analysis of the SABR model) applies for the shifted SABR model, and therefore formulas (3.7) to (3.17) can be used within a shifted SABR context with the pertinent modifications $f \to f + s, K \to K + s$. Analytical specification for the price of plain-vanilla derivatives is therefore maintained under this scheme, and calibration becomes straightforward from formulas (3.15), (3.17) (by previously adding the shift s).

4.5 Free boundary SABR (2015)

Free boundary SABR is an extension of the classic SABR model (3.7) firstly introduced in [30], which tries to both *avoid choosing a shift parameter s a priori* and *deal with negative rates* in a natural way. In this sense, it eliminates SABR's zero-boundary by assuming the form

$$\begin{aligned} dF(t) &= \sigma(t) \cdot |F(t)|^{\beta} \cdot dW(t), && (4.9a)\\ d\sigma(t) &= \alpha \cdot \sigma(t) \cdot dZ(t), && (4.9b)\\ E^{Q^T}(dW(t) \cdot dZ(t)) &= \rho\, dt. && (4.9c) \end{aligned}$$

As stated in [3], the condition $0 \leq \beta < \frac{1}{2}$ guarantees stable solutions.

The main problem of this model is not the lack of an analytical solution (except in some particular but not interesting cases). In fact, Bachelier's implied volatility[38] can be computed as shown in [3]:

$$\begin{aligned} &\sigma^n(T_{start}, K, f) = \frac{\sigma(0)(f-K)(1-\beta)}{f/|f|^{\beta} - K/|K|^{\beta}} \cdot \frac{c}{g(c)} \cdot \\ &\left[1 + T_{start}\left(\frac{-\beta(2-\beta)\sigma(0)^2}{24|sign(fK)\sqrt{|fK|}|^{2-2\beta}} + \frac{\alpha\beta\rho\sigma(0)sign(sign(fK)\sqrt{|fK|})}{4|sign(fK)\sqrt{|fK|}|^{1-\beta}}\right)\right], \end{aligned} \tag{4.10}$$

with:

$$\begin{aligned} c &:= \frac{\alpha(f-K)}{\sigma(0)|sign(fK)\sqrt{|fK|}|^{\beta}}, \\ g(c) &:= \log\left(\frac{\sqrt{c^2 - 2\rho c + 1} + c - \rho}{1-\rho}\right). \end{aligned} \tag{4.11}$$

[38] Again, since the free boundary SABR permits the forward rate to lie among the whole real line, a lognormal specification (even a shifted's lognormal) is not appropriate (values below the fixed boundary are not permitted), and this model can only be calibrated within Bachelier's implied volatility.

As claimed in [3], free boundary SABR's main drawback is that there exist implied probability densities which are *negative* for huge areas around zero (which, in fact, is the area we are interested in) for a given set of parameters. Therefore, it *can fail* precisely in the most inappropriate area for us, and has been generally discarded among industry firms, which have shown preference for the shifted SABR model as the natural candidate to replace the original SABR model when negative rates are considered.

4.6 Vasicek (1977) and Hull-White (1990). Short rate models

Short rate models differ *in essence* to the ones previously exposed and should therefore be treated in a different way.[39] Vasicek model, firstly introduced in [31], inherits its formulation from an Ornstein-Uhlenbeck process with constant coefficients under the risk-neutral measure:

$$dr(t) = k(\theta - r(t))dt + \sigma dW(t). \tag{4.12}$$

The first huge difference should be appreciated. While former models usually stated their initial formulation in terms of the *T-forward measure*, short rate models tend to propose a SDE under the *risk-neutral* (and sometimes even under the *objective*!) measure.

The Ornstein-Uhlenbeck process is *mean-reverting*, in the sense that the instantaneous short rate $r(t)$ tends to return to the *long-term value* θ on a rate specified by the *mean reversion speed* k. σ accounts for the instantaneous short rate volatility.

Hull-White stated in [32] that the instantaneous short-rate evolves according to the following SDE:

$$dr(t) = k(t)(\theta(t) - r(t))dt + \sigma(t)dW(t). \tag{4.13}$$

(4.13) extends Vasicek's model (4.12) by permitting its parameters to depend (deterministically) on the calendar time, and therefore is sometimes called the *exogenous* version of the *endogenous* Vasicek model (or *extended Vasicek model*). As it has been frequently done in previous literature (see, for instance, [7]), we analyse a restricted version of Hull-White's model which imposes the constraints $k(t) = k$, $\sigma(t) = \sigma$. Under this specification, $\theta(t)$ is chosen to guarantee that the currently observed market *term structure of interest rates* (from now on, *TSIR*) is fitted perfectly[40]. As it is shown in [7], under Vasicek spec-

[39] Models from sections 4.1 to 4.5 are devoted to deal with commonly-traded instruments of the market, instead of providing a full integrated scheme for the evolution of the instantaneous short rate from where prices for these instruments are consequently deduced. These second kind of models are usually called *short rate models*, and this section is devoted to them. Analytical formulation, if possible, usually becomes *far more* complicated under this new scheme, since these models are not focused in pricing the kind of derivatives we are interested in. In spite of this, they have been included in the survey for completeness.

[40] See [7] for an explicit expression of the calibration formula of $\theta(t)$ in terms of *market instantaneous forward rates* $F^m(0,t)$ and *market discount factors* $P^m(0,t)$.

ification the price of a cap/floor at time t with notional value N, strike rate K and set of times $T = [T_{start}, T_{mat}]$ can be computed as:

$$Cap(t,T,N,K) = N \sum_{i=T_{start}+1}^{T_{mat}} [P(t,T_{i-1})\phi(-h_i+\sigma_i) - (1+\delta_i K)P(t,T_i)\phi(-h_i)] \tag{4.14a}$$

$$Floor(t,T,N,K) = N \sum_{i=T_{start}+1}^{T_{mat}} [-P(t,T_{i-1})\phi(h_i-\sigma_i) + (1+\delta_i K)P(t,T_i)\phi(h_i)] \tag{4.14b}$$

where:

$$\begin{aligned}
P(t,T) &= A(t,T)\exp(-r(t)B(t,T)),\\
B(t,T) &= \frac{1-\exp(-k(T-t))}{k},\\
A(t,T) &= \exp\left((\theta - \frac{\sigma^2}{2k^2})(B(t,T)-T+t) - \frac{\sigma^2}{4k}B^2(t,T)\right),\\
\sigma_i &= \sigma\sqrt{\frac{1-\exp(-2k(T_{i-1}-t))}{2k}}B(T_{i-1},T_i),\\
h_i &= \frac{1}{\sigma_i}log\left(\frac{(1+\delta_i K)P(t,T_i)}{P(t,T_{i-1})}\right) + \frac{\sigma_i}{2}.
\end{aligned} \tag{4.15}$$

Regarding Hull-White model, formulae can be similarly deduced by incorporating the currently observed *TSIR* in the form of both *market instantaneous forward rates* $F^m(0,t)$ and *market discount factors* $P^m(0,t)$. As proved in [7], once Hull-White model has been calibrated to market data, pricing formulae read as:[41]

$$Cap(t,T,N,K) = N \sum_{i=T_{start}+1}^{T_{mat}} [P(t,T_{i-1})\phi(-h_i+\sigma_i) - (1+\delta_i K)P(t,T_i)\phi(-h_i)] \tag{4.16a}$$

$$Floor(t,T,N,K) = N \sum_{i=T_{start}+1}^{T_{mat}} [-P(t,T_{i-1})\phi(h_i-\sigma_i) + (1+\delta_i K)P(t,T_i)\phi(h_i)] \tag{4.16b}$$

[41]Note that, within this formulation, Hull-White model actually presents *less parameters* (σ, k) than Vasicek's (σ, k, θ). One could, in principle, let $\theta(t)$ be a free time-dependent parameter included to calibrate cap market prices *perfectly* via trinomial trees (see, for instance, [1]), and therefore Vasicek's would be a nested specification of Hull-White's. However, we understand that by doing so we are making an *unfair comparison* between the models of Chapter 4 in two different ways:

1. We understand that the original aim of *exogenous models* is betrayed then, since they are designed to fit today's TSIR *perfectly*. $\theta(t)$ covers this role. If we force it to participate in the cap calibration procedure, there is no guarantee that market observed TSIR is fitted within the model (indeed, it would not be fitted *at all*).

2. Every model presented in Chapter 4 of the Thesis accounts for time-independent parameters. From a mathematical point of view, introducing time-dependent parameters in the caps' calibration is equivalent to introduce an independent-time model with *infinite* parameters. The model is therefore guaranteed to fit caps' prices *perfectly*, and the comparison lacks of sense. In terms of comparability, every model under contrast should have a finite number of parameters. This is why we only deal with *analytical models* through the survey.

where:

$$
\begin{aligned}
P(t,T) &= e^{-r(t)B(t,T)}\frac{P^m(0,T)}{P^m(0,t)}\exp(B(t,T)F^m(0,t) - \frac{\sigma^2}{4k}(1-e^{-2kt})B^2(t,T)),\\
B(t,T) &= \frac{1-\exp(-k(T-t))}{k},\\
\sigma_i &= \sigma\sqrt{\frac{1-\exp(-2k(T_{i-1}-t))}{2k}}B(T_{i-1},T_i),\\
h_i &= \frac{1}{\sigma_i}log\left(\frac{(1+\delta_i K)P(t,T_i)}{P(t,T_{i-1})}\right) + \frac{\sigma_i}{2}.
\end{aligned}
\tag{4.17}
$$

Parameters underlying both Vasicek and market-calibrated Hull-White models (σ and k) can be calibrated by a straightforward non-linear least-squares comparison between theoretical cap/floor prices given by equations (4.14), (4.16) and market cap/floor prices obtained by introducing the (flat) cap/floor volatility in Bachelier/Black pricing formulae.

To end up with this section, a last pertinent comment shall be made. There exist *many other* short-rate models[42] which might (and *should*) be included in the analysis in the spirit of greater depth, and their incorporation into the survey in a consistent manner with the previous exposition is currently under research. Among the set of candidates, Hull-White model (and its nested specification: Vasicek's model) has been selected due to both pragmatic and theoretical reasons:

- According to several conversations with practitioners, Hull-White model is widely used among the industry, owing to the existence of analytical formulae (4.16), easy-calibration procedure and suitability for coping with a negative interest rates context.
- Hull-White model is markovian, which reduces the amount of time spent in numerical simulations, in case they were necessary (non-analytical version of Hull-White model).[43]

[42] In an extensive but not exhaustive list, we can enumerate the *Ho-Lee, Cox* or *Heath-Jarrow-Morton* models, among many others (in fact, HJM is not strictly considered a short-rate model, since it models the instantaneous forward rate.)

[43] Non-markovian processes need *non-recombining* lattices to be simulated. This feature hugely increases computational time (since the number of nodes in the tree will grow exponentially with the number of steps) and is particularly relevant when working in a HJM framework, since only some particular choices of the volatility structure are both consistent with the *absence of arbitrage opportunities* and the *markovianity* of the process (see the original paper from HJM, [33], for further details on this issue). A thorough discussion about HJM general framework is given in [7], who devote a full chapter of their book to this model. Its analysis, however, lies beyond the scope of this Thesis.

Chapter 5

The volatility cube

This chapter is fully devoted to characterize the object known as *the volatility cube*, which stands as a key input in every interest rates derivatives pricing software among industry firms. Our main innovative contribution is the development of a new fast completion method, based on the previous work by [2] and especially [34].

5.1 The third dimension

As explained in previous chapters, the constant-implied volatility hypothesis underlying (shifted) Black and Bachelier models ((4.7), (4.1)) is not supported by the markets. Until now, two different functional dependences[44] have been acknowledged in the Thesis:

1. **Strike (smiles/skews)**: It stands as the most relevant dimension to be calibrated in terms of accuracy. Smile calibration has been lately performed with stochastic interpolation methods, with special interest on SABR and its extensions to deal with negative rates.

2. **Maturity**: The maturity of the caplet under consideration is the second variable of interest for accurate pricing. It is unreliable to assume that the underlying forward rate of a caplet that matures in twenty years is as volatile as the forward of a one-year maturing caplet, and this dependence should be modelled somehow. As explained later[45], we adhere to the simplest functional form explored in previous literature (see [35]) for interpolating implied volatilities between maturities: a *piecewise constant* approach.

Market implied volatility surfaces, however, present a third relevant dependence which has not been stated yet. The phenomenon known as *tenor splitting* accounts for the change

[44] Attending to the classification of [35], interpolation procedures can be classified in *functional forms of interpolation* and *stochastic interpolation methods*. The former need an *a priori selected* functional form of calibration, while stochastic interpolation methods deduce the functional form for the implied volatility from an stochastic theoretical specification. This stochastic formulation might be performed over the forward rate *exclusively* (as it happens in the CEV model, for example) or over a more complex structure of correlated stochastic variables (as it happens in the SABR model and its extensions). From now on, we follow the classification of interpolation methods stated in [35].

[45] See Chapter 7.

in the observed implied volatility when the tenor of the underlying caplet is modified *ceteris paribus*. The intuition behind this empirical feature is obvious: the market should not assign the same volatility to two underlying forwards with similar characteristics but with different length of the investment period they are referred to.

The resulting implied volatility structure, therefore, depends on three underlying variables: strike, maturity and tenor of the underlying caplet[46]. This three-dimensional dependence is often known as *the volatility cube* structure.[47] **Completing the cube** is the process of filling the gaps along the three calibration directions by any appropriate procedure. Once the cube is fully calibrated, it can be used as an input by industry firms software for pricing any kind of interest rates derivative.

Given the two interpolation methods proposed for strike and maturity, it seems straightforward to think that extending any interpolation procedure to the tenor dimension is the natural way to complete the volatility cube. Unfortunately, the interpolation does not work empirically along this direction due to *the scarcity of data*. While several strikes and maturities are usually quoted in the markets, volatilities are standardized to a scarce range of tenors. This is the reason why the problem of completing the cube has aroused so much interest in previous literature[48]. Several authors have complained about the problem of scarce data and looked for alternative ways of interpolating the cube over its third edge without the explicit use of any interpolation method.[49]

Hopefully, the completion of the cube should attend to three main features:

1. Precision in the recovery of market data once the calibration has been performed (*in-sample* test).
2. Accuracy when pricing interest rates derivatives which have not been used during the calibration (*out-of-sample* test).
3. Continuity of the global four-dimensional structure, guaranteeing that pricing evolves smoothly in any given direction.

[46]Indeed, a four dimensional structure is really under consideration:
$\sigma^{(n)} = \sigma^{(n)}(Strike, Maturity, Tenor)$.

[47]Due to broker's (i.e., *ICAP*) standard quoting convention (see chapter 6), when practitioners talk about the volatility cube, they usually refer to the dependence of **swaptions'** implied volatilities with $(Strike, Maturity, Tenor)$. The term *volatility surface* is often used for caplets' implied volatilities dependence on $(Strike, Maturity)$, ignoring the tenor splitting effect for these instruments. The extension of the survey using swaptions is currently under research and, once completed, the term *volatility cube* would have its usual meaning. However, since tenor splitting *is a fact* for caplet quoting volatilities as well (it would be illustrated in chapter 8 of the Thesis), we prefer not to adhere to practitioners' standard nomenclature and use the expression *volatility cube* indistinctly, since we believe it expresses more accurately the whole dependency structure for both instruments. We presume that the essential no-arbitrage hypothesis underlying this approach can be extrapolated to deal with swaptions' volatility cubes, and we are currently working in the development of this idea.

[48]The treatment of [36] is especially recommended, as well as the analysis of [35] and [16].

[49]We encourage the interested reader to the standard treatments of [37] and [38] for explicit inter- and extrapolation methods in the process of attaining volatilities for arbitrary tenors. Their approach is fully different to the one shown here, and therefore their analysis lies beyond the scope of this Thesis.

Due to time constraints, only the second point is fully covered through this text. We leave the remaining issues for future research.

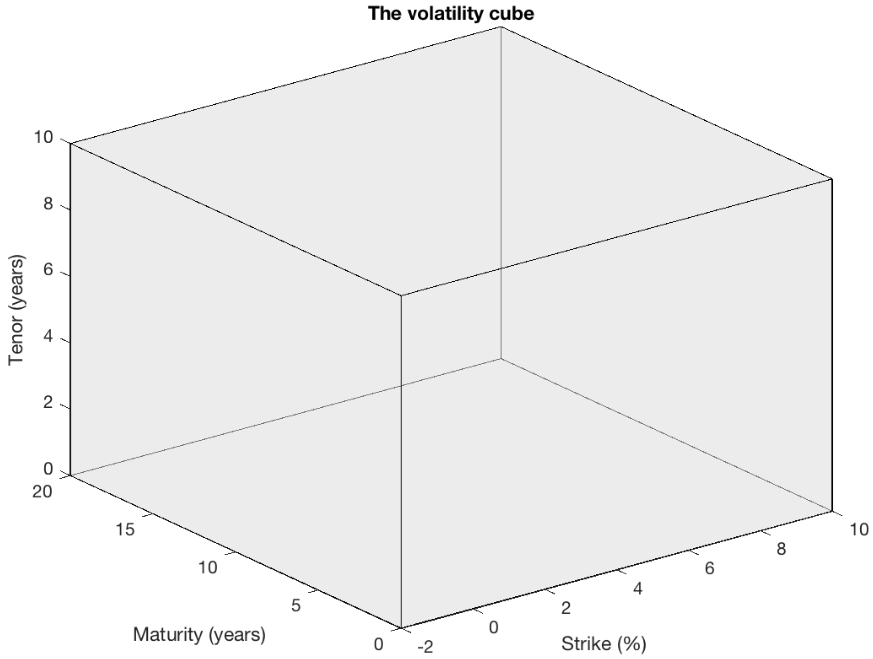

Figure 5.1: *The volatility cube. The implied volatility function $f : R^3 \to R$ assigns a unique value $f(x, y, z)$ to each point (x, y, z) of the cube.*

5.2 No-Arbitrage condition. Completing the cube

Previous work by [34] was pioneer in introducing the idea of using a no-arbitrage condition to derive volatilities for non-quoting tenors from the ones quoting within the markets. [2] extended this scheme to a negative rates environment. Both of them, however, limit its applicability to collapsing market data into a unique benchmark tenor or going the other way round (using a standard tenor to derive volatilities of non-standard tenors).

We claim, however, that this method is not limited to collapsing market non-standard data into unified-tenor data or going the other way round in the spirit of stripping appropriate data[50] for calibration purposes, but can be extended to face **the calibration of the volatility cube itself**. To the best of our knowledge, this possibility has not been explored in previous literature.[51]

[50]i.e., avoiding *tenor splitting* issue by collapsing the whole set of data into a unique tenor.

[51]Indeed, [34] seems to draw an equivalent scheme for transferring the *whole smile structure* for non-standard tenors (instead of transferring each volatility *point by point*), and call it *Transferring the smile*. The explanation of this alternative and the arguments to discard it are given in appendix C of the Thesis.

If we admit that the usual absence of arbitrage hypothesis between forward rates applies (at least, to a high order of approximation), volatilities of different tenors are necessarily related via this arbitrage-free relationship. Moreover; while the strike and maturity interpolation procedures are not reliable when extrapolating[52], this arbitrage-free condition can be exploited for extrapolating volatilities of several non-standard tenors from a unique implied volatility surface calibration *as far as it is needed to*, as long as the no-arbitrage condition still applies.[53]

During this section, we follow the reviewed approach for negative rates given by [2]. This methodology depends on the combination of maturity and tenor that is to be extrapolated[54], and would therefore be explained for a particular illustrative example. Any other volatility for a different $(maturity, tenor)$ pair can be obtained in a similar trend. Likewise, the procedure depends on whether extrapolating *a longer tenor volatility* from its shorter tenor homologues or going *the other way round*. Both cases are covered, to provide a fully-integrated calibration scheme of the volatility cube, independently of the benchmark tenor which quotes in the markets. Finally, the extrapolation depends on the nature of the quoting volatilities (shifted Black's or Bachelier's). We provide formulae for both situations.

5.2.1 Extrapolating longer tenor volatilities

We firstly examine the case given by figure (5.2) below. Let $X_{i,j}$ be today's shifted forward rate $F_{i,j} + s$[55] for the future investment period $[T_i, T_j]$. Its shifted (implied) Black volatility is denoted by $\sigma_{i,j}$, and its normal equivalent is $\sigma^n_{i,j}$. Let $\tau_{i,j} = T_j - T_i$ be the year fraction length of that future investment period.

Consider that, for a maturity time T_3 of one year ($T_3 = 1$) and a benchmark tenor of 3 months ($\tau_{2,3} = 0.25$), we have a quoting shifted Black (Bachelier) volatility $\sigma_{2,3}$ ($\sigma^n_{2,3}$), and the objective is to attain a longer tenor shifted Black (Bachelier) volatility for that given maturity (for example, a 6 month-tenor volatility for the 1 year maturity is requested; i.e., if $\tau_{1,3} = 0.5$, $\sigma_{1,3}$ ($\sigma^n_{1,3}$) is to be attained.

[52] When necessary, at most a *constant extrapolation* is usual market practice. Obviously, this is not desirable and leads to huge mistakes when extrapolating far away from the quoting data.

[53] To be precise, under this scheme not every implied volatility for any arbitrary tenor can be reached. We only aim to provide a method for implying volatilities for tenors which are *multiples* of the benchmark tenor. Some further considerations for the underlying forward rates are needed to extend this framework to any arbitrary tenor.

[54] The whole methodology is developed for any given strike K.

[55] Under Bachelier specification, $s = 0$.

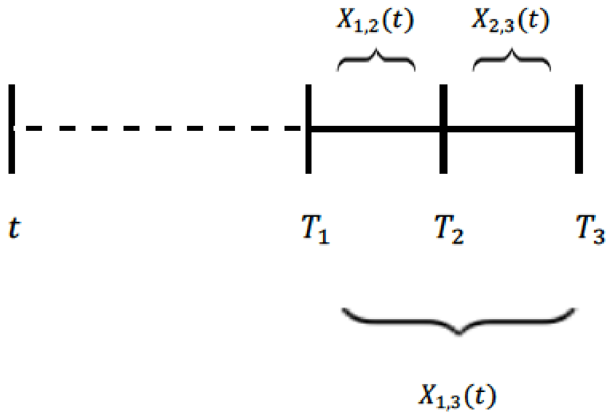

Figure 5.2: *Arbitrage-free condition. Relationship between the forward rates being considered.*

By piecewise constant hypothesis in maturity[56], $\sigma_{1,2}(\sigma^n_{1,2}) = \sigma_{2,3}(\sigma^n_{2,3})$. If the standard arbitrage-free relationship between forward rates for the investment periods applies today, and denoting $F_{i,j} = X_{i,j} - s$, it should be satisfied that:[57]

$$1 + \tau_{1,3}(X_{1,3} - s) = [1 + \tau_{1,2}(X_{1,2} - s)][1 + \tau_{2,3}(X_{2,3} - s)]. \tag{5.1}$$

Rearranging terms from (5.1) results in:

$$X_{1,3} = \frac{\tau_{1,2}X_{1,2} + \tau_{2,3}X_{2,3} + \tau_{1,2}\tau_{2,3}X_{1,2}X_{2,3}}{\tau_{1,3}} + s\left[1 - \frac{\tau_{1,2} + \tau_{2,3} + \tau_{1,2}\tau_{2,3}(X_{1,2} + X_{2,3} - s)}{\tau_{1,3}}\right]. \tag{5.2}$$

Under the T-forward measure, the following relationships must hold:

$$\begin{aligned} dX_{i,j} &= \sigma_{i,j}X_{i,j}dW_{i,j} \\ dX_{i,j} &= \sigma^n_{i,j}dW_{i,j} \end{aligned} \tag{5.3}$$

for shifted Black's and Bachelier's quoting procedures respectively. Particularizing (5.3) to the 6 month tenor (shifted) forward rate gives:

$$dX_{1,3} = \sigma_{1,3}X_{1,3}dW_{1,3} \tag{5.4}$$

or

[56]This assumption applies in maturity of the caps being considered and, therefore, forces caplets' implied volatilities to be constant from $T_0 = 0$ to $T_3 = 1Y$. See chapter 6 for further information on the nature of quoting data and chapter 7 to further explanation on the piecewise constant hypothesis.

[57]Remember that a LIBOR forward rate is being considered, which affects the compounding for the investment period it refers to.

$$dX_{1,3} = \sigma^n_{1,3} dW_{1,3} \tag{5.5}$$

respectively. Applying Ito's lemma to equation (5.2) provides an alternative expression for $dX_{1,3}$. Comparing this expression with formulae (5.4) or (5.5), we should be able to derive a relationship between quoting ($\sigma^{(n)}_{2,3} = \sigma^{(n)}_{1,2}$) and sought ($\sigma^{(n)}_{1,3}$) volatilities. Firstly, the application of standard Ito's formula to expression (5.2) results in:

$$dX_{1,3} = \frac{\tau_{1,2}[1+\tau_{2,3}F_{2,3}]}{\tau_{1,3}} dX_{1,2} + \frac{\tau_{2,3}[1+\tau_{1,2}F_{1,2}]}{\tau_{1,3}} dX_{2,3} + \frac{\tau_{1,2}\tau_{2,3}}{\tau_{1,3}} d[X_{1,2}, X_{2,3}], \tag{5.6}$$

where:

$$\begin{aligned} d[X_{1,2}, X_{2,3}] &= d[F_{1,2}, F_{2,3}] = \sigma_{1,2}\sigma_{2,3}X_{1,2}X_{2,3} d[W_{1,2}, W_{2,3}] = \sigma_{1,2}\sigma_{2,3}X_{1,2}X_{2,3}\rho dt, \\ d[X_{1,2}, X_{2,3}] &= d[F_{1,2}, F_{2,3}] = \sigma^n_{1,2}\sigma^n_{2,3} d[W_{1,2}, W_{2,3}] = \sigma^n_{1,2}\sigma^n_{2,3}\rho dt \end{aligned} \tag{5.7}$$

for shifted Black's and Bachelier's quoting procedures respectively, and ρ accounts for the correlation between the two Wiener processes of $F_{1,2}, F_{2,3}$. For implied shifted Black volatilities, taking quadratic variation in expressions (5.3) and (5.6) gives[58]

$$\sigma^2_{1,3} = \frac{a^2\sigma^2_{1,2}X^2_{1,2} + b^2\sigma^2_{2,3}X^2_{2,3} + 2ab\sigma_{1,2}\sigma_{2,3}X_{1,2}X_{2,3}\rho}{X^2_{1,3}} \tag{5.8}$$

where:[59]

$$a := \frac{\tau_{1,2}[1+\tau_{2,3}F_{2,3}]}{\tau_{1,3}}, b := \frac{\tau_{2,3}[1+\tau_{1,2}F_{1,2}]}{\tau_{1,3}}. \tag{5.9}$$

Regarding implied Bachelier's volatilities, applying quadratic variations to (5.3) and (5.6) results in:

$$\sigma^2_{1,3}(n) = a^2\sigma^2_{1,2}(n) + b^2\sigma^2_{2,3}(n) + 2ab\sigma_{1,2}(n)\sigma_{2,3}(n)\rho. \tag{5.10}$$

Making $\sigma^{(n)}_{1,2} = \sigma^{(n)}_{2,3}$ in equations (5.8), (5.10) gives the final expression for $\sigma^{(n)}_{1,3}$ in terms of today's forward rates, the volatility currently quoting in the markets $\sigma^{(n)}_{2,3}$ and the correlation between the Wiener processes of the forward rates $F_{1,2}, F_{2,3}$:

[58] Notice that the last term in equation (5.6) vanishes when quadratic variation is taken.

[59] An standard *freezing the drift* argument has been used in this step. More information regarding this technique can be found in [2], [7] or [34].

$$\sigma_{1,3}^2 = \frac{\sigma_{2,3}^2}{X_{1,3}^2}\left[a^2 X_{1,2}^2 + b^2 X_{2,3}^2 + 2abX_{1,2}X_{2,3}\rho\right] \tag{5.11}$$

$$\sigma_{1,3}^2(n) = \sigma_{2,3}^2(n)\left[a^2 + b^2 + 2ab\rho\right] \tag{5.12}$$

(5.11) replicates the formula previously obtained in [2]. To the best of our knowledge, equation (5.12) has not been found in existing literature, and therefore provides a new scheme for comparative purposes.[60]

5.2.2 Extrapolating shorter tenor volatilities

Now, consider the situation in which the 3 month tenor volatility ($\sigma_{2,3}$) is to be attained from the 6 month tenor quoting volatility ($\sigma_{1,3}$). Again, by piecewise constant assumption, $\sigma_{1,2} = \sigma_{2,3}$, although both are unknown. Rearranging terms from (5.11) or (5.12) provides the answer:

$$\sigma_{2,3}^2 = \frac{\sigma_{1,3}^2 X_{1,3}^2}{a^2 X_{1,2}^2 + b^2 X_{2,3}^2 + 2abX_{1,2}X_{2,3}\rho} \tag{5.13}$$

$$\sigma_{2,3}^2(n) = \frac{\sigma_{1,3}^2(n)}{a^2 + b^2 + 2ab\rho}. \tag{5.14}$$

At this point, it should be stated that both [2] and [34] do not adhere to the piecewise constant hypothesis, resulting in an ill determined system (both $\sigma_{1,2}$ and $\sigma_{2,3}$ are unknown) which is solved by minimizing the squared differences between market volatilities and those provided by a pre-specified parsimonious functional form (usually called *Rebonato like function*). However, we believe that this method presents *internal inconsistency* within the calibration procedure, since it uses this parsimonious functional form (which is far from being piecewise constant) for collapsing caplet volatilities into a unique tenor and then a piecewise constant functional form is assumed for stripping volatilities (see chapter 7 of the Thesis). The interested reader is redirected to the original references [2] or [34] for further discussion on this topic.

[60] We have disregarded the discussion about the appropriate choice of the only free parameter remaining both equations: ρ. Chapter 7 of the Thesis further analyses this aspect.

Part II

Practice

Chapter 6

Data

Through this chapter, we aim to describe briefly the different datasets included in the survey[61] . In a nutshell, these inputs are:

- **OIS zero-coupon curves:** They have been downloaded from *ICAP's* 24th May quoting data, via *Thomson Reuters Eikon* terminal. Both EUR (*EONIA*) and USD (*Fed Funds Rate*) curves (figure (D.1)) are used when discounting or forwarding. *Continuously-compounding* method is used for both of them, and the daily basis convention used for day-counting is *Actual/Actual.* Although both curves behave similarly (almost monotonically growing with the tenor underlying), they are shifted so that the lowest tenor rates are negative in *EONIA* but remain strictly positive for the *Fed Funds Rate.*[62]

- ***Standard* flat implied (shifted) Black volatilities[63]:** ICAP quotes the so-called *flat implied volatility*, defined as the *unique* volatility that should be introduced in the (shifted) Black formula (3.5) for every constituent caplet in order to recover the price of the cap that incorporates that whole set of caplets. Tables (D.1) and (D.2) respectively show what we call *standard tenor* EUR/USD cap flat (shifted, $s = 3\%$) Black volatilities for maturities on the range $[1, 20]$ years and closest-to-moneyness strikes (from $K = -0.75\%$ to $K = 10\%$ for EUR and $K = 0.50\%$ to $K = 4\%$ for USD data respectively). By the word *standard*, we mean that these volatilities respond to the ones implied by the most liquid traded instruments for every strike and maturity outstanding. Indeed, they have been provided by the standard broker (i.e., *ICAP*) via *Thomson Reuters Eikon* terminal. The quoting convention of these volatilities *depends on* the market (*EUR/USD*) being considered:

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[61] For the sake of continuity of the text, tables and figures are displayed in appendix D of the Thesis.

[62] In fact, this is the reason why data from both EUR and USD quoting caps have been included in the survey: comparing economies with/without negative rates.

[63] Since there exists an strict *one-to-one* correspondence between prices and implied volatilities for the models used in market quotations, one could fairly wonder about the reasons why so many traders prefer quoting instruments market prices *indirectly via their implied volatilities* (and an associate model) instead of *actually providing these market prices.* An insightful discussion is given in ([14]). To summarize, implied volatilities work better as a communication tool due to the fact that they tend to be **much more stable** than equivalent prices, which fluctuate sharply (non-stationary nature of prices). Implied volatilities filter the effect of many other variables that affect the option price better than market prices do.

– **EUR quoting process:** EUR caps from table (D.1) are classified attending to the *tenor of the constituent caplets*. While caps that expire before the $T = 3Y$ maturity ($T = 1, 1.5$ and $2Y$) quote with a 3-month tenor for the caplets underlying, caps with expiries equal or above $T = 3Y$ depend on 6-month tenor caplets. This split in the quotation process affects to the standard methodology designed to strip caplet volatilities from their cap's homologues, known as *caplet stripping* (see chapter 7 of the Thesis). Moreover, the first caplet of every cap is excluded from the quotation process, to guarantee that any currently-quoting cap accounts for random payoffs even for the first caplet underlying. In market's lingo, this is usually known as a *spot starting cap*, although there is not option until a period of time equal to the tenor being considered (3/6 months respectively) has elapsed. Figures (6.1), (6.2) below illustrate ICAP's EUR caps quoting processes.

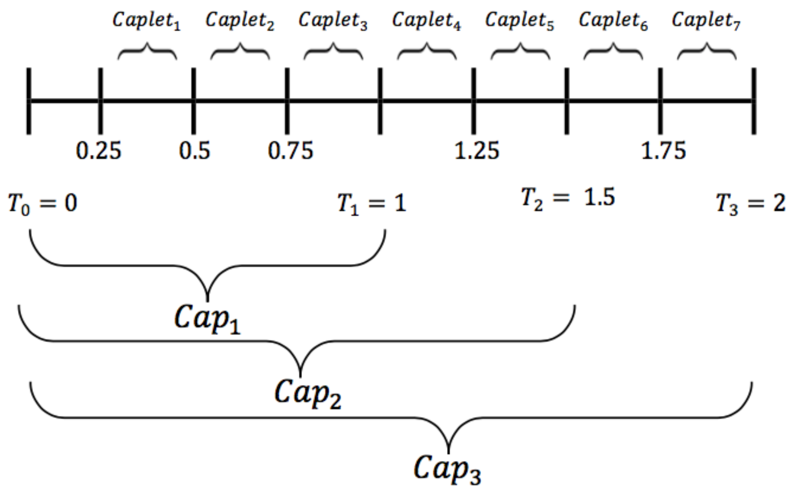

Figure 6.1: *EUR caps quoting procedure for maturities up to* $T = 2Y$.

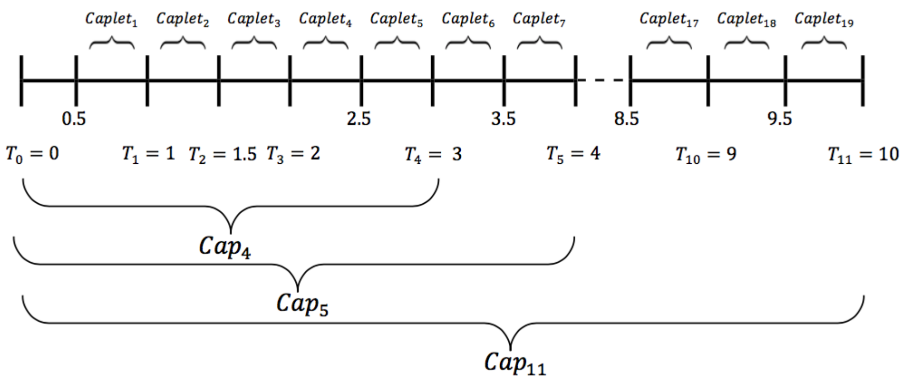

Figure 6.2: *EUR caps quoting procedure for maturities above* $T = 2Y$.

– **USD quoting process:** Quotation of USD caps from table (D.2) does not depend on the tenor of the caplets underlying. Therefore, they quote in a more consistent way for modelling purposes. The standard benchmark tenor is 3 months for any maturity outstanding, and again spot starting caps are being used. This quoting procedure is summarized in figure (6.3) below.

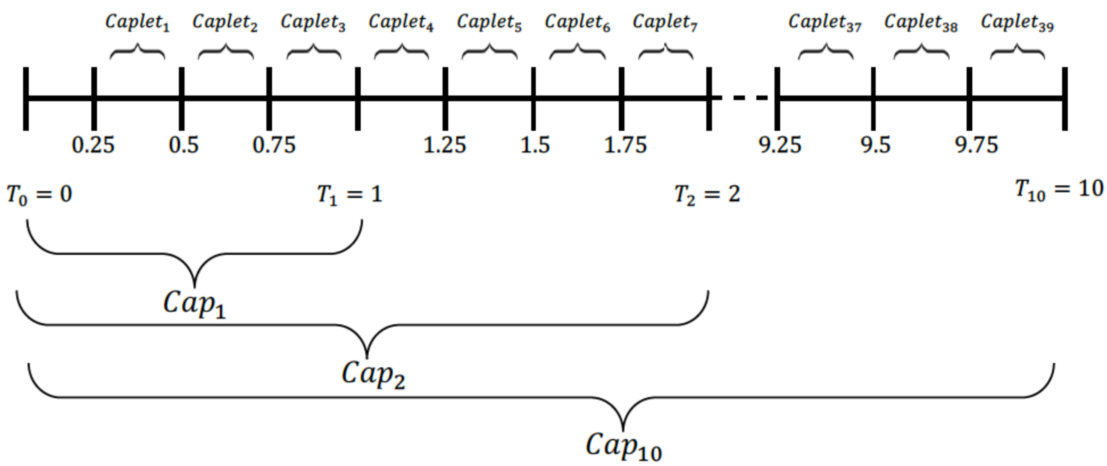

Figure 6.3: *USD caps quoting procedure.*

A last pertinent comment should be made about two additional standard inputs included in the empirical research. In order to study the dependence of the results within the *nature* of the quoting (implied) volatility, Bachelier's have been also included in the survey. Hopefully, one should download tables for Bachelier implied volatilities equivalents to table (D.1) for shifted Black's. However, several quoting inefficiencies were detected for these volatilities. The two most concerning are:

1. **Gaps for some specific cells:** Not every combination of (T, K) among the ranges $[1Y, 20Y]$, $[-0.75\%, 10\%]$ quotes a normal volatility, hindering subsequent comparisons.[64]
2. **Arbitrage is allowed:** It has been checked that introducing (shifted) Black-/Bachelier quoting volatilities for any given cell of the array (T, K) in their standard pricing formulae ((3.5), (4.5) respectively) does not result in the recovery of identical prices for the caps outstanding. Therefore, arbitrage is introduced if both datasets ((shifted) Black's from table (D.1) and Bachelier's) are jointly used.

[64]A thorough research has been conducted to ascertain the reasons for these gaps. They are mostly focused on the *high strike/short maturity* area, that coincides with the lowest caplet prices (see table (8.1) in chapter 8 of the Thesis). Implied Bachelier volatilities are more difficult to attain within this area, since the one-dimensional root-finders algorithms struggle to converge for these lowest prices. We believe, then, that this problem might have been noted by the standard broker as well, and this may be the reason why these volatilities are not usually quoted within the markets.

We cannot allow these quoting inefficiencies to affect our study, and therefore filter their influence by *avoiding using actual normal data*, but *implying it* from shifted Blacks' via **unique-price hypothesis**. That is to say that, for every possible combination of (T, K), we impose that the recovered prices via Black pricing formula (3.5) equals the ones attained via Bachelier's implied volatilities (formula (4.5)). Arbitrage is therefore forbidden, and no gaps quote in our (transformed) data. To imply a matrix of Bachelier caps volatilities similar to table (D.1) for shifted Black's, the following algorithm has been applied:

1. Imply cap prices for every cell of the matrix (D.1) from these shifted Black volatilities by the use of formula (3.5).
2. Use any one-dimensional root finder (Newton-Raphson has been chosen) to convert these market prices into Bachelier's implied flat volatilities by inverting Bachelier's cap pricing formula (4.5).

- **Non-standard flat implied shifted Black volatilities:** Two final inputs have been included into the survey to test the adequacy of our fast-approach calibration of the volatility cube. Tables (D.3) and (D.4) display the values of several EUR flat implied shifted Black volatilities for caps whose caplets quote with *non-standard tenors* of 3/12 months respectively. Available maturities equal the ones shown in tables (D.1), (D.2), and the range of strikes varies from $K = -1\%$ to $K = 9\%$. These data have been implied from *Totem IHS Markit* report. Several OTC cap market prices for a huge range of strikes, maturities and tenors quote on it. They are not as liquid as standard 3-6 month-tenor volatilities from table (D.1), but still liquid enough (attending to *IHS markit* claim) to be fully reliable. The algorithm used for the institution that has kindly provided these data to convert cap market prices from *Totem* report into non-standard flat implied shifted Black volatilities is summarized within the following steps:

1. Convert cap market prices (for a less than desirable number of strikes) into implied shifted Black volatilities inverting cap Black pricing formula (3.5).
2. Strip caplet volatilities from caps' by the *stripping algorithm* described in chapter 7 of the Thesis.
3. Calibrate (in strike) a shifted SABR model for every possible combination of maturity and tenor.
4. Interpolate caplet volatilities for any required strike by the use of formula (3.9).
5. Recover cap prices for those strikes via Black pricing formula (3.5) (every caplet enters with its own interpolated volatility from previous step).
6. Invert expression (3.5) to recover non-standard (3 or 12 months tenor) flat implied shifted Black volatilities (a unique value for each cap's maturity) from the prices outstanding for any given combination of strike, maturity and tenor. These are shown in tables (D.3), (D.4) for non-standard tenors of 3,12 months respectively.

Again, non-standard flat implied shifted Black volatilities from tables (D.3), (D.4) have been converted into Bachelier's (when necessary) via unique-price hypothesis.

Chapter 7

Methodology

The empirical research[65] conducted during the survey is structured into two main pillars.

Firstly, a thorough comparison about models' relative performance when pricing caplets[66] is performed for every model presented in chapter 4 of the Thesis, both in terms of **accuracy** of the calibration procedure (*in-sample* analysis) and capability of the models on **estimating prices** of caplets which have not been used during the calibration (*out-of-sample* analysis). **Shifted SABR** model (4.8) emerges as the best approach, supporting the industry standard choice and justifying it by the use of several empirical approaches.

Secondly, the new proposal for completion of the volatility cube once any standard tenor is calibrated by the use of a *no-arbitrage argument* among the implicated forward rates is tested by *out-of-sampling* volatilities for non-standard tenors. These predictions are compared with the values quoting in the markets. The test is **satisfactory**, validating the innovative methodology detailed in chapter 5 of the Thesis. Moreover, standard *Transferring the smile* technique is challenged by these same datasets, which clearly do not support its most fundamental hypothesis: **smiles' shapes are not conserved** with respect to moneyness when the tenor is modified *ceteris paribus*.

7.1 Models comparison

7.1.1 Caplet stripping

The standard calibration procedure proposed in formulae (3.15), (3.17) requires caplets (and not caps) implied volatilities. Due to its quoting nature, a previous treatment of the data is then needed. For every combination of tenor, maturity and strike, brokers quote the so-called *flat implied volatility*, defined as the *single* volatility that should be introduced in (shifted) Black/Bachelier formulae for every constituent caplet in order to recover the price of the cap that incorporates that whole set of caplets((3.5), (4.5)). It is, therefore, an averaged implied volatility concept that hardly can be extrapolated to every

[65]Empirical research has been fully accomplished using software *MATLAB*, version R2017a.

[66]An extension of the empirical part of the survey by using european swaptions is currently under research.

caplet constituting the cap. If the flat implied volatility was selected for the whole set of caplets of the cap, we would fall into excess of simplicity.[67]

The standard methodology designed to extract caplet implied volatilities from their cap homologues is known as *caplet stripping*.[68] Since any cap consists on more than one caplet, the bootstrapping technique is based on an assumption on the *functional dependence of the caplet implied volatilities, for both strike and tenor given, with the time to maturity of the caplet/cap.* In this sense, several hypothesis about this behaviour can be formulated, as long as they are *collectively* consistent.[69] For the sake of simplicity, we assume a **piecewise constant** functional form for the caplet implied volatility *between every cap maturity under consideration*, for any strike and tenor given.

Therefore, our methodology replicates the one of [2], and can be structured within the following steps:

1. Using formulae (3.3), (4.3) (whether quoting flat implied volatilities were (shifted) Black's or Bachelier's), every constituent caplet of the cap is priced with *the same flat implied volatility*, for every cap under study. The cap price is then obtained by aggregating individual caplet prices (see formulas (3.5), (4.5)).

2. For a given strike K, the n cap prices $Cap(t, T_1, N, K), Cap(t, T_2, N, K), ..., Cap(t, T_n, N, K)$ are sorted by ascending order of maturity (i.e., $T_{n,mat} > T_{n-1,mat} > ... > T_{1,mat}$).

3. The series of price differences for consecutive caps is computed for the strike K:

$$Cap(t, T_j, N, K) - Cap(t, T_{j-1}, N, K), j = 1, ..., n, \tag{7.1}$$

   where $Cap(t, T_0, N, K) := 0$.

4. Every price difference of the series is mapped to the corresponding number of caplets on that region.[70]

5. For a given strike K, every price difference would therefore be mapped with a given number of caplets on specific start and maturity dates that lie in the considered

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[67] For instance, think about the longest time to maturity caps of USD data (20 years). If the flat implied volatility was selected as the implied volatility corresponding to every caplet of the set, we would be assuming that every one of the 79 caplets that constitutes the cap has the same implied volatility. This idea is unsound, and the piecewise constant alternative (to be proposed) looks much more reasonable.

[68] This process is thoroughly summarized, for example, in [27], [35] and [2]. The treatment of [2] is particularly interesting, since several alternatives for the stripping procedure (attending to the authors classification: *Bootstrapping, Rebonato* and *Global Sabr*) are discussed in depth. Through this Thesis, and following their claim, we restrict ourselves to the first class of methods.

[69] For example, a linear assumption on time to maturity of the cap from year 5 to 6 is not simultaneously consistent with a piecewise constant hypothesis during the time interval [5,7] for the maturity of the caps. A thorough study about the influence of the selected functional form of the caplet implied volatility in the stripping procedure over subsequent calibrations remains an issue of obvious interest.

[70] For instance, the fourth term of USD series (corresponding to the difference between caps that mature in years 3 and 4) is mapped to the four underlying caplets for that period; from year 3 to year 4, within a tenor of 3 months.

region. Since a piecewise constant hypothesis of caplets' implied volatilities is assumed, the implied caplet volatility $\sigma(K,j)$ for every region between two consecutive cap maturities is constant (i.e., the same for every caplet of the region), and can be computed by applying a one-dimensional root finder to the equation[71]

$$Cap(t,T_j,N,K) - Cap(t,T_{j-1},N,K) = \sum_{i=j_1}^{n_j} Caplet(t,T_i,N,K,\sigma(K,j)), \qquad (7.2)$$

where n_j accounts for the number of caplets for that particular region.[72]

While the stripping procedure can be directly performed with USD cap volatilities, the process for EUR volatilities requires to carry out separately the stripping for *short* maturities (up to 2 years) from the *long* maturities procedure, since the tenor of the underlying forward rate changes from 3 months to 6 months in this case.

As it is mentioned in [27], extracting caplet ATM volatilities is trickier, since the location of the strike (i.e., the underlying forward rate) depends on *every maturity.* The previous algorithm is not valid anymore, since the difference $Cap(t,T_j,N,K_{ATM}) - Cap(t,T_{j-1},N,K_{ATM}), j = 1,...,n$ **does not provide** the ATM caplets on the sought region. The only calculation that can be identically repeated is step 1 of the algorithm, since ATM cap prices can be recovered for every maturity if both ATM strike K_{ATM} and ATM flat implied volatility σ_{ATM} are quoted for that maturity. The stripping algorithm with ATM caps is fully developed (with an illustrative example) in appendix E.

Several implied caplet volatility term structures (USD and EUR, both for shifted Black and Bachelier quoting conventions) are plotted once constructed via stripping, in order to compare their evolution with the maturity of the caps under research, for every strike included in the survey.

7.1.2 Discounting and forwarding. Further considerations

The OIS relevant curve for each market (*EONIA* and *Fed Funds Rate* for EUR and USD respectively) is used at the valuation date *24th May, 2017* for both computing discounting factors and implying forward rates when necessary. See equation (2.2) for the process of stripping implied *instantaneous* forward rates from market zero-coupon

[71]Standard Newton-Raphson algorithm has been used for every price difference (see, for instance, [39] for quick refresh on one-dimensional root finding methods). No convergence problems (such as seed-dependence, low computational speed or any other) have been acknowledged.

[72]Returning to the previous example: in the fourth element of USD series of price differences, four caplets lie in the region between year 3 and 4. We have to find the quantity $\sigma(K,4)$ that solves:

$$Cap(t,[T_{start},4],N,K) - Cap(t,[T_{start},3],N,K) = Caplet(t,[3,3.25],N,K,\sigma(K,4)) + \\ Caplet(t,[3.25,3.50],N,K,\sigma(K,4)) + Caplet(t,[3.50,3.75],N,K,\sigma(K,4)) + Caplet(t,[3.75,4.00],N,K,\sigma(K,4)) \qquad (7.3)$$

.

prices, which are respectively extracted from the discounting curve via $P^{market}(t,T) = \exp(-R(t,T) * \delta(t,T))$, where $R(t,T)$ stands for the OIS rate at date t, maturity T.

Additionally, as shown in formulae (4.15) and (4.17), Vasicek and Hull-White calibration procedures ask for a proxy of the *instantaneous short rate* at valuation date, $r(t)$. Attending to our own *heuristic* criterium, we have selected the corresponding OIS 1 week rate $r(t, 1\text{ week})$ as the optimal trade-off between avoiding *excess of market noise* (with notable influence over the shorter rates) and *representativeness* of the instantaneous short rate (worse as the time to maturity of the rate increases). Obviously, this *ansatz* can be discussed. Indeed, it has been thoroughly done among previous literature, and no consensus seems to have been reached yet[73].

7.1.3 Models calibration

- **(Shifted) SABR:** According to previous literature (see [24]) and our own experience during the calibration procedure, formula (3.15) offers more robust results than (3.17) for both SABR and shifted SABR calibration for every maturity under consideration and needs less time to converge. Consequently, the definitive results are computed via expression (3.15). To simplify the calibration process, β has been fixed at 0.5 for every maturity following the claim of several authors (such as [14]).

  MATLAB standard optimization with restrictions routine *fmincon* have been used, forcing α and $\sigma(0)$ to be positive and $-1 \leq \rho \leq 1$. No convergence problems have been detected during the process. Several plots illustrating the term structure for the calibrated parameters and the implied calibrated volatility surfaces/smiles for both EUR and USD data are shown next, for comparative purposes.[74]

- **Shifted Black/Bachelier:** Several shifted Black (4.7) or Bachelier (4.1) models have been calibrated for every maturity outstanding. Expression (3.15) is applied for both models to calibrate the unique parameter $\sigma(0)^{(n)}$ for any given maturity. The term structure of both parameters is plotted afterwards, as well as the resulting smiles for every given maturity. No convergence problems have been acknowledged within standard application of *fmincon* (the only restriction is $0 \leq \sigma(0)^{(n)}$).

- **Normal/Free boundary SABR:** As explained before, both models are calibrated with normal implied volatilities. Equation (3.15) is used together with formula (3.13) for the normal SABR model (fixing $\beta = 0$) or expression (4.10) for the free boundary SABR model (for the sake of comparability with the calibration of the shifted SABR model, β has been likewise fixed to an (arbitrary) close value to 0.5: $\beta = 0.49999$.[75]) No convergence problems have been detected within standard use of *fmincon* (the restrictions are similar to the ones imposed in shifted SABR's calibration). Parameters term structures are plotted afterwards.

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[73] See, for instance, [40], which reaches to a conclusion that fully faces our choice.

[74] This is the only model where USD data is used, in order to compare implied volatility surfaces from EUR and USD data. Since no negative rates have been yet observed in USD quoting instruments, and our main aim is to contrast several models' pricing behaviour when negative rates are permitted, from now on only EUR data are to be considered in the analysis.

[75] Recall that $0 \leq \beta < \frac{1}{2}$ is necessary within a free boundary SABR framework.

- **Vasicek/Hull-White:** The calibration of both models can be accomplished with either caplet or cap prices, since the former account for a particular case of caps with a single payment date:
  - If caps are selected, the calibration procedure is straightforward. Once cap market prices have been recovered from quoting flat implied shifted Black volatilities for any maturity-strike combination (T, K) given,[76] minimizing the sum of squared differences for every strike and a given maturity with respect to equations (4.14) or (4.16) gives the calibrated parameters for both models:

$$(\sigma, k, (\theta)) = \arg\min_{\sigma,k,(\theta)} \sum_i \left(Cap(t,T,N,K_i)^{market} - Cap(t,T,N,K_i) \right)^2. \quad (7.4)$$

  - Calibrating with caplets requires to proceed as follows:
    * Strip piecewise constant caplet volatilities from flat cap market volatilities as explained in subsection 7.1.1.
    * Recover caplet market prices from implied shifted Black caplet volatilities by using a standard shifted Black pricer (3.3).
    * Proceed as in the previous algorithm, minimizing the sum of squared errors for caplet pricing formulas (4.14), (4.16):

$$(\sigma, k, (\theta)) = \arg\min_{\sigma,k,(\theta)} \sum_i \left(Caplet(t,T,N,K_i)^{market} - Caplet(t,T,N,K_i) \right)^2. \quad (7.5)$$

In spite of needing further data transformation, the second procedure is preferable in terms of *consistency* within the previously-calibrated models. Since the former have been calibrated by minimizing caplets' pricing error, it is a more symmetric methodology. Not only that, but the *error introduced in the stripping procedure* should also be considered. Every one of the previous models needed from this pre-calibration technique for their calibration algorithms. Since the stripping results depend on the interpolation method being considered, this dependency might corrupt future calibration. In the spirit of *mutual compensation*, this effect shall be filtered by **introducing the stripping bias in every calibration to come**. Therefore, if Vasicek/Hull-White calibration is performed without prior stripping (formula (7.4)), there exists a competitive advantage for these models which is not exclusively due to the *nature of the models itself*, but to *the way the data is quoting in the markets*. Those are the arguments to opt for the second calibration algorithm (caplets' pricing, formula (7.5)) instead of an straightforward caps' comparison.

[76] By the *unique-price* hypothesis explained in Chapter 6, these prices should be equivalent to the prices obtained via flat implied Bachelier volatilities, and therefore it is *completely equivalent* to calibrate Vasicek/Hull-White with either one of both datasets.

The algorithm that uses (7.5) in combination with (4.14) or (4.16) for calibrating Vasicek/Hull-White models respectively struggles to converge for several seeds' combinations $(\sigma_0, k_0, (\theta_0))$. It often gets stuck in a particular region of the parametric space, or reaches to an arbitrary large number of iterations without finding a solution that satisfies the constraints given. A thorough analysis about the convergence problems of both algorithms is provided in appendix F.

7.1.4 Caplets pricing comparison

The test among the full set of models is accomplished by comparing their *caplets pricing accuracy*, which is analysed attending to both *in-sample* and *out-of-sample* criteria:

- **In-sample:** Once every model is calibrated, the matrix of caplet prices is recovered via each model pricing formula.[77] These arrays are compared with the matrix of caplet market prices (obtained whether by shifted Black/Bachelier market implied volatilities, since they are equivalent by unique-price hypothesis). Several plots of caplet prices term structures for some representative strikes as well as absolute and relative pricing errors are then plotted, so their main differences can be analysed.

- **Out-of-sample:** The out-of-sample research has been conducted both *in strike* and *in maturity* dimensions:[78]
  - *Strike:* An arbitrary column[79] is removed from the matrix of stripped caplet volatilities.[80] Each model is recalibrated without these data, and the price of a caplet for each maturity and the omitted strike is estimated with each model's standard pricing formula.[81] Several plots of the attained price and absolute/relative errors with respect to the market quoting prices are shown for some representative strikes, for comparative purposes.
  - *Maturity:* A full arbitrary row is wiped out from the stripped caplet volatilities matrix, and every model is recalibrated without these data. The price of a caplet for every strike and the selected maturity is then forecasted with each model's pricing formula.[82] Several plots of the attained price and absolute/relative errors with respect to the market quoting prices are shown afterwards.

[77] A standard notional of $N = 100$ has been considered in every pricing algorithm.

[78] Out of sampling in the *tenor* dimension is not considered at this point, since it is to be studied in the section to come (*Completing the cube*).

[79] This column shall not be neither the first nor the last one of the matrix, to avoid the acknowledged problem of extrapolating in strike.

[80] Note that the matrix of caplet volatilities accounts for maturities in its rows and strikes in its columns.

[81] In SABR alike models (shifted Black, Bachelier, shifted SABR, free boundary SABR and normal SABR), the implied volatility for the given strike is interpolated via smile (horizontal line for both shifted Black's and Bachelier's), and the price is recovered by standard shifted Black/Bachelier pricers ((3.3), (4.3)). Regarding short-rate models, the price is directly computed via pricing formulae (4.14) or (4.16) respectively. This distinction applies to maturity out-of-sampling as well.

[82] Previous distinction applies in maturity out-of-sampling. In this case, by piecewise constant hypothesis the relevant implied volatility for SABR-alike models is the one of the *previous maturity* for any strike being considered, and the smile interpolation is therefore substituted by a *constant interpolation*, which predictably accounts for bigger mistakes. However, note that, when applying shifted Black/Bachelier pricers ((3.3), (4.3)), the maturity of the caplet shall be the *actual maturity of the caplet being priced*, not the former one (i.e., the piecewise constant hypothesis applies **in implied volatilities**, not **in prices**!)

7.2 Completing the cube

We challenge the robustness of our proposal for the full completion of the volatility cube in the tenor dimension by recovering the price of caplets with a *non-benchmark tenor* (3 and 12 months, see tables (D.3), (D.4)) from implied volatilities of our standard calibration tenor (table (D.1)). We use several OTC caps to test the adequacy of this approach.

When the scheme for extrapolating implied volatilities was deduced[83], the analysis of the only free remaining parameter in formulae (5.11), (5.12), (5.13), (5.14), ρ, was disregarded. Appendix G of the Thesis focuses on this crucial aspect of our calibration proposal. An insightful study about the impact of ρ over volatility's extrapolation can be found in [2], and the interested reader is readdressed there for further information.

In a nutshell, to obtain any arbitrary implied volatility (shifted Black's or Bachelier's) for a caplet with maturity T_{market}, strike K_{market} and tenor τ_{market}, we proceed as follows:[84]

1. Map the maturity T_{market} to the preceding maturity T of the calibrating data. By piecewise constant hypothesis[85] , the sought volatility remains constant in the interval $[T, T_{market}]$, and can be therefore computed as if T was the *actual* maturity of the caplet.[86]

2. Use the previously calibrated shifted SABR parameters for maturity T to fit the smile and recover shifted Black's / Bachelier implied volatility for K_{market}, $\sigma^{(n)}(K_{market}, T, \tau)$.

3. Extrapolate $\sigma^{(n)}(K_{market}, T, \tau)$ to $\sigma^{(n)}(K_{market}, T, \tau_{market})$ via formulae (5.11), (5.12), (5.13) or (5.14).

Once caplet stripping algorithm has been applied to flat cap implied volatilities from tables (D.3), (D.4), non-standard tenor (3, 12 months) term structures are plotted for comparative purposes. Thereafter they are converted to (market) prices, and compared with non-standard tenor prices attained by our arbitrage-free formulation via absolute/relative pricing errors[87]. The section ends with an empirical research on the robustness of an alternative competitor when transforming standard-tenor volatilities into non-standard's: *tranferring the smile* technique (see appendix C).

[83]See chapter 5 of the Thesis.

[84]When creating the cube, we focus on shifted SABR model for smile fits, since chapter 8 of the Thesis illustrates that it systematically outperforms alternative competitors.

[85]Assuming piecewise constant hypothesis for pricing new derivatives guarantees *internal consistency* of the pricing scheme within the calibration procedure. If any other assumption was done at this point, both approaches would not be simultaneously compatible.

[86]Note that this hypothesis only applies to the implied volatility of the caplet to be priced. When applying standard pricing formulae ((3.3) or (4.3)), the maturity that appears as T_{mat} is T_{market}, not T.

[87]To filter the influence of maturity out-of-sampling errors (see chapter 8), volatilities with *identical time to maturity* than the ones quoting for the standard 6-month tenor have been used. In addition, maturities below $T = 3Y$ have been excluded from the comparison owing to *typical market quoting conventions*. Since the first caplet is excluded from the quoting process for EUR data, stripping volatilities from 12-month caps' results in identical caplet volatilities for maturities $T = 1, 1.5, 2Y$ (the first caplet of the quoting process expiries at $T = 2Y$). We overcome this inefficiency by excluding these short-maturity data from our analysis (for consistency in the comparisons, we have excluded them from the 3-month extrapolation as well). Consequently, a 6-month benchmark tenor is the only one included in the study.

Chapter 8

Empirical results

The main results of the empirical research are summarized within the next sections.

8.1 Volatility term structures

We devote this section to a descriptive analysis of market caplet stripped volatilities.

Figure (8.1) below shows the piecewise constant caplet volatility term structures resulting of stripping cap volatilities from tables (D.1)[88] and (D.2). From left to right, Black USD, shifted Black EUR and Bachelier EUR volatility term structures are plotted.

While EUR caplet volatilities exhibit a certain *mean-reverting* behaviour to a given long-term value as time to maturity of the underlying caplet increases (especially pronounced for shifted Black's), USD volatilities diverge with time to maturity, suggesting *higher variance in the uncertainty* for the underlying forward rate for a longer time to maturity. Therefore, a first pattern might be identified when comparing an economy with *strictly* positive rates (USD) with a market that permits both positive and negative rates (EUR).

The convergence to the long term value, however, is not accomplished in a similar manner for shifted Black and Bachelier quoting volatilities. The dynamic evolution of the former results in a rearrangement of the volatilities (lower strikes quote with greater volatility for the longest maturities), while Bachelier EUR's dynamic evolution is monotonous with time to maturity (different strike curves never cross each other).

Studying the figures the other way round gives insight about the behaviour of market caplet volatilities as the options get closer to maturity. For short maturities EUR volatilities increase with the strike[89], and USD volatilities manifest the opposite behaviour. This

[88]Remember that this table mixes the 3-month and 6-month tenors, and therefore is not valid for performing tenor-dependence analysis. This research is disregarded until *Testing the cube* section.

[89]In terms of caplet pricing, a trade-off between several magnitudes appears at this point. While increasing the strike results in a pricing drop (to guarantee that the term structure is arbitrage-free), higher implied volatilities increase the price of the caplet. Therefore, EUR volatilities term structure suggests a trade-off between higher strikes (lower prices) and higher volatilities (higher prices).

phenomenon applies for increasing time to maturity in USD and Bachelier EUR term structures (since different strike curves never cross each other), but the tendency is reverted for Shifted Black EUR volatilities.

Figure (8.2) splits in strike EUR term structures from figure (8.1) to illustrate the dependence of the dynamic evolution of the term structure with the strike being considered. Two different patterns are mainly observed:

- In the **low strike area** (negative, ATM and lowest positive strikes), implied volatilities *tend to increase* with time to maturity.

- For the **positive greater strikes**, the tendency is reverted and the volatilities fall with time to maturity.

This behaviour is shared by both shifted Black and Bachelier quoting volatilities.

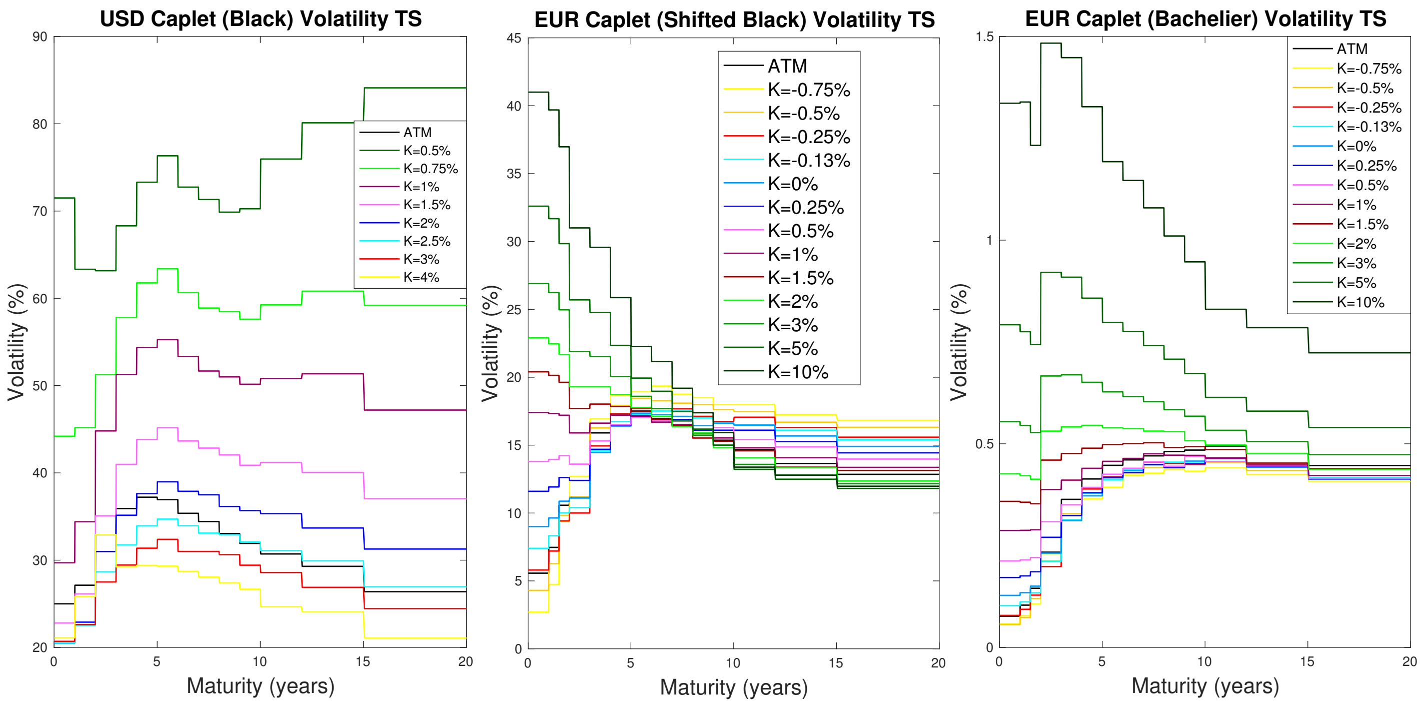

Figure 8.1: *USD and EUR market volatility term structures. These have been stripped from flat cap Black, shifted Black and Bachelier volatilities respectively.*

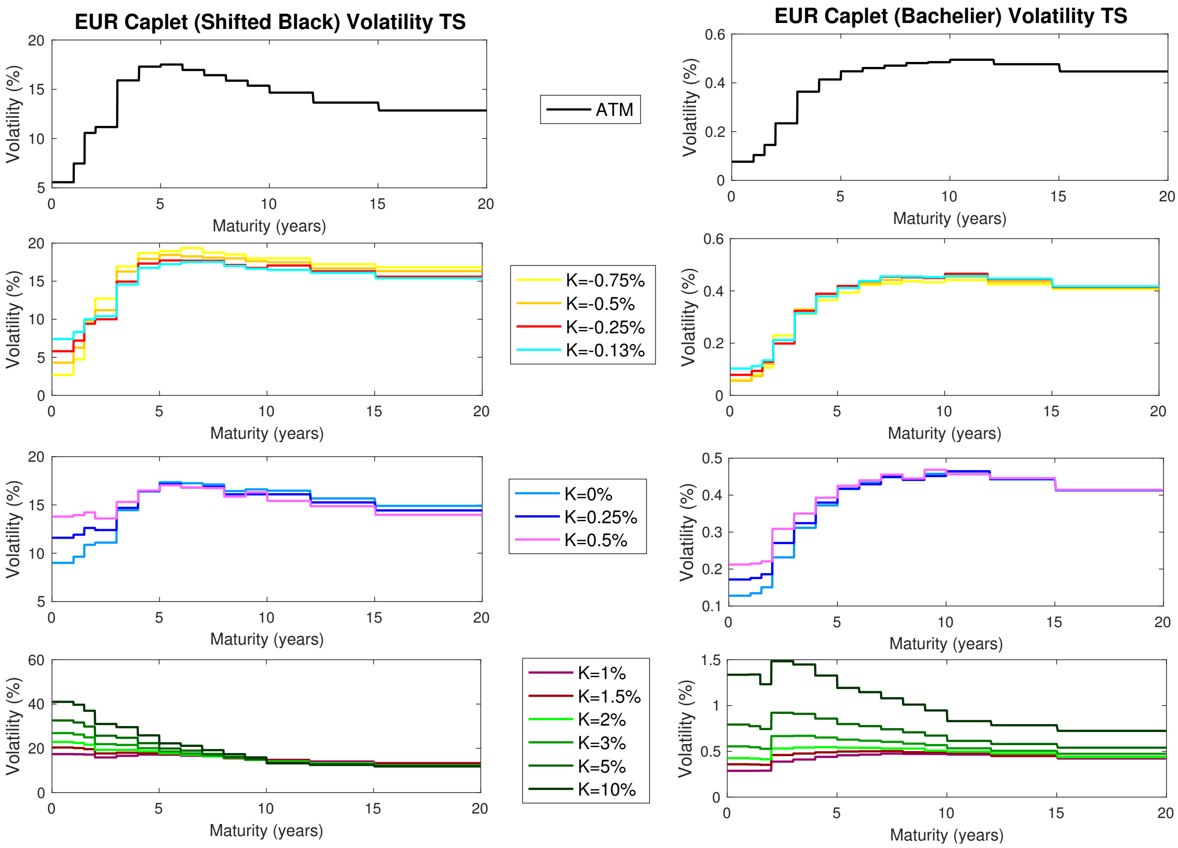

Figure 8.2: *Strike dependence of the dynamic evolution of EUR volatility term structures.*

8.2 Models calibration

Figure (8.3) below displays several parameters term structures obtained by fitting each model to the term structure shown in figure (8.1), for every maturity outstanding. In a nutshell, the models under study can be classified in *stable* and *unstable* categories:

- **Stable models:** For every parameter of the model, its term structure evolves *smoothly*. This characteristic is desirable in the sake of *continuity*, since it is more likely that these models were correctly specified (*overparameterization* is avoided). SABR, shifted SABR, shifted Black, Bachelier, normal SABR and free boundary SABR seem to fulfil these characteristics.

- **Unstable models:** At least one of the parameters term structure evolves *wildly*, with huge peaks and appearance of discontinuity. These models are more unreliable, since *misspecification* of the parameters might appear during the calibration procedure (several different combinations of parameters result in similar values of the objective function, and the algorithm struggles to optimize it in the parametric space). Attending to figure (8.3), Vasicek and Hull-White pertain to this class of models. Further discussion about numerical troubles involving Vasicek/Hull-White calibration procedures can be found in appendix F of the Thesis.

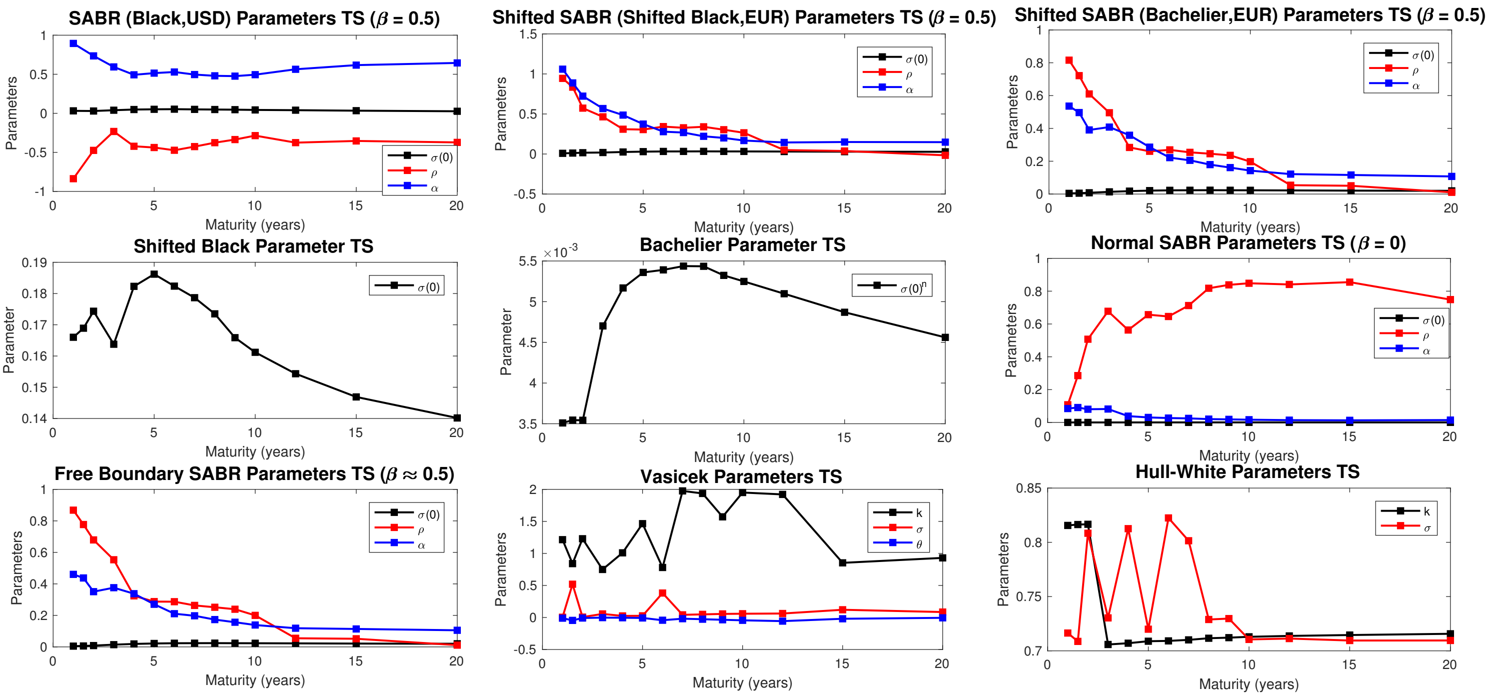

Figure 8.3: *Parameters term structures for the models included in the survey.*

SABR model and its negative-rates extensions share an almost zero value for today's forward rate volatility, $\sigma(0)$, irrespective of the maturity being considered. As illustrated in figure (3.2), small changes in this parameter cause remarkable shifts in the smiles. While the *volatility of the volatility* parameter, α, tends to decrease smoothly when time to maturity increases for every SABR extension under study[90], the dynamic evolution of the correlation for the Wiener processes of $F(t)$ and $\sigma(t)$, ρ, depends on the model being considered. It is (almost) monotonically decreasing for shifted SABR (with independence of the quoting volatility used in the calibration) and free boundary SABR, from close-to-one values at the shortest maturities to almost zero correlation for the longest being considered. It follows the opposite trend in the normal SABR model, and it is negative (and relatively steady) for USD SABR calibration.

Regarding Vasicek/Hull-White calibration, the parameter accounting for the *instantaneous volatility of the short rate*, σ, fluctuates wildly for the shortest maturities, stabilizing when time to maturity grows. The *mean reversion speed* of the short rate towards its long term value, k is lesser and much more stable in Hull-White than in Vasicek, where it evolves in discontinuous peaks. Finally, the long term value of the short rate, θ, is always small and negative for any maturity outstanding, in consonance with the current economic situation.

[90]In consonance with figure (8.1), which shows that volatilities tend to *converge to a long term value*, and therefore account for lesser variance with increasing time to maturity. Note that USD term structure diverges with T, in consonance with top-left subfigure in figure (8.3) (where α does not decrease even for the longest maturities being considered).

8.3 Volatility smiles and surfaces

This section aims to contrast 1-D (smiles) and 2-D (surfaces) fitted volatility structures, by the use of SABR/shifted SABR, against market actual data. This comparison is accomplished for both Black/Bachelier quoting volatilities.

Figure (8.4) below displays the shape of several market caplet volatility smiles for every maturity outstanding. It includes shifted SABR smile calibration via shifted Black volatilities (formula (3.9), using the calibrated parameters of figure (8.3)) and shifted Black calibrated volatility for every maturity of the survey. The **existence of smiles** in the markets is hardly arguable for any given maturity with figure (8.4) in mind. Also, it is manifested that shifted SABR is flexible enough to accommodate *many different smile shapes* in a really accurate fashion. Constant volatility hypothesis implied by shifted Black (and Bachelier) models is fully rejected.

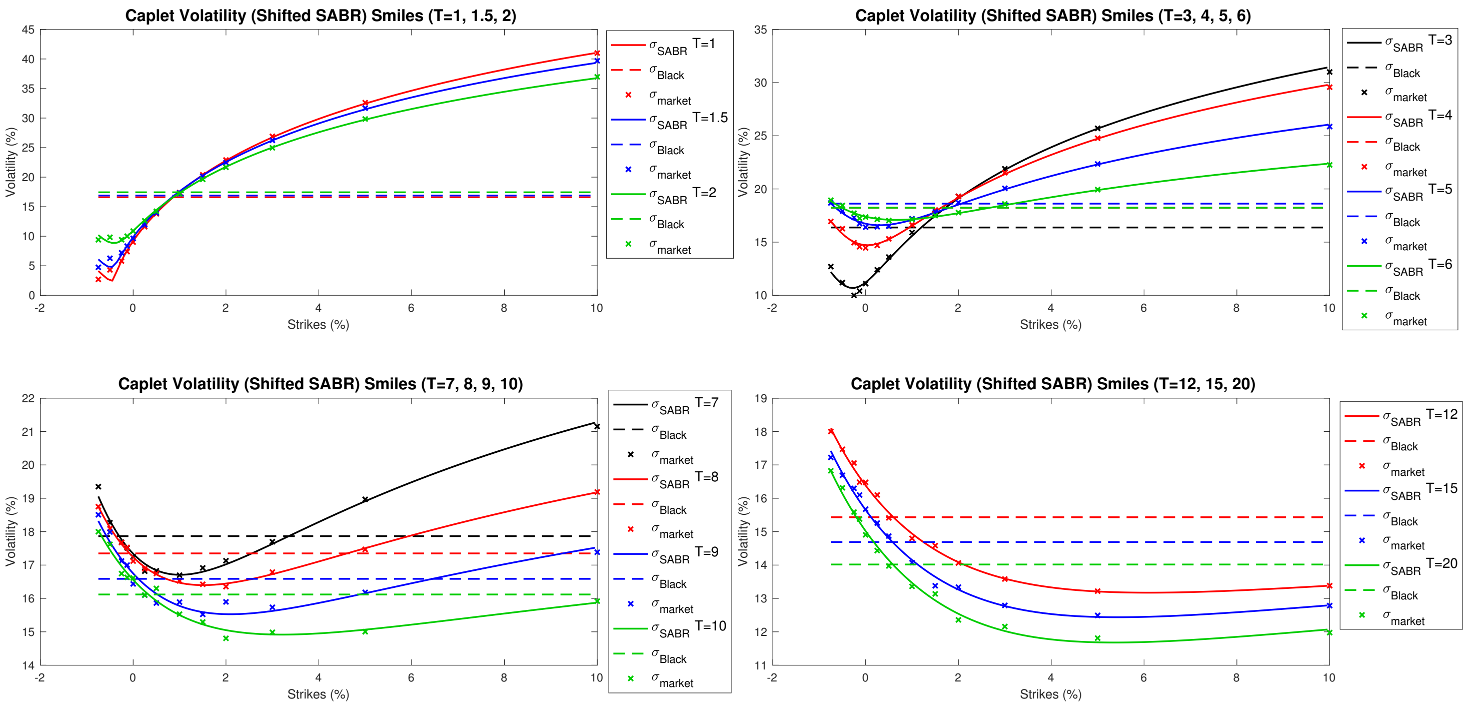

Figure 8.4: *Market, shifted Black and shifted SABR volatilities. The existence of smiles is clearly supported by the markets, irrespective of the maturity being considered.*

Figure (8.5) compares (shifted) SABR volatility surfaces[91] for USD (strictly positive) and EUR (positive and negative) interest rates data.

[91] The *implied volatility surface* is computed by interpolating (shifted) Black/Bachelier volatilities for every maturity and strike outstanding via formulas (3.9), (3.13) and plotting these volatilities against both variables. Again, the tenor dependence of the volatility cube is momentarily ignored.

Recalling the discussion of section 4.1, we claimed that a floorlet implied volatility within a Black context shall rise sharply when $K \to 0\%$ to guarantee that a non-zero price is attained. As shown in the left subfigure of figure (8.5), this behaviour is not only manifested by floorlets, but by caplets volatilities. Their increase when $K \to 0\%$ to values up to 80% implies a double effect for increasing prices (the rise in the implied volatility when the strike descends and the drop in the strike itself), and therefore match market quoting instruments. USD implied SABR volatility surface is then splitted between a **steady area** (strikes above $K = 1.5\%$) and a **high volatility zone**, below $K = 1.5\%$.

As regards EUR shifted SABR volatility surface, their values stand far below from the volatilities attained by their USD's homologues. In this case, the splitting occurs *in maturity.* The *stable* area is located above 5-6 years to maturity, while shorter maturities account for further variance in strike. The lowest volatilities (over 5%) are attained for the lowest strikes, shifting up to values near 45% for the highest strikes ($K = 10\%$). This behaviour is in consonance with the one shown in figure (8.4).

In figure (8.6), shifted SABR implied volatility surface via shifted Black volatilities (right subfigure in figure (8.5)) is compared with its Bachelier homologue (formula (3.13)). In broad terms, the surface shape is *robust to the nature of the quoting volatility*, since increasing volatilities when the strike rises are again observed for the shortest maturities. However, a permanent slope appears in the former stable area, shifting the whole volatility surface upwards when the strike ascends, irrespective of the maturity being considered. Moreover, the **tenor splitting** feature is much more evident for Bachelier quoting volatilities than it used to be with shifted Black's. Right subfigure exhibits a sharp drop in Bachelier's shifted SABR volatilities for the three shortest maturities, which account for the shortest tenor under study (3 months).[92]

Inquiring deeper on this aspect, figure (8.7) splits both volatility surfaces between the two underlying tenors. Left subfigures are consistent with the shape of right's, supporting the argument of robustness of the volatility surfaces with respect to the *nature* of the quoting volatility. However, an obvious shape difference is noticed between top and bottom subfigures. While 3-month tenor volatility surfaces grow monotonically with the strike for every maturity, 6-month's manifest the behaviour observed in figure (8.6). The tenor splitting feature for caplets implied volatilities is therefore plainly illustrated, and modelling it via volatility cube's completion becomes a must.

[92]This fall is not observed so clearly for shifted Black's volatility surface, and therefore makes us believe that modelling the tenor splitting via arbitrage-free cube's calibration might provide better results for Bachelier quoting volatilities than for shifted Black's. This conjecture is tested next (see section 8.7, *Testing the cube*).

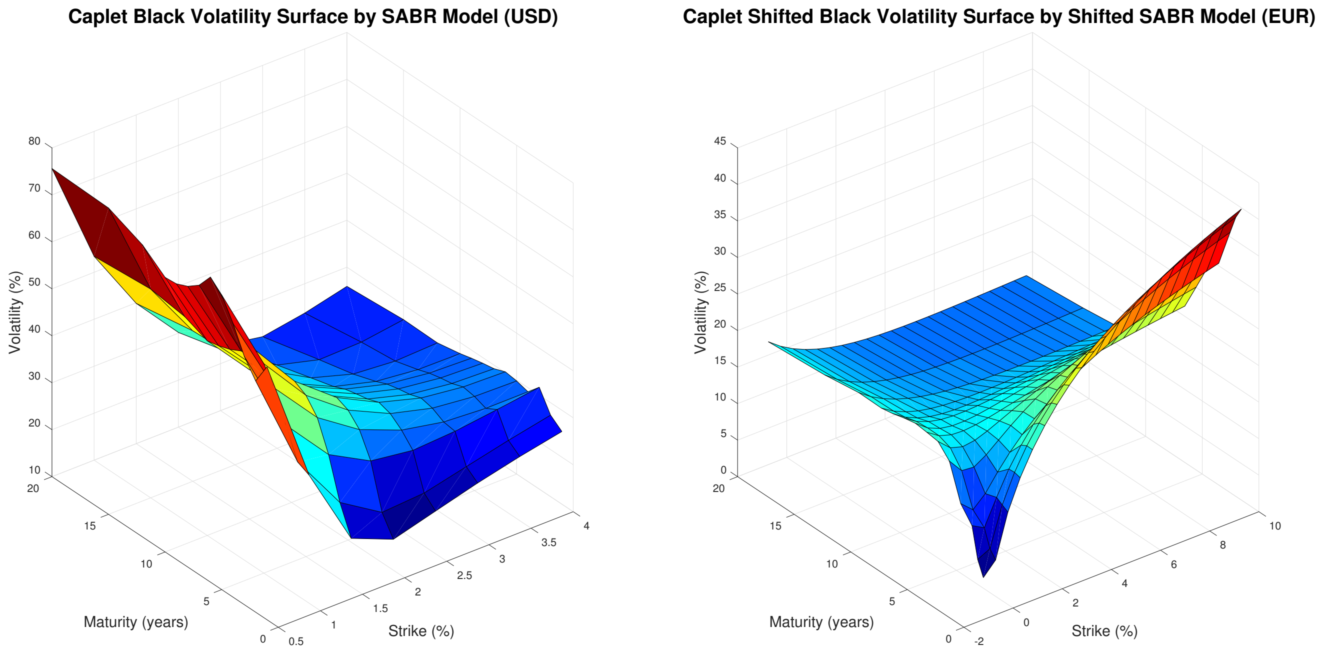

Figure 8.5: *SABR/Shifted SABR implied volatility surfaces. EUR structure mixes a tenor of 3 months for maturities up to two years with a 6 month-tenor onwards.*

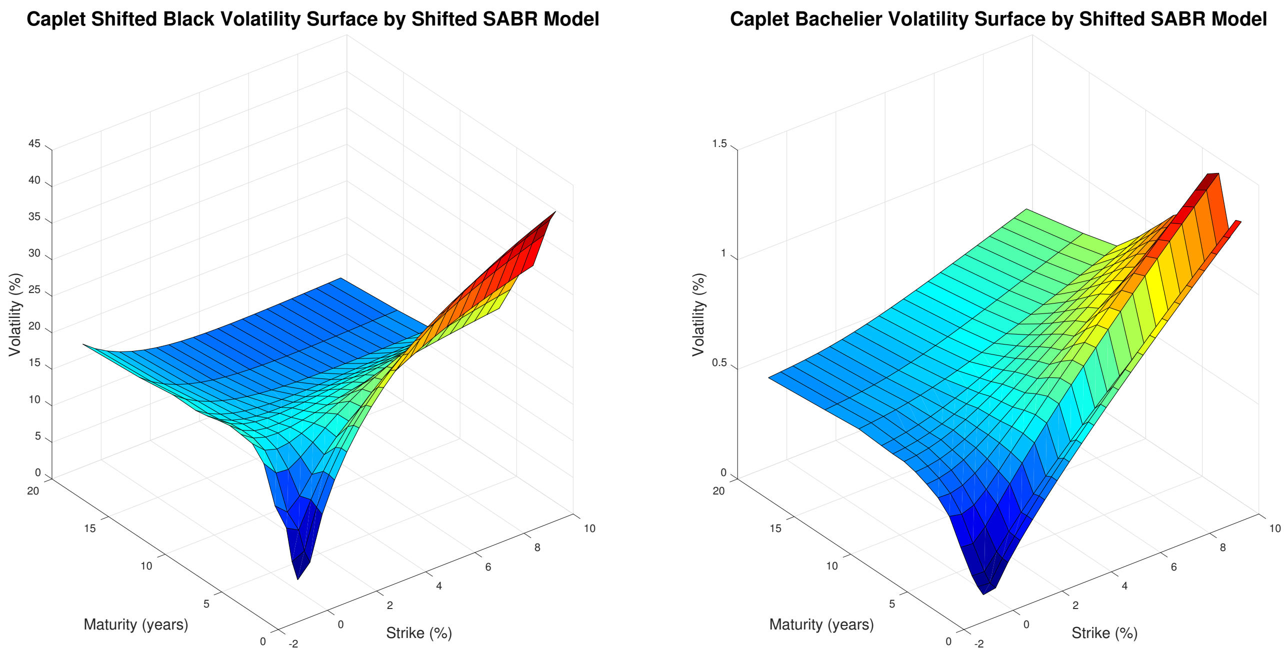

Figure 8.6: *Shifted Black/Bachelier shifted SABR implied volatility surfaces.*

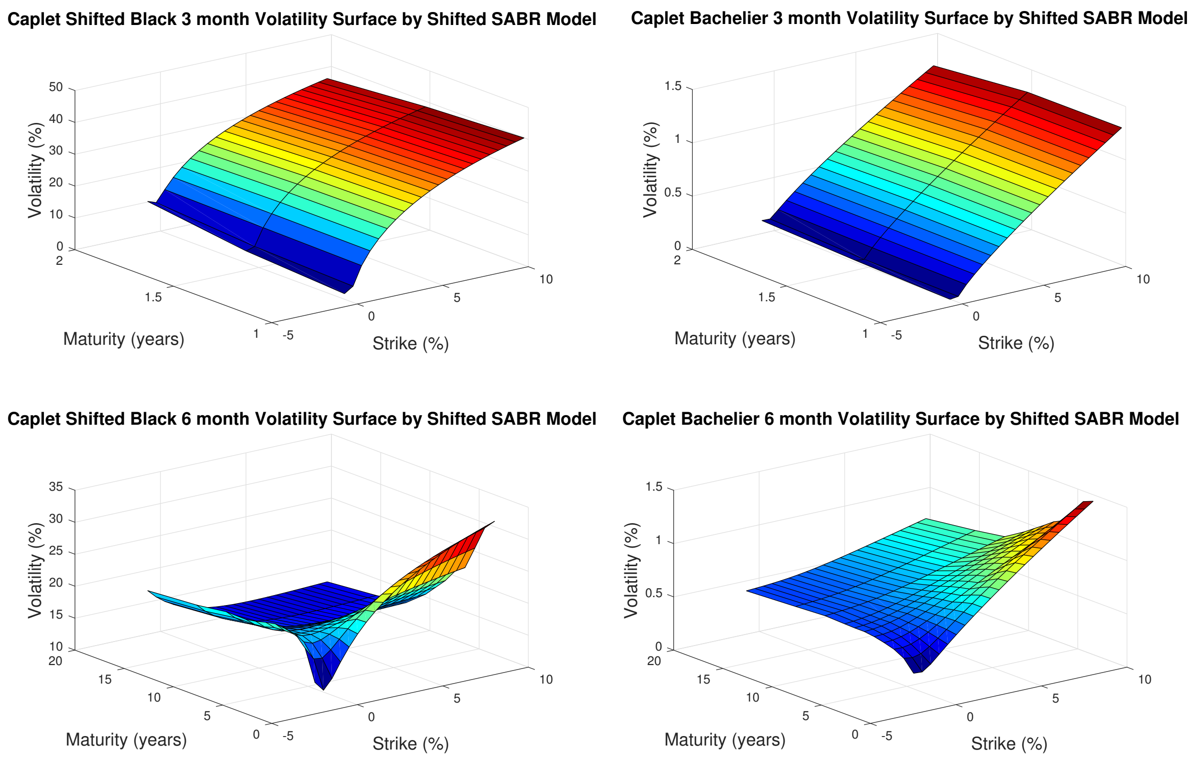

Figure 8.7: *Tenor splitting in figure (8.6).*

8.4 In-sample analysis

Models' accuracy when recovering the price of every caplet of the calibration process is tested through this section for every model under research. Therefore, this part should be understood as a thorough in-sample comparison among these models.

Figure (8.8) below is divided into two relevant sets of subfigures. Top rows compare caplet market prices term structure with caplet prices term structures implied by every calibrated model. Bottom row shows in conjunction caplets prices term structures for every model for comparison purposes within the market benchmark curve for some representative strikes of the survey.

Roughly speaking, top rows illustrate that every model fits the market benchmark term structure shape *reasonably well*, at least in qualitative terms. At a first sight, only Vasicek model tends to fail systematically when fitting the maturity $T = 6Y$. Both market prices term structure as well as every model term structure are **arbitrage-free**, since caplet prices are sorted in descending order in strike for any given maturity, never crossing each other's curve. For any given strike, caplet price tends to rise for longer maturities, although it stabilizes (and even drops) for the last maturity ($T = 20Y$). Tenor splitting phenomenon is plainly observed again (especially for the lowest strikes), since caplet prices increase sharply from almost negligible values at maturities $T = 1, 1.5, 2Y$ to appreciable values above $T = 2Y$. Bottom subfigures manifest that pricing accuracy of the models is mainly challenged for the highest strikes (lower prices). While negative and low-positive strikes prices curves closely resemble each other, $K = 5\%$ prices term structures differ significantly. In fact, only **shifted SABR** and **free boundary SABR** pricing curves follow market's caplets behaviour.

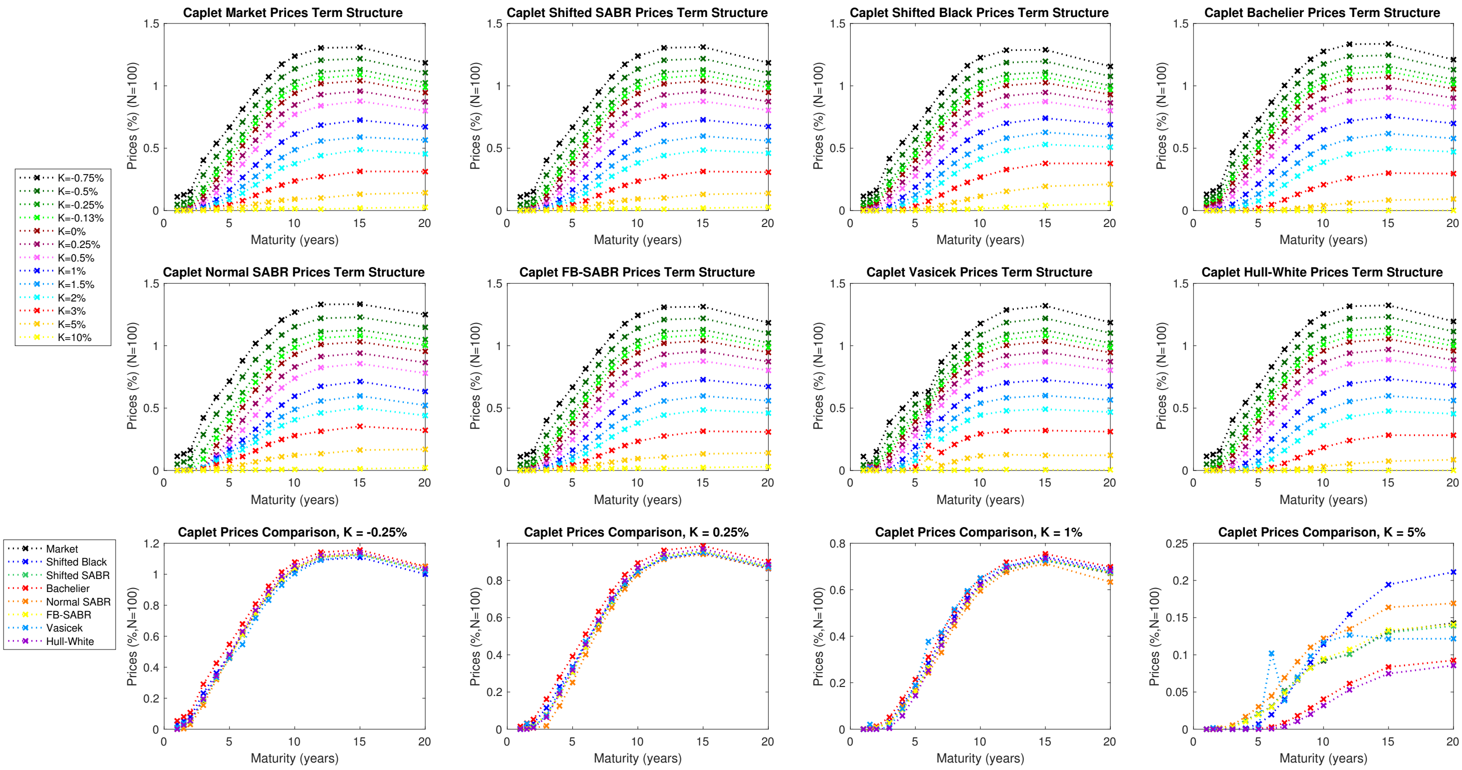

Figure 8.8: *In-sample caplets pricing analysis. First and second rows show caplets prices term structures implied by every previously calibrated model (see figure (8.3)), as well as caplet market prices term structure. Last row compares every model's pricing accuracy within the market benchmark for some representative strikes.*

Figure (8.9) inquires further on the (absolute) pricing accuracy of every model for the strikes chosen in bottom subfigures of figure (8.8). In consonance with the two top rows of figure (8.8), no model commits a high absolute pricing error for any maturity under consideration (it reaches 0.12% as maximum, for a standard notional of $N = 100$). The absolute pricing error tends to drop within the strike (which is reasonable to support that every model fits market prices reasonably well, since prices decrease within the strike). Shifted SABR and free boundary SABR arise again as the best models irrespective of the maturity or strike being considered, with slight preference for the former. Vasicek and Bachelier are generally the worst models in terms of absolute pricing accuracy. Hull-White, normal SABR and shifted Black work reasonably well for the negative/low-positive strike area, but tend to fail for higher strikes.

Figure (8.10) converts absolute pricing errors from figure (8.9) into relative's. Since caplet market prices are notably small (especially for the shortest maturities or the highest strikes; see figure (8.8) above), relative pricing errors might grow (almost) unboundedly. To provide some insight about the implied difficulty in fitting almost negligible caplet market prices, table (8.1) below displays their values (that, as illustrated, can reach to 10^{-7}). Except for shifted SABR and free boundary SABR models, every model outstanding fails for the shortest maturities ($T = 1, 1.5, 2Y$) under study[93]. Therefore, only these two models could be acceptable in the shortest-maturity (lower prices) region. Figure (8.11) compares shifted SABR and free boundary SABR within this close-to-maturity area, evidencing that shifted SABR's relative errors are considerably smaller (hardly reaching 10%[94]) than free boundary SABR's, that could reach to values near 30%. Shifted SABR is generally[95] preferred for the shortest maturities under consideration.

Finally, figure (8.12) aims to compare the best models[96] for the chosen representative strikes and the longest maturities area (from $T = 3Y$ above). Again, shifted SABR and free boundary SABR (in this order) clearly outperform alternative competitors. Shifted SABR is remarkably accurate, with relative errors within the range $[0, 5]\%$ even for the $K = 5\%$ strike (where free boundary SABR starts to fail, given that caplet market prices drop sharply). As regards the rest of the models, Hull-White, Vasicek and normal SABR are possibly the best candidates for the *low strike area*,[97] although they fail for the *high strike area*. Shifted Black arises as a reasonable candidate just for the $K = 1\%$ strike. Bachelier model is hardly recommended.

[93] We understand that a failure occurs when a relative error of 25% is exceed.

[94] Recall that, even being a high relative error, it is fitting almost negligible caplets (prices over 10^{-7}) and, therefore, it is inappreciable in absolute terms.

[95] With the possible exception of $K = 1\%$.

[96] Only models with lesser than 25% relative errors are included in the plots.

[97] We define the *low strike area* as the one which accounts for either *negative* or *low-positive* strikes, while *high strike area* stands for strikes above $K = 5\%$. Low strike area is much more concerning nowadays, given the current negative rates context. High strike area caps are usually traded in the markets with maturities much longer that the ones considered in the survey (30 years or more).

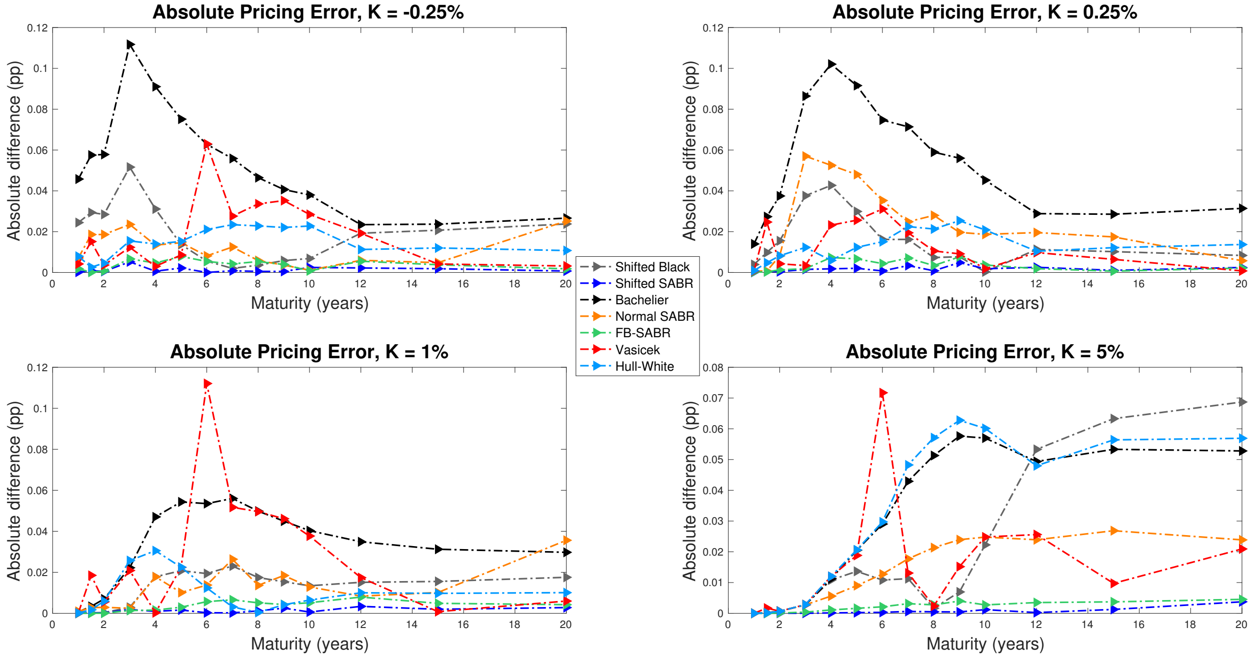

Figure 8.9: *In-sample caplets absolute pricing errors for every model and several representative strikes.*

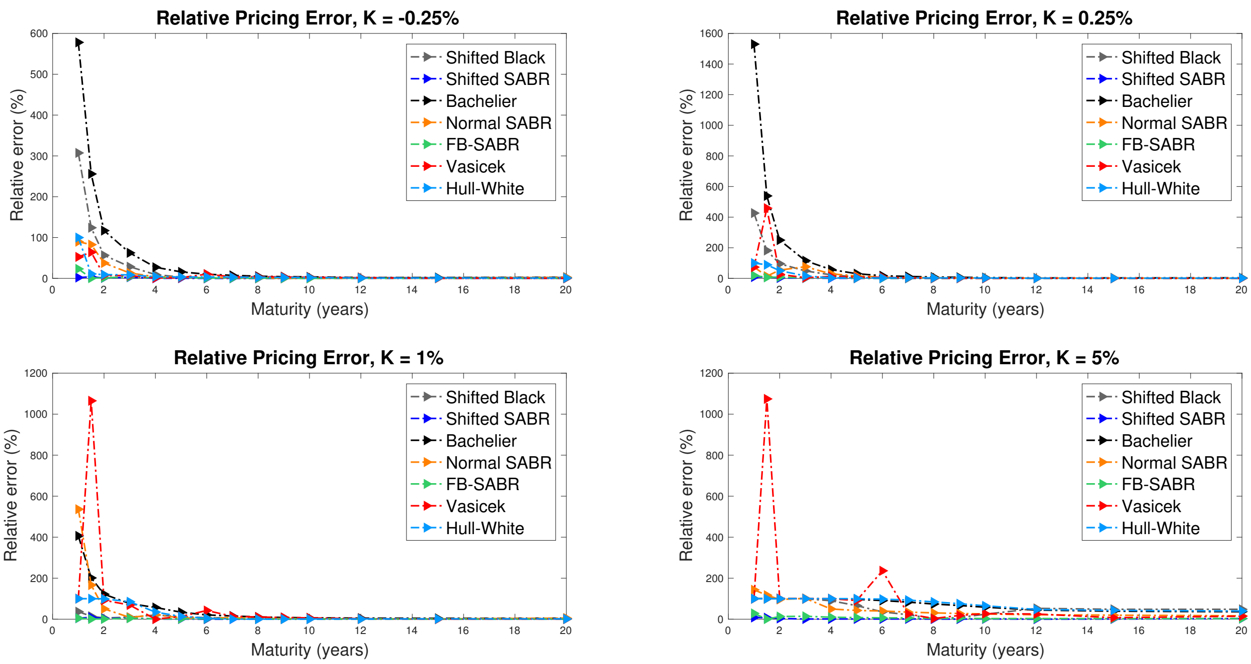

Figure 8.10: *In-sample caplets relative pricing errors for every model and several representative strikes.*

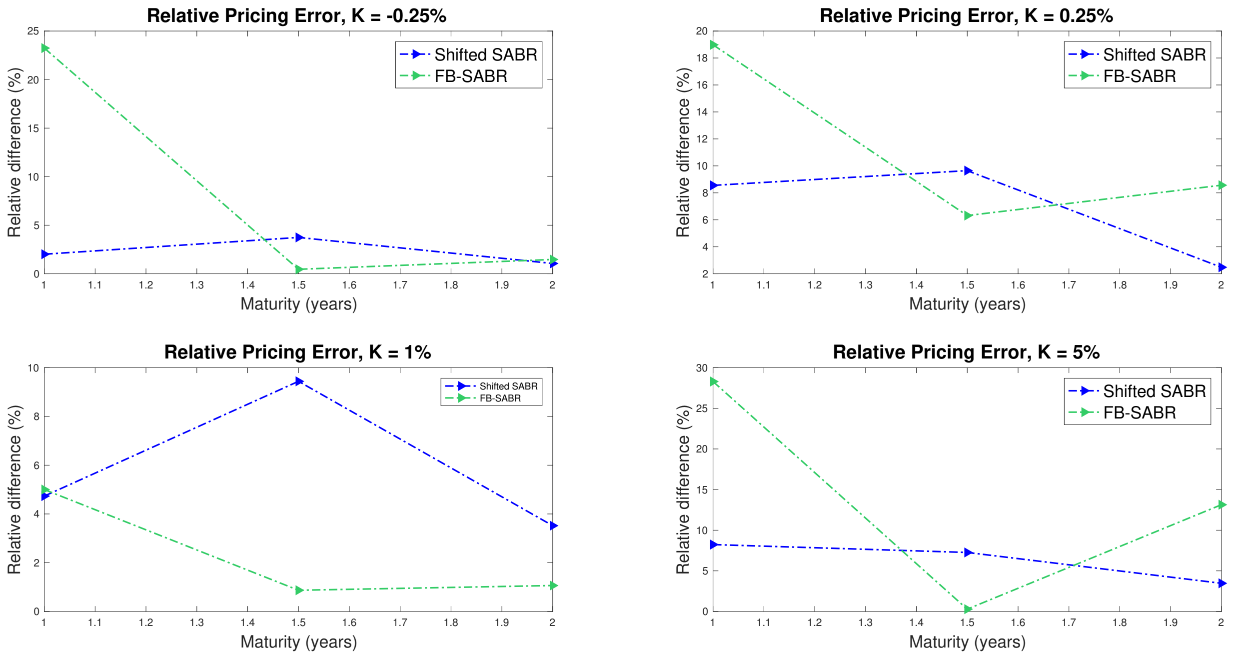

Figure 8.11: *In-sample comparison of shifted SABR and free boundary SABR models in terms of caplets relative pricing errors for the shortest maturities.*

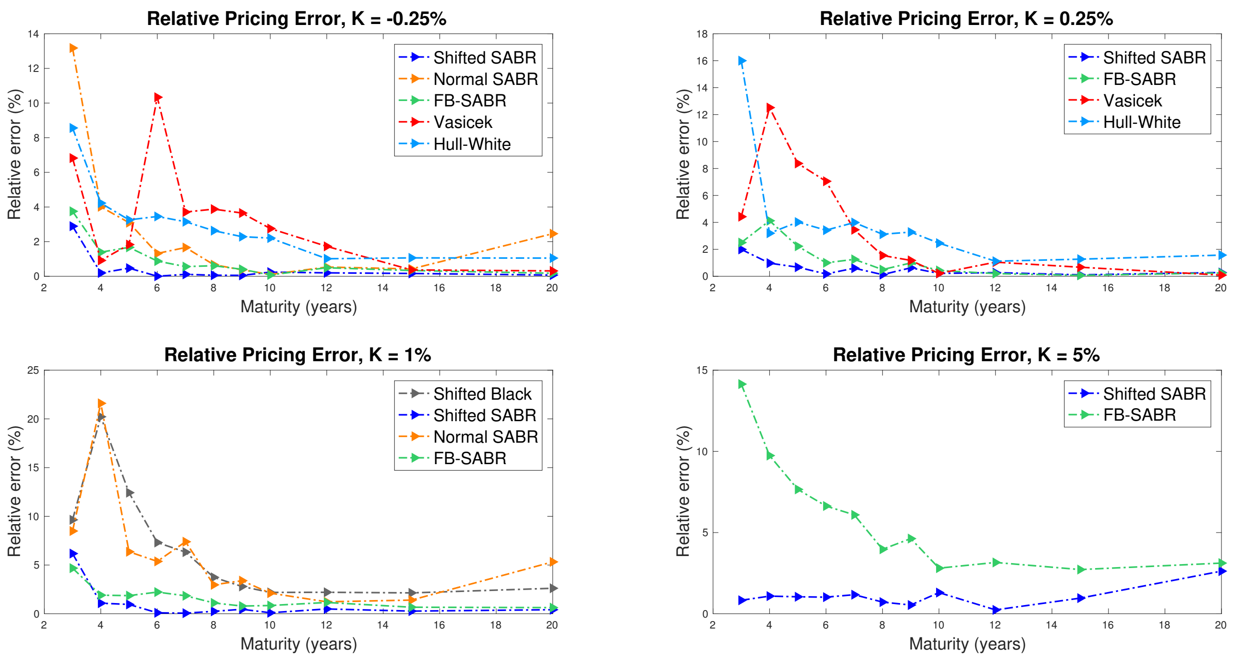

Figure 8.12: *In-sample caplets relative pricing errors for the best models of the longest maturities area for some representative strikes.*

T/K(%)	-0.75	-0.50	-0.25	-0.13	0.00	0.25	0.50	1.00	1.50	2.00	3.00	5.00	10.00
1Y	0.1103	0.0488	0.0080	0.0039	0.0022	0.0009	0.0005	0.0002	0.0001	5e-05	2e-05	5e-06	5e-07
18M	0.1270	0.0668	0.0237	0.0149	0.0099	0.0054	0.0034	0.0017	0.0011	0.0007	0.0004	0.0002	4e-05
2Y	0.1524	0.0964	0.0503	0.0366	0.0269	0.0163	0.0108	0.0060	0.0038	0.0026	0.0015	0.0007	0.0002
3Y	0.4028	0.2861	0.1807	0.1444	0.1152	0.0768	0.0535	0.0302	0.0187	0.0127	0.0068	0.0028	0.0006
4Y	0.5390	0.4334	0.3312	0.2877	0.2471	0.1853	0.1422	0.0881	0.0598	0.0431	0.0257	0.0120	0.0036
5Y	0.6677	0.5633	0.4673	0.4210	0.3760	0.3059	0.2471	0.1673	0.1152	0.0845	0.0480	0.0207	0.0051
6Y	0.8127	0.7092	0.6094	0.5620	0.5201	0.4410	0.3720	0.2651	0.1917	0.1391	0.0790	0.0304	0.0056
7Y	0.9536	0.8448	0.7442	0.6990	0.6506	0.5633	0.4902	0.3646	0.2740	0.2066	0.1223	0.0517	0.0111
8Y	1.0722	0.9662	0.8665	0.8210	0.7698	0.6827	0.6037	0.4666	0.3579	0.2725	0.1671	0.0678	0.0134
9Y	1.1734	1.0693	0.9647	0.9189	0.8648	0.7736	0.6893	0.5490	0.4247	0.3410	0.2049	0.0829	0.0143
10Y	1.2384	1.1363	1.0323	0.9868	0.9402	0.8472	0.7711	0.6132	0.4876	0.3757	0.2374	0.0920	0.0143
12Y	1.3039	1.2055	1.1112	1.0619	1.0176	0.9300	0.8401	0.6852	0.5579	0.4405	0.2720	0.1010	0.0111
15Y	1.3091	1.2172	1.1293	1.0879	1.0405	0.9572	0.8771	0.7255	0.5892	0.4871	0.3141	0.1311	0.0200
20Y	1.1843	1.1052	1.0244	0.9884	0.9454	0.8715	0.7999	0.6713	0.5654	0.4545	0.3123	0.1426	0.0268

Table 8.1: Caplet market prices (top left subfigure of figure (8.8). $N = 100$.)

In conclusion, every model fits market prices accurately in absolute terms, adhering to the *absence of arbitrage opportunities* implied by market prices. This feature changes considerably when relative errors are under concern, due to the extremely small values of the caplets to be fitted.[98] **Shifted SABR** and **free boundary SABR** systematically tend to outperform alternative competitors, with slight preference for the former. They are the only admissible models for the shortest maturities under study, and arise as really accurate models for the longest maturities as well. Far away from them in comparative terms, Hull-White, Vasicek and normal SABR rank reasonably well in the low strike area, although they fail when applied to higher strikes. One-parameter models (shifted Black and Bachelier) are hardly recommended.[99]

[98]Obviously, these relative errors would have been reduced if the comparison had been done between **cap market prices** (which are the instruments that *actually quote* within the markets) and **cap theoretical prices**, attained by aggregating (per model) the set of individual caplet prices for each cap outstanding (formulas (3.5), (4.5)). However, it should be noted that our main aim is not calibrating cap market prices *in the most accurate way*, but contrasting how several models relatively perform when pricing the instruments used in their respective calibration processes. It is sound to think that this *qualitative* ranking is conserved when the cap pricing problem is under concern with a *downward shift* in the scale of relative errors (since caps account for basket of caplets, and therefore permit mutual compensation of errors when aggregating). This issue is actually under research, and results are expected soon.

[99]Further discussion about the main characteristics of the parametric space in every model is highly interesting at this point of the survey. Although shifted SABR, free boundary SABR, normal SABR and Vasicek models account for the same number of parameters (three per maturity) to accommodate several smile shapes, they do not seem to do it *in the same way*. Shifted SABR and free boundary SABR fit market smiles accurately, resulting in precise caplet prices calibrations. Normal SABR is clearly outperformed by prior models, despite of accounting for the same number of parameters. In spite of having one additional degree of freedom, Vasicek model does not outperform systematically Hull-White's, and therefore Vasicek's might be overparameterizated, as suggested in appendix F of the Thesis. Shifted Black and Bachelier models are clearly surpassed by every alternative candidate. Even though fixing $\beta = 0.5$ does not allow us to conduct a standard *F-test* to show that both models are not parameterizated enough (since they are not strictly nested specifications of shifted SABR's when β is fixed to 0.5), it seems obvious that more general structures (such as shifted or free boundary SABR's) are needed for further accuracy.

8.5 Strike out-of-sampling

Through this section, several smile-fitting methodologies for **strike out-of-sampling** (interpolating in strike) are compared.

Firstly, figure (8.13) displays the conjunction of several models caplet prices term structures in comparison with benchmark market caplet curves when out-of-sampling some representative strikes of the survey. Again, a clear difference is observed between *low* and *high strike* areas. In the former, and excluding few minor divergences, every model resembles market curves consistently (in absolute terms). A higher variability is manifested for the $K = 5\%$ curve. Moreover, figure (8.13) strongly reminds of figure (8.8) bottom subfigures' shape. Therefore, it is claimed that the models outstanding **behave similarly** (at least in absolute terms) either when *in-sampling* or *out-of-sampling in strike*. Again, shifted and free boundary SABR are the only sound candidates for the $K = 5\%$ curve (lower prices).

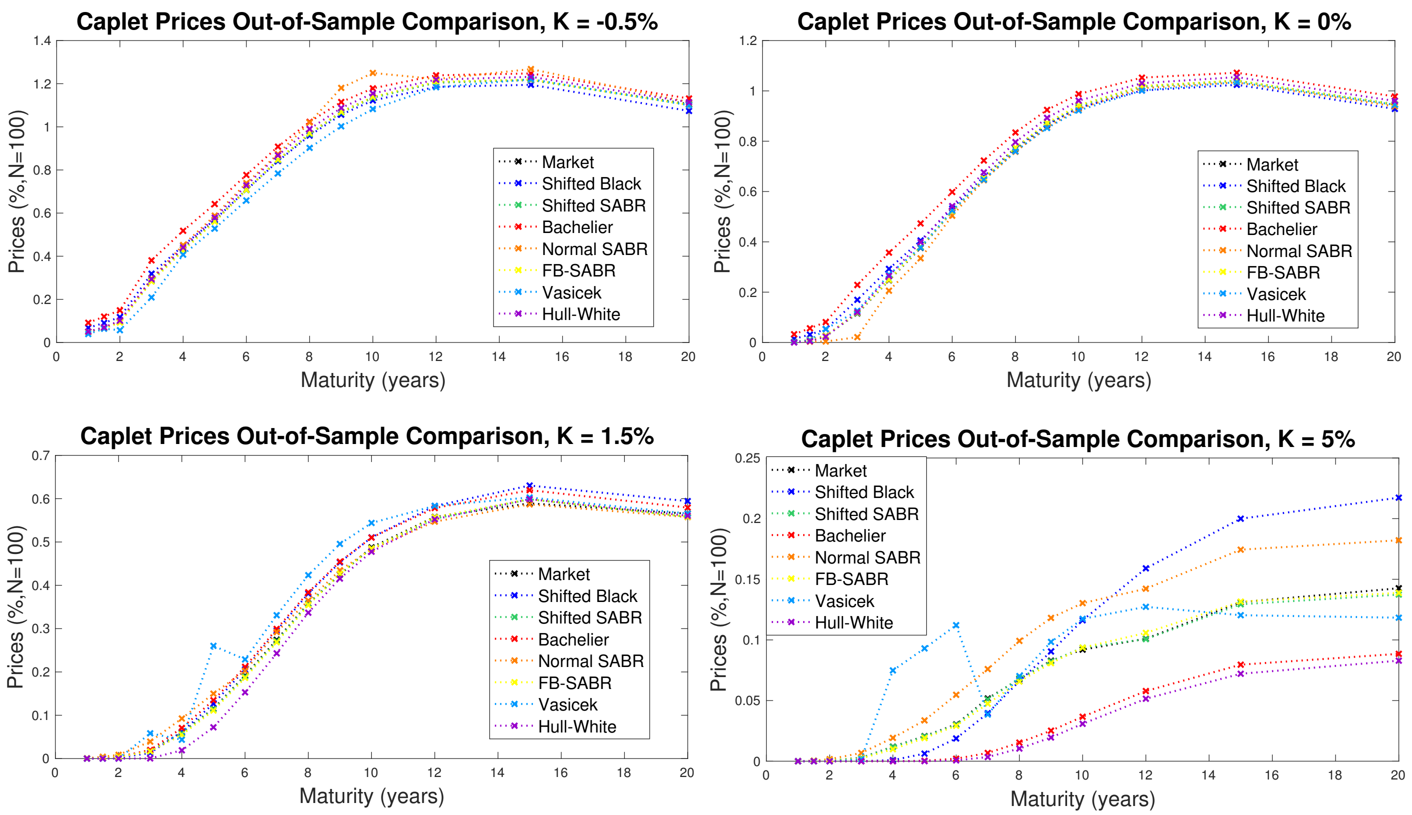

Figure 8.13: *Strike out-of-sample caplets pricing analysis. For some representative strikes, every model's pricing accuracy is tested against caplets market prices.*

In a similar trend, figures (8.14), (8.15), (8.16) and (8.17) below share their main features within their in-sample homologues (figures (8.9), (8.10), (8.11) and (8.12)). As shown in figure (8.14), again no huge (absolute) mistakes are committed. Shifted SABR and free boundary clearly outperform alternative competitors for every maturity under consideration in terms of accuracy (absolute pricing errors). In this case, only Hull-White and shifted Black arise as sound candidates for strike out-of-sampling in the low strike zone, failing when out-of-sampling the $K = 5\%$ strike. Normal SABR's accuracy depends on the pair (K, T) being considered[100], and Vasicek and Bachelier models are generally not recommended.

Once again, figure (8.15) manifests that, with exception of shifted SABR and free boundary SABR, every model fails for the shortest maturities for at least one of the strikes (in this case, being out-of-sampled). Figure (8.16) focuses on a deeper comparison of both models. It certifies the exceptional proficiency of shifted SABR's for fitting market smiles (even for the lowest prices, the relative error when out-of-sampling (interpolating) in strike never exceeds 10%). Free boundary SABR struggles to predict market prices accurately when extremely short maturities or high strikes are under consideration, being clearly outperformed by shifted SABR.

Figure (8.17) zooms in the longest maturities area, displaying only the best models when out-of-sampling some representative strikes. As in its in-sample homologue (8.12), shifted SABR and free boundary SABR's small relative errors rank them as the two best candidates. Hull-White, shifted Black and Vasicek arise as reasonable models for the low strike area, but their behaviour is not good enough for $K = 5\%$. Bachelier is the worst model among the ones being considered.

As shown through the section, the main features of in-sample's comparison are mostly maintained when out-of-sampling in strike. **Shifted SABR** and **free boundary SABR** clearly fulfil their function of fitting market smiles accurately, being also **the best models** for strike interpolations.

[100] At this point of the survey, a pertinent comment comparing shifted SABR with normal SABR model might be interesting. Fixing $\beta = 0.5$ in the overparameterizated general (shifted) SABR model **does not seem merely an aesthetic consideration**, attending to the results shown in figures (8.9), (8.10), (8.12), (8.14), (8.15), (8.17) or figures (8.19), (8.20) and (8.22) to come. In our scheme, normal SABR (which basically accounts for a different (arbitrary) choice of the parameter β) struggles to fit market caplet prices accurately, while shifted SABR does it precisely. Therefore, further research shall be conducted about the implied effects of choosing an arbitrary value for β. By the time on we follow the claim of [7], where an argument about the reason why markets adhere to the choice $\beta = 0.5$ is given. In any case, the results shown in figures (8.9), (8.10), (8.12), (8.14), (8.15), (8.17), (8.19), (8.20) and (8.22) support this claim.

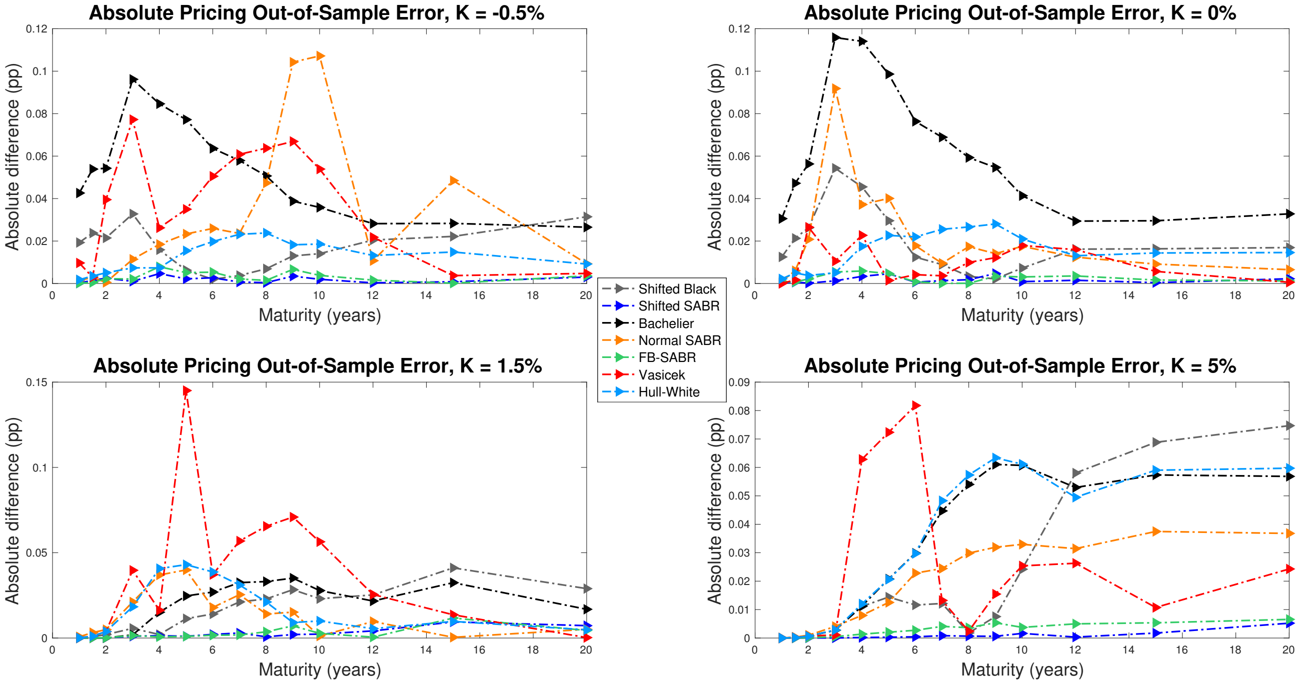

Figure 8.14: *Strike out-of-sample caplets absolute pricing errors of figure (8.13).*

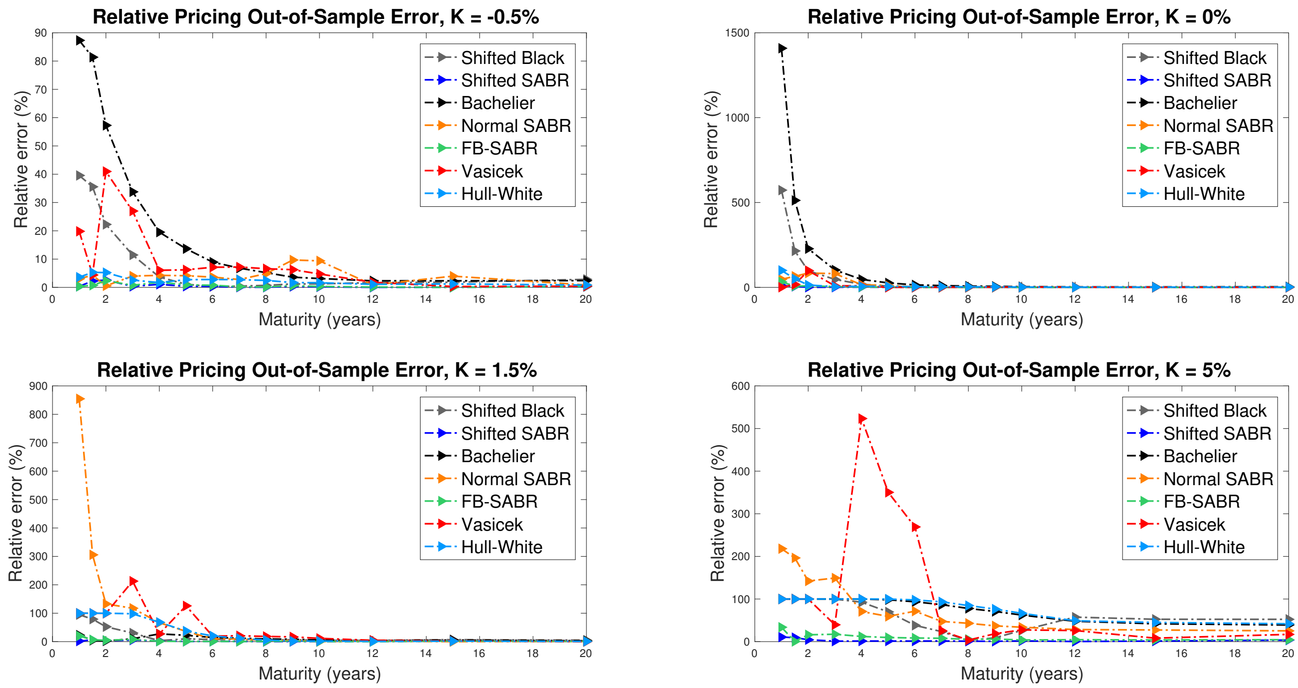

Figure 8.15: *Strike out-of-sample relative pricing errors of figure (8.13).*

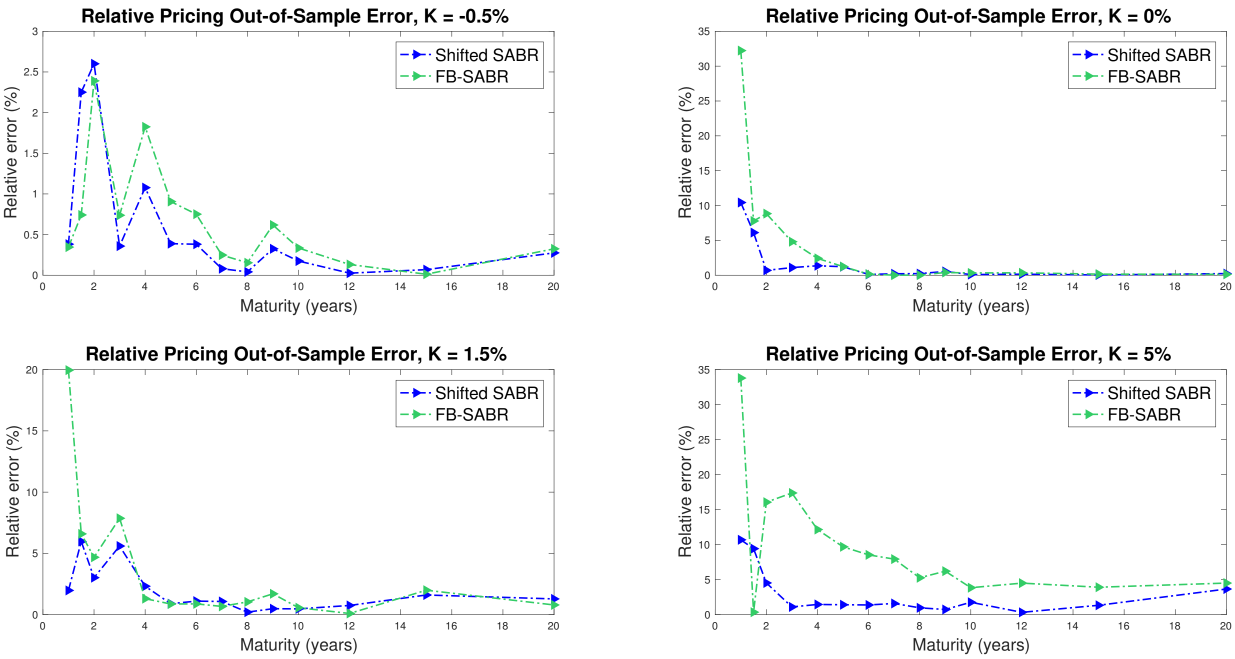

Figure 8.16: *Strike out-of-sample comparison of Shifted SABR and free boundary SABR models.*

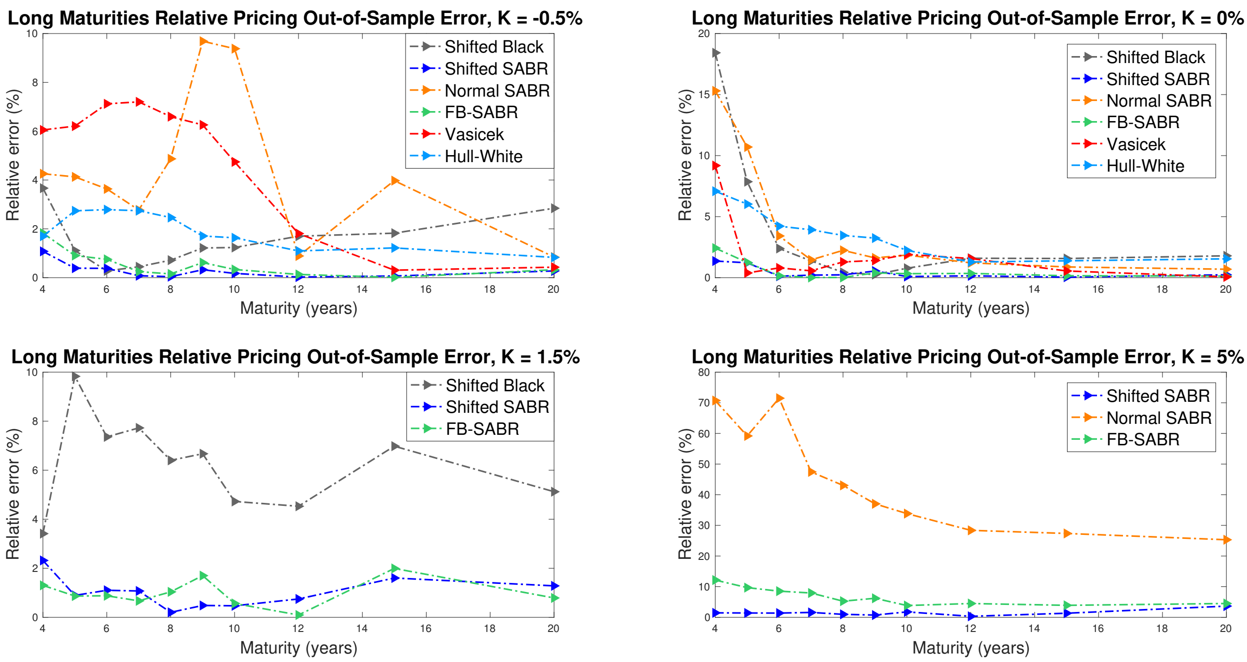

Figure 8.17: *Strike out-of-sample relative pricing errors for the best models for the longest maturities and some representative strikes.*

8.6 Maturity out-of-sampling

A thorough comparison between models' behaviour when forecasting caplets' prices for maturities removed from the calibration processes has been attained as well. Its main results are highlighted through this section.

Figure (8.18) below reproduces predicted caplet prices for some representative maturities of the survey[101], and compares them with caplet market prices data. Roughly speaking, it seems that every model fits market data *accurately*, although this precision worsens for closer-to-maturity caplets (prices drop). Again, both market prices curves and models curves *do not allow arbitrage*, since every pricing curve falls monotonically within the strike. Caplet prices tend to rise with time to maturity for any given strike, in consonance with the results shown in figure (8.8).

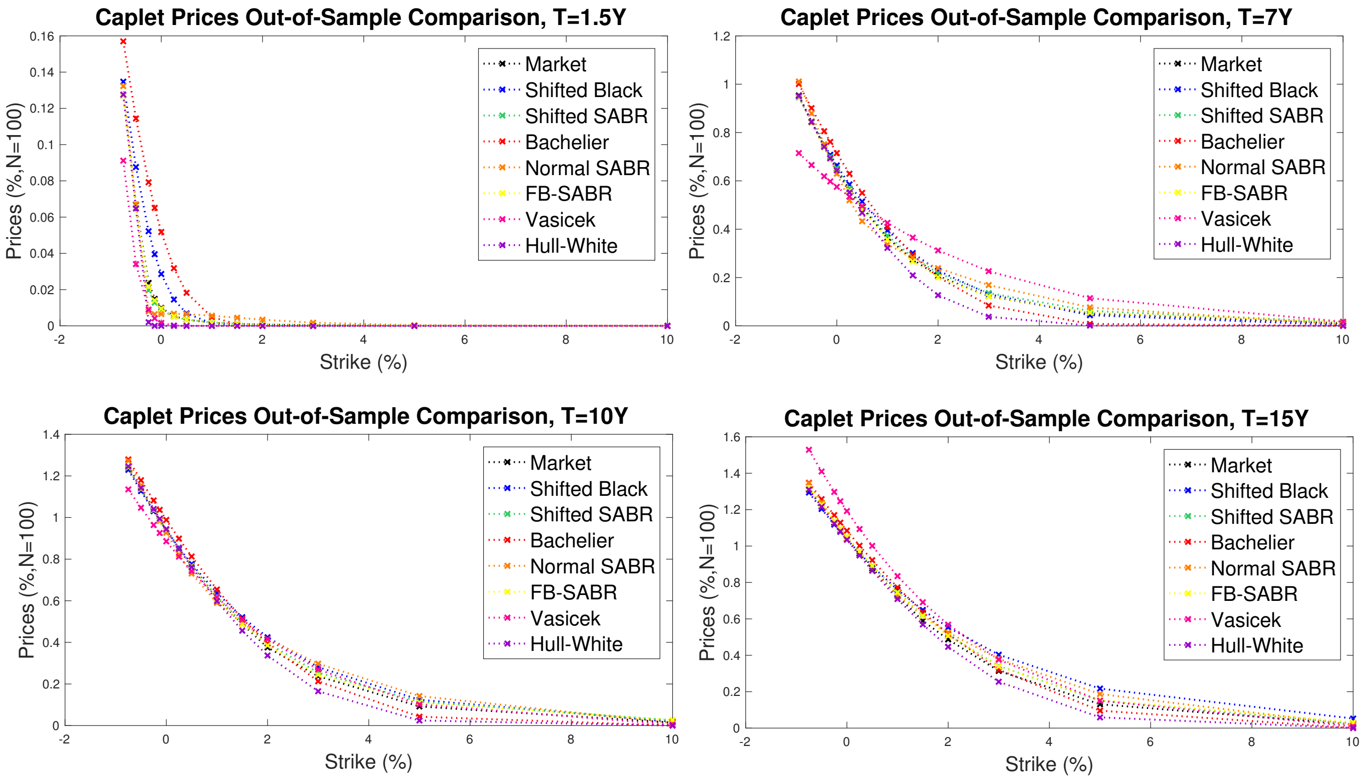

Figure 8.18: *Maturity out-of-sample caplets pricing analysis. For some representative maturities, every model's pricing accuracy is tested against caplets market prices.*

Figure (8.19) challenges every model of the survey by displaying absolute maturity out-of-sample pricing errors for some representative expiries. While *in-sample* and *strike out-of-sample* absolute pricing errors were close to each other's and did not exceed 0.15%

[101]Note that these figures resemble classic Black-Scholes calls' dependence on the strike.

(see figures (8.9) and (8.14)), maturity out-of-sample's might even attain 0.25% for Vasicek's model. The first difference, therefore, comes in terms of *scale* of the errors. By and large, (absolute) errors tend to decrease with the strike on the *high strike area* for every model outstanding (*low strike area* analysis is momentarily postponed). Although shifted SABR and free boundary SABR account for lesser absolute errors in average, they are outperformed for some particular combinations of (K, T). While Bachelier and Vasicek models are hardly recommended, Hull-White, normal SABR and shifted Black do not look unsound, especially in the *low strike area*.

Figure (8.20) transforms absolute pricing errors of figure (8.19) into relative's. Former shifted SABR and free boundary SABR's superiority over their competitors is somehow challenged within this figure, especially for free boundary SABR's (which is outperformed by several competitors for the highest strikes under consideration). However, since the *highest strike area* is not concerning nowadays (at least, not as much as the lowest strike's), this issue is not further analysed. Shifted SABR stands as the preferred approach in average. Except for shifted SABR and free boundary SABR, every model fails[102] for some representative maturity at a lower strike area (say, strikes below or equal to 5%).

Consequently, even though maturity out-of-sample relative pricing errors have significantly grown (especially for the $K = 10\%$ strike), we still consider that free boundary and especially shifted SABR models exhibit further *robustness* to the arbitrary combination (K, T), and therefore claim that they remain as our most accurate approaches for maturity out-of-sampling purposes. Figure (8.21) focuses on comparing both models' accuracy when interpolating caplet prices in maturity. It is plainly illustrated that free boundary SABR (typically) behaves worse than shifted SABR. Although high relative errors are committed in the *high strike area*, low strike area errors are acceptable when dealing with shifted SABR. In any case, notice that relative pricing errors for maturity out-of-sampling considerably surpass those attained either *in-sampling* (see figures (8.10), (8.11)) or *out-of-sampling in strike* (figures (8.15), (8.16)).

Lastly, figure (8.22) further analyses best models' behaviour (in terms of relative pricing errors) for each representative maturity in the *lowest strikes area*, since it is the one we care more about. For the shortest maturities ($T = 1.5Y$), only shifted SABR remains as an acceptable candidate. However, for longer expiries normal SABR, Hull-White, shifted Black and free boundary SABR emerge as sound alternatives. All of them, in conjunction with shifted SABR, typically account for relative errors lesser than 5%, which results in accurate predicted prices within this *low strike area*, irrespective of the model being considered.[103]

[102]Say that a failure occurs when relative error exceeds 30%.

[103]As usual, except for Vasicek and Bachelier models, which systematically account for higher relative pricing errors.

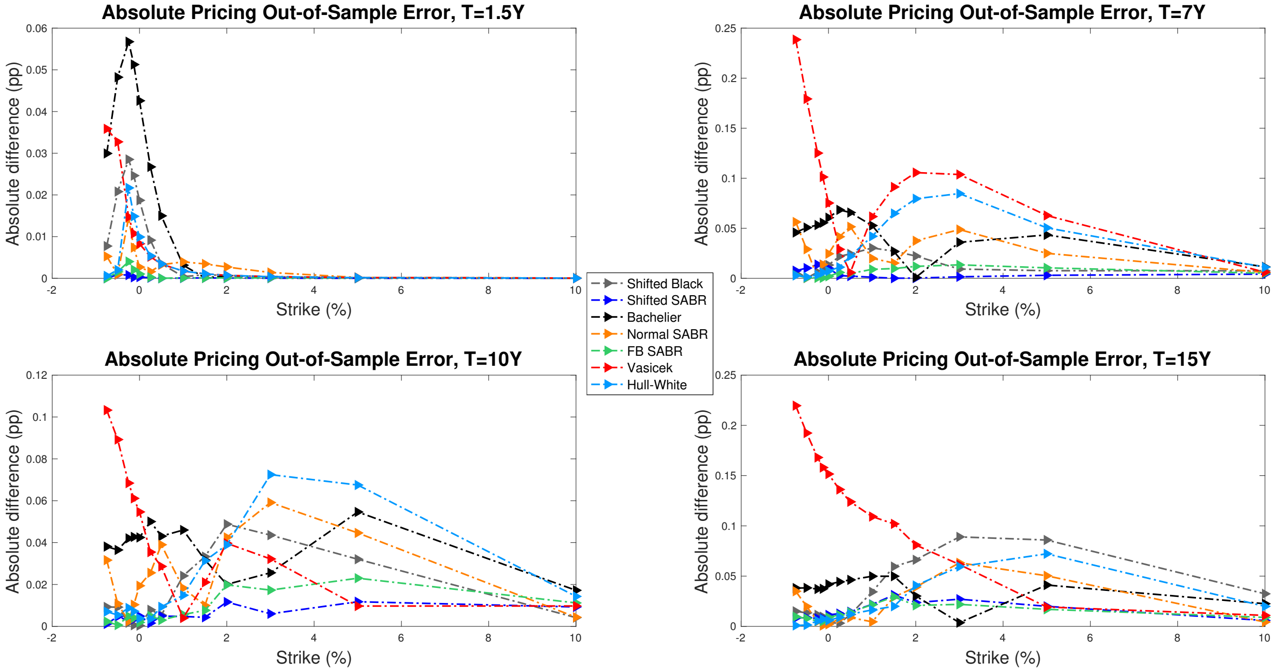

Figure 8.19: *Maturity out-of-sample absolute pricing errors for every model outstanding and some chosen maturities.*

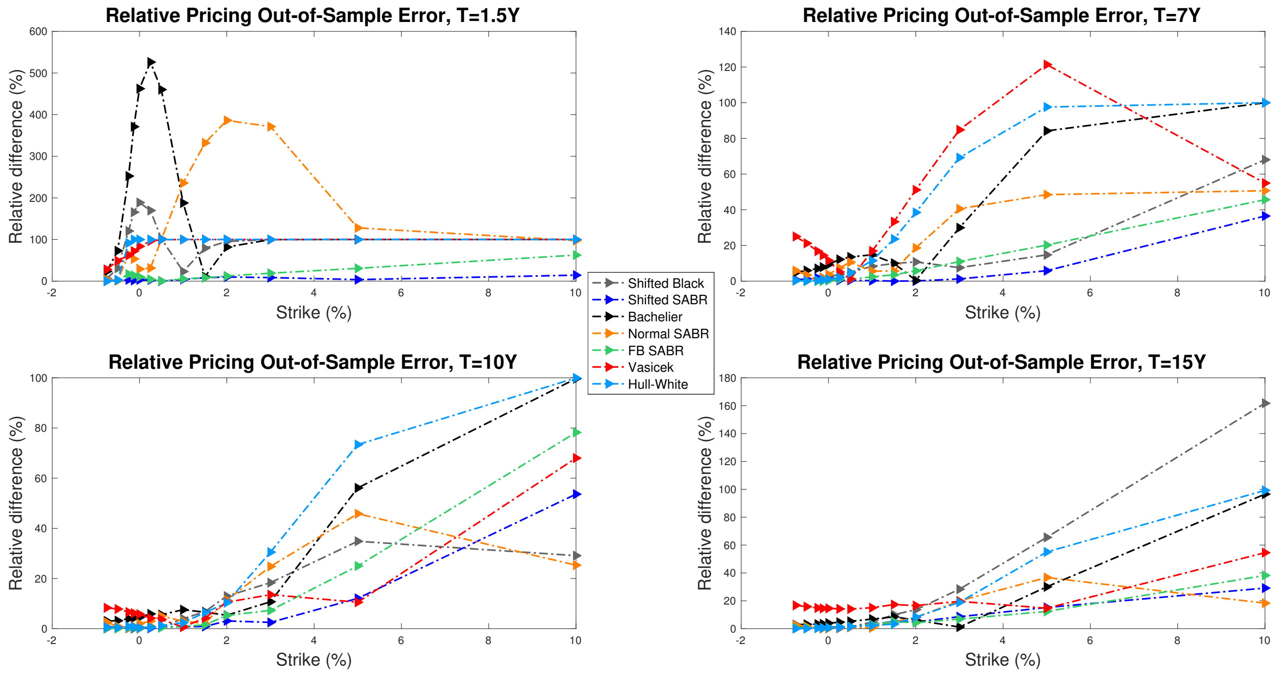

Figure 8.20: *Maturity out-of-sample relative pricing errors of figure (8.18).*

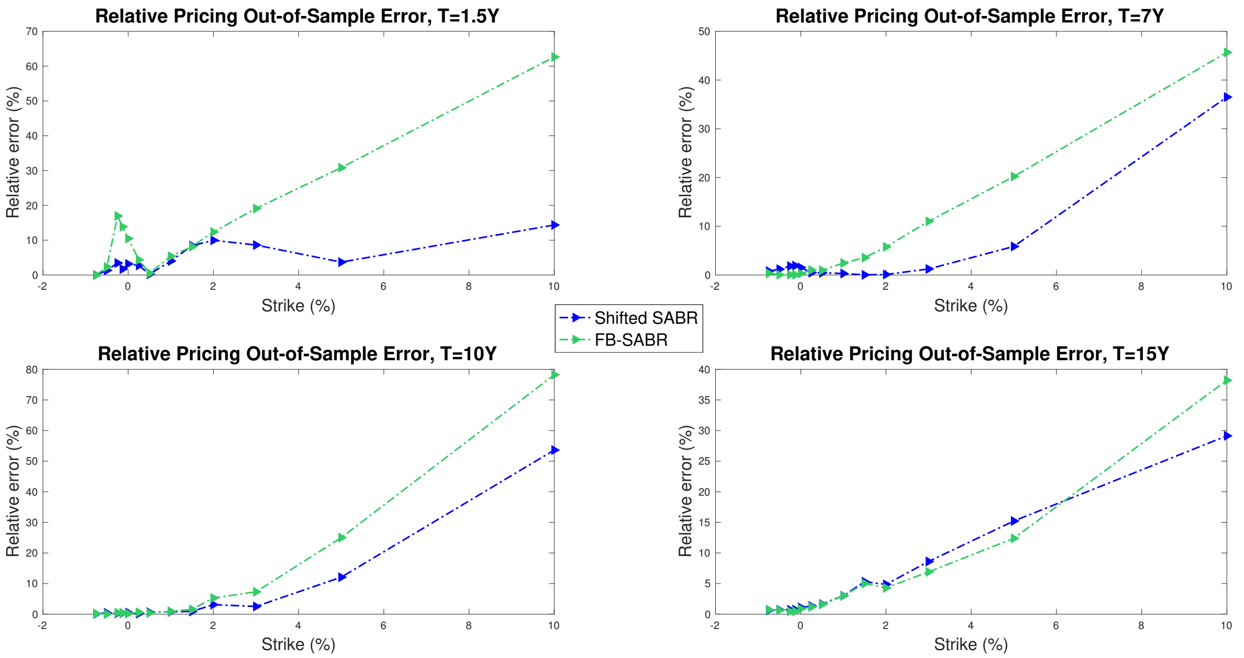

Figure 8.21: *Maturity out-of-sample comparison of Shifted SABR and free boundary SABR models.*

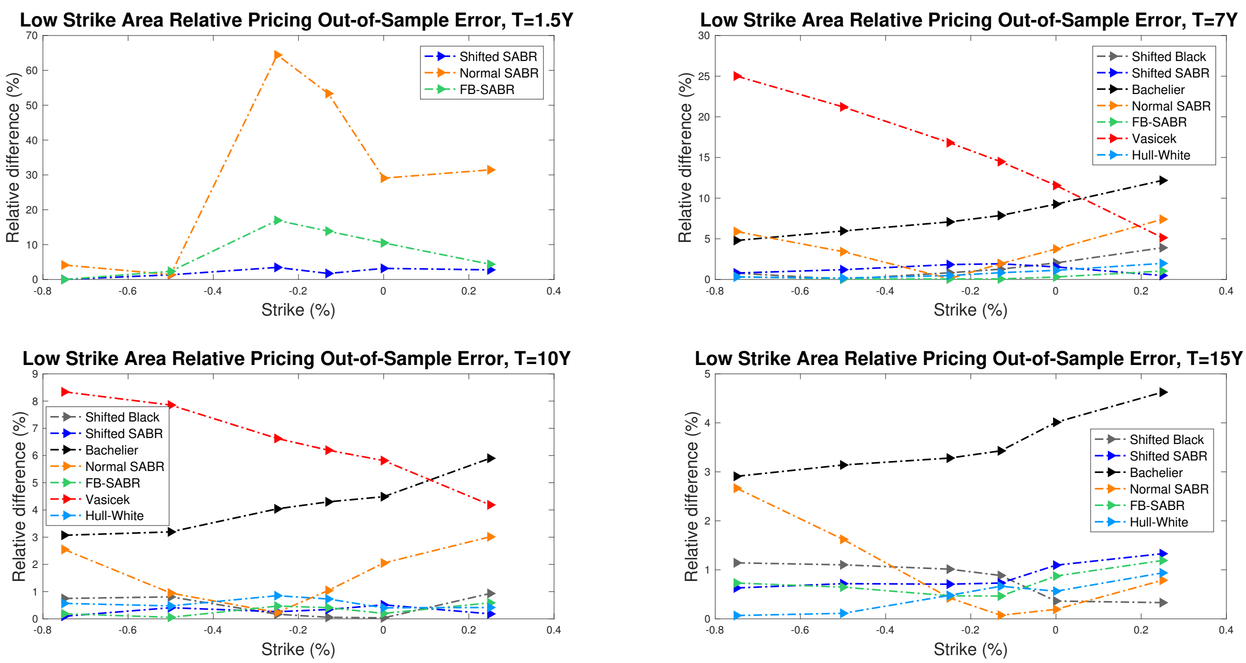

Figure 8.22: *Low strike area maturity out-of-sample relative pricing errors for the best models for each maturity under consideration.*

In conclusion, **out-of-sampling in maturity** mostly reproduces the results attained by either *in-sampling* or *out-of-sampling in strike*, with a noticeable shift in the magnitude of the errors being committed. We believe that this issue is due to *the nature of the models under research.*[104] Arbitrage opportunities do not arise, and the ranking of models depends on the *strike area* being considered. While maturity out-of-sampling in **high strike areas** results in high relative errors for every model outstanding (and shifted SABR arises as the more sound approach), **low strike area** accounts for lesser relative errors (below 5%) for any model outstanding (except for Vasicek and Bachelier models).

8.7 Testing the cube

This section analyses two clarifying examples[105] illustrating the accuracy of our proposed transfer algorithm for fast calibration of the volatility cube via no-arbitrage considerations. The hypothesis of [34] on *Transferring the smile* technique (see appendix C) is subsequently tested.

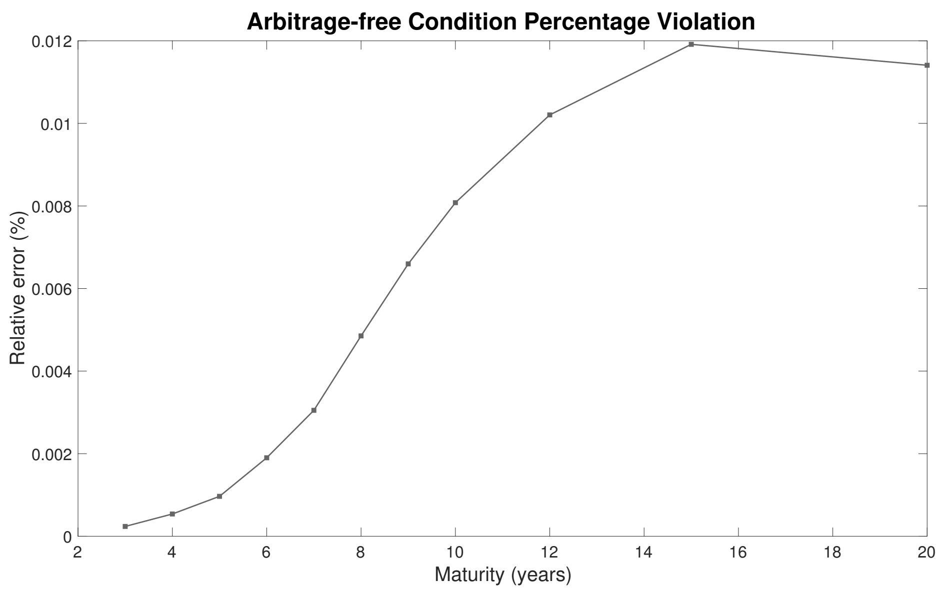

Figure 8.23: *Arbitrage-free relationship percentage violation.*

Firstly, to guarantee that arbitrage-free conditions *indeed apply* within our market

[104]Notice that these are basically designed for further accuracy in smile-fitting procedure. Interpolation in maturity has been conducted via piecewise constant hypothesis, which obviously attends for less precision. Moreover, if a maturity is removed for out-of-sampling purposes, *a full model* is being eliminated from the calibration (since we calibrate a different model for each maturity), and we are making the assumption that previous-maturity implied volatility applies for the maturity under concern. This assumption looks strong, and the results attained for maturity out-of-sampling support this asseveration.

[105]In our study, we consider both extrapolating *from a shorter to a longer tenor* as going *the other way round*. Both shifted Black (formulas (5.11), (5.13)) and Bachelier ((5.12), (5.14)) volatilities are included in the research (as usual, the last ones are attained via unique-price hypothesis.)

data, *relative error* of formula (5.1)[106] has been computed for every maturity outstanding (as forward rates depend on the maturity being considered). Figure (8.23) above manifests that the arbitrage-free condition applies within our data, since relative errors do not exceed 0.012% for any maturity under consideration.

Figure (8.24) is splitted into two relevant sets of subfigures. Top row displays several market caplet volatility term structures[107] for both Bachelier/shifted Black quoting volatilities and both non-standard tenors (3/12 months) to be extrapolated from our already calibrated (standard tenor) market term structures (see figure (8.1)). Bottom row converts these volatilities into prices term structures by the use of standard (shifted) Black (3.3)/ Bachelier (4.3) pricers. Tenor splitting phenomenon is clearly observed, both in volatilities and in prices. These are shifted upwards when a higher tenor is considered, going from 0.7% to levels above 3% when the tenor rises from 3 to 12 months. Except for some minor differences[108], volatility term structures mostly follow the behaviour described in figure (8.1). Market implied prices are arbitrage-free, and basically adhere to the same trend followed by their 6-month tenor homologues in top-left subfigure of figure (8.8).

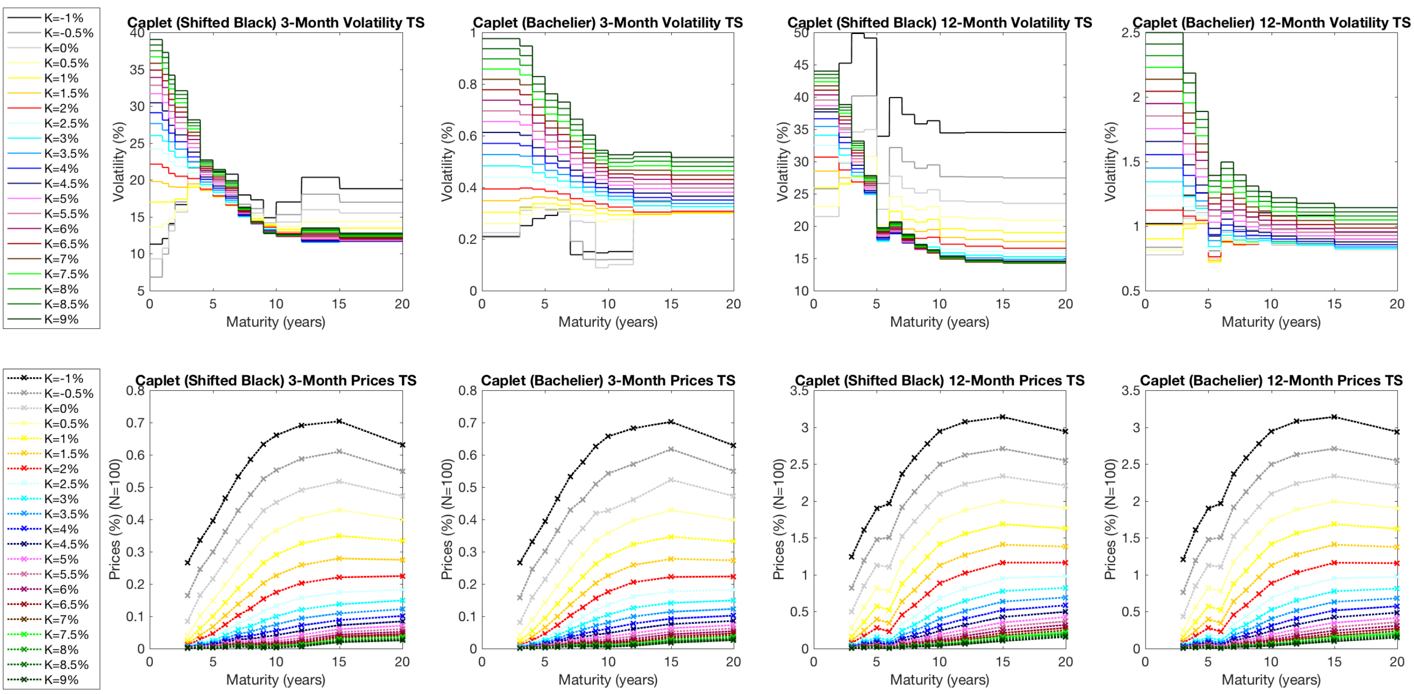

Figure 8.24: *Top row shows the stripped 3/12 month market volatility term structures. Bottom row converts these implied volatilities into caplet prices via formulae (3.3) or (4.3) .*

[106] I.e., $abs(LHS - RHS)/RHS(\%)$.

[107] As usual, these have been extracted from implied flat cap volatilities via caplet stripping algorithm.

[108] For instance, volatilities corresponding to negative strikes are not strictly inversely sorted for the shortest maturities and 12-month tenor shifted Black volatilities do not exhibit a mean-reverting behaviour for the longest expiries.

Figures (8.25) and (8.26) respectively compare the extrapolated 12-month/3-month caplet prices term structures with actual market data, both for shifted Black (top rows) and Bachelier (bottom rows) quoting volatilities. From left to right, we show: market prices term structure, extrapolated (arbitrage-free) prices term structure, absolute and relative error of the extrapolation.

As shown in the figures, the prices have been considerably shifted upwards/downwards from the 6-month prices within our benchmark tenor (top-left subfigure of figure (8.8)), accommodating *smooth* and *precisely* the shape of market prices actual data (irrespective of the nature of the quoting volatility being considered). In consonance with the nature of the method, no arbitrage has been introduced in the pricing scheme (different strike curves do not cross each other, with prices sorted in descending strike). Although tenor splitting seems to be consistently modelled, it is likely that some kind of *bias* exists within our approach, since we recover prices that systematically lie *slightly below market data* for the longer tenor (6 months to 12 months) extrapolation. This issue does not appear so explicitly for the shorter tenor extrapolation, where no dependence on the strike being extrapolated is detected and the errors appear to exhibit a *white noise* structure. This systematic bias is likely to be considerably reduced when the *forward correlation issue* is treated in a more consistent way.

As regards absolute pricing errors, these are significantly lower for the shorter extrapolation methodology (where they hardly reach a value of 0.02% for a standard notional of $N = 100$, and appear to manifest a white noise structure with no dependence on strike or maturity) than for the longer tenor's, which accounts for systematic errors of 0.35%. Moreover, these are somehow sorted in descending strike, and seem robust to the nature of the quoting volatility (which supports the idea of existence of a slight bias in the estimation). However, last column on both figures gives grounds for optimism (recalling, again, that market caplet prices lie in the ranges provided by table (8.1), and therefore low relative errors are hardly attainable).

Roughly speaking, Bachelier errors tend to be lesser than shifted Black's, for both extrapolation processes. In the longer tenor extrapolation, only the combination of high strikes and short maturities (which is irrelevant for standard quoting instruments nowadays) results in unacceptable relative errors. As soon as the strike drops into the low strike area and the maturity is above $T = 5Y$, the relative pricing errors do not exceed 10%, including some particular well-fitted strike curves. These errors are relatively comparable with the ones obtained both in-sampling and out-of-sampling in strike, and fairly better than the ones recovered from out-of-sampling in maturity.

As regards 3-month tenor extrapolation, the situation becomes even better. Relative pricing errors do not exceed 20% for any maturity or strike under study, and typically fluctuate randomly below the 10% barrier. The ideal situation is attained when the longest maturities and the lowest strikes are considered, since errors range from 0% to 5% irrespective of the nature of the quoting volatility. Moreover, the negative strike area ($K = -1\%$) is being fitted with lesser than 2% relative error for any maturity outstanding.

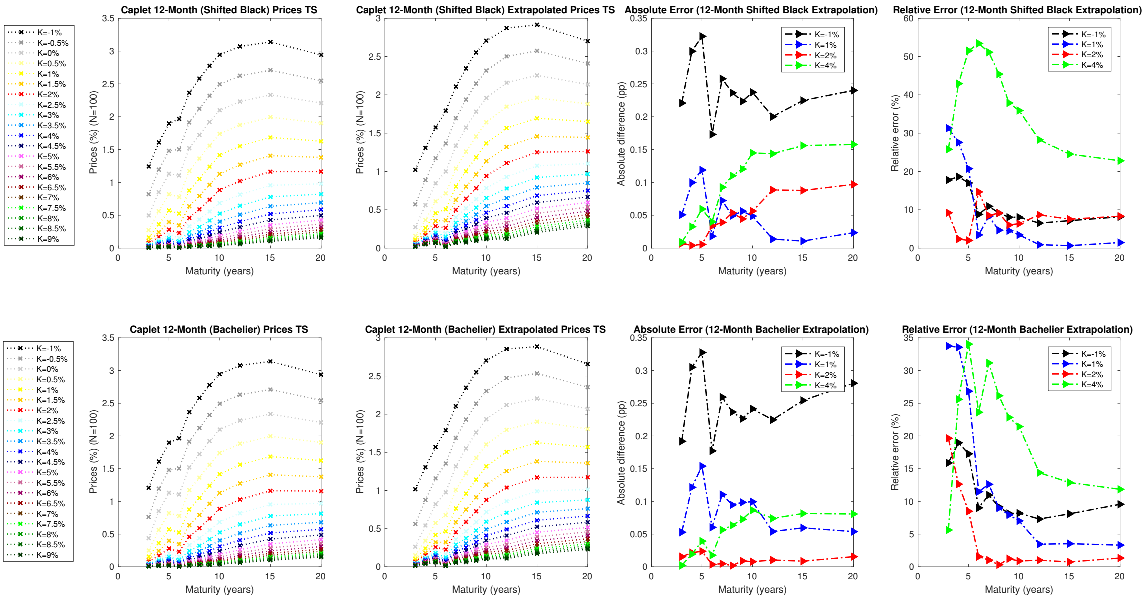

Figure 8.25: *Longer tenor (12 month) extrapolation from the 6 month implied volatility surface for both shifted Black (top row) and Bachelier (bottom row) quoting volatilities. Absolute and relative errors analysis.*

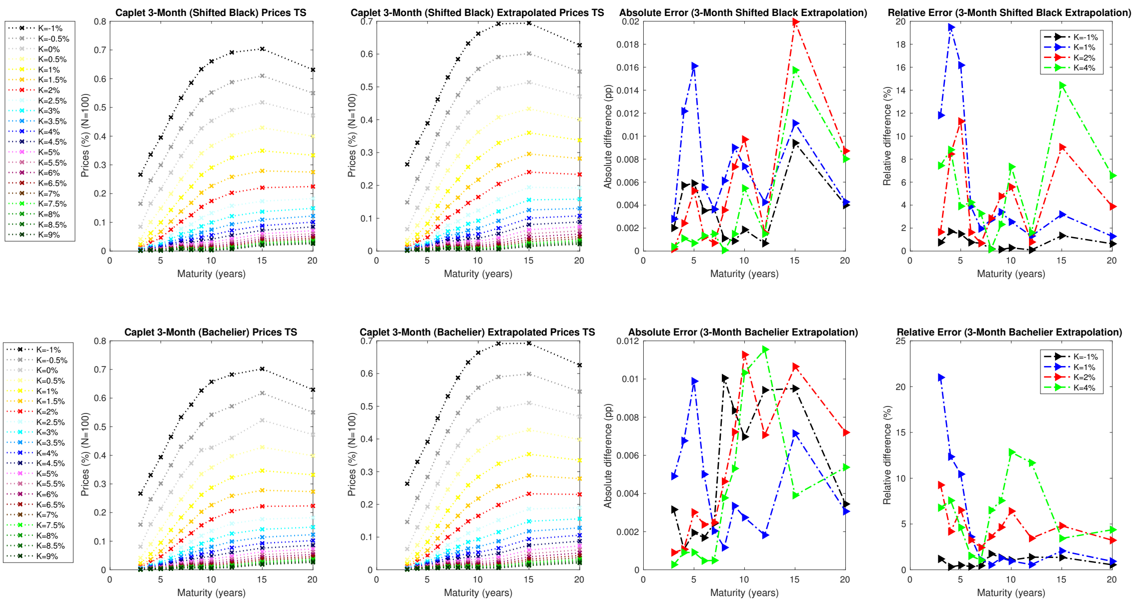

Figure 8.26: *Shorter tenor (3 month) extrapolation from the 6 month implied volatility surface for both shifted Black (top row) and Bachelier (bottom row) quoting volatilities. Absolute and relative errors analysis.*

The rest of the section is devoted to test the main hypothesis of one of market's standard methodologies when the conversion of volatilities among tenors is under concern: *transferring the smile* technique of [34] (see appendix C). Figure (8.27) displays the shape of several fitted smiles (either using 6-month tenor standard data (volatilities from figure (8.1)) or 3/12-month non-standard data (figure (8.24)) via shifted SABR for every maturity outstanding. Comparing by rows, it is clear that markets **do not support** this hypothesis, since although smiles shapes respect to moneyness are somehow maintained among 3-6 month tenors, there exists a break in these shapes for every maturity outstanding when the 12-month tenor structure arises. Figure (8.28) inquires further on this aspect, by testing one of the main conclusions implied by the smile-shape conservation assumption: parameters α, ρ are conserved when the tenor is modified *ceteris paribus*. Parameters term structures are plotted for every tenor under concern, and the **rejection of this assumption** by market data is evident, especially when the 12-month tenor is under consideration. We believe that market data do not support the technique given in [34], and therefore claim that an alternative methodology (such as our full no-arbitrage proposal) shall be applied.

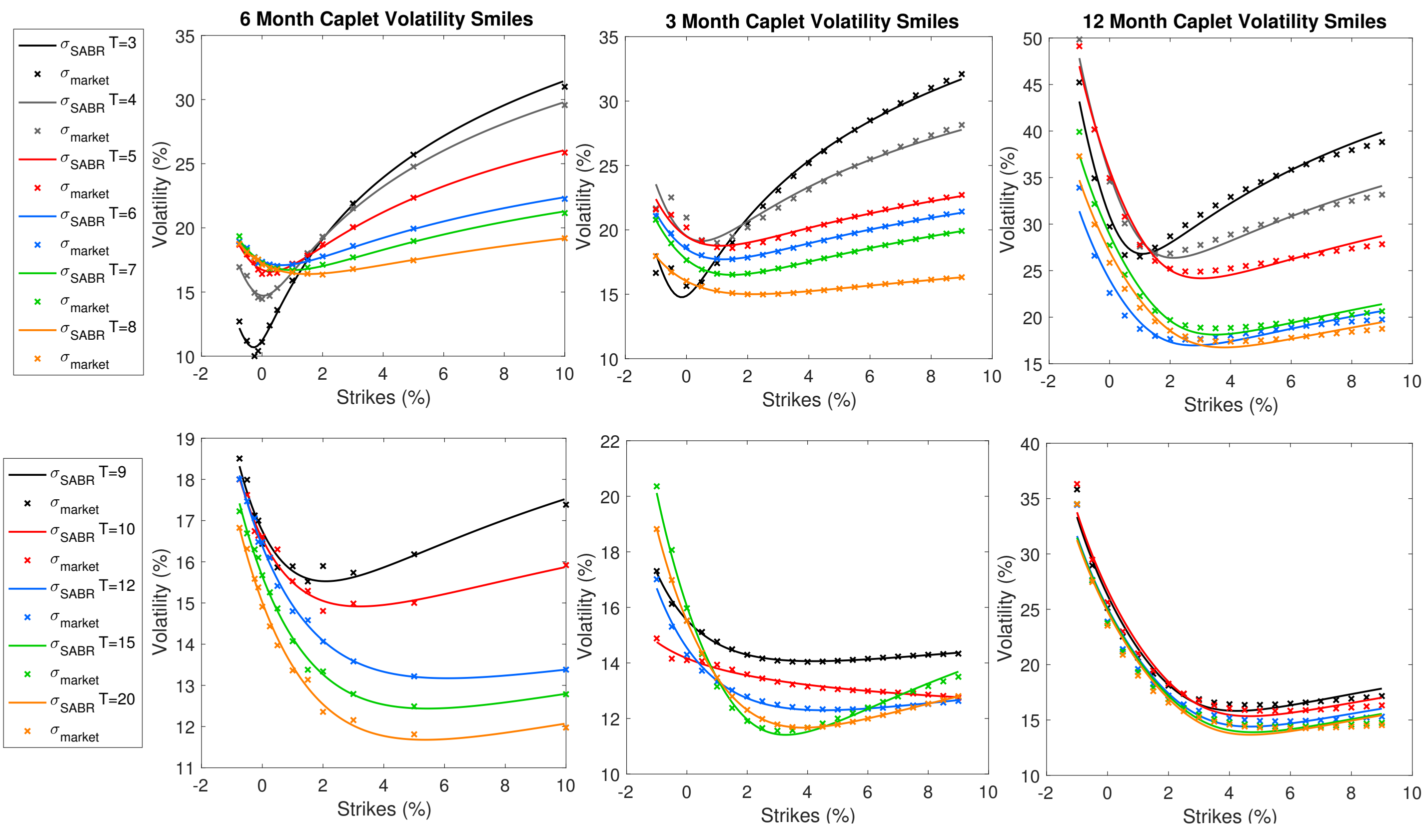

Figure 8.27: *Non-robustness of* ***transferring the smile*** *technique (see appendix C). Smile shape changes when the tenor is modified either to longer or shorter investment periods, for any maturity under consideration.*

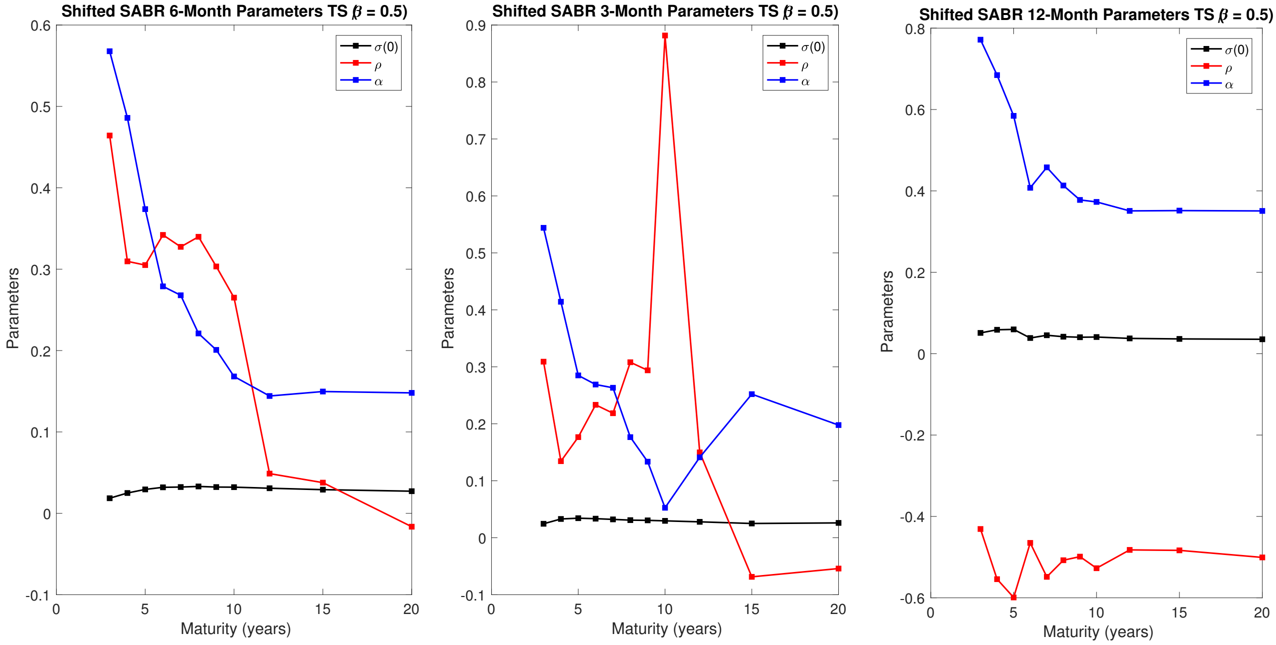

Figure 8.28: *Shifted SABR parameters term structures dynamic dependence on the calibrated tenor.*

To sum up, during this section it has been manifested that standard *transferring the smile* technique struggles to reproduce market's behaviour consistently due to its strong smile-shape conservation assumption. Given that arbitrage-free condition applies within our data, we have exploited it to propose a new full arbitrage-free scheme for calibrating the volatility cube based on previous work by [2] and especially [34], with some outstanding results (if a choice was possible, and basing on our empirical research, we recommend to extrapolate from longer to shorter tenor via Bachelier quoting volatilities). We consider that this method is quite promising, since it still has strong room for improvement within the *correlating the forward rates* issue (see appendix G). Moreover, some other (possible) sources of uncertainty have been identified within the calibration process.[109]

[109] For instance:

- Data transformation. The datasets provided as non-standard market data have suffered previous transformations from original caps' prices from *IHS Markit Totem* report.
- Different brokers have been used for models' calibration (*ICAP's quoting data*) and cube's extrapolation (*IHS Markit Totem report*). Also, the liquidity of OTC caps has not been checked.
- No data of the tenor to be extrapolated has been used, to *replicate market's worst possible situation*. Therefore, fixing $\rho = 0.9$ seems too arbitrary. As explained in appendix G, using market data when extrapolating may help to fit prices more accurately, interpreting ρ as a free parameter of the extrapolation process. This issue is currently under research.

Conclusion

This MSc Thesis aims to provide a common reference framework in which several interest rates derivatives pricing methodologies are challenged and compared under the new negative rates context. After a full revision of analytical pricing formulae implied by every model outstanding, these have been compared in terms of *accuracy* and *smoothness* of resulting caplet prices term structures. Every model is arbitrage-free and fits market pricing curves reasonably well, but not all of them are acceptable when absolute/relative errors are under concern. Shifted SABR and free boundary SABR clearly outperform alternative competitors, with strict preference for the former. This result is in consonance with the industry usual approach, supporting its choice. The model performs outstandingly for both in-sample and strike out-of-sample analysis, but its accuracy worsens when maturity out-of-sampling is considered.

The new fully arbitrage-free methodology for completion of the volatility cube has then been tested with non-standard OTC volatilities, and compared with a well-established technique such as *transferring the smile*. The results are quite promising (especially for the currently observed low rates situation). We understand that there is still strong room for improvement within the method.

We state, then, that both strike and tenor inter/extrapolations (via shifted SABR's smile fitting or arbitrage-free considerations) are promising methodologies when completing the volatility cube in a consistent way. Maturity interpolation standard approach (piecewise constant hypothesis) should clearly grow in complexity, since it is not accurate enough for industry's standard requirements.

Further research

In an extensive but not exhaustive list, the following topics are either under current research or left for future study:

1. Empirical research on the influence of the *shift parameter* s in the process of calibrating, pricing and hedging within a shifted SABR framework. ICAP's standard choice $s = 3\%$ is to be fully reviewed.

2. Some other important models (such as *Ho-Lee*, *HJM*, numerical version of *Hull-White* and many others) should be introduced in the survey in a consistent way with the previous exposition. These models are calibrated *numerically* by standard *tree approaches*. We aim, then, to open the survey to *non-analytical* models.

3. *European swaptions* should be included in the survey for the sake of completeness and comparability within the cap/caplets framework.

4. Complete *in-sample/out-of-sample* analysis on the accuracy of the proposal method for full completion of the volatility cube for both quoting volatilities outstanding. This study should include insights about the *smoothness* and *continuity* of the resulting output pricing four-dimensional structure.

5. Empirical research about the influence of the chosen functional form in maturity of the caplet implied volatility in the stripping process over the results of the stripping, subsequent calibrations and models' performances. Piecewise constant hypothesis does not seem accurate enough and a growth in complexity is required.

6. Empirical comparison of the three methods proposed in appendix G to compute parameter ρ within our *completing the cube* framework. More parameters could be included in the free-arbitrage extrapolation method (within ρ), resulting in a more complex methodology designed to gain further accuracy.

7. Conversion of the whole caplets pricing analysis into caps'. Relative errors are expected to be reduced then (since caps account for basket of caplets, and are therefore more expensive and permit mutual compensation of errors when aggregating). Moreover, caps are the instruments that *actually* quote within the markets (in form of flat implied volatilities), so the analysis would be of higher interest for the industry.

8. Empirical (further) research on the influence of the choice of parameter β within subsequent results. By the moment, we cannot claim that this choice *mainly attends to aesthetical reasons*, as stated previously by some authors (see [21], for instance).

9. Empirical contrast on *the influence of the calibration method* chosen for analytical version of Hull-White/Vasicek models (either doing it directly via caps' flat volatilities or pre-attaining caplets' volatilities via stripping algorithm and calibrating with caplets). Hopefully, the choice of the calibration method should not overly affect calibration results.

10. Inclusion of market's standard *multicurve framework* for decoupling forwarding and discounting in the sake of further accuracy.

11. Time-dependent extensions of the SABR model (such as the *SABR-LIBOR market model* of [14]) should be included in the survey, in the spirit of greater depth.

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Appendix A

Local volatility predicts the wrong dynamics of the volatility smile

For simplicity, consider the special case where *local volatility* σ_{loc} only depends on the current forward rate $F(t)$[110]:

$$dF = \sigma_{loc}(F)F\,dW(t), F(0) = f. \tag{A.1}$$

The authors had previously shown (see [41]) by singular perturbation methods that European call and put prices are given by Black's model with the implied (Black) volatility:

$$\sigma(K, f) = \sigma_{loc}\left(\frac{1}{2}|f + K|\right)\left\{1 + ...\right\} \tag{A.2}$$

in this particular local volatility specification, where the dots account for negligible higher approximation orders. Suppose that the forward price today was f_0, with an (observed) implied volatility-curve $\sigma^0(K, f_0)$. The calibration of the model to these market data forces the local volatility to be:

$$\sigma_{loc}(F) = \sigma^0(2F - f_0, f_0)\left\{1 + ...\right\} \tag{A.3}$$

for every forward rate F under consideration. Once the model is calibrated to market data, it is turn to examine its predictions. Assume that today's forward rate changed from f_0 to some new value f. Using (A.2) and (A.3), model's prediction for the new implied (Black) volatility curve reads as

$$\sigma(K, f) = \sigma^0(K + f - f_0, f_0)\left\{1 + ...\right\} \tag{A.4}$$

[110]i.e., its dependence on the calendar time is embedded into the forward rate dependence. There is no explicit dependence on the time. $F(t) := F$ for shorthand notation.

for an option with strike K for a given current forward rate f. Concretely, if the forward rate f_0 *increases* to f, the implied volatility curve moves *to the left*. If f_0 *decreases* to f, the curve moves *to the right*. Therefore, the prediction of local volatility models is clear: the smile/skew moves in the opposite direction to the price of the underlying asset. Figure (A.1) below illustrates this phenomenon. A theoretical perfect smile $\sigma^0(K, f_0) = \alpha + \beta(K - f_0)^2$ (*black line*: $\alpha = 0.2$, $\beta = 100$, $f_0 = -0.5\%$) is shifted to the left when f_0 grows to $f = 0\%$, and to the right when it drops to $f = -1\%$. Formula (A.4) has been applied in both cases to compute the new implied (Black) volatility curve (red and blue lines respectively). This hypothesis is invalidated by typical market behaviour, in which smiles and skews move in the *same direction* as the underlying.

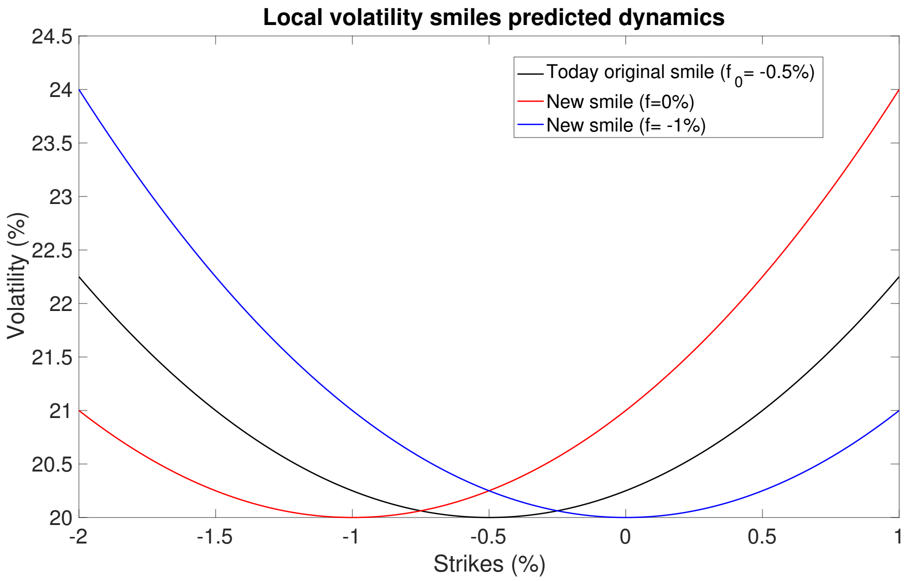

Figure A.1: *Smiles dynamic evolution when $f_0 \to f$ under local volatility specification.*

Additionally, and without too much effort, it can be shown that hedges calculated within a local-volatility context are certainly wrong. As proved in [21], the Δ-risk can be computed as:

$$\Delta_{loc} = \Delta + \nu \frac{\partial \sigma(K, f)}{\partial f}, \tag{A.5}$$

where Δ and ν denote naive Black's delta and vega risks respectively. As it has been proved before, $\frac{\partial \sigma(K,f)}{\partial f}$ has the *opposite* sign in local-volatility models that the one experienced in the markets. Therefore, and highly surprisingly, hedges calculated under naive's Black model are more accurate than the ones provided by local-volatility models. This feature is, without any doubt, local volatility models' main drawback, since they lead to unstable (and highly incorrect) hedges, although their capability to fit *current* smiles and skews is undeniably spectacular.

Appendix B

CEV model (1975)

SABR model (3.7) is the stochastic-volatility version of the CEV (*constant elasticity of variance model).* CEV was firstly introduced in [42], and postulates that the underlying instantaneous forward rate follows the process

$$dF(t) = \sigma \cdot F(t)^{\beta} \cdot dW(t), \tag{B.1}$$

where the constraint $0 \leq \beta \leq 1$ is usually imposed for the power parameter β. CEV model arises as the natural generalization of both Bachelier (4.1) and Black (3.1) models, since both of them are obtained as particular cases with $\beta = 0, 1$ respectively.[111] Consequently, it shares their fundamental drawbacks:

- **It cannot deal with negative rates**[112].
- **Volatility is constrained to be constant for every strike and underlying forward price:** Therefore, it cannot reproduce smile effects.
- **Analytical complication:** This feature, which is not shared by its nested (Black and Bachelier) specifications, is due to its more general structure. The formulae are expressed in terms of the cumulative function of the non-central χ^2 distribution (see, for example, [16]).

[111]Equivalently, CEV model has sometimes been addressed as a particular case of local volatility models by several authors (see [16], for instance).

[112]Unless it was shifted, in which case we would have a restricted version of the shifted SABR model (4.8) with constant volatility.

Appendix C

Transferring the smile

This method is fully detailed in [34], and this appendix should be understood as a quick summary of its main characteristics.

Transferring the smile aims to extrapolate the smile structure for any given maturity (and standard tenor) to the same maturity and any different tenor by displacing the smile as a whole, instead of transferring each volatility *point by point*. (Shifted) SABR model is taken as the guide of the process, which can be summarized within the following steps:

1. Calibrate standard (shifted) SABR parameters $(\sigma(0)_{\tau(0)}, \alpha_{\tau(0)}, \rho_{\tau(0)})$ for the benchmark tenor $\tau(0)$ by using usual calibration formulae ((3.15) or (3.17)) and market data for that tenor.

2. Assuming that *the smile shape respect to moneyness does not change when the tenor does*, $\alpha_{\tau(1)}$ and $\rho_{\tau(1)}$ are fixed to the previously calibrated values $(\alpha_{\tau(0)}, \rho_{\tau(0)})$ when the tenor changes from $\tau(0)$ to $\tau(1)$ without change in the maturity. Therefore, the only free parameter to adjust the new smile structure is $\sigma(0)_{\tau(1)}$. Since we have illustrated that this parameter accounts for the level of the smile mainly (see figure (3.2)), the smile shape is guaranteed to be conserved under any change in the tenor.

3. The new value of the ATM volatility within the change $\tau(0)$ to $\tau(1)$, $\sigma^{ATM}_{\tau(1)}$, can be computed by the standard methodology detailed in chapter 5 of the Thesis (changing $\sigma^{ATM}_{\tau(0)}$ to $\sigma^{ATM}_{\tau(1)}$ by applying equation (5.11), (5.12), (5.13) or (5.14)). This is the only volatility that is transferred by this procedure in the whole transforming the smile process.

4. Once $\sigma^{ATM}_{\tau(1)}$ is computed, its value is introduced in equation (3.16) (along with fixed parameters $\alpha_{\tau(1)}$ and $\rho_{\tau(1)}$) to recover the new value of $\sigma(0)_{\tau(1)}$ via root-finding algorithms. The new (shifted) SABR for the non-standard tenor is already calibrated, and therefore any implied volatility can be computed within this new tenor.

Although the method is quite simple[113] and guarantees conserving smile shapes for any tenor under consideration (which eases the continuity requirement for the cube), we

[113]In fact, it is much simpler than our methodological approach given in chapter 5, since our volatilities shall be transferred *point by point*.

do not adhere to this approach *precisely due to the smile-shape conservation guarantee*. Empirically, we have observed (see figure (8.4)) that the smile shape respect to moneyness is not conserved under maturity *ceteris paribus* changes. Then, we do not find reasons to believe that this assumption *does apply* for *ceteris paribus* tenor changes (and, in fact, we think that it is hardly admissible, given the wild shape changes observed within maturity modifications). We prefer to avoid this strong assumption and adopt a fully free-arbitrage approach instead of transferring the smile. In any case, we have tested this methodology through *empirical results* chapter (see figures (8.27), (8.28)). For further discussion about this topic, the interested reader is encouraged to the insightful (and brilliantly exposed) original reference [34].

Appendix D

The Data

T	K(%)	ATM	-0.75	-0.50	-0.25	-0.13	0.00	0.25	0.50	1.00	1.50	2.00	3.00	5.00	10.00
1Y	-0.20	5.57	2.70	4.30	5.80	7.40	9.00	11.60	13.80	17.40	20.40	22.90	26.90	32.60	41.00
18M	-0.20	6.83	6.70	6.30	6.90	8.30	9.70	12.00	14.00	17.30	20.00	22.20	25.80	31.00	38.60
2Y	-0.20	8.59	9.60	8.20	8.10	9.30	10.50	12.50	14.20	17.20	19.60	21.60	24.80	29.50	36.30
3Y	0.02	11.17	12.70	11.20	10.00	10.40	11.10	12.40	13.60	15.90	17.70	19.30	21.90	25.70	31.00
4Y	0.14	13.39	15.10	13.50	12.10	12.30	12.80	13.80	14.80	16.50	18.00	19.30	21.50	24.60	29.10
5Y	0.26	14.94	16.50	15.00	13.80	13.80	14.10	14.90	15.60	16.90	17.90	18.90	20.40	22.70	26.10
6Y	0.39	15.88	17.20	15.90	14.80	14.70	15.00	15.60	16.10	17.00	17.70	18.30	19.30	20.80	23.20
7Y	0.52	16.33	17.70	16.40	15.40	15.30	15.50	15.90	16.30	16.90	17.40	17.80	18.50	19.70	21.70
8Y	0.64	16.50	17.90	16.70	15.80	15.70	15.80	16.10	16.40	16.80	17.10	17.30	17.80	18.60	20.20
9Y	0.76	16.49	18.00	16.90	16.00	15.90	15.90	16.10	16.30	16.60	16.70	16.90	17.10	17.60	18.80
10Y	0.87	16.37	18.00	17.00	16.10	16.00	16.00	16.10	16.30	16.40	16.40	16.40	16.50	16.70	17.60
12Y	1.06	15.99	18.00	17.10	16.30	16.10	16.10	16.10	16.10	16.00	15.90	15.70	15.50	15.30	15.70
15Y	1.25	15.37	17.80	17.00	16.30	16.10	16.00	15.90	15.80	15.50	15.20	15.00	14.60	14.20	14.30
20Y	1.41	14.63	17.50	16.80	16.10	15.90	15.70	15.50	15.30	14.90	14.60	14.20	13.80	13.30	13.20

Table D.1: *EUR cap flat standard implied shifted Black volatilities (%). Maturities up to $T = 2Y$ quote within a 3-month tenor. Above $T = 2Y$, a 6 month-tenor is used in the quoting procedure.*

T	K(%)	ATM	0.50	0.75	1.00	1.50	2.00	2.50	3.00	4.00
1Y	1.40	25.00	71.50	44.20	29.70	22.80	20.50	20.50	20.70	21.10
2Y	1.58	27.86	62.10	46.70	36.30	28.30	27.00	26.60	26.40	33.50
3Y	1.70	31.78	64.50	51.40	42.40	33.10	30.60	28.90	27.80	31.40
4Y	1.81	34.05	67.00	54.70	46.30	36.50	33.00	30.90	29.30	29.60
5Y	1.90	35.40	69.80	57.30	48.90	38.80	34.70	32.20	30.40	29.50
6Y	1.98	35.94	71.50	58.80	50.40	40.30	35.80	32.90	31.00	29.40
7Y	2.05	35.87	71.70	59.10	50.90	40.90	36.20	33.10	31.00	29.00
8Y	2.11	35.62	71.50	59.00	51.00	41.20	36.40	33.10	31.00	28.70
9Y	2.16	35.19	71.20	58.90	51.00	41.30	36.30	33.00	30.80	28.30
10Y	2.21	34.69	71.10	58.70	50.80	41.20	36.20	32.80	30.50	27.90
12Y	2.29	33.81	71.90	58.80	50.80	41.20	36.00	32.40	30.00	27.00
15Y	2.36	32.74	73.40	59.20	50.90	41.00	35.50	31.80	29.20	26.10
20Y	2.43	31.11	75.30	59.20	50.30	40.20	34.50	30.60	27.90	24.60

Table D.2: *USD cap flat standard implied Black volatilities (%). The quoting tenor is 3 months.*

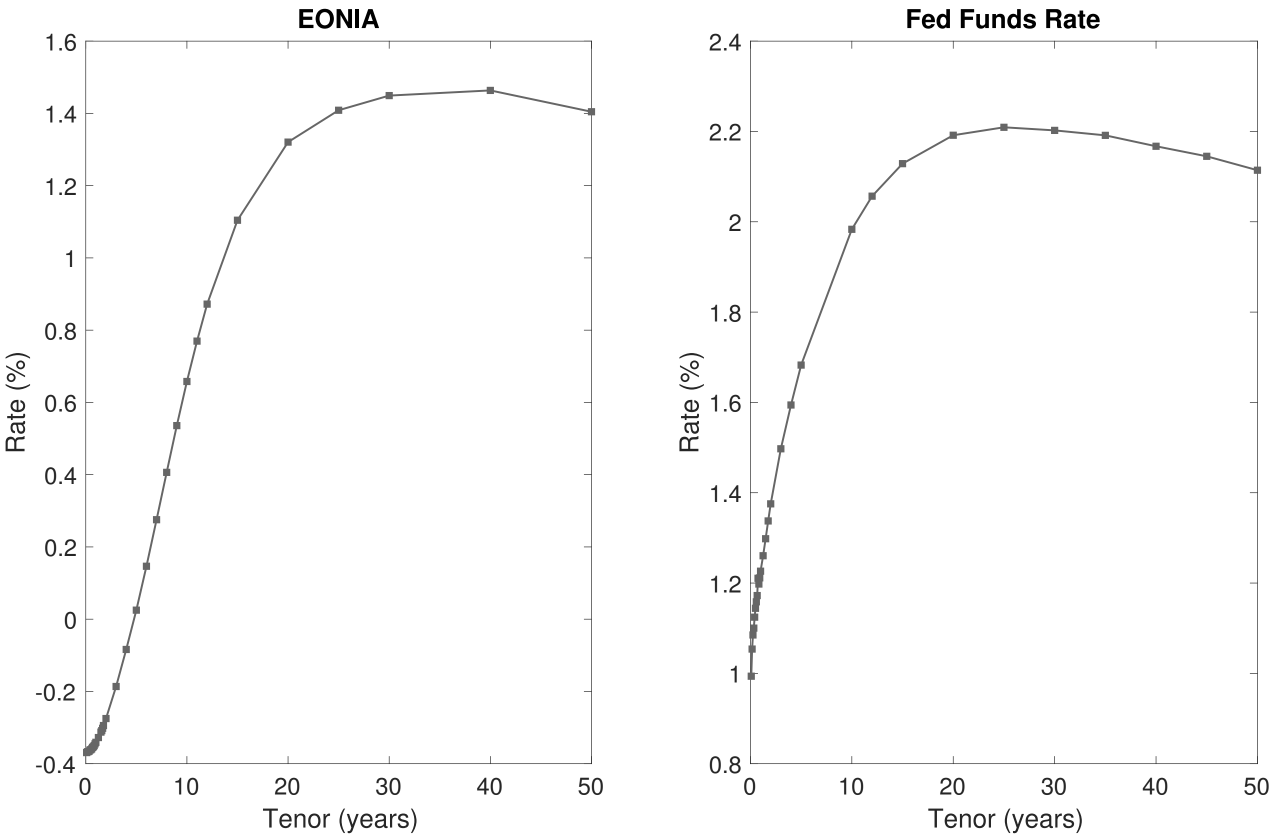

Figure D.1: *OIS zero-coupon curves. In our single-curve approach, these rates are used for both discounting and forwarding.*

T/K(%)	-1.00	-0.50	0.00	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00	4.50	5.00	5.50	6.00	6.50	7.00	7.50	8.00	8.50	9.00
1Y	11.30	6.84	9.31	13.61	17.00	19.78	22.15	24.20	26.02	27.64	29.11	30.46	31.69	32.83	33.89	34.88	35.81	36.68	37.51	38.29	39.03
18M	12.45	9.36	10.90	14.20	17.03	19.39	21.41	23.17	24.73	26.12	27.38	28.53	29.59	30.56	31.47	32.31	33.11	33.85	34.55	35.22	35.85
2Y	13.97	11.57	12.30	14.70	17.02	19.04	20.79	22.33	23.70	24.92	26.03	27.04	27.97	28.83	29.63	30.38	31.07	31.73	32.35	32.93	33.49
3Y	17.34	15.48	15.28	16.25	17.62	18.99	20.27	21.42	22.47	23.43	24.30	25.11	25.85	26.54	27.18	27.78	28.34	28.87	29.37	29.84	30.29
4Y	20.35	18.86	18.29	18.38	18.85	19.49	20.18	20.86	21.52	22.15	22.74	23.29	23.81	24.29	24.75	25.18	25.59	25.98	26.34	26.69	27.02
5Y	20.87	19.67	19.01	18.74	18.73	18.88	19.13	19.42	19.73	20.05	20.36	20.67	20.97	21.26	21.54	21.80	22.06	22.30	22.54	22.76	22.98
6Y	20.94	19.68	18.91	18.49	18.30	18.29	18.39	18.55	18.75	18.97	19.21	19.44	19.68	19.91	20.13	20.35	20.57	20.77	20.97	21.16	21.34
7Y	20.90	19.53	18.62	18.04	17.72	17.59	17.58	17.66	17.79	17.96	18.15	18.35	18.56	18.76	18.97	19.17	19.37	19.56	19.75	19.93	20.10
8Y	20.36	19.05	18.13	17.48	17.05	16.78	16.63	16.57	16.57	16.62	16.69	16.79	16.91	17.03	17.16	17.29	17.43	17.56	17.69	17.82	17.95
9Y	19.88	18.63	17.71	17.03	16.53	16.17	15.91	15.75	15.64	15.59	15.57	15.58	15.62	15.66	15.72	15.79	15.86	15.94	16.02	16.11	16.19
10Y	19.26	18.10	17.22	16.55	16.03	15.63	15.32	15.09	14.92	14.79	14.71	14.65	14.62	14.60	14.61	14.62	14.64	14.67	14.70	14.74	14.79
12Y	18.75	17.55	16.63	15.91	15.34	14.88	14.53	14.25	14.03	13.87	13.74	13.65	13.59	13.55	13.53	13.53	13.53	13.54	13.57	13.59	13.62
15Y	19.23	17.68	16.47	15.51	14.76	14.17	13.73	13.40	13.17	13.02	12.94	12.91	12.92	12.96	13.01	13.08	13.17	13.26	13.36	13.45	13.56
20Y	19.09	17.48	16.21	15.20	14.39	13.77	13.29	12.94	12.70	12.56	12.48	12.45	12.47	12.52	12.58	12.67	12.76	12.86	12.97	13.08	13.19

Table D.3: *3 month-tenor EUR non-standard shifted Black volatilities (%).*

T/K(%)	-1.00	-0.50	0.00	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00	4.50	5.00	5.50	6.00	6.50	7.00	7.50	8.00	8.50	9.00
1Y	33.56	19.17	16.15	22.49	27.52	31.32	34.34	36.81	38.91	40.72	42.30	43.71	44.98	46.13	47.17	48.14	49.03	49.85	50.62	51.34	52.02
18M	35.63	22.49	18.68	22.42	26.43	29.63	32.20	34.34	36.15	37.71	39.09	40.31	41.41	42.40	43.31	44.14	44.91	45.63	46.29	46.92	47.50
2Y	38.13	25.79	21.48	23.03	25.93	28.51	30.68	32.50	34.07	35.43	36.62	37.69	38.65	39.52	40.32	41.05	41.72	42.35	42.94	43.48	44.00
3Y	42.51	31.18	26.44	25.47	26.35	27.71	29.07	30.31	31.43	32.42	33.32	34.12	34.86	35.53	36.14	36.71	37.24	37.73	38.19	38.62	39.03
4Y	45.71	34.94	29.89	27.63	27.00	27.19	27.71	28.35	29.01	29.65	30.25	30.81	31.33	31.81	32.27	32.69	33.09	33.46	33.81	34.14	34.46
5Y	46.85	36.64	31.56	28.76	27.31	26.69	26.53	26.63	26.87	27.16	27.49	27.82	28.14	28.46	28.77	29.06	29.34	29.61	29.86	30.10	30.33
6Y	43.44	34.04	29.16	26.25	24.52	23.57	23.12	22.98	23.02	23.17	23.37	23.60	23.84	24.09	24.33	24.57	24.80	25.03	25.24	25.45	25.65
7Y	42.63	33.63	28.84	25.84	23.91	22.69	21.97	21.58	21.42	21.40	21.46	21.58	21.73	21.90	22.08	22.26	22.45	22.62	22.80	22.97	23.14
8Y	41.59	32.93	28.26	25.27	23.25	21.88	21.00	20.45	20.14	20.00	19.96	19.99	20.07	20.18	20.31	20.45	20.59	20.73	20.88	21.03	21.17
9Y	40.61	32.27	27.74	24.78	22.73	21.29	20.30	19.63	19.20	18.95	18.82	18.77	18.79	18.84	18.91	19.00	19.11	19.22	19.34	19.45	19.57
10Y	39.96	31.86	27.43	24.50	22.44	20.95	19.88	19.12	18.60	18.26	18.05	17.94	17.89	17.89	17.92	17.97	18.04	18.13	18.21	18.31	18.40
12Y	38.74	30.95	26.67	23.81	21.77	20.26	19.15	18.33	17.74	17.33	17.06	16.88	16.78	16.73	16.72	16.74	16.78	16.83	16.90	16.97	17.04
15Y	37.72	30.16	25.98	23.18	21.16	19.65	18.52	17.67	17.05	16.60	16.29	16.07	15.94	15.86	15.82	15.82	15.84	15.87	15.91	15.97	16.03
20Y	36.86	29.44	25.33	22.57	20.57	19.08	17.95	17.11	16.49	16.03	15.72	15.50	15.36	15.27	15.23	15.21	15.22	15.25	15.28	15.33	15.38

Table D.4: *12 month-tenor EUR non-standard shifted Black volatilities (%).*

Appendix E

ATM Caplet Stripping

[114]Consider that ATM USD caplet implied volatilities are to be stripped from ATM USD cap volatilities. For the first maturity (year 1), the stripping procedure is similar to the one explained for any fixed strike K, since the implied stripped volatility is constant during the first year (by hypothesis). The equation to be inverted is:

$$Cap(t, [T_{start}, 1], N, K_{ATM,1}) = Caplet(t, [0.25, 0.50], N, K_{ATM,1}, \sigma(K_{ATM,1}, 1)+ Caplet(t, [0.50, 0.75], N, K_{ATM,1}, \sigma(K_{ATM,1}, 1) + Caplet(t, [0.75, 1], N, K_{ATM,1}, \sigma(K_{ATM,1}, 1), \tag{E.1}$$

where $K_{ATM,1}$ stands for the strike (i.e., the underlying forward rate) at maturity $T = 1$. Once the equation is inverted, $\sigma(K_{ATM,1}, 1)$ is computed. The difference between stripping from any fixed strike and doing it for ATM strikes comes next. Since the underlying forward rate changes for every maturity, the ATM strike does, and therefore $K_{ATM,1} \neq K_{ATM,2}$. Consequently:

$$\begin{aligned}
&Cap(t, [T_{start}, 2], N, K_{ATM,2}) - Cap(t, [T_{start}, 1], N, K_{ATM,1}) = \\
&Caplet(t, [0.25, 0.50], N, K_{ATM,2}, \sigma(K_{ATM,2}, 1) + Caplet(t, [0.50, 0.75], N, K_{ATM,2}, \sigma(K_{ATM,2}, 1)+ \\
&Caplet(t, [0.75, 1], N, K_{ATM,2}, \sigma(K_{ATM,2}, 1) + Caplet(t, [1, 1.25], N, K_{ATM,2}, \sigma(K_{ATM,2}, 2)+ \\
&Caplet(t, [1.25, 1.50], N, K_{ATM,2}, \sigma(K_{ATM,2}, 2) + Caplet(t, [1.50, 1.75], N, K_{ATM,2}, \sigma(K_{ATM,2}, 2)+ \\
&Caplet(t, [1.75, 2], N, K_{ATM,2}, \sigma(K_{ATM,2}, 2) - Caplet(t, [0.25, 0.50], N, K_{ATM,1}, \sigma(K_{ATM,1}, 1)- \\
&Caplet(t, [0.50, 0.75], N, K_{ATM,1}, \sigma(K_{ATM,1}, 1) - Caplet(t, [0.75, 1], N, K_{ATM,1}, \sigma(K_{ATM,1}, 1) \neq \\
&Caplet(t, [1, 1.25], N, K_{ATM,2}, \sigma(K_{ATM,2}, 2) + Caplet(t, [1.25, 1.50], N, K_{ATM,2}, \sigma(K_{ATM,2}, 2)+ \\
&Caplet(t, [1.50, 1.75], N, K_{ATM,2}, \sigma(K_{ATM,2}, 2) + Caplet(t, [1.75, 2], N, K_{ATM,2}, \sigma(K_{ATM,2}, 2),
\end{aligned} \tag{E.2}$$

as it **did happen** with the fixed (arbitrary) strike K. The difference lies in the fact that caplets for both caps do not compensate each other during the overlapping period (until one year), since each set is quoted *with its own ATM strike.* As $\sigma(K_{ATM,2}, 1)$ is an

[114]The algorithm proposed was firstly stated in [43].

unknown quantity (the one we would like to find in previous equation is $\sigma(K_{ATM,2}, 2)$), knowing $\sigma(K_{ATM,1}, 1)$ do not help us with the next maturity since we have two variables for a single equation.

Instead of the previous difference, the algorithm of [43] proposes to compute the quantity $Cap(t, [T_{start}, 2], N, K_{ATM,2}) - Cap^{theoretical}(t, [T_{start}, 1], N, K_{ATM,2})$, where $Cap^{theoretical}(t, [T_{start}, 1], N, K_{ATM,2})$ stands for the theoretical price of a cap with that characteristics. Since $Cap^{theoretical}(t, [T_{start}, 1], N, K_{ATM,2}) =$

$$Caplet(t, [0.25, 0.50], N, K_{ATM,2}, \sigma(K_{ATM,2}, 1) + Caplet(t, [0.50, 0.75], N, K_{ATM,2}, \sigma(K_{ATM,2}, 1) + Caplet(t, [0.75, 1], N, K_{ATM,2}, \sigma(K_{ATM,2}, 1), \tag{E.3}$$

we have:

$$\begin{aligned} &Cap(t, [T_{start}, 2], N, K_{ATM,2}) - Cap^{theoretical}(t, [T_{start}, 1], N, K_{ATM,2}) = \\ &Caplet(t, [1, 1.25], N, K_{ATM,2}, \sigma(K_{ATM,2}, 2) + Caplet(t, [1.25, 1.50], N, K_{ATM,2}, \sigma(K_{ATM,2}, 2) + \\ &Caplet(t, [1.50, 1.75], N, K_{ATM,2}, \sigma(K_{ATM,2}, 2) + Caplet(t, [1.75, 2], N, K_{ATM,2}, \sigma(K_{ATM,2}, 2), \end{aligned} \tag{E.4}$$

and the stripping procedure can be performed as usual. The only necessary condition, then, is being able to find the quantity $\sigma(K_{ATM,2}, 1)$ to compute the theoretical price of the cap for the first period given by (E.3). $\sigma(K_{ATM,2}, 1)$ accounts for the implied caplet volatility during the first period at $K_{ATM,2}$, which is not quoted in the markets. However, it can be easily computed by interpolating over quoting implied volatilities by the standard use of the (shifted) SABR. Once the model is calibrated as explained in chapter 7 of the Thesis, the implied caplet volatility *for next maturity's ATM strike* is interpolated and introduced in equation (E.3). The theoretical price of the cap is used in equation (E.4) to obtain, via Newton-Raphson algorithm, the quantity $\sigma(K_{ATM,2}, 2)$) for the sought maturity. This bootstrapping procedure is performed in ascending order of maturity to obtain the term structure of ATM implied volatilities for every caplet under consideration.

The stripping methodology for ATM caps can be then summarized within the following scheme:

1. Using formulae (3.3), (4.3) (whether quoting flat implied volatilities were (shifted) Black's or Bachelier's), every constituent ATM caplet of the ATM cap is priced, for every ATM cap under study. The ATM cap price is then obtained by aggregating individual ATM caplet prices (see formulas (3.5), (4.5)).

2. Fixed-strike caps are stripped as described in chapter 7 of the Thesis.

3. A strike-interpolating model (such as (shifted) SABR) should be calibrated for every (cap) maturity.

4. Defining $K_{ATM,j}$ as the ATM strike for every maturity under consideration, a price difference series between the current cap market prices and the preceding cap theoretical prices for the current ATM strike is constructed, by the explicit use of

the interpolated implied volatility from the previous maturity at the current ATM strike:

$$Cap(t, T_j, N, K_{ATM,j}) - Cap^{theoretical}(t, T_{j-1}, N, K_{ATM,j}), j = 1, ..., n, \tag{E.5}$$

where $Cap^{theoretical}(t, T_0, N, K) := 0$.

5. Steps 4 and 5 of the fixed-strikes algorithm are repeated (mapping the price difference to the appropriate caplets and extracting the implied ATM caplet volatility by the explicit use of a one-dimensional root finder).

Within this formulation, the ATM caplet stripping procedure presents *intermaturity* dependence, since every stripping (excluding the first one) depends on the interpolation procedure of the previous maturity.

Appendix F

Pricing caplets under Vasicek/Hull-White. Numerical issues

This appendix tries to clarify which kind of numerical difficulties can be found when Vasicek/Hull-White models are calibrated via caplet pricing formulae (4.14), (4.16)[115] . In a nutshell, classical numerical routines troubles might be splitted between two different categories:

- **Getting stuck in a local critical point:** Stochastic optimization techniques have grown in importance during recent years due to their capability to deal with this classical problem. Appendix H is fully devoted to the *Simulated annealing* technique that has been implemented in our calibration process. No improvement has been detected, and therefore we assume that this is not the main concern for us.

- **Discontinuity of the pricing function:** To set ideas, figure (F.1) below shows the dependence on the parameter θ of Vasicek's caplet pricing formula (4.14)[116] for fixed k, σ^2 (to their respective calibrated values for $T = 1Y$) and $K = 0\%$, as an illustrative example. It can be seen that the pricing function is *remarkably discontinuous*, and therefore the calibration algorithm cannot fit properly any price between two given points of the 1-D parametric space for θ[117]. We believe that the arisen problems during the optimization procedure are due to the *nature of the caplets pricing functions*, not to the *optimization procedure itself.*

[115] MATLAB's internal procedure for calibrating Vasicek/Hull-White models based on caplets market data, *hwcalbycap*, has been also used as a benchmark to contrast the results of our own calibration algorithms based on formulae (4.14), (4.16). No better results have been attained.

[116] Since formulas (4.14), (4.16) present a similar structure, the pricing problems in (4.14) are reproduced in (4.16) , and therefore any comment made for (4.14) during the appendix applies for (4.16) as well.

[117] For instance, say that the actual market price of the caplet priced in figure (F.1) is 0.02%. As shown in the figure, every considered point of the parametric space for θ gives a different price respect to our market benchmark. Therefore, the step-size tolerance is not enough to guarantee a proper fit for small caplet prices, where the relative pricing error increases wildly as soon as the market price is not attained with high accuracy.

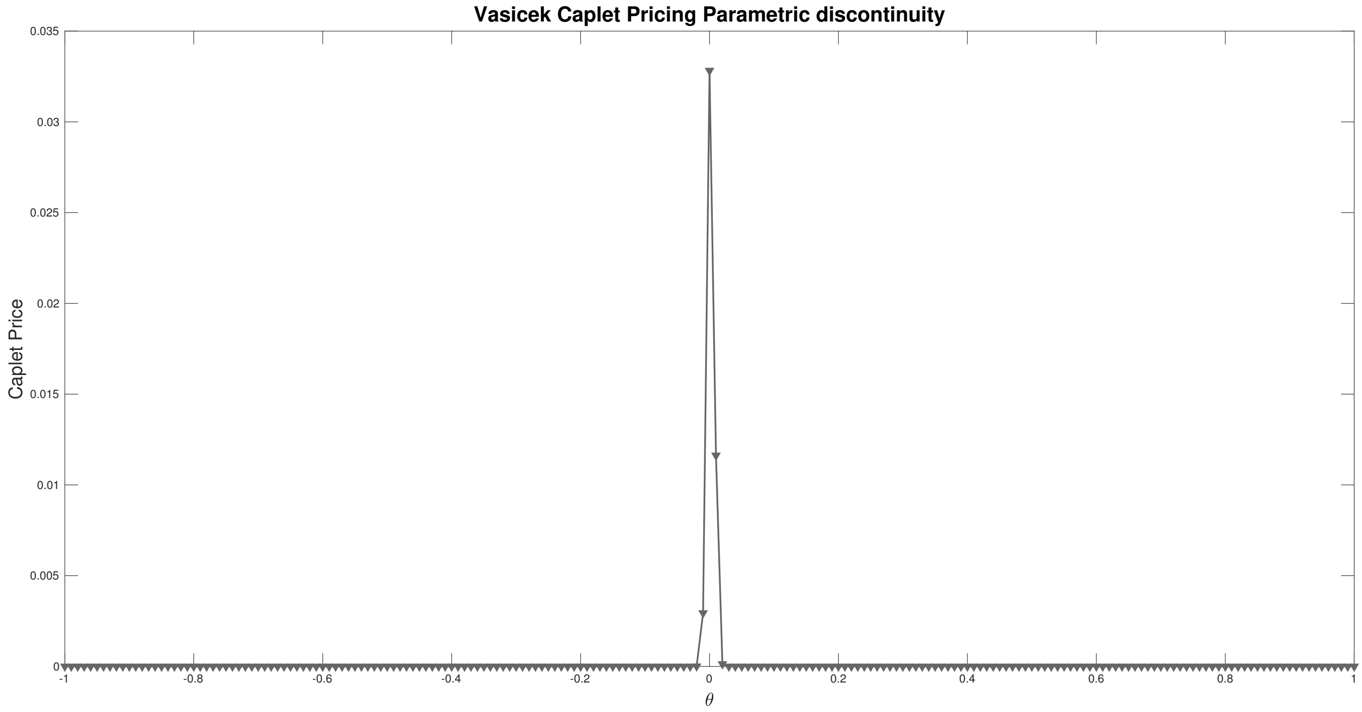

Figure F.1: *Vasicek caplet pricing formulae (4.14) as a function of the long term value* θ*. k and* σ^2 *have been fixed to* 1.2147 *and* $1.4546e-05$ *respectively. A standard notional of* $N = 100$ *and* $T = 1Y$ *have been chosen.*

Appendix G

Correlating the forward rates

There exist three different approaches to estimate parameter ρ from formulas (5.11), (5.12), (5.13), (5.14):

1. **Historical correlation of time series for both forward rates:** This econometric approach can be sophisticated to attend for *time varying* correlation, using DCC-GARCH alike models to reproduce several characteristics of both variables (leverage, assymetry, etc.)[118]. Even doing so, the classical problem of relying the estimation of a *by nature* forward-looking measure such as the implied volatility within a historical (*realized*) correlation appears. Abundant literature has been written about this topic, and the most optimistic recommended treatment is the one of [45]. The problem is obvious: we do not want to answer the question of whether these forward rates have been highly correlated (or not) in the past without an explicit use of any forward-looking model (which is the standard approach for historical measures), but to *forecast* how this correlation would be in the future for the given model (5.3). This is the main flaw of the historical econometric approach.

2. **Let ρ being a free parameter to be calibrated within the extrapolation method:** In our research, we act *as if we had no data for the non-standard tenors to be extrapolated*, to resemble the worst possible situation among the markets. If we had a reasonable dataset for any non-standard tenor we could, in principle, infer parameter ρ from a non-linear least squares comparison between formulas (5.11), (5.12), (5.13), (5.14) and market volatilities. Then, we could estimate out-of-sample volatilities in a more accurate fashion.

3. **Adhere to previous literature:** Following the claim of [7] (*high correlation among forward rates*), [2] fixes the correlation between any pair of arbitrary forward rates under study in $\rho = 0.9$. We follow this choice, and fix $\rho = 0.9$ for any pair of forward rates under consideration.

Obviously, the three methods shall be contrasted in terms of *accuracy* of the resulting calibration method. This issue is left for further research.

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[118]For quick refresh on standard econometric analysis of financial time series with particular interest on DCC-EWMA/GARCH correlation models, the brilliant treatment of [44] is always recommended.

Appendix H

Stochastic optimization. Simulated annealing

[119] **Stochastic optimization** attempts to overcome the classical problem of "traditional" optimization tecniques: *getting stuck in a local critical point.*

While traditional optimization improves towards the better *local* solution (i.e., **exploits**), stochastic's aims to wander on the full range provided for the parameters (i.e., **explores**). When traditional optimization gets close to a promising local optimal, the step size is not big enough to escape from local minimum barriers, and the convergence is finally attained over the local point, ignoring the possibility of further exploration of the parametric space.

Inspired by Physics' potential barriers, stochastic optimization techniques have a non-zero probability of attaining *any arbitrary point* of the parametric space, although this probability decreases in an a priori functional specified form with the number of iterations (that is why this method is called "simulated annealing"[120]). The applied version of simulated annealing for finding a local minimal of the function $f(\theta^n)$: $R^n \to R^1$ is structured in the following steps:

1. The full parametric space is collapsed into a R^n $[0x1]^n$ space.
2. An arbitrary seed θ_0^n is given. i is fixed to zero.
3. A new sample point of the parametric space $\theta_{i,alternative}^n$ is generated via standard one-dimensional uniform distributions in every dimension.
4. If $f(\theta_{i,alternative}^n) \leq f(\theta_i^n)$, we jump to the new sample point (i.e., $\theta_{i+1}^n = \theta_{i,alternative}^n$). Otherwise, there is still a non-zero probability of jumping, given by:

$$P(\theta_i^n, \theta_{i,alternative}^n, T) = \exp\left(\frac{f(\theta_i^n) - f(\theta_{i,alternative}^n)}{T}\right) \tag{H.1}$$

[119]Special thanks to Carlos A. Catalán García. This version of the simulated annealing algorithm is fully inspired in the slides he selflessly provided.

[120]Within this analogy, the *temperature* is an indirect measure of the *probability of the jump.*

If $P(\theta_i^n, \theta_{i,alternative}^n, T)$ has been computed in this step, another sample from a standard uniform distribution x_i is extracted and compared with $P(\theta_i^n, \theta_{i,alternative}^n, T)$. If $x_i \leq P(\theta_i^n, \theta_{i,alternative}^n, T)$, again $\theta_{i+1}^n = \theta_{i,alternative}^n$. Otherwise no jump has occurred, and $\theta_{i+1}^n = \theta_i^n$ In any case, $i = i + 1$ and the temperature variable T is reduced smoothly.[121]

5. Steps 3 and 4 are iterated until T reaches a pre-specified low minimal.

For any fixed number of iterations, the process guarantees that a huge region of the parametric space is explored, instead of finding a (possible) local minimum in an accurate fashion. For further information about simulated annealing technique, [46] is highly recommended.

[121] Notice that standard values of T cannot be pre-specified, since they depend on the optimization procedure, in order to make the difference $f(\theta_i^n) - f(\theta_{i,alternative}^n)$ relatively comparable with T.