# PRICING OPTIONS ON IBEX-35 UNDER THREE DIFFERENT MODELS OF VOLATILITY

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# PRICING OPTIONS ON IBEX-35 UNDER THREE DIFFERENT MODELS OF VOLATILITY

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Research work
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## 1. Introduction

The well known constant volatility model of Black and Scholes (1973), extended by Merton (1976), is the most often used option pricing model in financial practice.
This model explains that a purchase option (Call option), denoted as $C$, is equal to the price of the underlying asset ($S_0$) adjusted by a dividend yield rate ($q$) in a continuous way, multiplied by the Normal distribution function $N(d_1)$. Then, the strike price ($K$) is countinously discounted at the $r$ interest rate and multiplied by the Normal distribution function $N(d_2)$.

$$C = S_0 \mathrm{e}^{-qt} * N(d_1) - K\mathrm{e}^{-rt} * N(d_2)$$

$$d_1 = \frac{ln(\frac{S_0}{K}) + t(r - q + \frac{\sigma^2}{2})}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

This model makes certain assumptions, which are efficent markets, no transaction costs in buying the option, European options which can only be exercised at expiration, normally distributed returns on the underlying asset, constant risk-free rate and constant volatility of the underlying.

However, in reference to the last assumption mentioned, in the last decades evidences suggest that a constant volatility model is not appropriate.
Indeed, numerically inverting the Black-Scholes and Merton formula on real data sets, supports the notion of asymmetry with strike price ($K$) as well as dependence on time expiration ($T$). In general, this dependence is referred to as the volatility smile of an option reflecting that in practice, this implied volatility is not constant.
This variability of the volatility contradicts the assumptions of the Black and Scholes (1973) model mentioned before.

In order to overcome this problem, in this work different methods of modelling the volatility are analised and the methods' performance is compared to determine which one has the best answer to this problem.

Looking through the literature on this field, three different models of option pricing were decided to use in this work. These models focus the problem of constant volatility of the Black and Scholes (1973) model, changing its process in three different ways. A stochastic volatility with the Heston (1993) model, a conditional volatility with the Heston and Nandi (2000) model and a deterministic volatility with the Local Volatility model.

Following the steps of Ferreira, Gago, León and Rubio (2005) work, we compare the performance of the models mentioned before in the IBEX-35 pricing recent options on the Mini IBEX-35 Futures in sample and out of sample with the contribution of the Local Volatility model and its impact in pricing this kind of options.

The rest of the report is organized as it follows. Section 2 contains a discussion of the three competing models used in this research and Section 3 describes the option data used in this report. The results obtained applying these models to the data options are shown in Section 4 and Section 5 provides the final remarks and concludes.
This report ends with the Section 6 showing all the reference articles used to develop this work and Section 7 providing the results of the estimation of the models' parameters.

## 2. Competing pricing models and estimation

### *2.1. Heston stochastic volatility model*

In reference to the exposed in the introduction, the Heston stochastic volatility model is another answer to the known inconsistency problem of the constant volatility implemented in the Black and Scholes (1973) model.
This model allows the volatility to be stochastic rather than constant, changing to a more realistic model than Black-Scholes' one.

The Heston (1993) model assumes that the stock price follows the diffusion process:

$$dS_t = \mu_t S_t dt + \sqrt{\upsilon_t} S_t dZ_1 \tag{1}$$

where $\mu_t$ is the drift $((r-q))$ with dividends for risk-neutral dynamics, $\sqrt{\upsilon_t}$ is the diffusion term (i.e. the volatility of the stock price) and $dZ_1$ is a Wiener process (i.e. a Brownian motion).

The volatility of the model is stochastic, and follows the process

$$d\upsilon_t = \kappa(\theta - \upsilon_t)dt + \sigma\sqrt{\upsilon_t} dZ_2 \tag{2}$$

where $\kappa$ is the speed of mean-reversion, $\theta$ is the long term level of the variance, $\upsilon_t$ is the variance, $\sigma$ is the volatility of volatility, and $dZ_2$ is a Wiener process.

In this case, the two Wiener processes $dZ_1$ and $dZ_2$ are correlated by this way

$$dZ_1 dZ_2 = \rho dt \tag{3}$$

The formulation of (2) for the variance is a process as defined in Cox, Ingersoll and Ross (1985). The only difference is that here, it is applied to stochastic variance instead of being applied to interest rates.
The reason why the variance is modelled this way is that the drift term ensures that the variance $\upsilon_t$ is mean reverting towards the long-term mean $(\theta)$. The higher the value of $\kappa$, the quicker the model reaches the long term mean of the variance.

As shown in (2), the expression $\sigma\sqrt{\upsilon_t}$ ensures that the variance will be strictly non-negative for positive values of $\kappa$ and $\theta$.
If the variance gets close to 0, the stochastic process of the volatility (2) will be defined by the drift term and it will be pull upwards to the long term mean.
Furthermore, if $2\kappa\theta \geq \sigma^2$, the variance cannot become zero. This last expression is known as the Feller condition deduced from his work Feller (1951).

The partial differential equation of the Heston (1993) model is derived based on the Gatheral (2006) derivation of the PDE for this kind of models. The derivation of the PDE is based on an Itô process, where is assumed that the underlying asset's price and variance satisfy (1) and (2).

Doing the pertinent numerical calculations, and expressing the PDE in terms of the logarithmic stock price ($x = ln(S)$), the expression is the following

$$-\frac{\partial V}{\partial t} = -rV + \left(r - q - \frac{1}{2}\upsilon\right)\frac{\partial V}{\partial x} + \frac{1}{2}\upsilon\frac{\partial^2 V}{\partial x^2} + \kappa(\theta - \upsilon)\frac{\partial V}{\partial \upsilon} + \frac{1}{2}\sigma^2\upsilon\frac{\partial^2 V}{\partial \upsilon^2} + \rho\upsilon\sigma\frac{\partial^2 V}{\partial \upsilon \partial x}$$

In an option pricing model, if the characteristic function for the price process is known, with the use of the transforms shown in Fourier (1822), it is possible to price call options by finding the probabilities from the characteristic function. This was what Heston did in his paper. He got the characteristic functions of the risk-neutral probabilities as solutions for a second order PDE.

The most common approaches are those of Carr and Madan (1999) and Lewis (2000).
The issue with the Carr and Madan's approximation is that the Fourier transform is needed for different options that are to be priced. On the other hand, with Lewis' technique only the Fourier transform is needed for the option payoff. As the options' payoffs are defined explicitly in the contract, it is more straightforward to get the Fourier transforms of the option payoffs than of the prices.

Consider a function $f(x)\colon \mathbb{R} \to \mathbb{C}$. Then the generalized Fourier transform of $f$ is defined as

$$\widehat{f}(z) = \int_{-\infty}^{\infty} \mathrm{e}^{izx} f(x) \mathrm{d}x$$

where $i$ is the imaginary unit and $z \in \mathbb{C}$ for which $\mathrm{e}^{izx} f(x)$ is integrable.

If $f(x)$ is the density function for a real valued $x$, then $\widehat{f}(z)$ is the generalized characteristic function

$$\widehat{f}(z) = \phi(z) = \mathbb{E}[\mathrm{e}^{izx}]$$

and its domain contains the real axis. There exists a one to one correspondence between the characteristic function and the probability density function.
Then $f(x)$ can be recovered from $\widehat{f}(z)$ via de inversion formula

$$f(x) = \frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty} \mathrm{e}^{-izx}\widehat{f}(z)\mathrm{d}z \tag{4}$$

where $z \in \mathbb{C}$: $z = z_r + iz_i$.

### *2.1.1. Pricing formula of the Heston (1993) model*

The expression for call prices in this model is derived following the technique of Lewis (2000).

Defining $X_T = log(S_T)$ as the log price at maturity $T$, its generalized characteristic function is $\phi(z)$ and its density is $q(x)$ and the payoff function of the log price is $w(x)$. So, the Generalized Fourier transform would be $\widehat{w}(z)$.

The $t = 0$ price $V(S_0)$ of the payoff $w(x_T)$ is given by

$$V(S_0, K, T) = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}[w(x)]$$

where $\mathbb{Q}$ denotes the risk-neutral probability.
Making use of the invertion formula, we obtain

$$V(S_0, K, T) = \frac{\mathrm{e}^{-rT}}{2\pi}\left[\int_{iz_i-\infty}^{iz_i+\infty} \phi(-z)\widehat{w}(z)\mathrm{d}z\right]$$

So, this is the expression of the option price getting the density function from the characteristic function of the payoff. One of the properties of the characteristic function is that $\phi(-z) = \overline{\phi(z)}$. Therefore, the integrand is well defined along the integration line.

From here, using Lewis' technique the expression for call prices is derived. One thing to remark is that the call price can be expressed in terms of the spot price and a covered call. This was shown by Rouah (2013).

The payoff of a covered call ($CC$) is given by

$$CC = min(S_T, K) = K - (K - S_T)^+ \tag{5}$$

This leads to express the price of a call option as a function of the strike price in the following way

$$C(K) = S_0\mathrm{e}^{-qT} - CC \tag{6}$$

After applying the technique mentioned before and taking into account the expression of the call option with the covered call, the resulting formula for the call price is

$$C(S_0, K, T) = S_0 \mathrm{e}^{-qT} \left( 1 - \frac{\sqrt{\frac{K}{F_T}}}{\pi} \int_0^{\infty} \frac{\mathrm{d}u}{u^2 + \frac{1}{2}} RE \left[ \phi_0 \left( u - \frac{i}{2} \right) \mathrm{e}^{-iuk} \right] \right) \tag{7}$$

where $F_T$ is the forward stock price $F_T = S_0 \mathrm{e}^{(r-q)T}$, and $k = log(\frac{K}{F_T}) = log(\frac{K}{S_0 \mathrm{e}^{(r-q)T}})$.

Now the characteristic function $\phi_0(z)$ or equivalently $\phi(z)$ can be found. The characteristic function is model dependent.
Assuming that there is not drift term in (1) and taking as a function $X_t = log(S_t)$ and applying the Itô lemma, the following characteristic function is obtained

$$\phi(z, X, \upsilon, t, T) = \mathrm{e}^{C(z,\tau) + D(z,\tau)\upsilon + izX} \tag{8}$$

where

$$D(z, \tau) = r_- \frac{(1 - \mathrm{e}^{(-d\tau)}}{1 - g\mathrm{e}^{(d\tau)}}$$

$$C(z, \tau) = \kappa\theta \left( r_- \tau - \frac{2}{\sigma^2} ln \left( \frac{1 - g\mathrm{e}^{-d\tau}}{1 - g} \right) \right)$$

$$d = \sqrt{b^2 - 2a\sigma^2}$$

$$r_{\pm} = \frac{b \pm d}{\sigma^2}$$

$$g = \frac{r_-}{r_+}$$

After having developed and explained the Heston (1993) stochastic volatility model, one can notice that the parameter $\rho$ affects the skewness of the distribution. For negative values of $\rho$ the distribution becomes positively skewed and for positive values, it becomes negatively skewed.
Furthermore, $\sigma$ affects the kurtosis of the distribution. When $\sigma$ increases, the kurtosis of the distribution increases.

*2.1.2. Computation of the Heston stochastic volatility model*

The purpose of the calibration is to make the Heston (1993) model prices fit as closely as possible with the observed market prices. The parameters needed to calibrate this model are

$$parameters(p) = \kappa, \theta, \upsilon, \sigma, \rho$$

where, as described before, $\kappa$ is the speed of mean-reversion, $\theta$ is the long term level of the variance, $\upsilon$ is the variance, $\sigma$ is the volatility of volatility and $\rho$ is the correlation parameter.

For this purpose, we use the non-linear least squares method. Taking as a reference the work done in Mikhailov et al. (2003), the non-linear least square optimization method is performed in the following way:

$$min_p SSE(p) = min_p \sum_{i=1}^{N} [C_i^H(K_i, T_i) - C_i^M(K_i, T_i)]^2 \tag{9}$$

where $C_i^H(K_i, T_i)$ is the price of the call option depending of the strike price $(K)$ and maturity $(T)$ obtained with the initial values of the parameters in the Heston (1993) model and $C_i^M(K_i, T_i)$ is the price of the call option in the market.

In essence, we are minimizing the sum of squared errors. To be able to solve correctly the otpimization problem, we have to define the lower and upper bounds of the parameters, which are:

$$0 < 2\kappa\theta - \sigma^2 < 10$$

$$0 < \theta < 1$$

$$0 < \sigma < 5$$

$$-1 < \rho < 1$$

$$0 < \upsilon_0 < 1$$

These restrictions are necessary in order to avoid solutions that would not make sense from an economical point of view. The first restriction is the Feller condition, which is given by $2\kappa\theta - \sigma^2 \geq 0$. It has the lower bound set to 0 in order to ensure the variance process will be positive and never reach 0.

Giving initial values for the parameters of the model, we calculate the prices of the options following the development showed in section 2.1.1.
Then, we minimize the errors with the market values using the non-linear least squares method receiving the optimum parameter values. With these values,

we recalculate the option prices.

There is another method that fits very well with this optimization problem. This method is the ASA method (Adaptive Simulated Annealing method). The advantage of this calibration method is that it does not stop its search in the first minimum obtained. However, this method does not guarantee a global minimum .
As at the time of computing this method, a lot of problems appeared, the ASA method was dismissed for this work.

The results of the parameter estimations for the in sample and out of sample context in this model can be seen in the Section 7.1 of the Appendix.

*2.2. Heston and Nandi GARCH(1,1) model*

The next model developped by Heston and Nandi (2000) and presented in this work has a main difference between the others, which is the ability to explain the path of the volatility that determines the price of the underlying asset. It extends the model of Black and Scholes (1973) and Merton (1976) to solve the problem exposed in the introduction by adapting the implied volatility estimation.

In their work, Heston and Nandi (HN from now on) considered the Generalized Autoregressive Conditional Heteroscedasticity model (GARCH) to capture the path dependence in volatility as well as the negative correlation of the volatility with the index returns.

GARCH models have been used to model time varying variances of assest prices. These models where developed by Duan (1995) and Heston and Nandi (2000). The great contribution of these last two was that they derived a semi-closed form option pricing formula. Later, many others used this idea with some other changes in order to price options with more accuracy.
Barone-Adesi, Engle and Mancini (2008) proposed an option pricing method based on a GARCH with no-normal innovations and Byun and Min (2010a) and Byun and Min (2010b) refined the work of Barone-Adesi, Engle and Mancini (2008) letting physical and risk-neutral one-day ahead GARCH volatilities to be different.

The Heston and Nandi model used in this work is discrete and the parameters can be estimated by Maximum Likelihood. As it will be seen later, we consider here an asymmetric GARCH model, where the parameter $\gamma$ reflects the skewness in it.
This asymmetric GARCH version has a continuous limit identified as Heston's model with perfect correlation between the underlying asset and the volatility. This fact implies that in this framework neither the volatility risk premium nor

the jump-risk premium exist.

The HN's GARCH(1,1) expresses the dinamics of the asset by

$$ln(S_t) = ln(S_{t-1}) + r + \lambda h_t + \sqrt{h_t} z_t \tag{10}$$

which can be written in terms of the log-return of the asset in this way

$$R_t = ln\left(\frac{S_t}{S_{t-1}}\right) = r + \lambda h_t + \sqrt{h_t} z_t \tag{11}$$

where $S_t$ indicates the price of the underlying asset at time $t$, $r$ is the risk-free rate countinously compunded and $R_t$ is the log-return of the asset price. $h_t$ is the conditional variance of the log-return between $t-1$ and $t$, known at time $t-1$. The expression can be shown here

$$h_t = w + \beta h_{t-1} + \alpha(z_{t-1} - \gamma\sqrt{h_{t-\Delta}})^2 \tag{12}$$

$\lambda h_t$ is the risk premium of the underlying asset. It is embedded in returns. $z_t$ is the standard normal random variable and, as indicated before, $\gamma$ is the parameter which controls the skewness of the distribution.

This GARCH model, as being a first-order model, remains stationary with finite mean and variance if $\beta_1 + \alpha_1\gamma_1^2 < 1^2$.
It can be deduced the expression of the conditional variance of the log-return underlying assert at time $t$ by clearing $z(t-1)$ from (12) and replacing its value in (11). These steps make one reach at the following expression

$$h(t+\Delta) = \omega + \beta_1 h(t) + \alpha \frac{(log(S(t)) - log(S(t-\Delta)) - r - \lambda h_t - \gamma_1 h(t))^2}{h(t)} \tag{13}$$

It can be noticed with this expression that the parameter $\alpha$ determines the kurtosis of the distribution and, being zero, results in a deterministic time varying variance.

As it can be seen, the Heston and Nandi (2000) model can be considered as a model with predictable volatility, since volatility can be estimated from the past information of the underlying return path.
Another characteristic is that it enables us to appreciate the correlation between the asset returns and the variance by this way

$$cov_{t-\Delta}[h(t+\Delta), lnS(t)] = -2\alpha\gamma h(t)$$

where, given a positive $\alpha$, positive values for $\gamma$ result in negative correlation between returns and variance.

### *2.2.1. Pricing formula of the Heston and Nandi GARCH(1,1) model*

After having estimated the parameters of the GARCH(1,1) process, in order to being able to price options, the process must be represented in terms of the risk-neutral measure

$$c = \mathrm{e}^{-r\tau}[F(t)P_1 - XP_2] \tag{14}$$

where $P_2$ is the risk neutral probability of the asset being greater than $X$ at maturity and $P_1$ corresponds to the delta of the call value. The estimation of the $P_1$ and $P_2$ in this case requires the estimation of the risk-neutral parameters of the GARCH model. To do so, the values that HN determined in their paper are used, with which the risk-neutral process takes the same GARCH form as (12) and (11), but with a variance in the skewness parameter $\gamma$.

The expressions of these risk neutral parameters determined by HN are

$$\lambda^* = -\frac{1}{2}$$

$$\gamma_1^* = \gamma_1 + \lambda + \frac{1}{2}$$

The variant employed in this report is identified with the parameter $\gamma$. Instead of applying the expression shown before, it is calculated as the argument that minimizes the squared error between the market options and the options calculated inserting the estimated parameters in the model with the real messure as Ferreira, Gago, León and Rubio (2005) did in their work.
It is said, the expression to define the $\gamma$ parameter in this work is

$$\gamma_t^* = argmin_\gamma \sum_{i \in t} (c_{HN,i}(\gamma) - c_i)^2 \tag{15}$$

where $c_i$ is the price observed in the market, and $c_{HN,i}(\gamma)$ denotes the price resulting from the HN formula. The rest of the parameters are those obtained estimating the GARCH model of the volatility from the underlying log-return series.

Heston and Nandi (2000) derive an almost closed-form option pricing formula for the European call option ($C$) with strike price ($K$) and maturity ($T$)

$$C = \frac{1}{2}S_t + \frac{\mathrm{e}^{-r(T-t)}}{\pi}\int_0^\infty E\mathrm{d}\phi - K\mathrm{e}^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty F\mathrm{d}\phi\right)$$

being $E$ and $F$ defined as

$$E = Re\left(\frac{K^{-i\phi}f^*(i\phi+1)}{i\phi}\right)$$
$$F = Re\left(\frac{K^{-i\phi}f^*(i\phi)}{i\phi}\right)$$

where $Re()$ denotes the real part of a complex number, $f^*(i\phi)$ is the conditional characteristic function of the logarithmic asset price using the risk neutral probabilities and $i$ is the imaginary number.

Heston and Nandi (2000) also shows that the conditional generating function of the asset price takes the following log-lienar form of the GARCH(1,1) process

$$f(\phi) = E_t[S_T^\phi] = S_t^\phi \mathrm{e}^{A_t + B_t h_{t+1}}$$

which is also the moment generating function of the logarithmic asset price.

The coefficients $A_t$ and $B_t$ can be calculated recursively by working backward from the maturity date using the terminal conditions

$$A_t = A_{t+1} + \phi r + B_{t+1}w - \frac{1}{2}log(1 - 2\alpha B_{t+1})$$
$$B_t = \phi(\lambda+\gamma) - \frac{1}{2}\gamma^2 + \beta B_{t+1} + \frac{\frac{1}{2}(\phi-\gamma)^2}{1-2\alpha B_{t+1}}$$
$$A_T = B_T = 0$$

*2.2.2. Calibration of the Heston and Nandi GARCH(1,1) model*

To fit the option prices with the Heston and Nandi (2000) model to the market prices, the parameters must be estimated in a risk neutral enviroment. To do so, first these parameters are estimated from the conditional variance model (GARCH(1,1)) by Maximum Likelihood.
After that, the prices are calculated with the Heston and Nandi model in the real measure, it is said, without changing the values of the parameters obtained in the estimation of the GARCH.

Once the option prices are obtained with the model under the real measure, the risk neutral $\gamma$ parameter (skewness parameter) is estimated minimizing the distance between the option market prices and the option HN prices respect that parameter $\gamma$ as in (15).
Then, with the risk neutral value of $\gamma$ and the risk neutral value of $\lambda$ $\left(-\frac{1}{2}\right)$ given in the Heston and Nandi (2000), and the initially estimated values of the parameters $\alpha$ and $\beta$, the option prices are recalculated under the risk neutral measure following the procedure explained in the Section 2.3.1.

The results of the parameter estimations for the in sample and out of sample context in this model can be seen in the Section 7.2 of the Appendix.

### *2.3. Dupire's Local Volatility model*

The mentioned implied volatility in the introduction usually varies with both the strike price (K) an the time to maturity (T). To be able to match market quotes with the Black and Scholes (1973) and Merton (1976) models described in the introduction, we would have to use a different volatility for each maturity-strike combination, implying a different model in each occasion.
This fact leads to problems with options whose payoff depends on the level of the forward rate at different points in time. Pricing these kind of options is not possible with the Black and Scholes (1973) and Merton (1976) models.

One solution for this problem was found by Dupire (1993).
Given the prices of European Calls $C$ of all strike prices $K$ and maturities $T$ $(C(K,T))$, he got a risk neutral process for the underlying asset $(S)$ in the form of a diffusion

$$\frac{dS}{S} = r(t)dt + \sigma(S,t)dW \tag{16}$$

where the instantaeous volatility $\sigma$ is a deterministic function of the spot $(S)$ and of the time $(t)$. By this way, the underlying asset price follows a one dimensional difussion process and the model would be complete.

To illustrate this idea, for a collection of option prices $C(K,T)$ of different strike prices, a risk neutral density function $\varphi_T$ of the spot price of the underlying asset $S$ at time $T$ is yield as it can seen in the following relationship

$$C(K,T) = \int_0^\infty (S-K)^+ \varphi_T(S)\mathrm{d}S \tag{17}$$

and, differentiating twice with respect to the strike price $K$, the risk neutral densitiy function of the spot price of the underlying asset is obtained as function of the strike price:

$$\varphi_T(K) = \frac{\partial^2 C(K,T)}{\partial K^2} \tag{18}$$

This expression of the density function is which must be calculated to get a risk-neutral process for the underlying asset.
In order to know if there is a unique diffusion proces which generates these densities, a conversion problem is posed. The notation $(K,T)$ is changed into $(x,y)$ to generalize the conversion.

$$dx = a(x,y)dt + b(x,y)dW$$

where $W$ is a Wiener process. From the coefficents $a$ and $b$ the conditional distributions $\varphi_T$ can be deduced using the Fokker-Planck equation

$$\frac{1}{2}\frac{\partial^2(b^2 f)}{\partial x^2} - \frac{\partial(a,f)}{\partial x} = \frac{\partial f}{\partial y} \tag{19}$$

where $f(x,y)$ is used to denote $\varphi_T(x)$. Restricting to risk-neutral diffusions, we can recover a unique difussion process from the $f(x,y)$.
As Dupire assumes the interest rate $(r(t))$ is 0, in this development the Fokker-Planck equation results as the following

$$\frac{1}{2}\frac{\partial^2(b^2 f)}{\partial x^2} = \frac{\partial f}{\partial y}$$

and, as $f$ can be written as $\frac{\partial^2 C}{\partial x^2}$, the previous equation can be written in this sense

$$\frac{1}{2}\frac{\partial^2(b^2 f)}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial C}{\partial y}\right)$$

where the function $f$ is known but not $b$.

Integrating twice in $x$ for a constant $y$ and taking the only possible candidate, the following expression is deduced

$$\frac{1}{2}b^2\frac{\partial^2 C}{\partial x^2} = \frac{\partial C}{\partial y}$$

and clearing away $b$

$$b(x,y) = \sqrt{\frac{2\frac{\partial C(x,y)}{\partial y}}{\frac{\partial^2 C(x,y)}{\partial x^2}}}$$

Once this expression is obtained, the instantaneous volatility can be deduced with the spot price of the underlying asset's ($S$) process

$$\sigma(S,t) = \frac{b(S,t)}{S}$$

and taking into account that the expressoin of the risk neutral densitiy function of the spot price of the underlying asset ($\varphi_T$) has been expressed in terms of the strike price ($K$) and maturity ($T$), the Dupire's Local Volatility formula is given by

$$\sigma_{LV}(K,T) = \frac{1}{K}\sqrt{\frac{2\frac{\partial C(T,K)}{\partial T}}{\frac{\partial^2 C(T,K)}{\partial K^2}}} = \sqrt{\frac{2\frac{\partial C(T,K)}{\partial T}}{K^2\frac{\partial^2 C(T,K)}{\partial K^2}}} \tag{20}$$

So, calculating the Dupire's formula, we calculate the instantaneous volatility function which enables the estimation of the process of the underlying asset $S$ under a risk-neutral framework.

Even if the assumption of continuity of the given option prices holds, problems arise in the implementation of the second derivative of the option price from the strike price $\frac{\partial^2 C(T,K)}{\partial K^2}$.

Numerical approximations for the derivatives have to be made, which are imperfect by they nature.
On the one hand, problems can arise when the values are very small and small absolute errors in the approximation can lead to big relative errors, perturbing the estimated quantity. When the disturbed quantity is added to other values, the effect will be limited. This does not happen in Dupire's formula where the second derivative with respect to the strike in the denominator stands by itself. Small errors in the approximation of this derivative will get multiplied by the strike value squared resulting in big errors at these values, sometimes even giving negative values, resulting in negative variances and complex Local Volatilities.

On the other hand, the continuity assumption of option prices is not very realistic. In practice, option prices are known for certain discrete points. Usually, option maturities correspond to the end of a certain fixed period, like the end of a month. So the number of different maturities is always limited. The result of this is that in practice, the inversion problem has not a unique solution and is unstable. An extra problem for the implementation of Dupire's formula.

An easier and more stable method to obtain the Local Volatility surface is to obtain it from the implied volatility surface. This is the method chosen for this work.
The Local Volaitlity can be described as a function of the implied volatility if a change of variables is made in (17) by using $C$ as a function of some other variable. Because of the non disposal of a closed form formula for $C$ to be transformed, this is not possible. But we can make use of the Black-Scholes formula and the concept of implied volatility.

Adapting the method proposed in Gatheral (2006), the option price under the Black and Scholes (1973) and Merton (1976) framework has the following expression

$$C_{BS}(S_0, t_0, K, T, r, \sigma_{imp}) = S_0 e^{-\int_{t_0}^{T} q_s \mathrm{d}s}[N(d_1) - e^y N(d_2)] \tag{21}$$

where

$$y = ln\left(\frac{K}{S_0}\right) + \int_{t_0} T(q_s - r_s)\mathrm{d}s$$

$$d_1 = -\frac{y}{\sqrt{\sigma^2_{imp}(K,T)(T-t_0)}} + \frac{\sqrt{\sigma^2_{imp}(K,T)(T-t_0)}}{2}$$

$$d_2 = -\frac{y}{\sqrt{\sigma^2_{imp}(K,T)(T-t_0)}} - \frac{\sqrt{\sigma^2_{imp}(K,T)(T-t_0)}}{2}$$

Taking derivatives of the expression (21) from the maturity ($T$) and from the strike price ($K$) once and twice, and inserting these results in (20), results in

$$\sigma^2_{LV} = 2\frac{-q_T C + (q_T - r_T)\frac{\partial C}{\partial y} + \frac{\partial a}{\partial T}\frac{\partial C}{\partial a} + (r_t - q_T)\frac{\partial C}{\partial y} + (r_T - q_T)K\frac{\partial a}{\partial K}\frac{\partial C}{\partial a} + q_T C}{\left(\frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y}\right) + 2K\frac{\partial a}{\partial K}\frac{\partial^2 C}{\partial a \partial y} + K^2\left(\frac{\partial^2 a}{\partial K}\frac{\partial C}{\partial a} + \left(\frac{\partial a}{\partial K}\right)^2\frac{\partial^2 C}{\partial a}\right)}$$

where $a = \sigma^2_{imp}(K,T)(T-t_0)$.

Simplifying this expression and putting it in terms of the variable $y$, the Local Volatility formula from the Implied Volatility is obtained

$$\sigma^2_{LV} = \frac{\Sigma^2 + 2\Sigma(T-t_0)\left(\frac{\partial\Sigma}{\partial T} + (r_T - q_T)K\frac{\partial\Sigma}{\partial K}\right)}{\left(1 - \frac{Ky}{\Sigma}\frac{\partial\Sigma}{\partial K}\right)^2 + K\Sigma(T-t_0)\left(\frac{\partial\Sigma}{\partial K} - \frac{1}{4}K\Sigma(T-t_0)\left(\frac{\partial\Sigma}{\partial K}\right)^2 + K\frac{\partial^2\Sigma}{\partial K^2}\right)} \tag{22}$$

where $\Sigma = \sigma_{imp}(K,T)$.

The transformation of Dupire's formula into one which depends on the implied volatility ensures that there no longer is a lone second derivative in the denominator as in (20). Small errors in it will not necessarily lead to large errors in the local volatility function.
But there is still the matter that the implied volatility is not a continuous function of strike and maturity, but only known at certain points. To get the Local Volatility function, some method has to be used to interpolate and extrapolate the given data points into a surface, and we have to acknowledge since obtaining the local volatility out of the data involves taking derivatives, the extrapolated surface cannot be too rough to avoid irregularities in the Local Volatility surface.

*2.3.1. Pricing formula of the Local Volatility model*

The Local Volatility model takes the same variables as the Black and Scholes (1973) and Merton (1976) models with the exception of the implied volatility, which is a correction of the Black Scholes implied volatility in terms of the Local Volatility.

Those variables are, the current underlying price ($S_0$), the strike price ($K$), risk-free interest rate ($r$), amended implied volatility ($\sigma_a$), time until option exercise ($\tau = T - t_0$) and the dividend yield ($q$).
So, the pricing formula of the call option ($C$) is

$$C(S_0, K, T, r, \sigma_a, q) = S_0 \mathrm{e}^{-qt} * N(d_1) - K\mathrm{e}^{-rt} * N(d_2)$$

where

$$d_1 = \frac{ln\left(\frac{S_0}{K}\right) + t\left(r - q + \frac{\sigma_a}{2}\right)}{\sigma_a\sqrt{t}}$$

$$d_2 = d_1 - \sigma_a\sqrt{t}$$

This amended implied volatility is obtained in function of the Local Volatility function as Hagan, Kumar, Lesniewski and Woodward (2002) did in their work using perturbation methods

$$\sigma_a(K,S_0) = \sigma_{LV}\left(\frac{[S_0+K]}{2}\right)\left(1+\frac{1}{24}\frac{\sigma''_{LV}\left(\frac{[S_0+K]}{2}\right)}{\sigma_{LV}\left(\frac{[S_0+K]}{2}\right)}(S_0-K)^2\right) \tag{23}$$

where $\sigma_{LV}$ is the quadratic Local Volatility function that will be described in section 2.3.2 and which depends on the strike price ($K$) and the maturity ($T$).

*2.3.2. Computation of the Local Volatility model*

The first step to obtain the Local Volatility model is to calculate the market Implied Volatility from the given market option prices. To do so, the bisection method is implemented. This scheme is based on the intermediate value theorem for continuous functions that was exposed by Burden and Faires (2000).

Once the Implied Volatility is obtained, the Local Volatility is calculated from it with (22). Since the number of data points is always many times less than the number of grid points for the surface, there are many degree of freedom in the fitting of the surface. Although it cannot be said with certainty which method of fitting and smoothing the Local Volatility surface is the best, in this report the Thin Plate Splines (TPS) method is used, which is considered to be a natural candidate for this type of problem.

The TPS is the two-dimensional equivalent of the cubic spline. It is constrained to go through all the data points and it is the fit with the least amount of curvature. The name of this method comes from the physical process of bending a thin plate methal. If the spline function is denoted by $f(x,y)$, and the bending energy function by

$$J = \int\int_{\mathscr{R}^2}\left(\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 f}{\partial y^2}\right)^2\right)\mathrm{d}x\mathrm{d}y \tag{24}$$

the TPS is found by minimising the bending energy function

$$E = \frac{1}{n}\sum_{i=1}^{n}(f(x_i,y_i)-z_i)^2 + \lambda J \tag{25}$$

where $z_i$ are the $n$ data points at coordinates $(x_i, y_i)$ and $\lambda$ is the smoothing parameter. For $\lambda$=0 the procedure simply finds the interpolation spline. When $\lambda$ >0, the resulting function is smoothed to reduce the function's curvature. By adjusting the value for $\lambda$, the amount of smoothing can be controlled. This procedure ensures the TPS agrees with the original data as good as possible.

From the original paper by Duchon (1976) and the work by Meinguet (1979), it is deduced that there's an unique solution to this problem, which is

$$f(x_i, y_i) = \sum_{j=1}^{n} a_j A_{i,j} + \sum_{j=1}^{3} b_j B_{i,j} \tag{26}$$

Here, $A$ is an $[nxn]$ matrix and $B$ an $[nx3]$ matrix, where $n$ denotes the number of data points.
This function $f$ is the necessary one to interpolate and smooth via splines the Local Volatility function.

When the smoothed Local Volatility surface is obtained, a quadratic function is fitted to it in order to get the volatility in terms of the strike price and maturity to obtain the amended implied volatility. It takes the following form

$$\sigma_{LV} = b_1 + b_2 K + b_3 T + b_4 KT + b_5 K^2 + b_6 T^2$$

Since Local Volatility surface is obtained by using the implied volatility as an input, the right value of these options can be calculated from the Black Scholes equation. However, this implied volatility inserted in the Black and Scholes equation is not the same as its model's. That implied volatility must be adjusted via the Local Volatility function as explained in (23), where the second term inside the brackets is a small correction to the first term. More terms follows in this equation, but they are so small that we don't consider them in this work.

Once we obtain the amended implied volatility, we insert it in the Black and Scholes formula and we get the option prices with this model.

## 3. Option data description

The Spanish IBEX-35 index is a value-weighted index comprising the 35 most liquid Spanish stocks traded in the continuous auction market system. The official derivative market for risky assets, which is known as MEFF, trades a Future contract on the IBEX-35, the corresponding option on the IBEX-35 Mini Future contracts for calls and puts, and individual option contracts for blue-chips stocks.

In the case of the options on this index, there is only an option contract, "'the small". That is, there are only options on the Mini Future contract con the IBEX-35 index.
These contracts give you the right but not the duty of buying or selling a Mini Futurecontract on the IBEX-35 index at a date and with a change previously determined.

The nominal of these contracts is a Mini Future on the IBEX-35 and the multiplier of the contract is 1. In other words, if the index is quoting in 10.800 points, the Future Mini on the IBEX-35 will have as value 10.800€.
This kind of contracts quote in integer points of the Mini Future on the IBEX-35 index with a minimum fluctuation of one point. It is said, if the quote of an option premium is 800€, its inmediatly bottom and mayor quote would be 799€ and 801€ respectively.

Its time to maturity is monthly, the third Friday of each month, as same as the Mini Future on the IBEX-35. In this work, when we refer to the maturity date only giving the month, actually we are refering to July, 21st, August, 18th and September 15th.

These options are European options, so they can only be exercised at maturity. However, if the investor decides to materialize his inversion, it is always possible to close the position with an opposite operation to the initial one.
The risk-free rate used in this work has the value of $0,01563$ which is the corresponding rate to the Government Bonds.
The dividend yield corresponding to the dates used in this work is $0,04$.

The database[1] used for this report comprise call options on the IBEX-35 index Mini Futures traded daily during the period of May 2th and June 28th of 2017 with maturities at the third friday of July, August and September of 2017. All of them have six different strike prices, which go from 9.000€ to 11.500€ in intervals of 500€.
We have 233 observations for the option prices expiring in the month of July, 144 for the month of August and 252 for the month of September.

[1] The data used for this work have been provided by the quantitative department of BBVA.

The underlying asset used for this work, the Mini Future on the IBEX-35, takes a range of values from 10.600€ to 10.900€. This will be taken into account when analysing the performance in reference to the strike prices, taken the last two as being out of the money and the strike price of 10.500€ as reference for the at the money options.

One remarkable fact is that in the case of the Local Volatility model, as it has to be defined a volatility function for each stock price and different strike prices and maturities are needed to do so, the data has been modified taking into account those underlying pices of the Mini Future on the IBEX-35 which have three different maturity dates and excluding those which have two or less maturity dates in order to define as well as possible the Local Volatility function.

This is why we analyse 504 observations for the in sample framework and 336 observations in the out of sample framework.

## 4. Pricing Performance

This section reports the pricing performance of the three different models used in this analysis taking as reference, the market prices provided in the data set. These results are divided in three sections. The first one will show the results in a global manner for each of the three maturities, the second one will focus the results on each of the maturity dates and the third one will focus on the performance of the models in every strike price.
Each of the analysis will be provided for the in sample and the out of sample contexts.

In order to be able to make these comparisons, the Relative Absolute Error (RAE errors from now on) and the Mean Absolute Relative Error or Mean Absolute Percentage Error (MAPE errors from now on) will be deployed. These errors are defined as it follows

$$RAE = \frac{|y_i - \hat{y}_i|}{y_i}$$
$$MAPE = \frac{1}{n}\sum_{i=1}^{n}\frac{|y_i - \hat{y}_i|}{y_i}$$

where $y_i$ is the price of the option observed in the market an $\hat{y}_i$ is the price of the option calculated with the model.
RAE errors will not be displayed in tables, but they are used in order to build the boxplots that will be shown in the following sections.

### *4.1. Global pricing performance*

When analysing the performance of the models in sample and out of sample, we obtained these results[2] expressed in a global vision:

[2]The results of the errors given in all sections are expressed as so much per one. In order to express them as percentage they would be multiplied by 100.

Table 1: Global Mean Absolute Percentage Errors

| | *In Sample* | *Out of Sample* |
|---|---|---|
| *Heston* (1993) | 0,063 | 0,1332 |
| *Heston and Nandi* (2000) | 0,4648 | 0,859 |
| *Local Volatility* | 1,163 | 1,966 |

This table shows the MAPE errors in a global context for the three different models analysed in this work.

According to the results, the Heston (1993) model has the minimum error pricing the options on the Mini IBEX-35 Futures in the two context, in sample and out of sample.
It can be seen that changing from the in sample context to the out of sample context, the Heston (1993) model increases its errors in a 111,43% while the Heston and Nandi (2000) model does the same in a 84,81% and in a 69,05% does the Local Volatility model.

Comparing the three models taking the Heston (1993) as the one which best fits the option prices on the Mini Future on the IBEX-35, in the in sample context the Heston and Nandi (2000) model fits 6 times worse than the stochastic volatility model and 17 times worse does the Local Volatility model.
With regard to the out of sample context, the Heston and Nandi (2000) model performs 5 times worse than the sotchastic volatility model and the Local Volatility worsens 14 times the Heston (1993) model.

In order to examine in more detail and understand these global results, in the following sections they will be analysed from others perspectives.

### *4.2. Pricing performance at different maturity dates*

The errors obtained in the previous Section 4.1 can be summarised in the following graphics and in the table 2 depending on each maturity date for the in sample and out of sample context.

Table 2: Mean Absolute Percentage Error at each maturity in sample context

| | *July* | *August* | *September* |
|---|---|---|---|
| *Heston* (1993) | 0,083 | 0,077 | 0,028 |
| *Heston and Nandi* (2000) | 0,11 | 1,219 | 0,063 |
| *Local Volatility* | 1,424 | 0,74 | 1,33 |

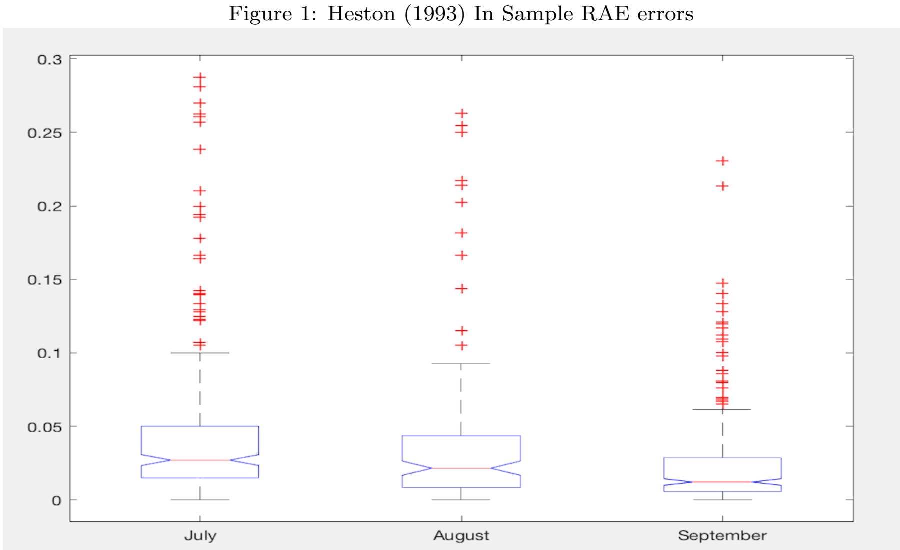
This table shows the MAPE errors when pricing the options with each of the three models in the in sample performace at each maturity date.

With the results of the table 2, it can be detected that the accuracy of the Heston (1993) model is greater when the maturity increases.
Not the same behaviour happens in the Heston and Nandi (2000) model, where on the maturity dates of July and September, the MAPE errors are relatively small, increasing in a 1008,18% from July to August and dicreasing in a 94,83% from August to September. This evolution of the MAPE error in the Heston and Nandi (2000) model suggests a bad fit with the option prices belonging to this maturity date which will be analysed in the following section.

In the case of the Local Volatility model, the maturity date where the model fits the best is on the August date, increasing its accuracy a 48% with regard to the July maturity date and worsening a 79% from August to September.

In order to show this table's results in a graphic display, we show how the RAE errors behave at each maturity with boxplots. This kind of charts visualizes the distribution of a data set. The red line points the median of the distribution and the upper an lower lines of the box point the third quartile and the first quartile of the distribution respectively. The first quartile ($Q_1$) shows until which value does the 25% of the errors reach and the third quartile ($Q_3$) shows until which value does the 75% of the errors reach. The space between $Q_1$ and $Q_3$ is called the interquartile range.
The lines extending from the box reach the maximum or minimum value of the data set or until 1,5 times the interquartile range. If any value exceeds these lines, it will be represented with a red cross indicating atipic values of the data set.

Figure 1: Heston (1993) In Sample RAE errors

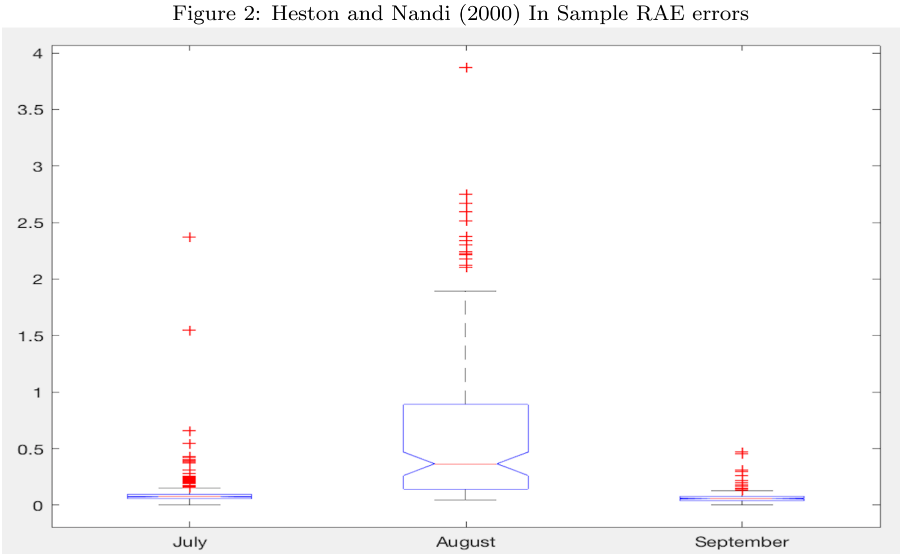



This figure shows the RAE errors in the In Sample performance of the Heston 1993 model and how they are distributed.

As it can be appreciated in figure 1, in the Heston (1993) model the value of the errors decreases as the maturity date moves away and they come together in a more accurate range as shown in table 2.

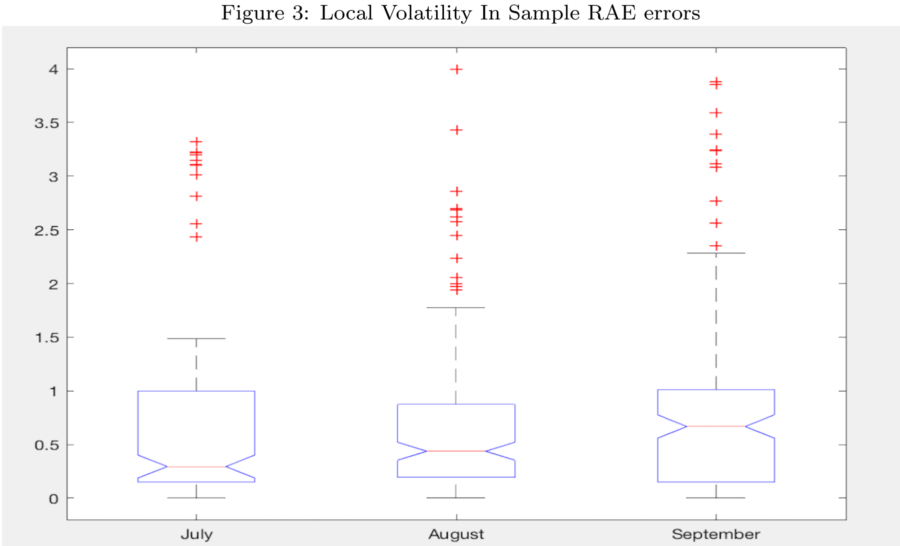
In the case on the Heston and Nandi (2000) model, figure 2 clarifies the evolution mentioned before refering to the August maturity date. As it can be seen, while the 75% of the RAE errors in July and September are consolidated in a small area next to 0, in the case of the August maturity date the RAE errors have a 75% of the values between a value next to 0 to 1, extending the maximum value till 2 and having a considerable amount of extreme errors above 2. This means a bad accuracy of the model and it is reflected in its global performance in table 1 raising the MAPE error up to 0,4648 (46,48%) in a global context.

Figure 2: Heston and Nandi (2000) In Sample RAE errors




This figure shows the RAE errors in the In Sample performance of the Heston and Nandi 2000 model and how they are distributed.

Figure 3: Local Volatility In Sample RAE errors




This figure shows the RAE errors in the In Sample performance of the Local Volatility model and how they are distributed.

In the case of the Local Volatility model, we can see that although the maximum errors increase their value as the maturity date moves away, the median mark raises with the maturity date meaning that inside the interquartile range, the RAE values behave in a different way at each maturity.
On the July maturity date, most of the RAE errors are located between the median and the third quartile while on the September maturity date, these errors are located between the first quartile and the median.
This is why, although the maximum error increases with the maturity date, the RAE errors at September are less than at July as the table 2 shows.
In figure 3 it can be appreciated that the RAE errors at August are less because the interquartile range is the smallest.

Refering to the out of sample performance of the models, the results analysing the MAPE errors in reference to each of the maturities are the following

Table 3: Mean Absolute Percentage Error at each maturity out of sample context

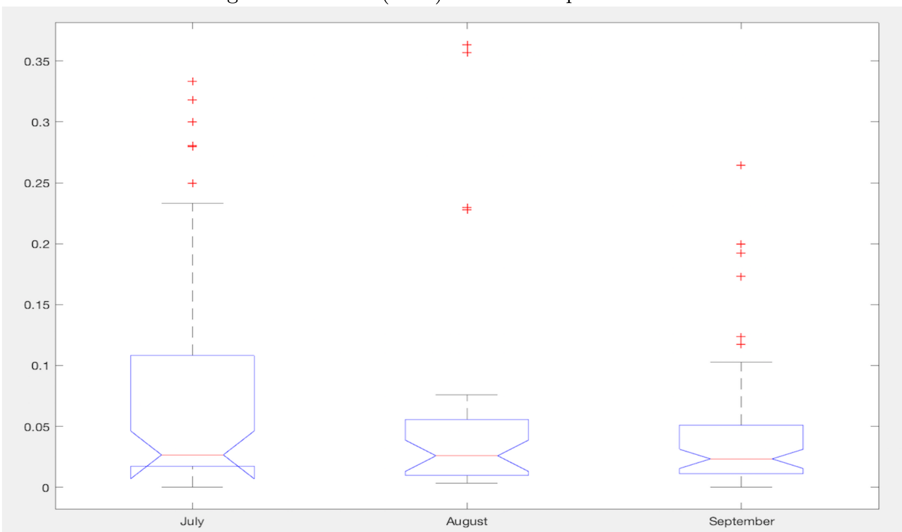
| | *July* | *August* | *September* |
|---|---|---|---|
| *Heston* (1993) | 0,160 | 0,196 | 0,043 |
| *Heston and Nandi* (2000) | 0,251 | 2,2323 | 0,0936 |
| *Local Volatility* | - | - | 1,966 |

This table shows the MAPE errors when pricing the options with each of the three models in the out of sample performace at each maturity date.

As the table 3 shows, the MAPE errors of the Heston (1993) and Heston and Nandi (2000) models are divided in the three maturity dates, but the Local Volatility's errors are only reflected at September. This is because for the out of sample performance with the Local Volatility, we had to get a function depending on all maturities and strike prices for the Local Volatility from each underlying asset. To do so, we used the maturities of July and August and the six different strike prices in order to get that function, and then implement it with the maturity of September and see how it worked.

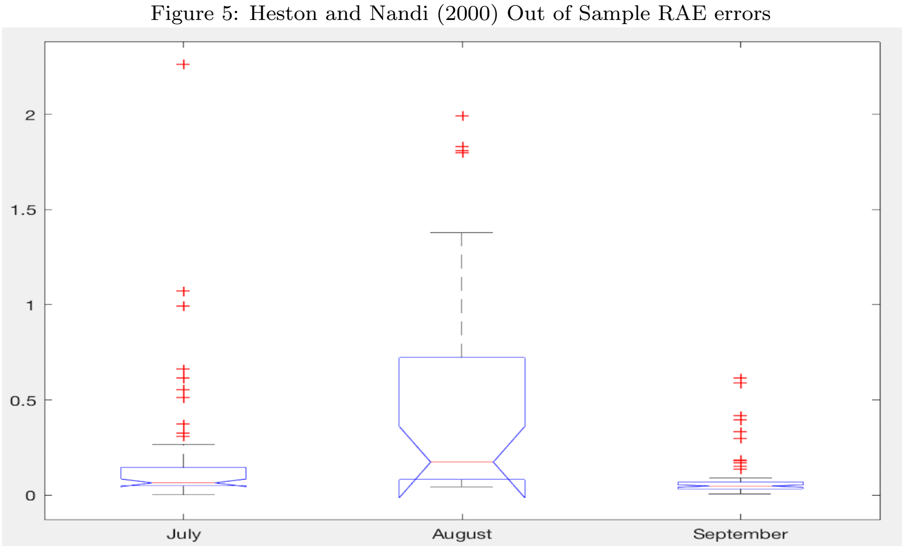
The Heston (1993) model is again the one with the smaller MAPE errors on the three maturities. In contrast to the results obtained in table 2, it can be appreciated a slight increase in the error from July to August of a 22,5%. But it is still at the September's maturity where the model fits the best.

Figure 4: Heston (1993) Out of Sample RAE errors




This figure shows the RAE errors in the Out of Sample performance of the Heston 1993 model and how they are distributed.

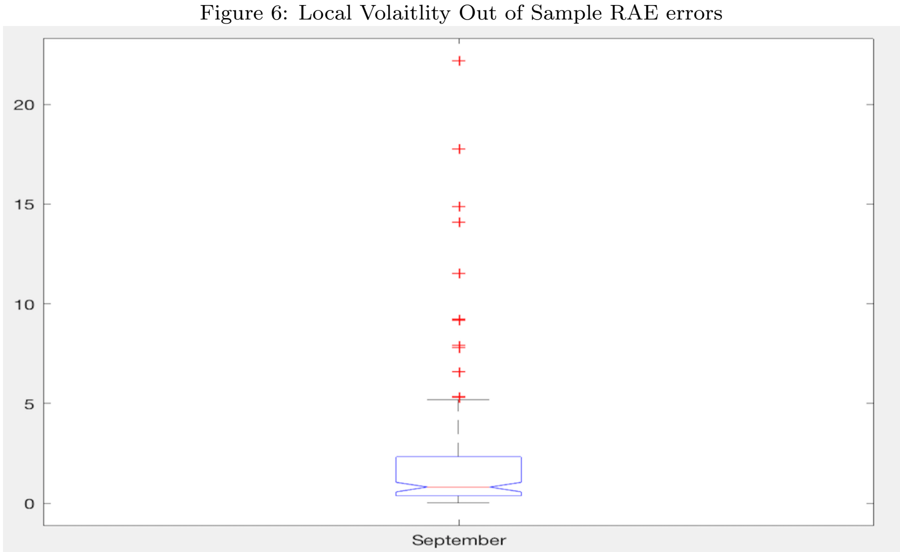
Figure 5: Heston and Nandi (2000) Out of Sample RAE errors




This figure shows the RAE errors in the Out of Sample performance of the Heston and Nandi 2000 model and how they are distributed.

Highlighting the evolution mentioned in the in sample context, it is remarkable that the Heston and Nandi (2000) model has the same behaviour as the one described in figure 2. In this occasion, from July to August the MAPE errors increase in a 789,36%.This high increase of the errors is understandable since the model has a worsening of 84,81% from in sample to out of sample performance. As it happens in the Heston (1993) model, it is on the farthest maturity date where the model fits the best.

Figure 6: Local Volaitlity Out of Sample RAE errors



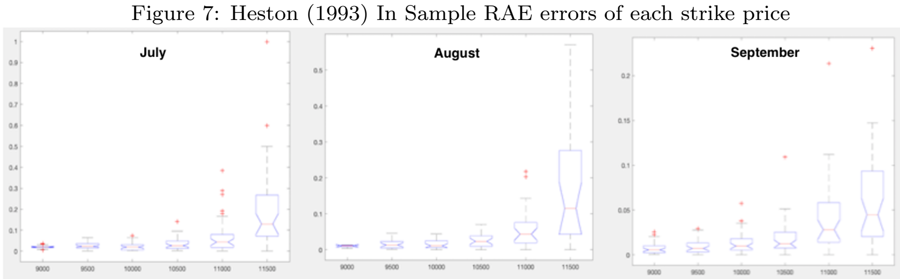

This figure shows the RAE errors in the Out of Sample performance of the Local Volatility model and how they are distributed.

Refering to the Local Volatility model , it can be appreciated the high RAE errors and the high number of atypical errors given in the out of sample context, worsening the performance of the model in a 47,82%.
It is by far the model which fits the worst on the September maturity date in the out of sample context, adjusting 20 times worse than the Heston and Nandi (2000) model and 44 times worse than the Heston (1993) model.

### *4.3. Pricing performance in different strike prices*

In order to be able to explain bad performances at each maturity dates, different boxplots are built pointing out the Relative Absolute Errors given at each strike price of each maturity date.

Figure 7: Heston (1993) In Sample RAE errors of each strike price

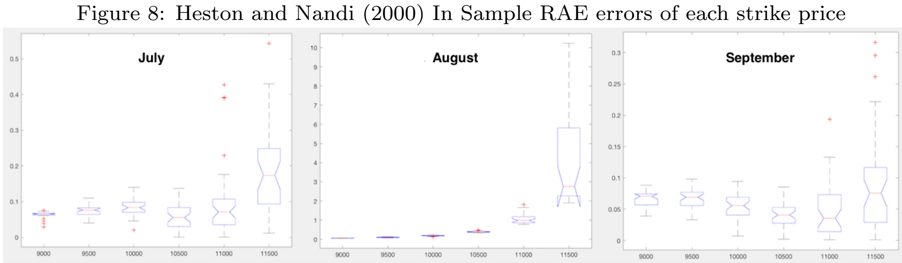



This figure shows the RAE errors of each strike price in the In Sample performance of the Heston 1993 model and how they are distributed.

In figure 7, it can be appreciated that the highest values of the errors in the Heston (1993) model are located in the last strike price (11.500€) of the three maturity dates.
In fact, the values of the errors become higher as we move towards higher values of the strike price. This means that the model responds in a worse way as the strike price takes a higher value.

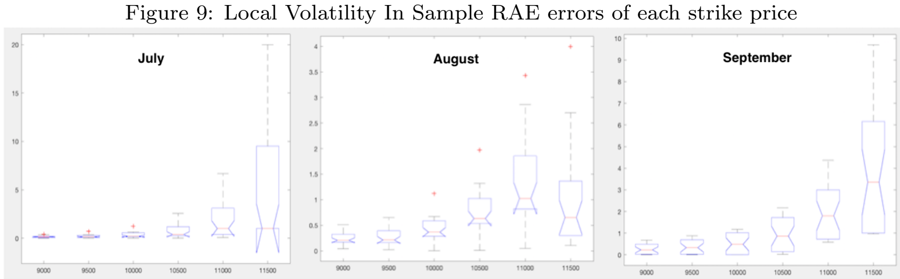
A similar behaviour occurs in the Heston and Nandi (2000) model, where the greatest values of the RAE errors come about in the strike price with value 11.500€.
Unlike the case of the Heston (1993) model, the behaviour of these errors is not ascendent in the three maturity months. In the month of September, form the strike price of 9.000€ to the strike price of 9.500€ is when the 75% of the errors is stored in the highest values until the strike price of 11.000€, when they take greater values. Even so, this difference of values is not too large.
This time, the worst performance of the model is located in the highest value of the strike price too.
As a remark, the scale of errors obtained for the maturity of August is larger than in the others' maturities, reaching error values such as 10 (which, being relative errors, means errors of 1000% relative to the real market option prices). This bad performance in the last strike price of the maturity date of August is reflected in the huge MAPE error value described in table 2.

Figure 8: Heston and Nandi (2000) In Sample RAE errors of each strike price




This figure shows the RAE errors of each strike price in the In Sample performance of the Heston and Nandi 2000 model and how they are distributed.

Figure 9: Local Volatility In Sample RAE errors of each strike price

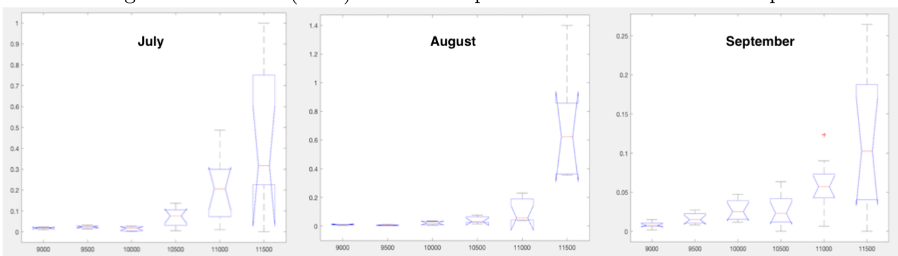



This figure shows the RAE errors of each strike price in the In Sample performance of the Local Volatility model and how they are distributed.

In figure 9, it can be seen that the values of the RAE errors in the Local Volatility model increase upwardly as the strike price value gets higher. At the maturities of July and September, when the models fits worse in sample, the higher values are given in the last strike price. This does not happen in the case of August, where the higher values of these errors, on average, are located in the strike price of 11.000€, being the errors of the last strike price a little smaller.
So, as in the other cases, the performance of the Local Volatility model becomes worse in the last strike prices at each of the maturities.

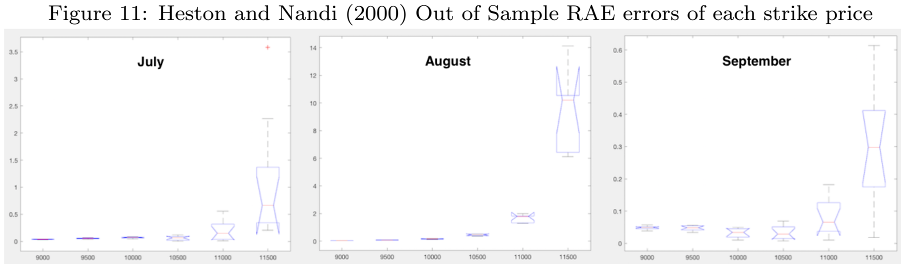
In the context of out of sample, the results in the Heston (1993) model is quite similar to the results given in the in sample context. As it can be seen in figure 10, the value of the Relative Absolute Errors increases as the strike prices become higher, obtaining the highest values in the last strike price again (11.500€). This fact is present at all maturity dates, refuting that this model fits the worst in the strike price of 11.500€ on all maturity dates.

Figure 10: Heston (1993) Out of Sample RAE errors of each strike price




This figure shows the RAE errors of each strike price in the Out of Sample performance of the Heston 1993 model and how they are distributed.

Figure 11: Heston and Nandi (2000) Out of Sample RAE errors of each strike price

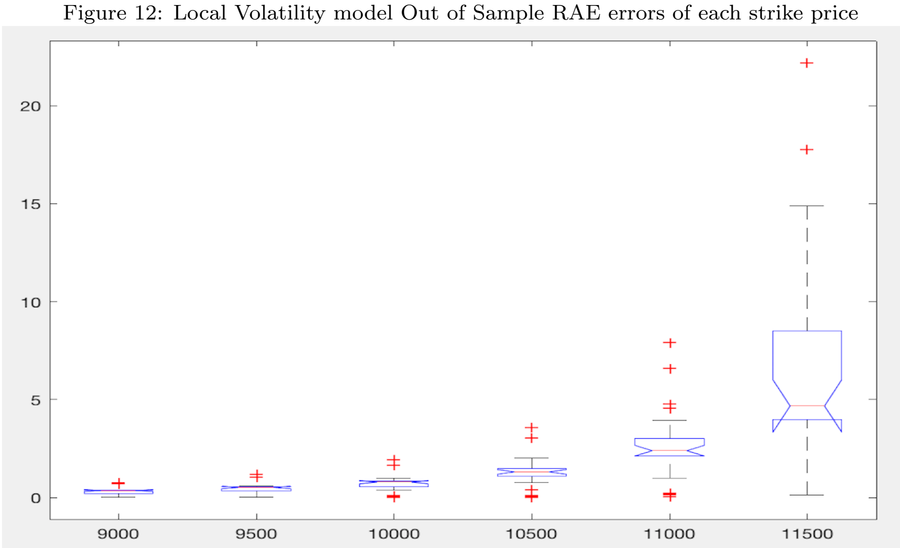



This figure shows the RAE errors of each strike price in the Out of Sample performance of the Heston and Nandi 2000 model and how they are distributed.

In the case of the Heston and Nandi (2000) model, the same behaviour is perceived in figure 11 as in the in sample context. The results obtained in the three maturities clarify that the model fits the worst in the strike price of 11.500€, reflecting again the big difference of the August maturity date, reaching errors of 1400% regarding the real option market prices, what distorts the accuracy observed in table 1.

Regarding to the deterministic volatility model, the out of sample performance in figure 12 shows that the Relative Absolute Errors increase in value as the strike price becomes higher, taking the 75% of these errors the highest values in the two last strike prices (11.000€ and 11.500€).
This fact means that the Local Volatility model fits the worst in the higher strike prices as the models of Heston (1993) and Heston and Nandi (2000) do, and having the same behaviour as in the in sample context.

Figure 12: Local Volatility model Out of Sample RAE errors of each strike price




This figure shows the RAE errors of each strike price in the Out of Sample performance of the Local Volatility model and how they are distributed.

## 5. Conclusions and Remarks

The main objective of this work was to analyse which model solved in the best way the problem of the constant volatility implemented in the Black and Scholes (1973) model. To do so, three different models with three different volatility modelling have been presented. A stochastic volatility model with the Heston (1993) model, a conditional volatility model with the Heston and Nandi (2000) model and a deterministic model with the Local Volatility model.

The results obtained in the previous section suggest that the Heston (1993) model is the one which solves the best the problem discussed at the three maturities, meaning that modelling the volatility with a stochastic behaviour is the best solution of the three exposed in this work.
This suggestion reveals that the volatility modelled with a stochastic behaviour is more realistic, since it takes into consideration what is observed in financial markets, namely the volatility's mean reversion, the leverage effect, volatility clustering and negative correlation between stock returns and volatility.

Indeed, the model permits a fast and easy calibration to the market data since it provides a semi closed form solution and if the volatility weren't mean reverting, it would either go to infinity or go to 0 and stay there. It would be non-stationary.
The model also allows for non-Gaussianity, unlike the Black and Scholes (1973) model, the Heston and Nandi (2000) model and the Local Volatility model (based in the Black and Scholes (1973) model). This fact is consistent with empirical researches which have shown that the distribution of returns is most often non-Gaussian and it is in fact characterized by heavy tails and sharper peaks, as it was seen in Cont (2001).

More over, the leverage effect is modelled in the Heston (1993) model as the correlation $\rho$ and it fits the implied volatility surface realtively well.

Finally, we have shown that extending the model of Black and Scholes (1973) and Merton (1976) but still letting the volatility being deterministic, the performance is still not adjusted to the reality of the market, resulting in huge biases which suggest that a deterministic behaviour is not an advisable way of modelling the volatility.

But, although the results obtained after developing the analysis of the work indicate that the suggestions previously mentioned could be authoritative, we are not able to assert them since the results obtained are not which we were expecting.
It is clear that with the data set employed in this work, the Local Volatility model results being the worst fitting the prices of the option market prices, but the bias produced specially in the out of sample context is too high.

Similarly, it is comprehensible that the errors produced in the Heston and Nandi (2000) model can be greater than the errors given with the Heston (1993), but they are a 637,77% higher in the in sample context and a 545% in the out of sample context.
Although the most amount of the global errors in the Heston and Nandi (2000) model is located at August maturity date, these differences are too high of what is usually given between these two models.

These discrepancies and oddities can be considered for the small size of the data set at each maturity and specially in the August maturity date data set. As described in Section 3, the number of observations at each maturity does not exceed the 300 observations.

In this way, the bad accuracy of the Local Volatility model can arise because we are force to build the Local Volatility function with only two maturities. Having more observations in order to build that function, the accuracy could be improved and the errors reduced.

This is why we suggest in order to obtain a more reliable conclusions, to acquire this analysis with a larger data set.

Another fact that must be taken into account is that, if the error given by the Heston and Nandi (2000) model on the August maturity date was not so big, at the time of choosing between the Heston (1993) model and the Heston and Nandi (2000) model, we would have to take into account that although the errors increase a 70% on average with the Heston and Nandi (2000) model (which continue being relatively small because of the great accuracy of the Heston (1993) model), the time it takes to computationally solve the Heston (1993) model and get the option prices is 4,8 times greater (92,147sec. versus 15,943sec.) than computationally solving the Heston and Nandi (2000) model and get those option prices with it.

This fact leads us to pose that in the choice of a model to price these options, in practice it may be better loosing a little bit of precission but gaining greater speed, getting the results in a more inmediate way.

As final three remarks, the three models show a bad accuracy pricing these options on the Mini Future on IBEX-35 in the last two strike prices at each maturity. Taking into account that the price of the Mini Future on IBEX-35 on the dates of the data set moved between 10.600€ and 10.900€, we can suggest that the options with strike prices of 9.000€ and 9.500€ were in the money (ITM), the options with strike prices of 10.000€ and 10.500€ were at the money (ATM) and those with strike prices of 11.000€ and 11.500€ were out of the money (OTM).
So, in this work, it is revealed that in the three cases, the models fits the worst at each of the maturities the options which are out of the money.

The second remark deals with the $\gamma$ parameter of the Heston and Nandi (2000) model. As it can be seen in the Section 7.2 of the Appendix, at the maturities of July and September the value of this parameter is really high, pointing out the great asymmetry of the distribution that is collected on these maturity dates, meaning that large negative shocks ($z(t)$) raise the variance more than a large positive shock.
This fact wouldn't have been reflected by the model if we had not defined this parameter as in (15), producing higher biases in the model if we had taken the risk neutral $\gamma$ parameter defined by the Heston and Nandi (2000) model.

Finally, the third and last remark goes with the idea that the reason why the Heston and Nandi (2000) model has a poorer performance than the Heston (1993) model is that it does not incorporate the information contained in the cross-section of option prices, in spite of the fact that the asymmetric GARCH parameter is estimated implicity from option data. It cannot generate the skewness and the kurtosis needed to price these options because the volatility inferred from the history of the index returns is not high enough.

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## 7. Appendix

### *7.1. Parameter estimation of the Heston (1993) model*

Table 4: Parameters of Heston (1993) model In Sample

| | *July* | *August* | *September* |
|---|---|---|---|
| $\kappa$ | 22,37 | 1008,255 | 0,486 |
| $\rho$ | 0,385 | -0,7403 | -0,565 |
| $\sigma$ | 0,004 | 4,99 | 0,28 |
| $\theta$ | 0,0205 | 0,0173 | 0,0808 |
| $\upsilon$ | 0,0038 | 0,0000000196 | 0,0189 |

This table shows the estimation of the Heston's (1993) model in sample context.

Table 5: Parameters of Heston (1993) model Out of Sample

| | *July* | *August* | *September* |
|---|---|---|---|
| $\kappa$ | 15,825 | 1014,69 | 0,0477 |
| $\rho$ | 0,5247 | -0,305 | -0,585 |
| $\sigma$ | 0,026 | 4,99 | 0,249 |
| $\theta$ | 0,0243 | 0,0172 | 0,999 |
| $\upsilon$ | 0,00009 | 0,000584 | 0,015 |

This table shows the estimation of the Heston's (1993) model out of sample context.

*7.2. Parameter estimation of the Heston and Nandi (2000) model*

Table 6: Parameters of Heston and Nandi (2000) model In Sample

| | *July* | *August* | *September* |
|---|---|---|---|
| $\alpha$ | -0,000000137 | 0,0001997 | -0,000000137 |
| $\beta$ | 0,7965 | 0,39984 | 0,7965 |
| $\gamma$ | 2100 | 5,999 | 1700 |
| $\lambda$ | -0,5 | -0,5 | -0,5 |

This table shows the estimation of the Heston and Nandi's (2000) model in sample context.

Table 7: Parameters of Heston and Nandi (2000) model Out of Sample

| | *July* | *August* | *September* |
|---|---|---|---|
| $\alpha$ | -0,000000708 | 0,00000568 | -0,000000844 |
| $\beta$ | 0,4001 | 0,39995 | 0,282 |
| $\gamma$ | 1399,99 | 5 | 1199,99 |
| $\lambda$ | -0,5 | -0,5 | -0,5 |

This table shows the estimation of the Heston and Nandi's (2000) model out of sample context.