Arbitrage-Free SABR: Partial Differential Equation Approach to Fix Negativity Density Function Issue

Julián Arce Arellano

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Directores: Dr. Manuel Moreno Fuentes
Gregorio Vargas Martínez

Universidad Complutense de Madrid

Universidad del País Vasco

Universidad de Valencia

Universidad de Castilla La Mancha

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Julián Arce Arellano †

Academic supervisor: Manuel Moreno Fuentes ‡
Industry supervisor: Gregorio Vargas Martínez ‡‡

†andarce@ucm.es
‡Associate professor at University of Castilla La Mancha
‡‡Market Risk Manager at Nfq Advisory Services

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Abstract

Over the last few years, after the credit crunch, interest rates derivatives pricing has become an arduous task due to the negative interest rates policy introduced by the European Central Bank. The benchmark SABR model has evolved to cope with this environment. As is widely known, SABR formulas have approximation errors for extra low or even negative strikes leading to arbitrage opportunities. Thesis's aim to provide a detailed description about the Arbitrage-Free SABR framework to deal with the arbitrage trouble including a full review of previous approaches, partial differential equations and market's caps and swaptions are used to test the Arbitrage-Free SABR context.

Contents

Introduction

The negative interest rates policy issued by the European Central Bank is a very controversial issue that currently affects banks and financial markets in general. With this policy, banks must face to make profits due to their business model and, on the other hand, all models in order to value interest rate derivatives were designed for positive interest rates. An interest rate is the price of money, and therefore it was never thought that the price of money could be negative. In this sense, all valuation models had to be rethought to see how the context of negative interest rates would affect the price of financial derivatives on interest rates.

The SABR model designed by Hagan et al. in 2002 is the star model for valuing financial products such as caps, floors or swaptions. Its motivation was to try to correct the well-known problem of the volatility smile. In this way, the objective of this model is to fit the volatility smile for the simplest options on the market, such as vanilla options. Therefore, in this model an implied volatility can be obtained from market prices.

The approach provided by them presents a problem related to such implied volatility. This matter comes from Taylor's approximations, and they did not take into account that for certain parameters of the SABR model, such as the expiration of an option or very low strikes (even negative), the implied volatility obtained was not correct leading to arbitrage opportunities. The way to determine the failure is by means of the probability density function of the interest rate at maturity. That is, for options with long maturity and with strikes close to zero or even negative as is the current context, this density function is negative leading to arbitrage opportunities in the market.

The main objective of this Thesis is to develop a complete understanding of this issue and to present in detail the solution to the problem which consists of an approach based on solving a partial differential equation using the Crank-Nicholson scheme. This approach was developed by Hagan et al. (2014) and this Thesis aims to test this approach with market data for caps and swaptions.

This MSc Thesis is divided into two main parts. The first one focuses on the theory. Chapter 1, places the reader in the current context of interest rates,the policy adopted by the European Central Bank and the impact on the interest rate derivatives valuation. Chapter 2 introduces the mathematical concepts necessary to develop the interest rate models which will be presented in Chapter 3, reviewing different previous models. Finally, in Chapter 4 the Arbitrage-Free SABR model proposed by Hagan et al. in 2014 is discussed in detail.

The second one is devoted to the practical part, starting with chapter 5 describing the data used in this Thesis. Chapter 6 comments on some detailed procedures that are prior to solving the partial differential equation (PDE). All calculations in this Thesis have been made in Python 3.7 programming language, including the calibration procedure and the resolution of the PDE. In addition, the Crank-Nicholson scheme is tested with market data for caps and swaptions. Conclusions are presented in the final part where the work performed is commented and suggestions of possible lines of research or extensions to this work are discussed, followed by two appendices with some explanatory theoretical notes.

Part I

Theoretical Background

Chapter 1

Context of Negative Interest Rates

In this chapter we will review some history about why a negative interest rate environment has been created in Euro zone. This is a concept which was developed a long time ago and due to the credit crunch in mid 2007, the European Central Bank decided to implement this negative interest rate policy.

1.1 History

A first idea of imposing a negative interest rate comes from an economist called Silvio Gesell (1862-1930) who was born in Germany and grew up in Argentina. The main concept of his theory is about fixing taxes on money, specifically, he created a special money apart from the legal tender in circulation with the aim of spurring the economy and the important issue is that the special money had an expiration date. In this way, as time was going by, the money was becoming worthless and therefore merchants had to keep money circulating.[1]

By 1970s, the Swiss National Bank introduced negative interest rates to weaken swiss francs for the first time. Switzerland had a stable currency while the U.S and other countries had very unstable currencies. In this manner, investors saw the swiss franc as a safe-haven currency, they started buying francs and it was strengthened making exports less competitive. For that reason, a negative interest rate environment was necessary to make the swiss franc an unstable currency and purchases of francs less desirable.[2]

In recent years, because of the financial crisis, several central banks of different

[1] See [1] for further information.
[2] The interested reader can find more details in (for example) [2] or [3].

countries have adopted negative rate policies. Some of those banks will be chronologically listed below:[3]

1. **Riksbank:** Sweden's central bank was the first bank to try out with negative interest rates policy by fixing the rate payed on commercial bank depositis to -0.25% in 2009.

2. **Danmarks Nationalbank (DNB):** In July 2012 this bank set the interest rate to -0.20% and a few years later in 2015 it went this rate down to -0.75%.

3. **Swiss National Bank (SNB):** It introduced the negative interest rate policy (-0.25%) by December 2014 and one month later the rate was reduced down to -0.75%.

4. **European Central Bank (ECB):** By 2014, the European Cental Bank decided to fix the interest rate to -0.2% and it has been decreasing until it reached the minimum historical of -0.4% since 2015.[4] Currently, the ECB announced on June 6 2019[5] that this rate will keep the level of -0.4% at least through the first half of 2020.

5. **Bank of Japan (BoJ):** Japan was the last country to adopt the negative interest rate policy by 2016 and it was set to -0.1%. Currently, BoJ keeps its monetary policy to -0.1%.[6]

1.2 Causes and impact on the economy

It is a fact that financial crisis, which occurred in August 2008, was due to a combination of factors affecting the whole financial system. One of the main factors was the credit risk, a significant element that had not been taken into account at that point. That is, banks created huge sums of new money by making loans without taking into consideration whether the loan was going to be repaid by the counterparty. As a result, largest financial companies collapsed and the credit standards became a key

[3]See https://www.bankofgreece.gr/Pages/en/Bank/News/Speeches/DispItem.aspx?Item_ID=347&List_ID=b2e9402e-db05-4166-9f09-e1b26a1c6f1b for additional information about decisions has been made.

[4]It is necessary to bear in mind that ECB has three official interest rates: the interest rate on the main refinancing operations, the interest rates on the marginal lending facility and the interest rates on the deposit facility. The interest rate at which we refer here in the text is on the deposit facility.

[5]The interested reader can find the press release at https://www.ecb.europa.eu/press/pressconf/2019/html/ecb.is190606~32b6221806.en.html.

[6]The press release issued on March 15 2019 can be found at https://www.cnbc.com/2019/03/15/asia-markets-us-china-trade-boj-decision-brexit-in-focus.html.

factor of the market risk.

To avoid a similar situation in the future, policy makers took matter into their own hands by imposing negative interest rates to banks for depositing its money at the central banks. These exceptional measures were managed by the ECB (among other central banks reviewed in the previous section) and it was about fixing a negative value (since 2014) for the deposit facility rate, below the theoretical lower bound of zero percent.[7] The main goal of this measure consists of simulating the economy growth by encouraging banks to lend or invest excess reserves rather than experience a guaranteed loss. Namely, a negative interest rate environment will reduce the costs to borrow for companies and households, driving demand for loans and incentivizing investment and consumer spending.[8]

Due to this measure, some banks (for instance, Royal Bank of Scotland in 2016,[9] some German banks in 2016,[10] BBVA and Santander Banks in 2016[11]) have also charged a negative interest rate to big and corporative clients with large balances.

1.3 Impacts on interest rate derivatives pricing

The use of financial derivatives has experienced a significant growth as shown in BIS.[12] The trading total amount (mainly in Europe) was about $72 trillion in 1998 in terms of notional amount for FX, equity and interest rate derivatives and it rose up to $594 trillion in the half first of 2018.

[7] In a sense it is a theoretical lower bound because this means that banks would have to pay to keep their excess reserves stored at the central bank rather than receive positive interest income. In other words, a lender can choose not to lend or just participate on any funds and this is comparable with getting an interest rate of zero. This is not the best idea, but better than an interest rate below zero.

[8] Currently, this is a controversial issue becuase there are some risk associated with a negative interest rate environment. If banks penalize households for saving, that might not necessarily encourage retail consumers to spend more cash. Instead, they may store cash at home. On the other hand, since it is logistically difficult and costly to transfer and store huge sums of physical cash, some banks are paying negative interest on their deposits. In [4], recent literature that deals with how negative interest rates can badly affect banks is reviewed and an empirical study about the significance of the various channels through which negative interest rates may cause an adverse effect on net interest income within the euro area is summarised. This is just for further information and it is a topic that goes beyond the scope of this Thesis.

[9] See for more information `https://www.theguardian.com/business/2016/aug/22/rbs-charging-customers-cash-negative-interest-rates`

[10] The briefing can be found at `https://business.financialpost.com/news/economy/german-bank-starts-charging-customers-to-hold-their-cash-in-negative-interest-rate-world`

[11] The press information is at `https://www.elperiodico.com/es/economia/20160824/banca-espanola-cobrar-depositos-grandes-clientes-5341519`.

[12] `https://www.bis.org/statistics/about_derivatives_stats.htm`

The new context has several effects on the financial system. Since an interest rate is a price, none expected it would have reached negative levels, hence none needed to explicit a shared way to handle negative rates in collaterals,[13] derivative contracts, bonds and savings.

The consolidated Black (76)[14] framework has become unfeasible for interest rate option valuation, since its assumptions of a log-normal distribution of the underlying interest rates to be modeled implies positive values which does not capture the current situation. Moreover, new models not only has been used to deal with negative interest rates but also to deal with the "smile problem". The trouble with these new models is that in negative interest rates environment, the derivatives pricing reveals arbitrage opportunities.

The previous paragraph contains a strong motivation for the development of this Thesis: derivatives pricing within a negative interest rate environment. In Chapter 3 a full detailed list of models which deal with negative interest rates is studied. Problems and benefits of each model in Chapter 4 will be discussed and an approach (Arbitrage-Free SABR) that copes with arbitrage opportunities will be studied in detail.

1.4 EURIBOR/LIBOR role

We have been talking about negative interest rates, but it is time to clarify what the negative interest rate really is in the market and why it is relevant in global economy. They are the EURIBOR and LIBOR rate.[15]

The **EURIBOR** is the acronym for Euro Interbank Offered Rate and it is based on the average interest rates at which main european banks borrow funds from one another for short-term loans. In total there are 6 different Euribor rates depending

[13] A collateral is very briefly adding an extra money to the price of a derivative in order to mitigate the credit risk.

[14] Model which will be studied in Section 3.2.

[15] These interest rates, in general, the Interbank Offered Rates (IBORs) will be gradually replaced from October 2019 by a set of overnight risk-free rates (RFRs) in order to achieve more robust and credible reference rates. The transition will start with overnight rates, which will be replaced by the definitive euro short-term rates (ESTER). On the website `https://www.bis.org/publ/qtrpdf/r_qt1903e.htm` there is a full report dated March 2019 about this (controversial) issue and the reader can refer to this page for further information.

on its matuirities: 1 day, 1 week, 1 month, 3 months, 6 months and 12 months. Currently (on date 16 June of 2019), they are all in negative levels and these data are published every day morning.

The **LIBOR** stands for London Interbank Offered Rates and it is the rate of interest that a set of major banks change each other for short-term loans. It is an indication of the average rate at which contributor banks can borrow money in the London interbank market for a particular period and currency.

As we can note, EURIBOR and LIBOR are comparable rates.[16] The main difference is that LIBOR is calculated for 5 currencies (the US dollar, the euro, the British pound, the Japanese yen and the Swiss franc). The Libor (euro) rate is (on date 17 June 2019) in negative levels for all its maturities. The Euribor/Libor rates are used worldwide in a extensive variety of financial products such as interest rate swaps, interest rate futures/options and swaptions.[17] Banks also use the Libor interest rates as the base rate when setting the interest rates for loans, savings and mortgages.[18] In such a way, Euribor/Libor rate will be modelled throughout this Thesis by means of different models that allow negative interest rates until we reach the final approach Arbitrage-Free SABR. From now on, if we refer to Libor rate, we will actually refer to any of these interbank rates.

In the following chapter, we will introduce some key mathematical aspects in order to develop the rest of the Thesis.

[16]For more information about the group of banks that participate in the formation of the EURIBOR/LIBOR rates and more precise details about these rates the interested reader can find it at, for example, `https://es.euribor-rates.eu/que-es-el-euribor.asp`

[17]Some of these instruments will be explained later in Section 2.2.

[18]See `https://www.global-rates.com/interest-rates/libor/libor-information.aspx` for more information.

Chapter 2

Preliminaries

In this chapter we present some basic mathematical definitions which are needed to develop the main argument of this Thesis. To carry it out we have mainly based on [5] with some support of [6]. We review some basic definitions about different kind of interest rates, the single-curve and multi-curve approach, the main interest rate derivatives, stochastic processes focus on change of numeraire and derivatives valuation under a certain measure and finally some concepts about arbitrage and the risk neutral probability density function which plays a significant role in the development of this Thesis.[1]

2.1 Basic definitions

We will start introducing the first mathematical object that represents a riskless investment: a bank account or money market account.

- **Bank account (Money market account).** We define $B(t)$ to be the value of a bank account at time $t \geq 0$ as

$$B(t) = B(0) \exp\left(\int_0^t r(s)ds\right), \tag{2.1}$$

  which is the solution of the differential equation $dB(t) = r(t)B(t)dt$ and $B(0)$ is the initial amount invested at time $t = 0$.

- **Numeraire.** A numeraire $N(t)$ is an asset on the market having a strictly positive value and not paying any dividend. For instance, the bank account $N(t) = B(t)$ is a numeraire. In other words, a numeraire can normalize the price of any financial product as follows: if $P(t)$ is the price of a financial product, then its price expressed in terms of the numeraire is $\frac{P(t)}{N(t)}$.

[1] An experienced reader can skip this chapter and a standard reader should follow (for example) [7] for some extra details.

The discounted prices $\frac{P(t)}{B(t)}$ correspond to prices expressed in a particular choice of numeraire: the bank account. As it is well known, in the Black-Scholes-Merton framework, under the risk neutral measure $\mathbb{Q}$ the discounted prices of assets are martingales. Measure $\mathbb{Q}$ is actually the martingale measure associated to the numeraire choice $B(t)$. We make special emphasis in this issue, becuase the choice of a specific numeraire is key to price different interest rate derivatives in a simple way such as caps, floors or swaptions.[2]

- **Zero-coupon bond.** A T-maturity zero-coupon bond (pure discount bond) is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t < T$ is denoted by $P(t, T)$. Clearly, $P(T, T) = 1$ for all T.

- **Stochastic discount factor.** The (stochastic) discount factor $D(t, T)$ between two time instants t and T is the amount at time t that is equivalent to one unit of currency payable at time T, and is given by
$$D(t, T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r(s)ds\right). \tag{2.2}$$
The randomness comes from the stochastic distribution of $r(t)$.[3]

- **Time to maturity.** The time $\tau = T - t$ is the amount of time (in years) from the present time to the maturity time $T > t$. There are several ways of measuring the remaining time between dates t and T. For this reason, the following concept comes below.

- **Day-count convention.** The day-count convention $\delta(t, T)$ is defined as the measure between t and T. The most common day-count conventions are: Actual/365, Actual/360 and 30/360. It is understood that the reader is familiar with this concept but if needed to refresh, the reader could consult, for example, [8]. However, otherwise speaking, we could say that day-count convention is a way to count days among two future payments.

- **Tenor.** We define the tenor of an interest rate derivative as the time to maturity for the underlying fixed income product. Therefore "maturity" is reserved for the time to maturity of the derivative.

[2] We will see this statement in Section 2.3.

[3] This interest rate is known in the literature as short rate or instantaneous spot rate and it is important to bare in mind that this is different from the forward rate, which will be under consideration during the whole thesis and hence an explicit definition of both short and forward rate will be given later on.

2.1.1 Simply-compounded forward interest rate

The symply-compounded forward interest rate prevailing at time t for the expiry $T_1 > t$ and maturity $T_2 > T_1$ is denoted by $F(t; T_1, T_2)$ and is defined by[4]

$$F(t; T_1, T_2) := \frac{1}{\delta(T_1, T_2)} \left(\frac{P(t, T_1)}{P(t, T_2)} - 1 \right). \tag{2.3}$$

We will see that this rate is closely linked to a FRA, a derivative which will be explained in Section 2.2.

We will introduce the instantaneous forward interest rate below whose maturity S is very close to its expiry T, namely $F(t, T, T + \Delta T)$ with ΔT adequately small. In other words, it will be the interest rate prevailing at time t for a infinitesimal period $[T, T + \Delta T]$, with $T > t$.

2.1.2 Instantaneous forward interest rate

The instantaneous forward interest rate prevailing at time t fot the maturity $T_1 > t$ is denoted by $F(t, T_1)$ and is defined by

$$F(t, T_1) := \lim_{T_2 \to T_1^+} F(t; T_1, T_2) = -\frac{\partial \ln P(t, T_1)}{\partial T_1}. \tag{2.4}$$

The previous definition is obtained as follows:

$$\begin{aligned}
F(t, T_1) &:= \lim_{S \to T^+} F(t; T_1, T_2) \\
&= \lim_{T_2 \to T_1^+} \frac{1}{\delta(T_1, T_2)} \left(\frac{P(t, T_1)}{P(t, T_2)} - 1 \right) \\
&= -\lim_{h \to 0} \frac{1}{P(t, T_1 + h)} \frac{P(t, T_1 + h) - P(t, T_1)}{h} \\
&= -\frac{1}{P(t, T_1)} \frac{\partial P(t, T_1)}{\partial T_1} \\
&= -\frac{\partial \ln P(t, T_1)}{\partial T_1}.
\end{aligned}$$

We reiterate that the instantaneous forward rate is the rate to be modelled throughout this Thesis and it is important to take into account that this rate is a theoretical construction (therefore it does not exist in markets) used in literature which allows us to obtain closed pricing formulae in continuous time or, failing that,

[4] An interested reader can find how the forward rate is obtained from bond prices in [6].

analytical expressions[5] which will allow us to price different derivatives.

We can observe that the short rate $r(t)$ is a particular case of the instantaneous forward rate. In fact, as shown in [6], $r(t)$ may be defined as follows:

$$r(t) := F(t,t) = \lim_{T_1 \to t^-} F(t, T_1). \tag{2.5}$$

Another important concept linked to the instantaneous forward rate is the ***forward curve*** which just consists of the graph $T \mapsto F(t,T)$ and it will be mentioned shortly.

Below, we are going to review two key concepts which are closely linked to interest rate derivatives pricing and to carry it out we will mainly follow [9].

2.1.3 Single-curve framework

Before the credit crunch in the second half of 2007 the traditional approach to be used in order to price an interestt rate derivative was the so called *single-curve approach*.

In the light of the forward's denition (2.3), the concept of single-curve approach consists in selecting the most convenient (**liquid**) plain vanilla interest rate instruments traded on the market to build a single curve to be used both as a discounting curve and as a forwarding curve. In this case, the single curve is given by the graph $T \mapsto P(t,T)$ for every maturity T and is used for both discounting the future cash flows and to build/estimate the forward rate. A common choice in the EUR market is a combination of short term EUR deposits, medium-term Futures/FRA on Euribor 3M and medium/long term swaps on Euribor 6M.[6].

We will not go into detail, but will give the main reasons reasons why a change of approach was needed. An **Overnight Index Swap** (OIS) is a common swap in which a fixed leg (a fixed rate) is exchanged against a floating leg (a variable rate) whose value is calculated as the geometric mean of a daily overnight rate (Eonia for EUR). The OIS market is currently liquid and there are OIS swaps with different maturities and due to its daily tenor (the shortest available in the market) the credit risk is mitigated. Contracts including the OIS rate (Eonia for EUR) are considered

[5] As is the case of SABR model where an implied volatility is obtained from asymptotic expansions in order to be able to work in continuous time. In any case, all these aspects such as SABR models and implied volatility will be seen in detail in the next chapter.

[6] FRA and swaps instruments will be defined in Section 2.2

less risky than those including Euribor/Libor rate for the reason just explained. The key factor is that the spread (difference or gap) between the OIS rate and the Libor was negligible (10 bp[7]) before 2007. This spread is a risk indicator of the money market becuase if there is a wide spread , then it shows that banks are in panic and they are afraid of lending money to other banks and if they do it, they will do in exchange for a high premium, that is, a high rate, as it happened in 2008 when the spread dramatically climbed up to 365 bp (see Figure 2.1).

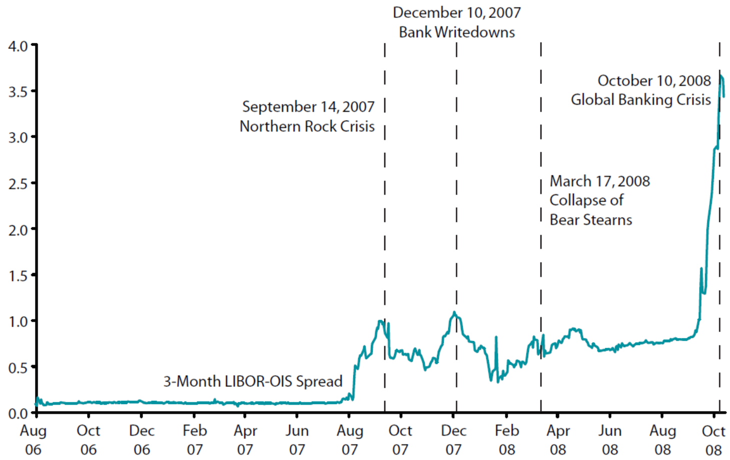

Figure 2.1: 3-Month Libor-OIS Spread in the financial crisis. Source: See [10].

To sum up, before 2007 Euribor was used as a risk free rate and only one curve was needed for generating the future cash flows and for discounting them. After 2008, this practice does not hold anymore and it became necessary to switch to the construction of a new risk-free discounting curve, denoted in this Thesis as $T \mapsto P(t,T)$ for every maturity T, and in the market, it refers to the OIS curve (the best proxy for a risk free rate).[8]

The discounting and forwarding curves are crucial for interest rate derivates pricing. In such a way, in markets, the OIS curve used as a single-curve for discounting and forwarding is not held anymore since 2008. All interest rate derivatives such as caps/floors and swaptions are priced under the *multi-curve approach* and that is why

[7] Basis points (bp).

[8] To be precise, as stated in [9], the construction of the discounting curve is currently a controversial issue. There are "two types" of discounting curves that may be encountered in the market: a) the bootstrapping procedure based on the selection of the most liquid instruments in the market (deposits, FRAs and swaps) and b) the OIS curve, based on the overnight rate (Eonia for EUR), considered as the best proxy for a risk free rate available on the market becuase of its 1-day tenor, justified with collateralized (riskless) counterparties. The specific construction is a technical topic that goes beyond the scope of this Thesis.

we decided to follow the multi-curve approach in this Thesis with the aim of achieving better fits of the parameters of the benchmark SABR model.

Following this line, the multi-curve approach will be reviewed below.

2.1.4 Multi-curve framework

In the single-curve approach, the construction of the discounting curve is based on the most liquid intruments, regardless of their tenor. It is a fact, that for instance, a 6-month Euribor-based swap is riskier than a 3-month Euribor-based swap. Therefore, the main change that includes the multi-curve approach is about the selection of the instruments to construct a forward curve, now depending on the underlying rate of the interest rate derivative that we want to price. For example, if we want to price a 6-month Euribor-based swap, we have to construct a specific 6 month Euribor forward curve by using Euribor-based instruments consistent with that tenor. This implies that we need as many forward curves as the tenors available in the market are. That is why the new pricing approach is called "multi-curve approach".

As stated in [9], the selection of instruments is a complicated task because there is not an unique financially sound recipe for selecting the bootstrapping instruments and rules. The procedure includes non trivial algorithms to pruduce smooth curves, multiple bootstrapping instruments implies multiple sensitivities[9] and about technology, the pricing libraries, platforms, etc. must be extended, configured, tested and release to manage multiple and separated yield curves for forwarding and discounting.[10]

We reiterate that we follow in this Thesis the multi-curve approach and hence, in practice, we are not going to compute fhe forward rate as in formula (2.3), but as follows:

$$F(t;T_1,T_2) := \frac{1}{\delta(T_1,T_2)}\left(\frac{P^e(t,T_1)}{P^e(t,T_2)} - 1\right), \quad t < T_1 < T_2, \tag{2.6}$$

where $P^e(t,T)$ is the estimated discount factor at maturity T in the multi-curve framework only used to compute the forward curve and $P(t,T)$ will be the (OIS) discount factor to compute the cash flow's present value.

[9] A sensitivity is a measure that indicates how the derivative price changes while an infinitesimal change is produced by a parameter (e.g. volatility).

[10] In [9] the full construction procedure of the forward curve is described and an interested reader can see it there.

2.2 Interest rate derivatives

In this section we describe some interest rate derivatives (FRA, swap, cap/floor and swaption) that will be used along this Thesis.

- **Forward rate agreement (FRA).** A forward interest rate contract gives to its holder the possibility to lock an interest rate at present time t for a loan to be delivered over a future period of time $[T_1, T_2]$, with $t \leq T_1 \leq T_2$. Here is the link between the forward interest rate as we mentioned in the previous page because the locked interest rate is precisely $F(t; T_1, T_2)$. Consequently, $F(t; T_1, T_2)$ is the value of the fixed rate that makes a FRA for the period $[T_1, T_2]$ a fair contract at time t.

- **Interest rate swaps (IRS).** An IRS can be interpreted as a portfolio including several FRAs. More explicitly, it is an agreement between two parties that exchanges future payments. Generally, one party pays a fixed interest rate and the other one pays a floating interest rate based on a reference forward rate (Euribor for EUR).[11] The future payments are periodic during the start date of the swap and the end date of the swap. To clarify notation, a swap with tenor 6 months and maturity 2 years, means that the life of the swap is 2 years and the floating leg is based on (for example) the 6 months Euribor.

  When the fixed leg is paid and the floating leg is received the IRS is termed Payer IRS (PFS), whereas in the other case we have a Receiver IRS (RFS).

  Given a set of n pre-specified payment dates $T_1, T_2, \ldots, T_n$, let N be the total notional agreed in the contract, $\delta(T_{i-1}, T_i)$ is the year count fraction between dates T_{i-1} and T_i, K the fixed rate designed by the contract and $L(T_{i-1}, T_i)$ [12] the floating reference rate resetting at the previous instant T_{i-1} for the maturity

[11] Actually, there are more types of IRS such as floating-floating swaps, based on two different floating rates or on the same floating rate but different tenor, and fixed-fixed swaps, in which both counterparties pay a (different) fixed interest rate and thus this sort of swaps just make sense when the interest is applied in one currency for the principal and the another (or the same applied before) interest in another currency, also called currency swaps. For further information, in [8] a full development of this topic is presented.

[12] It is necessary to clarify the subtle difference between $L(T_{i-1}, T_i)$ and $F(t; T_{i-1}, T_i)$. The notation $L(T_{i-1}, T_i)$ comes from the Libor rate and it refers to the (deterministic) spot rate prevailing in the period $[T_{i-1}, T_i]$. On the other hand, $F(t; T_{i-1}, T_i)$ is an estimation at time t of the Libor rate prevailing in the period $[T_{i-1}, T_i]$ and so $F(t; T_{i-1}, T_i)$ just indicates that $L(T_{i-1}, T_i)$ is random at time t. In fact, in Section 2.3 will be shown (see for example [5] or [9]) that the forward rate $F(t; T_{i-1}, T_i)$ is the expectation of $L(T_{i-1}, T_i)$ under a suitable probability measure.

given by the current payment instant T_i. For every fixing date T_i, $i = 1, \dots, n$, the fixed left party pays the amount

$$N\delta(T_{i-1}, T_i)K,$$

while the floating leg pays[13]

$$N\delta(T_{i-1}, T_i)L(T_{i-1}, T_i).$$

As mentioned in the previous page, from the swap definition a new interest rate is defined: **forward swap rate** $S(t, T_1, T_n)$, which is the rate in the fixed leg of the above IRS that makes the IRS a fair contract at the present time t:

$$S(t, T_1, T_n) := \frac{P(t, T_1) - P(t, T_n)}{\sum_{j=2}^{n} \delta(T_{j-1}, T_j)P(t, T_j)}, \tag{2.7}$$

where the denominator is usually called the annuity. Based on the forward swap rate a swaption is defined.

- **Caplets/floorlets.** A **caplet** for the future period $[T_1, T_2]$ is a call option on a floating rate, typically Libor $L(T_1, T_2)$ with strike K and therefore the payoff at date T_1 is given by

  $$N \cdot \delta(T_1, T_2) \cdot (L(T_1, T_2) - K)^+,$$

  where N is the nominal amount.

  Let us to clarify some important concepts about the key dates of a caplet. **The expiry** of a caplet is the date when the Libor rate is determined, namely, at time T_1. In other words, T_1 accounts for the time where the randomness of $F(t; T_1, T_2)$ ends. After T_1, $F(t; T_1, T_2)$ becomes the Libor rate $L(T_1, T_2)$ to be applied in the period $[T_1, T_2]$ and hence the cash flow on this caplet is received at time T_2. This means that there is no uncertainty about the caplet's cash flow after the Libor rate is set at time T_1. Very often, in financial markets these dates are known as **reset date** for T_1 and **payment date** for T_2.

  Analogously, a **floorlet** is a put option on the Libor rate. Thus, its payoff at time T_1 is given by

  $$N \cdot \delta(T_1, T_2) \cdot (K - L(T_1, T_2))^+,$$

  and it will be received at the payment date T_2.

[13]For the sake of simplicity, it is considered that the payment dates are the same for both parties.

- **Caps/Floors.** A **Cap** is a portfolio of caplets. Let us consider a set of payment dates $\{T_1, \ldots, T_n\}$ and a set of reset dates $\{T_0, T_1, \ldots, T_{n-1}\}$. With this notation the cap's maturity stands for T_n (the last payment date). The cap's strike K is the strike of every underlying caplet. In such a way the cap's discounted payoff is given by the sum of the caplet's discounted payoffs as follows:

$$N \cdot \sum_{j=2}^{n} P(t, T_j) \cdot \delta(T_{j-1}, T_j) \cdot (L(T_{j-1}, T_j) - K)^+. \tag{2.8}$$

where $t < T_0$ accounts for the valuation date (or today's date). Note that the starting index in the previous sum is $j = 2$, because the Libor rate $L(T_0, T_1)$ is already known considering that $t = T_0$, which is a common practice in cap's pricing for the sake of simplicity. Therefore, excluding the first caplet from the cap's payoff makes sense because an investor is not going to pay for a product "today" where its first future payment is already known. In any case, we will go into more detail in Chapter 6.

Similarly, a **floor** is a set of floorlets. Following the previous notation, its discounted payoff is given by:

$$N \cdot \sum_{j=2}^{n} P(t, T_j) \cdot \delta(T_{j-1}, T_j) \cdot (K - L(T_{j-1}, T_j))^+. \tag{2.9}$$

- **Swaptions.** A payer swaption is an option to enter into an IRS at a later date, paying fixed rate. A receiver swaption is an option to enter into an IRS, receiving fixed. In other words, a payer (receiver) swaption is a call (put) on forward swap rate. Following the previous notation for an IRS, $\{T_1, T_2, \ldots, T_n\}$ is the set of n payment dates. Then, usually the swaption expiry T_{ex} coincides with the first date T_1 of the underlying IRS. The underlying IRS lenght, i.e., from T_1 to T_n, is called the tenor of the swaption. For example, a $1Y \times 5Y$ payer swaption with strike K gives the holder the right to pay a fixed rate K on a 5 year swap starting in 1 year.[14] In this way, the payoff of a payer swaption of strike K at expiration date T_{ex} is given by:

$$\max(S(T_{ex}, T_2, T_n) - K, 0) \tag{2.10}$$

and the payoff of a receiver swaption of strike K at expiration date is given by:

$$\max(K - S(T_{ex}, T_2, T_n), 0). \tag{2.11}$$

Here, we are considering that the first payment is made at time T_2 and then $T_1 \equiv T_{ex}$.

[14] This type of swap is usually called a forward starting swap.

2.3 Mathematical basis

In this section we introduce some fundamental concepts about stochastic processes, risk-neutral pricing, forward measure, risk neutral probability density function and arbitrage. This will not only allow us to price any derivative under consideration in this Thesis but also to have a full comprehension of the main objective of this Thesis, which is the problem of the forward's density negativity at expiry.

No-arbitrage pricing. As mentioned in the previous numeraire definition, the risk-neutral measure $\mathbb{Q}$ not only is the martingale measure associated to the numeraire $B(t)$, but also is an equivalent martingale measure,[15] statement that is proved in [11].

In line with the previous definition, if $\mathbb{Q}^N$ is an equivalent martingale measure (fair and unique), associated to a numeraire $N(t)$, the price $V(t)$ of any contingent claim (derivative) is obtained by taking an (conditional) expected value under the measure $\mathbb{Q}^N$, such that the price $V(t)$ measured in terms of the numeraire is a martingale under $\mathbb{Q}^N$, i.e.,

$$\frac{V(t)}{N(t)} = \mathbb{E}^{\mathbb{Q}^N}\left[\frac{V(T)}{N(T)} \mid \mathcal{F}_t\right], \tag{2.12}$$

with $t \leq T$ and $\{\mathcal{F}_s\}_{s=0}^{t}$ is the natural filtration.[16]

We will see below key particular cases of the formula (2.12).

- **Risk-neutral measure ($\mathbb{Q}$).** The risk-neutral measure $\mathbb{Q}$ has the bank account $B(t)$ as a numeraire. Under $\mathbb{Q}$, and in the absence of arbitrage, a contingent claim is valued as

  $$V(t) = B(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{V(T)}{B(T)} \mid \mathcal{F}_t\right]. \tag{2.13}$$

  This choice of numeraire guarantees that the discounted value of any asset is a martingale, and is thus widely used in equity derivatives pricing.

[15] Recall that if $\mathbb{P}$ is a probability measure and $\mathbb{Q}$ is another probability measure, they are equivalent probability measures if they have the same null sets. However, if a stochastic process $X(t)$ is a martingale under $\mathbb{P}$, $X(t)$ is not in general a martingale under $\mathbb{Q}$. Hence, in short, a measure probability that preserves the martingale property is called an equivalent martingale measure. Several technicalities have been omitted for the sake of continuity and further details and a formal definition can be found in [5] or [7].

[16] The natural filtration for a stochastic process is the family of non-decreasing σ-algebras generated by the process itself $\{\sigma(X(s)),\ s \in [0,t]\}, t \geq 0$ and is a formal way to characterize the history of the process up to time t. All the stochastic processes considered throughout this text are $\mathcal{F}_t$-measurebles (adapted processes), which means that all the information is known at time t. These topics are beyond the scope of this thesis and we refer the interested reader to [7] for extra information.

- **T-forward measure ($\mathbb{Q}^T$).** The T-forward measure $\mathbb{Q}^T$ has the T-maturity zero coupon bond $P(t,T)$[17] and the relevant thing here is that $P(T,T) = 1$. This fact eases the valuation as follows:

$$V(t) = P(t,T)\mathbb{E}^{\mathbb{Q}^T}\left[\frac{V(T)}{P(T,T)} \mid \mathcal{F}_t\right] = P(t,T)\mathbb{E}^{\mathbb{Q}^T}[V(T) \mid \mathcal{F}_t]. \tag{2.14}$$

  Moreover, this measure is particularly interesting in the interest-rates world since there are several important result related to it:

  1. Under the T_2-forward measure, any simply-compounded forward rate accounting for a future investment period which ends at T_2 is a martingale. This statement is proved in [5] or [6], and the interested reader is referred there for extra details. As a consequence, we have that:

  $$\mathbb{E}^{\mathbb{Q}^{T_2}}[F(t;T_1,T_2) \mid \mathcal{F}_u] = F(u;T_1,T_2) \tag{2.15}$$

     for every $t \geq 0$ such that $0 \leq u \leq t \leq T_1 \leq T_2$. If we choose suitable dates, we can get that

  $$F(t;T_1,T_2) = \mathbb{E}^{\mathbb{Q}^{T_2}}[F(T_1;T_1,T_2) \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}^{T_2}}[L(T_1,T_2) \mid \mathcal{F}_t], \tag{2.16}$$

     namely, the forward rate is the expected value of $L(T_1,T_2)$ under the T_2-forward measure.[18]

  2. The instantaneous forward rate $F(t,T)$ is equal to the expected value of the future instantaneous spot rate $r(T)$ under the T-forward measure:[19]

  $$F(t,T) = \mathbb{E}^{\mathbb{Q}^T}[r(T) \mid \mathcal{F}_t]. \tag{2.17}$$

  3. Under the T-forward measure, the volatility of the instantaneous forward rate, $\sigma(t)$, is driftless. This feature is explicitly mentioned in the chapter 4 of [12], and makes the T-forward measure a really convenient tool when dealing with stochastic volatility models, which allow the volatility to follow its own stochastic process.

  The process of changing the numeraire between these particular choices (risk-neutral and T-forward measure) via Radon-Nikodym is fully reviewed in [13].

[17] It is called that way because this choice of the numeraire depends on the maturity T.

[18] It is good to remind the reader that the T_2-forward measure is always linked to the payment date of the derivative. In this case, the maturity T_2.

[19] The proof can be found in [5].

- **Swap measure** ($\mathbb{Q}^A$). As claimed in [14], since the forward swap rate $S(t, T_1, T_n)$ is given by a market tradable asset $(P(t, T_1) - P(t, T_n))$ denominated in the annuity numeraire, it is a martingale under a measure $\mathbb{Q}^A$ (called the swap measure) associated with the annuity numeraire which is denoted as

$$A_t^{T_1, T_n} = \sum_{j=2}^{n} \delta(T_{j-1}, T_j) P(t, T_j), \tag{2.18}$$

where the dates $T_1 < T_2 < \ldots < T_n$ are the reset dates of a swap. This measure is crucial in order to value swaptions.

2.3.1 Risk neutral probability density function

We dedicate a subsection to this issue beacause it is the main tool to understand the key problem of the SABR model mentioned later on in the Subsection 3.4.5 and the reason why another approach is followed in this Thesis to deal with this problem.

As mentioned in [15], in 1978, Breeden and Litzenberger [16] presented a paper in which they showed how to compute the risk neutral probability density function from a set of quoted option prices. To present the result of their paper in a simple way caplets prices are considered. Therefore the notation is as follows:

- Let T_{j-1} and T_j be the reset date and the payment date of a caplet, respectively.
- Let $V_{caplet}(t, T_{j-1}, T_j, K)$ be the value of a caplet at time t, with fixing date T_{j-1} and payment date T_j.
- $(F(T_{j-1}; T_{j-1}, T_j) - K)^+$ is the caplet's payoff for a given expiry date T_{j-1} and strike K.
- Let us define $Q_{F_{j-1}}(x) := Q_{F(T_{j-1}; T_{j-1}, T_j)}(x)$ as the risk neutral probability density function of the random variable $F(t; T_{j-1}, T_j)$ at its expiration date T_{j-1}. It is important to notice that after date T_{j-1}, $F(t; T_{j-1}, T_j)$ is not longer random.

According to (2.14), the value of the caplet at time t is given by

$$\begin{aligned} V_{caplet}(t, T_{j-1}, T_j, K) &= P(t, T_j) \cdot \delta(T_{j-1}, T_j) \cdot \mathbb{E}^{\mathbb{Q}^{T_j}} [(L(T_{j-1}, T_j) - K)^+ \mid \mathcal{F}_t] \\ &= P(t, T_j) \cdot \delta(T_{j-1}, T_j) \cdot \int_K^{\infty} (x - K) \cdot Q_{F_{j-1}}(x) \, dx. \end{aligned} \tag{2.19}$$

Derivating twices respect to the strike K yields

$$\frac{\partial^2}{\partial K^2} V_{caplet}(t, T_{j-1}, T_j, K) = P(t, T_j) \cdot \delta(T_{j-1}, T_j) \cdot Q_{F_{j-1}}(K). \tag{2.20}$$

Note that we are considering in (2.20) the caplet's value as a function of K. Therefore, a numerical computation of $Q_{F_{j-1}}(K_i)$, $i = 1, \ldots, n$, can be done using a series of caplet prices

$$\begin{array}{c} V_{caplet}(t, T_{j-1}, T_j, K_1), \\ V_{caplet}(t, T_{j-1}, T_j, K_2), \\ \vdots \\ V_{caplet}(t, T_{j-1}, T_j, K_n), \end{array} \tag{2.21}$$

corresponding to a set of options with strikes $K_1, K_2, \ldots, K_n$. This argument will be followed in Subsection 3.4.5 to show the flaw of the SABR model.

2.3.2 Arbitrage

There is a strong non-arbitrage property that must be fulfilled by the relation (2.20). Since $Q_{F_{j-1}}(K)$ is a probability density function, by definition, the condition

$$Q_{F(T_j)}(K) = \frac{\partial^2}{\partial K^2} V_{caplet}(t, T_{j-1}, T_j, K) \geq 0 \tag{2.22}$$

must be reached, otherwise, the option value would not be a convex function in strike and an arbitrage can be found. Specifically, a butterfly arbitrage may be obtained as it is discussed in [15]. This problem is also dealt in Section 4.1. There is another condition for being a density function such as it must integrate one.

As we will see in Section 3.4, the above condition (2.22) is not satisfied by the SABR model. In concrete terms, for some specific conditions such as long run options with very low strikes (or even negative as today's environment) the SABR model presents a significant drawback due to the density function negativity.

We have reviewed the context of negative interest rates in Chapter 1 and main mathematical tools in this chapter needed to develop the rest of the Thesis and the basic interest rate models will be checked below.

Chapter 3

Interest Rate Models

In this chapter we introduce different interest rate models that cope with negative interest rates. The basic models are presented as well as its main strenghts and drawbacks. Then, we introduce the benchmark SABR model and we discuss its major disadvantage which is the main reason why the different approach (AF-SABR) is reviewed in the next chapter. Finally, we comment some modifications of the SABR model that also have the same problem. We start with the simplest approach to model negative interest rates.

3.1 Bachelier (Normal) model (1900)

The Bachelier (or normal) model was introduced in 1900 by L. Bachelier [17]. The model is given by the following stochastic differential equation, under the T-forward measure, for the instantaneous forward rate $F(t)$:[1]

$$dF(t) = \sigma_N \cdot dW(t), \tag{3.1}$$

where $W(t)$ is a Wiener process, σ_N is the instantaneous forward rate (constant) volatility under normal specification, namely, the solution of (3.1) is given by

$$F(t) = f + \sigma_N \cdot W(t), \quad \text{where } f = F(0). \tag{3.2}$$

We can then note that $F(t)$ is normally distributed with mean f and variance $\sigma_N^2 t$. This means that negative interest rates can be modelled in a natural way. However, there are some disadvantages which are summarized below:

- The instantaneous forward rate volatility σ_N is constant which means that, for instance, options with different strikes for a given maturity have the same

[1] We defined in (2.4) the instantaneous forward rate with the notation $F(t,T)$. However, note that since one only future date is needed, we can simplify the notation as $F(t)$, namely, $F(t) := F(t,T)$.

volatility and it is well known that this fact does not happen in markets. This phenomenon is the so-called *volatility smile* or *skew*. In other words, implied volatilities have a strong dependence on their strikes.

- $F(t)$ is normally distributed. This implies that it may take arbitraly negative values with positive probability. This fact is unusual under typical circumstances since rates are not assumed to go far away below the zero-barrier.

Under Bachelier model (3.1), closed formulae for pricing caplets/floorlets can be obtained. Indeed, if the expiry date of a caplet is T_1 and the payment date is T_2, with $T_1 < T_2$, under the T_2-forward measure the value at time t of a caplet/floorlet on the Libor rate $L(T_1, T_2)$ with strike K, today's value forward $f = F(t; T_1, T_2)$ and notional $N = 1$, is given by:

$$V_{caplet}^{Bachelier}(t, T_1, T_2, K, \sigma_N) = \delta(T_1, T_2) \cdot P(t, T_2) \cdot B_{call}^N(T_1, K, f, \sigma_N), \tag{3.3a}$$

$$V_{floorlet}^{Bachelier}(t, T_1, T_2, K, \sigma_N) = \delta(T_1, T_2) \cdot P(t, T_2) \cdot B_{put}^N(T_1, K, f, \sigma_N), \tag{3.3b}$$

where

$$B_{call}^N(T_1, K, f, \sigma_N) = (f - K)\Phi(d_+) + \sigma_N \sqrt{\delta(t, T_1)}\ \phi(d_+), \tag{3.3c}$$

$$B_{put}^N(T_1, K, f, \sigma_N) = (K - f)\Phi(d_-) + \sigma_N \sqrt{\delta(t, T_1)}\ \phi(d_+), \tag{3.3d}$$

$$d_\pm = \pm \frac{f - K}{\sigma_N \sqrt{\delta(t, T_1)}}, \tag{3.3e}$$

where Φ and ϕ are the normal cumulative distrubution function and the probability density function, respectively.

Now, we consider a cap/floor with maturity T_n, reset dates $T_0, T_1, \ldots, T_{n-1}$ and payment dates $T_1, T_2, \ldots, T_n$ so that the cap is a portfolio of n caplets. The initial forward for every (expiry, payment date)= (T_{j-1}, T_j), is $f = F(t; T_{j-1}, T_j)$.[2] In such a way, the prices for the cap/floor are given by[3]

$$V_{cap}^{Bachelier}(t, T_n, K, \sigma_N) = \sum_{j=2}^{n} P(t, T_j) \cdot \delta(T_{j-1}, T_j) \cdot B_{call}^N(T_{j-1}, K, f, \sigma_N), \tag{3.4a}$$

[2]We recall the reader that $f = F(t; T_{j-1}, T_j)$ is computed under the multi-curve framework and then the formula (2.6) must be used.

[3]In these formulae, it is actually $\sigma_N \equiv \sigma_N(K)$.

$$V_{floor}^{Bachelier}(t, T_n, K, \sigma_N) = \sum_{j=2}^{n} P(t, T_j) \cdot \delta(T_{j-1}, T_j) \cdot B_{put}^{N}(T_{j-1}, K, f, \sigma_N). \quad (3.4b)$$

If we consider a payer/receiver swaption with expiration date $T_{ex} \equiv T_1$ on underlying swap with reset dates $T_1, T_2, \ldots, T_n$, its value at time t can be computed as:

$$V_{swaption}^{payer}(t, T_1, T_n, K) = N \cdot A_t^{T_1,T_n} \left[(S(t, T_1, T_n) - K)\Phi(d_+) + \sigma_N \sqrt{\delta(t, T_{ex})}\phi(d_+) \right], \quad (3.5a)$$

$$V_{swaption}^{receiver}(t, T_1, T_n, K) = N \cdot A_t^{T_1,T_n} \left[(K - S(t, T_1, T_n))\Phi(d_-) + \sigma_N \sqrt{\delta(t, T_{ex})}\phi(d_+) \right], \quad (3.5b)$$

where

$$d_{\pm} = \pm \frac{S(t, T_1, T_n) - K}{\sigma_N \sqrt{(t, T_1)}} \quad (3.6)$$

3.2 Black (Lognormal) model (1976)

The Black's model was presented by Fischer Black [18] in 1976. This model is widely used for modeling European options on commodities, forwards or futures. It is also used for pricing interest rate caps and floors. It is truth that in the current negative interest rates environment its use is not held longer anymore, instead, the Bachelier model is used, or even the shifted black model, which will be presented in Section 3.7. However, we will review the Black model because it has been a benchmark model utilized along many years.

In accordance with the foregoing, the model (under the T-forward measure) for the instantaneous forward rate $F(t)$ is

$$dF(t) = \sigma_B \cdot F(t) \cdot dW(t), \quad (3.7)$$

where $W(t)$ is a Wiener process, σ_B is the instantaneous forward rate (constant) volatility under lognormal specification, i.e., the solution to this stochastic differential equation 3.7 is as follows

$$F(t) = f \cdot e^{\sigma W(t) - \frac{1}{2}\sigma_B^2 t}, \quad f = F(0). \quad (3.8)$$

As we can note, $F(t)$ is lognormally distributed which means that this model can not cope with negative interest rates.

As it can be seen in Appendix A of [19], closed formulae for pricing caps/floors can be found. Hence, the Black formula states that the prices at time t, for a caplet/floorlet on the Libor rate $L(T_1, T_2)$ with strike K and notional $N = 1$ is given by

$$V_{caplet}^{Black}(t, T_1, T_2, K, \sigma_B) = \delta(T_1, T_2) \cdot P(t, T_2) \cdot B_{call}^{Black}(T_1, K, f, \sigma_B), \tag{3.9a}$$

$$V_{floorlet}^{Black}(t, T_1, T_2, K, \sigma_B) = \delta(T_1, T_2) \cdot P(t, T_2) \cdot B_{call}^{Black}(T_1, K, f, \sigma_B), \tag{3.9b}$$

where

$$B_{call}^{Black}(T_1, K, f, \sigma_B) = f\Phi(d_+) - K\Phi(d_-), \tag{3.10a}$$

$$B_{put}^{Black}(T_1, K, f, \sigma_B) = K\Phi(-d_-) - f\Phi(-d_+), \tag{3.10b}$$

and

$$d_\pm = \frac{\log\left(\frac{f}{K}\right) \pm \frac{1}{2}\sigma_B^2 T_1}{\sigma_B\sqrt{T_1}}. \tag{3.10c}$$

Adding the Black prices of caplets/floorlets we can get caps/floors prices under de Black model as follows:

$$V_{cap}^{Black}(t, T_n, K, \sigma_B) = \sum_{j=2}^{n} P(t, T_j) \cdot \delta(T_{j-1}, T_j) \cdot B_{call}^{Black}(T_{j-1}, K, f, \sigma_B) \tag{3.11a}$$

$$V_{floor}^{Black}(t, T_n, K, \sigma_B) = \sum_{j=2}^{n} P(t, T_j) \cdot \delta(T_{j-1}, T_j) \cdot B_{put}^{Black}(T_{j-1}, K, f, \sigma_B) \tag{3.11b}$$

where $f = F(t; T_{j-1}, T_j)$ is the estimated initial forward for the period $[T_{j-1}, T_j]$.

We have reviewed two fundamental pricing models in finance history in the interest rate world. The Bachelier's model has the volatility smile problem as well as the Black's model. In Sections 4.1 and (6.3.2) smile volatilities for caps and swaptions are shown, respectively.

An important aspect to remark is that both Bachelier and Black formulae have a *one-to-one* correspondence between the price of the option and the volatility parameter. This means if we introduce the Bachelier/Black implied volatility into formulae (3.4) and (3.11), we can then get the price of the cap/floor.

To be more precise, the instruments that quote (in volatility terms or basis points premium) in markets are caps/floors. Then, the process of extracting the caplet's implied volatility from cap's premium is called **stripping**. This procedure is going to be treated in detail in Section 6.1.

If we consider a payer/receiver swaption following the notation already seen, its value under the black model is given by

$$V_{swaption}^{payer}(t,T_1,T_n,K) = N \cdot A_t^{T_1,T_n}\left[S(t,T_1,T_n)\Phi(d_+) - K\Phi(d_-)\right], \tag{3.12a}$$

$$V_{swaption}^{receiver}(t,T_1,T_n,K) = N \cdot A_t^{T_1,T_n}\left[K\Phi(-d_-) - S(t,T_1,T_n)\Phi(-d_+)\right], \tag{3.12b}$$

where,

$$d_\pm = \frac{\log\left(\frac{S(t,T_1,T_n)}{K}\right) \pm \frac{1}{2}\sigma_B^2 T_1}{\sigma_B\sqrt{T_1}}. \tag{3.13}$$

It is important to make a subtle clarification. In a swaption there is only one decision to be made and, once taken, is valid for all cash flows in the period $[T_2, T_n]$. For this reason it differs from a cap/floor, which requires a decision at each reset date. In addition, we recall the reader that caplet's volatilities are not quoted in markets and it is necessary to make a stripping procedure. In contrast, a swaption is an option on forward swap rate (also called swap par rate) which is quoted directly in markets. Hence, this is an important aspect to be taken into account when a swaption is calibrated using the SABR model

3.3 Local volatility models (1994)

In this section we summarize the main features of these models provided in the paper of Hagan et al. [20]. Local volatility models were firstly introduced by Dupire, Derman-Kani (see [21] and [22]). Based on work by Black-Scholes (BS) (1973), Dupire asked himself whether it was possible to find a driftless process (i.e., martingale) such that the assumptions of the BS model continue to be fulfilled and he concluded that one way to accomplish that was by making the instantaneous volatility depending on time and underlying asset.[4]

In the same vein, the general model for the forward rate $F(t)$ suggested by Dupire was (under the T-forward measure)

$$dF(t) = C(F(t),t) \cdot dW(t), \tag{3.14}$$

[4]For further details see [21].

where $C(F(t), t)$ is a deterministic volatility coefficient.[5] Hence, the so-called **local volatility models** come when we set $C(F(t), t) = \sigma_{loc}(F(t), t) \cdot F(t)$. This model is able to capture the volatility smile (it was a pioneer in this topic) and therefore it was widely used over the industry. However, the dynamic evolution for the smile was opposed (the smile moves in the opposite direction as the underlying) to the one observed within the markets (the smile moves in the same direction as the underlying), producing unstable hedges. Hagan et al. explain this argument in an extensive way.

Under these circumstances, Hagan et al. (2002) define the Stochastic Alpha-Beta-Rho **(SABR)** model which is a stochastic volatility model that addresses the previous problem. The SABR model has some important flaws but despite all that, it has been broadly used by financial industry. This model will be presented below in the following section.

3.4 The SABR model

3.4.1 Introduction

The SABR model was proposed by Hagan et al. (2002) in [20]. This model arises to correct the bad prediction of volatility smile of local volatility models. Hagan introduced a stochastic process for the forward rate and another stochastic process for the volatility of the forward rate, denoted as $\sigma(t)$. Then, the SABR model is given by the following stochastic differential equations:

$$\begin{aligned} dF(t) &= \sigma(t) \cdot F(t)^{\beta} \cdot dW(t), \quad f = F(0), \\ d\sigma(t) &= \nu \cdot \sigma(t) \cdot dZ(t), \\ \mathbb{E}^{\mathbb{Q}^T}[dW(t) \cdot dZ(t)] &= \rho dt, \end{aligned} \tag{3.15}$$

where $\nu \geq 0$ is the volatility of the forward's volatility (also called the vol-vol parameter), $0 \leq \beta \leq 1$ is called the power parameter and $-1 < \rho < 1$ is the correlation between the two Wiener processes $W(t)$ and $Z(t)$. The Wiener processes are under the T-forward measure and hence they depend on the maturity of each forward.[6]

The model (3.15) is actually an extension of the Constant Elasticity Variance

[5]At this point, to refresh the reader's knowledge, notice that since $F(t)$ is a martingale under the T-forward measure, the Martingale representation theorem states that any martingale can be re-write as an Itô integral, namely, $F(t)$ must follow the driftless Itô process (3.14), where $C(F(t), t)$ is an adapted process. This comment has been done in [20] and for technicalities about Martingale representation theorem the interested reader can see [7].

[6]To be more pricese should be $dW(t)^T$ and $dZ(t)^T$ as presented in [12].

(CEV) models.[7] One important aspect is that SABR model does not cope with negative interest rates and thus in recent years there have been a lot of changes based on this model (some of them will be seen in the following sections). On the other hand, the SABR model is not a model to price, but it is a model which fits the implied volatility given by the markets for any single expiry date reasonably well and it is used as an input in the Bachelier or Black model to price. Due to that good fit, many traders choose the SABR model to price and hedge their fixed income plain-vanilla (single exercise date) derivatives, such as caplets, floorlets and swaptions. Later on, we will discuss some virtues and drawbacks in a summarised way.

3.4.2 Parameter's behaviour

Let $\alpha = \sigma(0)$ be the initial forward volatility. Each of the 4 parameters ν, β, ρ and α have different effects on the smile/skew for a given maturity. These effect were also studied by Hagan [20] and in [15] they are described in a nice way. We follow the latter reference to introduce them here.

- α shows the level of the volatility smile.
- β is the power parameter and it is usually taken between 0 and 1. The reason is beacuse the SABR model is a martingale only if $0 \leq \beta < 1$ or as long as $\rho \leq 0$ for $\beta = 1$.[8]
- ν is the volatility of the volatility and its effect on the smile is to decrease or increase its curvature.
- ρ stands for the slope of the smile.

Some of these parameters have second and even third smaller effects on the volatility curve as studied in detail in [12].

3.4.3 Implied volatilities

Implied Black and normal volatilities obtained by Hagan et al. were the main result of their paper. As mentioned in [15], Hagan et al. deduced an analytical formula (see (3.16)), which returns the price of vanilla options (such as caplets, floorlets, swaptions)

[7] These models are given by the stochastic differential equation

$$dF(t) = \sigma F(t)^{\beta} dW(t),$$

where σ is constant and $0 \leq \beta \leq 1$. For more information about these kind of models the reader can see (for example) [19].

[8] We refer the interested reader to [23] for more detailed information.

under the SABR model, in terms of the Bachelier/Black implied volatilities. This is an excellent feature for traders bacause of the one-to-one relation between prices and volatilities as it can be seen in formulae (3.4) and (3.11). Due to the negative interest rate environment, the above-mentioned vanilla options are now (commonly) quoted in terms of normal volatility units instead of quoting in Black volatility units.

Using asymptotic expansions valid for short maturities and small values of ν, Hagan obtained approximated analytical solutions for the Black's implied volatility that should be introduced in Black's formula (3.9) to price a caplet/floorlet for a future investment period $[T_1, T_2]$, with T_1 the expiry date, T_2 the payment date, strike K, notional amount $N = 1$ and an initial forward rate $f = F(t; T_1, T_2)$. This formula is usually called Hagan's formula, and in 2008 it was corrected by Obloj because Hagan made a mistake in his deduction.[9] Henceforth, the Hagan's formula is presented with the Obloj's correction:

$$\sigma_B(T_1, K, f) = \frac{\nu \cdot \log\left(\frac{f}{K}\right)}{x(z)} \cdot \left[1 + \left(\frac{\alpha^2(1-\beta)^2}{24(Kf)^{1-\beta}} + \frac{\nu\beta\rho\alpha}{4(Kf)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24}\nu^2\right) T_1\right] + O((\nu^2 T_1)^2), \tag{3.16}$$

where

$$z = \frac{\nu \cdot (f^{1-\beta} - K^{1-\beta})}{\alpha(1-\beta)}, \qquad x(z) = \log\left(\frac{\sqrt{z^2 - 2\rho z + 1} + z - \rho}{1-\rho}\right). \tag{3.17}$$

For at-the-money options ($f = K$), Hagan's formula reduces to

$$\sigma_B^{ATM}(T_1, f, f) \approx \frac{\alpha}{f^{1-\beta}} \left[1 + \left(\frac{\alpha^2(1-\beta)^2}{24 \cdot f^{2-2\beta}} + \frac{\nu\beta\rho\alpha}{4 \cdot f^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2\right) T_1\right]. \tag{3.18}$$

There is also a formula for the Bachelier model to calibrate a **normal implied volatility** smile. As demonstrated in [20], the following formula should be introduced in Bachalier's formula (3.9) in order to price a caplet/floorlet for a future investment period $[T_1, T_2]$, strike K, notional $N = 1$ and current observed forward rate $f =$

[9]See [24] for more detail about the Obloj's correction.

$F(t;T_1,T_2)$ and it is given by[10]:

$$\begin{aligned}\sigma_N(T_1,K,f) &= \frac{\alpha(1-\beta)(f-K)}{f^{1-\beta}-K^{1-\beta}}\cdot\left(\frac{z}{x(z)}\right)\\ &\quad\cdot\left[1+\left(\frac{\beta(\beta-2)\alpha^2}{24(fK)^{1-\beta}}+\frac{\alpha\beta\rho\nu}{4(fK)^{\frac{1-\beta}{2}}}+\frac{2-3\rho^2}{24}\nu^2\right)T_1\right]+O((\nu^2T_1)^2),\end{aligned} \tag{3.19}$$

with

$$\begin{aligned} z &= \frac{\nu}{\alpha}\frac{(f-K)}{(fK)^{\frac{\beta}{2}}},\\ x(z) &= \log\left(\frac{\sqrt{z^2-2\rho z+1}+z-\rho}{1-\rho}\right).\end{aligned}$$

Some special cases are presented below. For the normal SABR model ($\beta=0$ in (3.15))[11] the implied normal volatility can be written as:

$$\sigma_N(K)=\alpha\left(\frac{z}{x(z)}\right)\left(1+\frac{2-3\rho^2}{24}\nu^2T_1\right), \tag{3.20}$$

where

$$z=\frac{\nu}{\alpha}(f-K).$$

For the lognormal SABR model ($\beta=1$ in (3.15)), the normal implied volatility is given by[12]:

$$\sigma_N(K)=\alpha\left(\frac{f-K}{\log\left(\frac{f}{K}\right)}\right)\left(\frac{z}{x(z)}\right)\left[1+\left(\frac{-\alpha^2}{24}+\frac{\alpha\rho\nu}{4}+\frac{2-3\rho^2}{24}\nu^2\right)T_1\right], \tag{3.21}$$

where

$$z=\frac{\nu}{\alpha}\log\left(\frac{f}{K}\right).$$

Finally, we present the normal implied volatility for the at-the-money case ($K=f$)[13] below:

$$\sigma_N^{ATM}=\alpha f^{\beta}\left[1+\left(\frac{\beta(\beta-2)\alpha^2}{24f^{2-2\beta}}+\frac{\alpha\beta\rho\nu}{4f^{1-\beta}}+\frac{2-3\rho^2}{24}\nu^2\right)T_1\right]. \tag{3.22}$$

[10] These formulae can be found in [20]

[11] This model is presented in Section 3.5.

[12] See formula (A.71a) in [20].

[13] The limit of (3.19) as $K\to f$ is computed noting that

$$\lim_{K\to f}\frac{\alpha(1-\beta)(f-K)}{f^{1-\beta}-K^{1-\beta}}=\frac{0}{0}\overset{L'Hopital}{=}\lim_{K\to f}\frac{-\alpha(1-\beta)}{-(1-\beta)K^{-\beta}}=\alpha f^{\beta}.$$

We have outlined more in detail the implied normal volatility because those formulae (3.19)-(3.22) will be used for a market's calibration example and to show the main SABR's model flaw in Section 4.1.

The above formulae hold for any value of $\beta \in [0, 1]$. Moreover, notice, that a shift is necessary due to the presence of logarithms of the strike and forward in either Black or normal implied volatility. When $\beta = 0$, there is no presence of logarithms for the normal implied volatility and hence negative strikes or forwards can be modelled without the need to introduce a shift.

3.4.4 Calibrating the SABR model

The SABR model is calibrated to a set of option prices (volatilities) for a given expiration date. Hence, the calibration is carried out from caplet's market volatilities. In this sense, it is very relevant to bare in mind that the SABR model is not calibrated to a cap's smile but to a caplet's smile, as stated in [25] and [15].

There are two methods to carry the calibration out and in both of them the parameter β may be fixed to 0.5.[14] A comprehensive study about this choice is done in [12].

The two mentioned methods are explained below.

Method 1. Estimating ν, ρ and α directly: The method consists of minimizing the error between the Bachelier implied volatility (3.19) and market caplet volatilities $\{\sigma^{MKT}(T_{ex}, K_i)\}_{i=1}^{n}$ where T_{ex} is the caplet's expiration date. Therefore the parameters for that expiry may be obtained with any standard non-linear optimizer so that the sum of the quadratic error is minimised:

$$(\hat{\nu}, \hat{\rho}, \hat{\alpha}) = \underset{\nu,\rho,\alpha}{\arg\min} \sum_{K \in \mathcal{S}} \left(\sigma^{MKT}_{caplet}(T_{ex}, K) - \sigma_N(T_{ex}, K, f)\right)^2, \tag{3.23}$$

where $\mathcal{S}$ is the set of strikes to be employed in the calibration. This method will be described in detail in Section 6.2. Different weights $w_i \in [0, 1]$ can be allocated to the set of the market volatilities according to the analyst criteria.

Method 2. Estimating ρ and ν by implying α from at-the-money volatility: The aim is to reduce the number of parameters to be calibrated. If market data for

[14] The two methods can be found in [26].

ATM implied volatilities are available, we can use equation (3.18) (or its Bachelier equivalent) to obtain α by inverting that formula and taking into account that this paramater is the root of the following cubic polynomial which must be numerically solved

$$\left(\frac{(1-\beta)^2}{24 \cdot f^{2-2\beta}} T_1\right) \alpha^3 + \left(\frac{\beta\rho\nu}{4 \cdot f^{1-\beta}} T_1\right) \alpha^2 + \left(1 + \frac{2-3\rho^2}{24} \nu^2 T_1\right) \alpha - \sigma^{ATM} f^{1-\beta} = 0. \tag{3.24}$$

The method 1 is commonly faster than method 2. Obviously, the parameters will have to be recalibrated with a frequency which depends on how fast the smile shape changes. The previous equation (3.24) has 3 roots and hence in [27] it is recommended to choose the smallest one. In the minimization algorithm, at every iteration, α is found in terms of ρ and ν, namely $\alpha = \alpha(\rho, \nu)$, by solving (3.24). Then, the equation (3.23) becomes

$$(\hat{\alpha}, \hat{\rho}, \hat{\nu}) = \arg\min_{\alpha,\rho,\nu} \sum_{K \in \mathcal{S}} \left(\sigma_K^{Market} - \sigma(T_1, K, f; \alpha(\rho, \nu), \rho, \nu)\right)^2. \tag{3.25}$$

3.4.5 Strong and weak points

Here, we review the main advantages and disadvantages of the SABR model. Some of them have been mentioned in the previous paragraphs. However, in [15] a comprehensive summary is provided and we synthesize them below.

Advantages

- It allows to model the volatility smile.
- There exists an approximated analytical formula for the implied volatility that can be used for pricing vanilla options such as caplets, floorlets and swaptions.
- The implied volatilities computations are reasonably accurate

Disadvantages

- The forward process (3.15) is driftless and thus it is not mean reverting.
- Every forward rate process is under a T-forward measure which means the parameters of the process can only be calibrated to a single expiry. Hence, exotic products depending on more than one forward rate can not be priced.

- The development of the Bachelier/Black formulae is under the condition that $\nu^2 T \ll 1$, this is, for short-term options. This also means that as it was explained in advance in Section 2.3.2, the risk-neutral probability density function around negative strikes and for long expiries, can become negative implying arbitrageable option prices. In Section 4.1, an example of the density function negativity is shown.

  A solution to this issue is not about finding an approximated analytical implied volatility, but solving a Partial Differential Equation numerically which will have as a solution the Risk Neutral Probability Density Function (RNPDF) studied in the point 2.3.1 with a certain boundary conditions. This way, we can get option prices by integrating this RNPDF. In any case, this procedure will be seen in detail in the Chapter 4. This approach is called **Arbitrage-Free SABR (AF-SABR)**.

- For high values of β, the approximations have an exploding behaviour, returning too high implied volatility values for high strike options.

In the following sections we will review some modifications of the SABR model to deal with negative interest rates although some of them still having the problem of RNPDF's negativity.

3.5 Normal SABR model (2002)

The normal SABR model is an extension of the Bachelier model (3.1) and it is the only version of the SABR model, obtained by fixing $\beta = 0$ in equation (3.15), which can capture negative forward rates. It is defined by (under the T-forward measure)

$$\begin{aligned} dF(t) &= \sigma(t) \cdot dW(t), \quad f = F(0), \\ d\sigma(t) &= \nu \cdot \sigma(t) dZ(t), \\ \mathbb{E}^{\mathbb{Q}^T}[dW(t) \cdot dZ(t)] &= \rho dt. \end{aligned} \tag{3.26}$$

Given a set of market normal volatilities for a given expiry, the parameters are calibrated by using equations (3.23) or (3.24).

As mentioned in [28], one advantage is that this model gives a relatively good approximation to the volatility smile with a few parameters. The normal implied volatility was presented in formula (3.19).

One flaw of this model is that the dynamics of the forward may not be really consistent with observable market data.

3.6 Shifted Black model (2012)

Shifted Black model is an extension of the Black's model allowing to model negative forward rates. Namely, it is maintained the lognormal specification and it is added a term in the drift. Therefore the model for the intstantaneous forward rate obeys the following process (under the T-forward measure):

$$dF(t) = \sigma \cdot (F(t) + s) \cdot dW(t), \tag{3.27}$$

where $s > 0$ is a constant displacement parameter, which should be chosen a priori by the analyst. To sum up, the shifted model allows rates larger than $-s$ to be modelled. The main drawback is that the process of fixing s should be done accurately. Its value should be high enough to avoid the magnitudes $F(t) + s$ and $K + s$ going below zero for any given time, but it should not be extremely high because in this case it may be obtained arbitrarily negative values for the forward interest rate. This fact is the main flaw of this model: the choice of the shift s.

3.7 Shifted SABR model (2014)

As mentioned in Section 3.4, the SABR model is defined for positive interest rates. As discussed in [28], the shifted SABR model is similar to the SABR model except a shift parameter is introduced in the drift, i.e., the shifted SABR model is defined by:

$$\begin{aligned} dF(t) &= \sigma(t) \cdot (F(t) + s)^{\beta} \cdot dW(t), \quad f = F(0), \\ d\sigma(t) &= \nu \cdot \sigma(t) \cdot dZ(t), \\ \mathbb{E}^{\mathbb{Q}^T}[dW(t) \cdot dZ(t)] &= \rho dt. \end{aligned} \tag{3.28}$$

The same results for the SABR model (3.15) can be applied to this new model, changing in (3.15) $F(t) \to F(t) + s$ and $f \to f + s$. Also, this model can be seen as an extension of the shifted Black model, adding dynamic to the forward's volatility term $\sigma(t)$, i.e., it is the shifted black model with stochastic volatility.

One important benefit of this model is that the RNPDF's negativity problem is now around $-s$ and not around zero. However, it has the same problem that the shifted Black: the choice of s and it the interest rates go below s, a recalibration of the model paramareters is necessary.

3.8 Free boundary SABR model (2015)

Free boundary SABR is another extension of the SABR model to deal with negative interest rates. It was originally introduced by Antonov et al. [29]. Under this model the forward rate follows (under the T-forward measure) the following system of stochastic equations:

$$\begin{aligned} dF(t) &= \sigma(t) \cdot |F(t)|^{\beta} \cdot dW(t), \\ d\sigma(t) &= \nu \cdot \sigma(t) \cdot dZ(t), \\ \mathbb{E}^{\mathbb{Q}^T}[(dW(t) \cdot dZ(t)] &= \rho dt, \end{aligned} \tag{3.29}$$

with $0 \leq \beta \leq \frac{1}{2}$ so that the solution is stable as stated in [19].

There exists an exact solution for Bachelier's implied volatility which can be found in (for instance) [19] or [28].

Some strenghts and weaknesses are described below.[15]

Advantages

- Negative forward rates are allowed.
- No extra parameter is needed.
- The explicit solution exists in the zero correlation case.

Disadvantages

- The parameter β is restricted to the interval $[0, \frac{1}{2}]$.
- As stated in [19], the main flaw is that its RNPDF is negative for a large area around zero, which in fact is the most relevant area in a low rate environment. Hence, this model is not commonly used in financial markets.

[15]See [28] for further details.

Chapter 4

Hagan's Arbitrage-Free SABR Approach

In this chapter, the RNPDF's negativity issue will be reviewed following the Hagan's paper (2014) [30]. The purpose here is not to derive the intermediate results, but rather present them in a comprehensive way with some remarks to achieve a better understanding to the proposed solution by Hagan.

4.1 Hagan's formula arbitrage

The normal implied volatility (3.19) is (frequently) used to capture negative rates (strikes or forward). In Figure 4.1 we can observe how the caplet's market normal volatility smile looks like and the implied normal volatilities for different values of β when the rest of parameters are fixed.

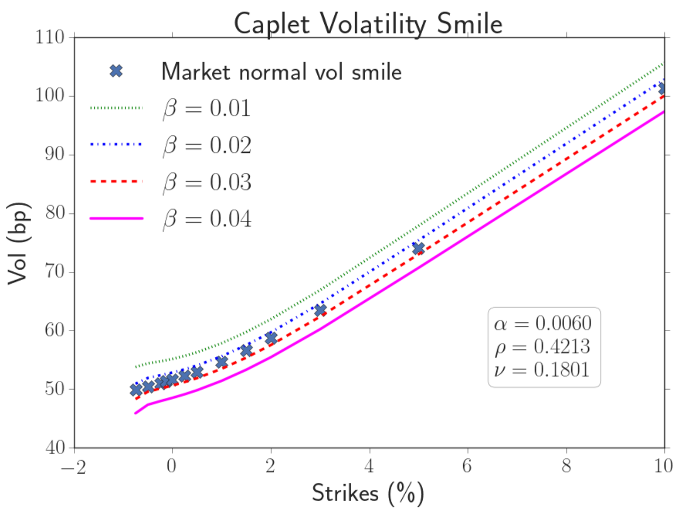

Figure 4.1: *Market normal volatility and implied normal volatilities for a caplet on Euribor 6 months with expiration date* $T_{ex} = 10$ *years. The parameters* α, ρ *and* ν *are fixed.*

As shown in Figure 4.1, the market's quotes fit is not accurate when the parameter β is close to 0. In fact, in Figure 4.2 an actual calibration has been done for the same quotes.[1] The calibration to the market normal volatilities was done via formula (3.23) adding the parameter β to be estimated. We can notice that we have obtained a better estimation with the free-paramater β. For this reason, we have chosen to carry out the calibration using the four parameters: α, β, ρ and ν.

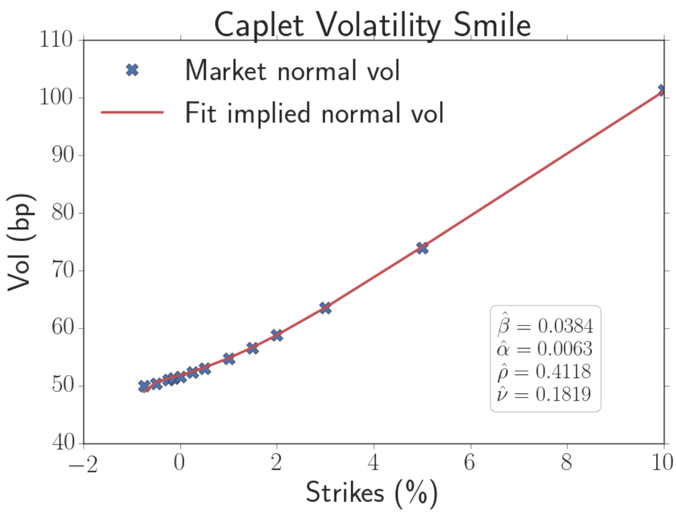

Figure 4.2: *Market normal volatility and its fit implied normal volatility for a caplet on Euribor 6 months with expiration date* $T_{ex} = 10$ *years.*

As we can see in Figure 4.2, the fit is highly accurate. This is the main reason why most traders (and the overall market) use this model to price interest rate derivatives. Another reason is because of its strong performance in risk management.[2] We can also observe in Figure 4.2 that the beginning of the curve is on the left of zero. This is becuase the set $\mathcal{K}$ of strikes used in calibration is

$$\mathcal{K} = \{-0.75, -0.50, -0.25, -0.13, 0, 0.25, 0.50, 1.00, 1.50, 2.00, 3.00, 5.00, 10.00\}$$

measured in percentage. As a consequence, in this calibration the shifted SABR model was used.

To improve the understanding about why an arbitrage is possible under the SABR

[1] We recall the reader that market's caplet volatilities are not quoted. The stripping procedure to get the caplet's volatilities will be explained in detail in Section 6.1. Therefore, some technicalities are now ommitted for the sake of continuity.

[2] For more detailed information about sensitivites under the SABR model we refer the reader to [25].

model, we are going to explain what a butterfly strategy is.

Call butterfly spread strategy

It consists of a specific combination of the same type of calls options with the same characteristics varying the strike, i.e.,

$$\begin{cases} V_c(t,K_1) \equiv V_{caplet}^{Bachelier}(t,T_{ex},T_p,K_1,\sigma_N(K_1)), \\ -2V_c(t,K_2) \equiv -2V_{caplet}^{Bachelier}(t,T_{ex},T_p,K_2,\sigma_N(K_2)), \\ V_c(t,K_3) \equiv V_{caplet}^{Bachelier}(t,T_{ex},T_p,K_3,\sigma_N(K_3)), \end{cases} \tag{4.1}$$

with $K_1 < K_2 < K_3$, where T_{ex} is the expiration date and T_p is the payment date. That is, buying a call with strike K_1, selling 2 calls with strike K_2 and buying a call with strike K_3. This strategy has a positive payoff at the expiration date T_{ex} and it looks like as shown in Figure 4.3.

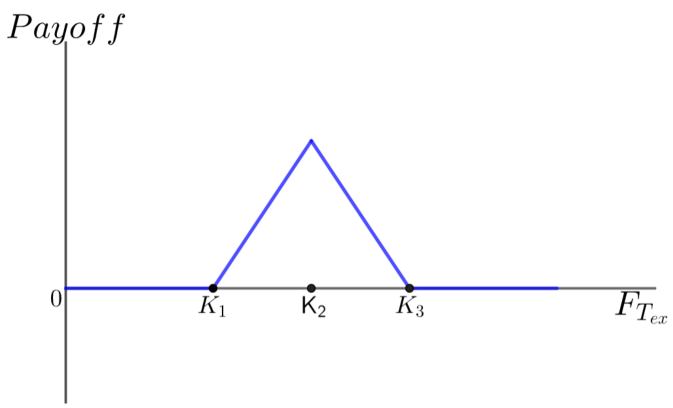

Figure 4.3: *Call's butterfly spread payoff at expiry.*

Hence, to avoid arbitrage opportunities, since the payoff

$$V_c(T_{ex},K_1) - 2V_c(T_{ex},K_2) + V_c(T_{ex},K_3) \geq 0 \tag{4.2}$$

is always positive at expiration date, that strategy should have a cost "today", this is, the price of this strategy should be:

$$V_c(t,K_1) - 2V_c(t,K_2) + V_c(t,K_3) > 0. \tag{4.3}$$

At this point we recall the formula (2.20) and taking into account the above notation, we can now re-write it as[3]

$$\frac{1}{P(t,T_{ex})\delta(T_{ex},T_p)}\frac{\partial^2}{\partial K^2}V_c(t,K) = Q_{F_{T_{ex}}}(K) \geq 0. \tag{4.4}$$

This second derivative may be approximated using a central difference scheme as follows:[4]

$$\frac{\partial^2}{\partial K^2}V_c(t,K) \approx \frac{V_c(t,K+\Delta K) - 2V_c(t,K) + V_c(t,K-\Delta K)}{(\Delta K)^2}, \quad \Delta K \to 0. \tag{4.5}$$

In order to ensure the relation (4.4) is held, must happen that (4.5) is positive. In other words, the caplet as a function of strike must be convex.

As discussed before in Subsection 3.4.5, the Hagan's formulae (3.16) and (3.19) have an approximation error for very low strikes leading to a violation of the condition (4.4). Therefore, let us suppose that there exists a very low strike K^* such that

$$V_c(t,K^*+\Delta K) - 2V_c(t,K^*) + V_c(t,K^*-\Delta K) < 0^5$$

$$\Downarrow$$

$$V_c(t,K^*+\Delta K) + V_c(t,K^*-\Delta K) < 2V_c(t,K^*) \tag{4.6}$$

This strategy is precisely a call butterfly spread and what we are saying is that with the sale of 2 calls with strike K^* we might buy 2 calls with strikes $K^*+\Delta K$ and $K^*-\Delta K$ and get a profit at time t. Since a call butterfly spread has a positive payoff at expiration date we would be incurring an arbitrage opportunity.

Figure 4.4 shows how the probability density function becomes negative for very low strikes (near -0.75%) and given parameters. In this case the implied normal volatility has been used to get the caplet's prices and then derivating like in formula (4.5), the RNPDF is obtained.

In the following points we deal the negativity problem and show how it is fixed by solving numerically a partial differential equation.

[3]The notation $F_{T_{ex}}$ refers to the underlying forward rate at the expiration date T_{ex}, namely,

$$F_{T_{ex}} \equiv F(T_{ex};T_{ex},T_p) = L(T_{ex},T_p).$$

[4]Appendix B provides a quick overview about finite difference schemes.

[5]This condition implies a negative density function.

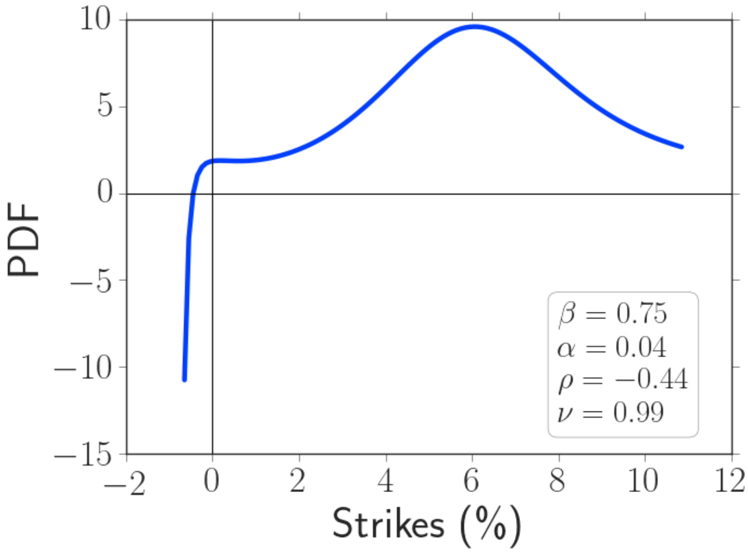

Figure 4.4: PDF at expiry based on the caplet's prices using the implied normal volatility. $T_{ex} = 34.5$Y and initial forward $f = 0.04$.

4.2 PDE context

Let us consider the SABR model (3.15) changing the expression $F(t)^\beta$ for a general one $C(F(t))$:

$$\begin{aligned} dF(t) &= \sigma(t) \cdot C(F(t)) \cdot dW(t), \quad f = F(0), \\ d\sigma(t) &= \nu \cdot \sigma(t) \cdot dZ(t), \\ \mathbb{E}^{\mathbb{Q}^T}[dW(t) \cdot dZ(t)] &= \rho dt, \end{aligned} \tag{4.7}$$

Let $p(T, F, A; t, f, \alpha)$ be the joint conditional probability density function of the random vector $(F(T), \sigma(T))$, conditioning to $F(t) = f$ and $\sigma(t) = \alpha$ at time t, namely

$$\begin{aligned} p(T, F, A; t, f, \alpha) dF dA &= \mathbb{P}\left(F' < F(T) < F' + dF,\ A < \sigma(T) < A + dA \mid F(t) = f,\right. \\ &\quad \left. \sigma(t) = \alpha\right) \end{aligned} \tag{4.8}$$

Notice that, in (4.8) the joint conditional density function is defined with a product of the infinitesimal amounts dF and dA equals to a joint conditional probability distribution $\mathbb{P}$. The reason of that can be found in Appendix A.1.

In the above definition, it is supposed that the market's economy is in the state $F(t) = f$ and $\sigma(t) = \alpha$. In such a way, the marginal (conditional) density function is defined as follows:

$$Q(T, F) = \int_0^\infty p(T, F, A; t, f, \alpha) dA. \tag{4.9}$$

Note that $Q(T, F)$ is actually the short notation for $Q_F(T, F \mid t, f, \alpha)$. One of the key factors of the development of the PDE is that $p(T, F, A; t, f, \alpha)$ satisfies the forward Kolmogorov equation (also called Fokker-Planck equation)[6]. In other words, as stated in [31], $p(T, F, A; t, f, \alpha)$ is also called the transition density function and it indicates that if the process starts at a point $(t, F(t), \sigma(t))$ what the probability density function will be of the position $(T, F(T), \sigma(T))$ of the diffusion (4.7). For this reason, $p(T, F, A; t, f, \alpha)$ is a function of the future state (T, F, A). The transition density function is the solution of the forward Kolmogorov equation and in the original Hagan et al. paper (2002) it was solved by constructing explicit asymptotic solutions in order to obtain implied volatilities (3.16) and (3.19).

In this new approach, $Q(T, F)$ also satisfies the Backward Kolmogorov equation and in [30] Hagan et al. used singular perturbation methods[7] to show that the $Q(T, F)$ satisfies what they call the **effective forward equation**. They aimed to solve this effective forward equation in a finite domain $[F_{min}, F_{max}]$ and hence some boundary conditions must be established. They do not consider any models in which paths are below F_{min} and above F_{max}. Thus, the reasoning is that the density function $Q(T, F)$ at $F = F_{min}$ must be accumulated around F_{min} so that the probability do not scape from the endpoint F_{min}. This concept can be represented by a delta function[8] centered at F_{min}. Hence, they define

$$Q(T, F) = Q^L(T)\delta(F - F_{min}) \quad \text{at } F = F_{min}. \tag{4.10}$$

[6]Very briefly, the forward Kolmogorov equation is one the Kolmgorov's equations. The another one is called the Backward kolmogorov equation. These are partial differential equations that characterize difusion processes of the form

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t).$$

The goal is to answer the question how the probability that a stochastic process is in a certain state changes over time. For more details and a nice derivation of these equations the reader is refered to [31].

[7]This theory is often applied in the context of partial diffferential equations and is used to develop a PDE solution around a small amount ϵ and allowing Taylor expansions. This topic is beyond the scope of this Thesis and the interested reader can see (for example) [32] for further information.

[8]The Dirac delta function $\delta(x - x_0)$ can be defined as follows:

$$\delta(x - x_0) = \begin{cases} \infty & \text{si } x = x_0 \\ 0 & \text{si } x \neq x_0 \end{cases} \qquad x \neq 0$$

with the property

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, dx = 1.$$

There are many other alternative definitions but this reflects that it may be understood as a density function since it integrates one. In addition, its whole probability is accumulated at $x = x_0$.

This way, the flux of probability will not be allowed below F_{min}. In the same way, it is defined

$$Q(T,F) = Q^R(T)\delta(F - F_{max}) \quad \text{at } F = F_{max}. \tag{4.11}$$

We recall that the above two restrictions are because of the necessity of solving a PDE in a finite domain. In such a way, the conditional density function $Q(T,F)$ is actually defined by:

$$Q(T,F) = \begin{cases} Q^L(T)\delta(F - F_{min}) & \text{at} \quad F = F_{min} \\ Q^c(T,F) & \text{for} \quad F_{min} < F < F_{max} \\ Q^R(T)\delta(F - F_{max}) & \text{at} \quad F = F_{max} \end{cases} \tag{4.12}$$

where $Q^c(T,F)$ accounts for the continuous part of the density. The so called **effective forward equation** is given by:

$$\frac{\partial Q^c}{\partial T} = \frac{1}{2}\alpha^2 \frac{\partial^2}{\partial F^2}\left[D^2(F)Q^c\right], \tag{4.13}$$

where

$$\begin{aligned} D(F) &= \sqrt{1 + 2\rho\nu z(F) + \nu^2 z^2(F)}e^{\frac{1}{2}\rho\nu\alpha\Gamma(F)(T-t)}C(F) \\ z(F) &= \frac{1}{\alpha}\int_f^F \frac{df'}{C(f')} \\ \Gamma(F) &= \frac{C(F) - C(f)}{F - f} \end{aligned} \tag{4.14}$$

A very important condition that will help us to set the boundary conditions is that the total probability has to be 1, this is

$$Q^L(T) + \int_{F_{min}}^{F_{max}} Q^c(T,F)dF + Q^R(T) = 1, \quad \forall\, T, \tag{4.15}$$

and this means that its derivative has to be zero:

$$\frac{d}{dT}\left[Q^L(T) + \int_{F_{min}}^{F_{max}} Q^c(T,F)dF + Q^R(T)\right] = 0. \tag{4.16}$$

If we derive (4.16), we have to use the Leibniz formula[9] and then substituing (4.13) for $\frac{\partial Q^c}{\partial T}$, we get:

$$\frac{dQ^L}{dT} + \frac{1}{2}\alpha^2 \int_{F_{min}}^{F_{max}} \frac{\partial Q^c}{\partial T}\, dF + \frac{dQ^R}{dT} = 0,$$

[9]**Leibniz formula:** Given the function

$$F(x) = \int_{g(x)}^{h(x)} f(t,x)\, dt,$$

the Leibniz formula to compute the derivative $F'(x)$ is as follows:

$$F'(x) = \int_{g(x)}^{h(x)} \frac{\partial f}{\partial x}\, dt + f(h(x),x)\cdot h'(x) - f(g(x),x)\cdot g'(x).$$

$$\frac{dQ^L}{dT} + \frac{1}{2}\alpha^2 \frac{\partial}{\partial F}\left[D^2(F)Q^c\right]\Bigg|_{F=F_{min}}^{F=F_{max}} + \frac{dQ^R}{dT} = 0. \tag{4.17}$$

Thus, conservation of probability requires that

$$\frac{dQ^L}{dT} = \lim_{F\to F_{min}^+} \frac{1}{2}\alpha^2 \frac{\partial}{\partial F}[D^2(F)Q^c], \tag{4.18a}$$

$$\frac{dQ^R}{dT} = \lim_{F\to F_{max}^-} -\frac{1}{2}\alpha^2 \frac{\partial}{\partial F}[D^2(F)Q^c]. \tag{4.18b}$$

Equations (4.18) will be used later. On the other hand, for that $F(T)$ to be a martingale, the expected value has to be constant:

$$\begin{aligned}\mathbb{E}\left[F(T) \mid F(t) = f, \sigma(t) = \alpha\right] &= \int_{-\infty}^{\infty} FQ(T,F)\; dF \\ &= F_{min}Q^L(T) + \int_{F_{min}}^{F_{max}} FQ^c(T,F)\; dF + F_{max}Q^R(T) \\ &= f.\end{aligned} \tag{4.19}$$

Note that in the first equality of equation (4.19) we are doing an abuse of notation.[10] Derivating the above second equality, we get

$$F_{min}\frac{dQ^L(T)}{dT} + \int_{F_{min}}^{F_{max}} F\frac{\partial Q^c(T,F)}{\partial T}\; dF + F_{max}\frac{dQ^R(T)}{dT} = 0. \tag{4.20}$$

Substituing (4.13) for $\frac{\partial Q^c}{\partial T}$, integrating by parts twice and using equations (4.18) leads to

$$D^2(F)Q^c(T,F)\Bigg|_{F_{min}}^{F_{max}} = 0. \tag{4.21}$$

Hence, **the boundary conditions** can be summarized as follows:

$$D^2(F)Q^c \to 0 \quad \text{as } F \to F_{min}^+, \tag{4.22a}$$

$$D^2(F)Q^c \to 0 \quad \text{as } F \to F_{max}^-. \tag{4.22b}$$

With regard to the initial conditions we have that

$$Q^c(T,F) \to \delta(F-f) \quad \text{as } T \to t, \tag{4.23}$$

[10] The abuse of notation comes from denoting the random variable $F(T)$ as F in the integrand. We recall the reader that if X is a random variable, its expected value is given by:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)\; dx,$$

where $f(x)$ is the density function of X.

namely, when the expiration date is very close to "today", we certainly know the today's forward. In other words the today's probability of being at $F = f$ is quite high and as we mentioned before, the Dirac delta function reflects that all probability is accumulated, in this case, around f.

On the other hand, as $Q(T, F)$ must be integrate 1 for all T and (4.23) already integrates 1, then must be happen that

$$Q^L(T) \to 0, \quad Q^R(T) \to 0 \quad \text{as } T \to t. \tag{4.24}$$

To sum up, the PDE to be solved is given by

$$\frac{\partial Q^c}{\partial T} = \frac{1}{2}\alpha^2 \frac{\partial^2}{\partial F^2}\left[D^2(F)Q^c\right], \quad F_{min} < F < F_{max}, \tag{4.25a}$$

with probability at the boundaries

$$\frac{dQ^L}{dT} = \lim_{F \to F_{min}^+} \frac{1}{2}\alpha^2 \frac{\partial}{\partial F}[D^2(F)Q^c], \tag{4.25b}$$

$$\frac{dQ^R}{dT} = \lim_{F \to F_{max}^-} -\frac{1}{2}\alpha^2 \frac{\partial}{\partial F}[D^2(F)Q^c], \tag{4.25c}$$

boundary conditions

$$D^2(F)Q^c \quad \text{as } F \to F_{min}^+, \quad D^2(F)Q^c \to 0 \quad \text{as } F \to F_{max}^- \tag{4.25d}$$

and the initial conditions are

$$Q^L(0) = 0, \quad Q^c(T, F) \to \delta(F - f), \quad Q^R(0) = 0, \quad \text{as } T \to t^+. \tag{4.25e}$$

In the next section we are going to sketch out a Crank-Nicholson scheme in order to solve (4.25).

4.3 PDE solution scheme

We will start simplifying notation. We define

$$M(T, F) = \frac{1}{2}\alpha^2 D^2(F). \tag{4.26}$$

The scheme to be applied is the Crank-Nicholson scheme and in Appendix B.2 the reader can find more information about this procedure.

Let N and J be the number of time steps and the number of steps in the forward F, respectively. Let Δt and h be the temporary increase and h the spatial increase. In this way, we define

- $\Delta t = \frac{T_{ex}}{N}$
- $t_n = n \cdot \Delta t, \quad n = 0, 1, \dots, N$
- $F_j = F_{min} + \left(j - \frac{1}{2}\right) h, \quad j = 0, 1, \dots, J + 1$
- The spatial step J satisfies $F_{max} = F_{min} + Jh$.
- $f = F_{min} + \left(j_0 - \frac{1}{2}\right) h$, for some $j_0 \in \{1, 2, \dots, J\}$.

Let $Q_j^n \equiv Q^c(t_n, F_j)$ be the density function evaluated at the grid point (t_n, F_j). Moreover, in the case of the SABR model, we set $C(F) = F^\beta$ and therefore

$$M_j^n \equiv M(t_n, F_j) = [1 + 2\rho\nu z_j + \nu^2 z_j^2] e^{\nu\alpha\Gamma_j t_n} F_j^{2\beta}, \tag{4.27}$$

where

$$z_j = \frac{F_j^{1-\beta} - f^{1-\beta}}{\alpha(1-\beta)}, \quad \Gamma_j = \frac{F_j^\beta - f^\beta}{F_j - f}.$$

Hence, if we apply the Crank-Nicholson scheme to (4.25a) we get the following discretisation:

$$\underbrace{\frac{Q_j^{n+1} - Q_j^n}{\Delta t}}_{\text{Forward difference in time}} = \frac{1}{2h^2} \Bigg[\underbrace{M_{j+1}^{n+1} Q_{j+1}^{n+1} - 2M_j^{n+1} Q_j^{n+1} + M_{j-1}^{n+1} Q_{j-1}^{n+1}}_{\text{Central difference at the (n+1)-th row}}$$

$$+ \underbrace{M_{j+1}^{n} Q_{j+1}^{n} - 2M_j^{n} Q_j^{n} + M_{j-1}^{n} Q_{j-1}^{n}}_{\text{Central difference at the n-th row}} \Bigg].$$

We can re-write it as

$$Q_j^{n+1} - \frac{\Delta t}{2h^2} \left[M_{j+1}^{n+1} Q_{j+1}^{n+1} - 2M_j^{n+1} Q_j^{n+1} + M_{j-1}^{n+1} Q_{j-1}^{n+1} \right] =$$

$$Q_j^n + \frac{\Delta t}{2h^2} \left[M_{j+1}^{n} Q_{j+1}^{n} - 2M_j^{n} Q_j^{n} + M_{j-1}^{n} Q_{j-1}^{n} \right] \quad \text{for } j = 1, 2, \dots, J. \tag{4.28}$$

The points $j = 0$ and $j = J + 1$ are used to generate the boundary conditions (4.25d) as follows:

$$M_0^{n+1} Q_0^{n+1} = -M_1^{n+1} Q_1^{n+1} \quad \text{at } j = 0, \tag{4.29a}$$

$$M_{J+1}^{n+1} Q_{J+1}^{n+1} = -M_J^{n+1} Q_J^{n+1} \quad \text{at } j = J + 1. \tag{4.29b}$$

We can interpret these conditions in the following way: the boundary conditions (4.25d) mean that as F is very close to F_{min}^{+}, $D^2(F)Q^c \to 0$, i.e., speaking in terms of discretisation this means that the first two terms have to sum 0. In other words, one term has to be the opposite of the other like in (4.29a). The same argument is repeated for the case when $F \to F_{max}^{-}$.

If we define

- $Q_L^n \equiv Q^L(t_n)$,
- $Q_R^n \equiv Q^R(t_n)$,

then the initial conditions are given by

$$Q_L^0 = 0, \quad Q_j^0 = \begin{cases} 0 & \text{for } j \neq j_0 \\ \dfrac{1}{h} & \text{for } j = j_0 \end{cases}, \quad Q_R^0 = 0. \tag{4.30}$$

At each time step, after solving for $Q_0^{n+1}, Q_1^{n+1}, \ldots, Q_J^{n+1}, Q_{J+1}^{n+1}$, applying forward difference in time, an average of the forward differences of the row $(n+1)$-th and the row n-th to (4.25b), (4.25c) and taking into account (4.29) we get

$$Q_L^{n+1} = Q_L^n + \frac{\Delta t}{2h}\left(2M_1^{n+1}Q_1^{n+1} + 2M_1^nQ_1^n\right), \tag{4.31a}$$

$$Q_R^{n+1} = Q_R^n + \frac{\Delta t}{2h}\left(2M_J^{n+1}Q_J^{n+1} + 2M_J^nQ_J^n\right). \tag{4.31b}$$

The next section shows how the equations system looks like.

4.4 Probability density as a solution of a tridiagonal system

The coefficients matrix will be defined by equations (4.28) for $j = 2, \ldots, J-1$. For the boundary conditions we have that

- For $j = 1$,

$$(2 + 3\Delta t M_1^{n+1})Q_1^{n+1} - \Delta t M_2^{n+1}Q_2^{n+1} = (2 - 3\Delta t M_1^n) + \Delta t M_2^n Q_2^n.$$

- For $j = J$,

$$-\Delta t M_{J-1}^n Q_{J-1}^n + (2 + 3\Delta t M_J^{n+1})Q_J = \Delta t M_{J-1}^n Q_{J-1}^n + (2 - 3\Delta t M_J^n)Q_J^n$$

In such a way, the equations system is given by:

$$\tilde{M}q = \tilde{q}, \tag{4.32}$$

where

$$\tilde{M} = \begin{pmatrix} 2+3\Delta t M_1^{n+1} & -\Delta t M_2^{n+1} & 0 & \cdots & 0 \\ -\Delta t M_1^{n+1} & 2+2\Delta t M_2^{n+1} & -\Delta t M_3^{n+1} & & \\ & \ddots & \ddots & \ddots & \\ & & -\Delta t M_{J-2}^{n+1} & 2+2\Delta t M_{J-1}^{n+1} & -\Delta t M_J^{n+1} \\ 0 & \cdots & 0 & -\Delta t M_{J-1}^{n+1} & (2+3\Delta t M_J^{n+1}) \end{pmatrix}_{J\times J} \tag{4.33}$$

the unknown vector q is given by

$$q = \left(Q_1^{n+1}\ Q_2^{n+1}\ \cdots\ Q_J^{n+1}\right)'_{1\times J}, \tag{4.34a}$$

and the independant term $\tilde{q}$ is given by

$$\tilde{q} = \begin{pmatrix} (2-3\Delta t M_1^n)Q_1^n + \Delta t M_2^n Q_2^n \\ \Delta t M_1^n Q_1^n + (2-2\Delta t M_2^n)Q_2^n + \Delta t M_3^n Q_3^n \\ \vdots \\ \Delta t M_{J-2}^n Q_{J-2}^n + (2-2\Delta t M_{J-1}^n)Q_{J-1}^n + \Delta t M_J^n Q_J^n \\ \Delta t M_{J-1}^n Q_{J-1}^n + (2-3\Delta t M_J^n)Q_J^n \end{pmatrix}_{J\times 1} \tag{4.34b}$$

In this procedure, we must solve the system (4.32) for every time step n. When the process ends, we will have a solution matrix Q which is the conditional probability density function evaluated at each grid point (t_n, F_j). Hence, the matrix Q would look like

$$Q = \begin{pmatrix} Q_L^0 & Q_1^0 & Q_2^0 & \cdots & Q_J^0 & Q_R^0 \\ Q_L^1 & Q_1^1 & Q_2^1 & \cdots & Q_J^1 & Q_R^1 \\ \vdots & & \ddots & & \vdots & \vdots \\ & \vdots & & Q_{J-1}^{N-1} & Q_J^{N-1} & Q_R^{N-1} \\ Q_L^N & \cdots & \cdots & Q_{J-1}^N & Q_J^N & Q_R^N \end{pmatrix} \tag{4.35}$$

We are interested in the last row of Q because we need the conditional density function at expiration $Q(T_{ex}, F)$ in order to be able to price plain vanilla options. Next section shows how call (put) options can be priced based on this probability density function at expiratation.

4.5 Option pricing

The vanillas option's value at time t with expiry T_{ex} is given by:

$$V_{call}(t, T_{ex}, K) = P(t, T_{ex}) \int_K^{\infty} (F - K) Q(T_{ex}, F)\, dF, \tag{4.36a}$$

$$V_{put}(t, T_{ex}, K) = P(t, T_{ex}) \int_{-\infty}^{K} (K - F) Q(T_{ex}, F)\, dF. \tag{4.36b}$$

The above equations can be approximated by[11]

$$V_{call}(t, T_{ex}, K) = f - K \quad \text{for } K < F_{min} \tag{4.37a}$$

$$V_{call}(t, Tex, K) = \tfrac{1}{2}\left(F_{min} + j_k h - K\right)^2 Q^N_{j_k} + \sum_{j=j_k+1}^{J} (F_j - K) h Q^N_j + (F_{max} - K) Q^N_R \quad \text{For } F_{min} < K < F_{max} \tag{4.37b}$$

$$V_{call}(t, T_{ex}, K) = 0 \quad \text{For } K < F_{min} \tag{4.37c}$$

For the put prices we have that:

$$V_{put}(t, T_{ex}, K) = 0 \quad \text{for } K < F_{min} \tag{4.38a}$$

$$V_{put}(t, Tex, K) = (K - F_{min}) Q^N_L + \sum_{j=1}^{j_k - 1} (K - F_j) h Q^N_j + \frac{1}{2}\left(F_{min} + j_k h - K\right)^2 Q^N_{j_k} \quad \text{For } F_{min} < K < F_{max} \tag{4.38b}$$

$$V_{put}(t, T_{ex}, K) = K - f \quad \text{For } K < F_{min} \tag{4.38c}$$

where $j_k \in \{1, 2, \ldots, J\}$ is such that $F_{min} + (j_k - 1)h < K < F_{min} + j_k h$.

In Part 2, we will present and describe the data with which we will work on this Thesis, as well as determine the probability density function at expiry for caps and swaptions using the Crank-Nicholson scheme.

[11] For simplicity we do not take into account the discount factor $P(t, T_{ex})$.

Part II

Empirical Analysis

Chapter 5

The Data

In this chapter the datasets used on this Thesis are described.

5.1 Discount factors curves

We start with the graph of the discount factor $P(t,T)$. It is shown in Figure 5.1 and we can see that due to the context of negative interest rates, short-mid term discount factors are greater than 1. Data can be found in the Appendix C.

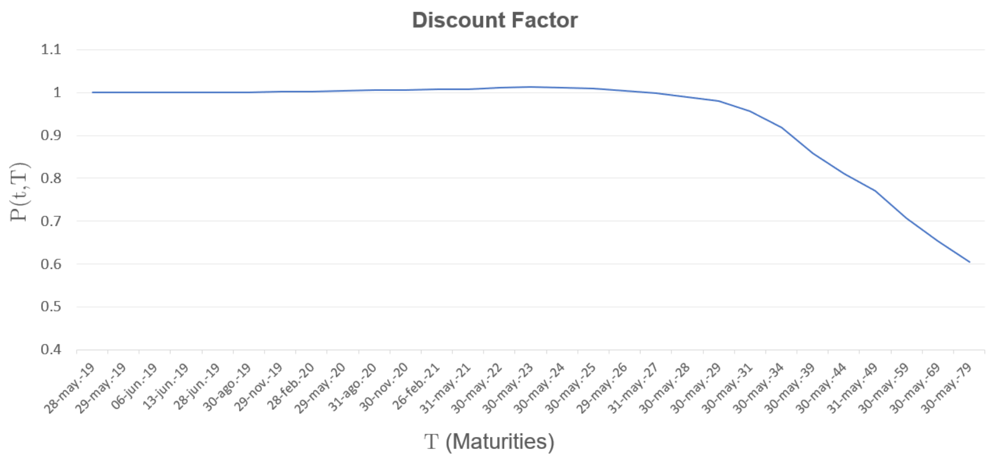

Figure 5.1: *Graph $T \to P(t,T)$ where t is 28 May 2019 and T accounts for the maturities shown on the horizontal axis of the graph.*

Below, we present in Figure 5.2 the estimated discount factor curve used for forwarding. Data can be found in the Appendix C.

Figure 5.2: *Graph* $T \to P^e(t,T)$ *where* t *is 28 May 2019 and* T *accounts for the maturities shown on the horizontal axis of the graph.*

5.2 Caps/Floors volatilities

In financial markets the cap/floor quoting is usually done in terms of implied (Black or normal) volatilities. In our case, normal volatilities are used and an explanation will be given of how they are quoted.

Some concepts to consider:

- **Flat volatility:** The flat normal (constant) volatility $\bar{\sigma}_N$ is defined as the implied volatility that matches the cap/floor price given by formulae (3.4), namely,

$$V_{cap}^{Bachelier}(t,T_n,K,\bar{\sigma}_N) = \sum_{j=2}^{n} P(t,T_j)\cdot\delta(T_{j-1},T_j)\cdot B_{call}^N(T_{j-1},K,f,\bar{\sigma}_N), \tag{5.1a}$$

$$V_{floor}^{Bachelier}(t,T_n,K,\bar{\sigma}_N) = \sum_{j=2}^{n} P(t,T_j)\cdot\delta(T_{j-1},T_j)\cdot B_{put}^N(T_{j-1},K,f,\bar{\sigma}_N). \tag{5.1b}$$

- **Spot start cap/floor:** This cap/floor is defined as in formulae introduced in Section (3.1), i.e., it is the cap's (floor's) value for the period from the valuation date t until the maturity date T_{ex}.

- **Forward start cap/floor:** This cap/floor is defined as the cap/floor that starts in a future date. Let us give an example. Suppose we have a cap with maturity 1 year and another one with maturiry 2 years. Therefore, the forward start cap that starts in 1 year and ends in 2 years (from the valuation date t) is defined by:

$$V_{cap}^{Bachelier}(t, 1Y, 2Y, K, \bar{\sigma}_N^{1\times 2}) = V_{cap}^{Bachelier}(t, 2Y, K, \bar{\sigma}_N^{1\times 2}) - V_{cap}^{Bachelier}(t, 1Y, K, \bar{\sigma}_N^{1\times 2}). \tag{5.2}$$

The cap's flat forward start normal volatility $\bar{\sigma}_N^{1\times 2}$ accounts for the constant volatility for the period $[1Y, 2Y]$.[1] This concept is crucial for the stripping caplet volatility procedure. The cap volatility $\bar{\sigma}_N^{1\times 2}$ is quoted in the market and is the one that we are going to use in this Thesis.

The cap/floor forward start normal volatility quoted in market at date 28 May 2019[2] is given by the Table 5.1.

	-0.75%	-0.50%	-0.25%	-0.13%	0.00%	0.25%	0.50%	1.00%	1.50%	2.00%	3.00%	5.00%	10.00%
1X2	17.9	17.4	20.5	23	25.7	30.9	35.7	44.6	53	60.9	78.5	121.8	235.4
2X3	27.2	28.3	30.2	31.3	32.6	35.3	38.1	43.9	49.6	55.2	65.9	92.9	179.4
3X4	35.1	36.3	37.9	38.7	39.8	41.9	44.2	49.1	54.2	59.3	69.4	89	149.9
4X5	40.5	41.5	42.8	43.6	44.4	46.3	48.3	52.7	57.4	62.2	71.9	91	137.3
5X6	43.9	44.7	45.8	46.4	47.1	48.7	50.4	54.3	58.6	63.1	72.4	91	135.1
6X7	46.1	46.8	47.7	48.1	48.7	50	51.5	55	58.9	63.1	71.9	89.8	132.6
7X8	47.6	48.2	49	49.4	49.9	51	52.2	55.1	58.4	62	69.6	85.6	124.6
8X9	48.5	49	49.6	49.9	50.3	51.1	52.1	54.5	57.4	60.6	67.7	82.8	120.5
9X10	50.1	50.5	51	51.3	51.7	52.4	53.3	55.4	57.9	60.8	67.2	81.3	116.9
2X4	31.8	32.6	34.3	35.3	36.6	39.2	42	47.6	53.1	58.5	69	89	156
5X7	45	45.8	46.7	47.3	47.9	49.4	51	54.7	58.8	63.1	72.1	90.2	133.3
5X10	47.3	47.9	48.6	49.1	49.6	50.7	52	54.9	58.2	61.7	69	84.2	121.3
10X15	49.9	50.4	51	51.3	51.6	52.3	53	54.7	56.6	58.8	63.5	74	101.3
10X20	48.6	49.1	49.7	50	50.4	51.1	51.8	53.4	55.2	57.1	61.4	70.7	94.9
15X20	47.6	48.1	48.7	49	49.3	50	50.7	52.2	53.9	55.7	59.6	68.4	91.8
20X30	46	46.6	47.1	47.4	47.7	48.3	48.9	50.3	51.8	53.5	57.1	65.2	86.8

Table 5.1: *Cap/Floor normal volatility forward starting on EUR 6M. The coloured area refers to floor's volatilities and the white area is for cap's volatilities. Volatilities are in basis points.*

The forward start normal volatilities in Table 5.1 are introduced in formula (3.23) in order to calibrate the SABR model. We will return to this point later in Section 6.2.

Other necessary dataset for the stripping procedure is the premium forward start. Namely, instead of quoting in volatility's terms they are quoted in premium which is basis points to be applied to a nominal amount. They are shown in Table 5.2.

The cap premium forward start $1y \times 2y$ with strike (for example) $K = -0.25\%$ is 9 bp, i.e., the price is $\frac{9}{10000} \cdot N$ €, where N is a nominal amount in euros. Blank spaces mean that those premiums are not quoted.

[1]For more detailed information, these concepts can be found at `http://www.smileofthales.com/financial/cap-floor-pricing-stripping-the-basics/` or [14].

[2]We have used Bloomberg from the source ICAP.

	-0.75%	-0.50%	-0.25%	-0.13%	0.00%	0.25%	0.50%	1.00%	1.50%	2.00%	3.00%	5.00%	10.00%
1X2			9	5.8	3.4	1.3							
2X3	1.1	4.4	12.7	18.8	14.4	7.8	4.2	1.3					
3X4	3.3	7.7	15.6	20.7	27.4	21.4	14.4	6.5	3	1.5			
4X5	5	9.6	16.7	21.2	26.9	38.3	28.6	15.8	8.9	5.1	1.9		
5X6	6.1	10.4	16.7	20.6	25.5	36.9	43.9	27	16.7	10.6	4.5	1.1	
6X7	6.6	10.5	16.2	19.6	23.8	33.7	45.9	39.1	25.7	17.1	8.1	2.3	
7X8	6.9	10.5	15.7	18.7	22.5	31.3	42	52	35.5	24.2	11.7	3.3	
8X9	6.9	10.2	14.7	17.4	20.7	28.4	37.8	62.3	44.4	31.1	15.6	4.6	
9X10	7.7	11.1	15.5	18.1	21.2	28.4	37.2	59.9	53.8	38.7	20.2	6.3	
2X4	4.4	12.1	28.2	39.5	45.8	29.1	18.5	7.8	3.5	1.6			
5X7	12.7	20.9	32.9	40.2	49.3	70.6	96.7	66.1	42.5	27.7	12.6	3.4	
5X10	34.1	52.7	78.8	94.4	114	159	214	256	176	122	60.1	17.6	1.9
10X15	43	59.4	80.4	92.4	107	139	178	275	358	268	149	47.8	4.7
10X20	103	139	183	208	238	303	380	572	745	568	327	112	12.1
15X20	60.1	79.2	103	116	131	164	203	297	386	299	178	63.8	7.4
20X30	215	267	326	358	394	471	558	758	706	564	359	152	26.5

Table 5.2: *Cap/Floor premiums. The coloured area refers to floor's premiums and the white area is for cap's premiums. Premiums are in basis points.*

5.3 Swaption volatilities

Swaption volatilities can be quoted in terms of the normal or Black volatilities. In this Thesis we will use normal volatities. The information is collected in Table 5.3.

Swaption normal volatilities are quoted in ATM differences which means that, for instance, the $1Y \times 2Y$ swaption volatility with strike $K = -25bp$ is $22.77bp - 4.55bp = 18.22bp$ and for strike $K = 25bp$ the volatility is $22.77bp + 5.45bp = 28.22bp$.

	-200	-150	-100	-50	-25	ATM	25	50	100	150	200
1y2y	7.71	2.64	-2.28	-5.76	-4.55	22.77	5.45	10.84	21.09	30.83	40.14
1y5y	13.35	7.31	1.64	-1.67	-1.65	32.31	3.13	7.09	15.86	24.89	33.87
1y10y	16.26	9.67	3.65	0.01	-0.67	38.11	2.15	5.37	13.39	22.65	32.76
1y20y	16.65	10.68	5.58	1.9	0.48	38.57	0.75	2.49	8.2	15.18	22.5
1y30y	17.5	11.71	6.82	2.78	0.99	37.56	0.15	1.22	5.88	12.1	18.87
2y2y	7.6	2.98	-0.79	-2.38	-1.81	32.14	2.78	6.11	13.41	20.87	28.31
2y5y	2.62	-0.82	-3.21	-2.66	-1.57	38.58	2	4.31	9.56	15.22	21.04
2y10y	1.86	-1.17	-2.99	-2.23	-1.35	41.29	1.72	3.6	8.03	12.97	18.93
2y20y	2.78	0.17	-0.94	-0.85	-0.65	40.54	0.98	2	4.81	8.37	12.35
2y30y	3.07	0.76	-0.16	-0.27	-0.34	39.46	0.6	1.08	2.69	5.06	7.96
5y2y	-6.43	-6.66	-5.15	-2.89	-1.55	47.93	1.67	3.35	6.95	10.76	14.67
5y5y	-4.57	-5.52	-4.83	-2.7	-1.51	48.1	1.63	3.16	6.5	10.1	13.87
5y10y	-4.96	-5.67	-4.66	-2.71	-1.63	47.43	1.76	3.16	6.23	9.58	13.1
5y20y	-2.37	-3.46	-2.83	-1.92	-1.41	43.93	1.64	2.64	5.11	8.07	11.36
5y30y	-1.85	-2.96	-2.46	-1.65	-1.28	42.47	1.54	2.33	4.31	6.74	9.51
10y2y	-6.25	-5.32	-3.89	-2.2	-1.24	54.86	1.32	2.49	5	7.72	10.58
10y5y	-5.56	-5.17	-3.64	-2.05	-1.21	53.88	1.28	2.31	4.56	7.02	9.66
10y10y	-5.25	-4.91	-3.62	-2.27	-1.54	52.12	1.66	2.64	4.81	7.22	9.82
10y20y	-4.58	-4.65	-3.33	-2.3	-1.85	47.15	2.05	2.77	4.47	6.49	8.78
10y30y	-4.63	-4.83	-3.56	-2.32	-1.83	45.62	2.16	2.99	4.84	6.95	9.29
15y2y	-4.83	-4.01	-2.93	-1.68	-0.98	53.93	1.04	1.92	3.86	5.99	8.27
15y5y	-4.86	-4.47	-3.12	-1.79	-1.09	52.66	1.15	2.02	3.92	6.03	8.33
15y10y	-4.45	-4.34	-3.19	-2.06	-1.46	51.35	1.58	2.39	4.2	6.25	8.51
15y20y	-5.19	-5.02	-3.66	-2.45	-1.99	46.89	2.19	2.86	4.37	6.13	8.14
15y30y	-5.62	-5.4	-4.16	-2.69	-2.14	44.93	2.51	3.38	5.23	7.26	9.47
20y2y	-3.84	-3.31	-2.42	-1.42	-0.86	51.47	0.92	1.66	3.31	5.17	7.21
20y5y	-3.89	-4.02	-2.9	-1.63	-1	51.05	1.06	1.81	3.46	5.33	7.39
20y10y	-4.17	-4.15	-3.16	-1.97	-1.41	49.73	1.51	2.22	3.78	5.55	7.51
20y20y	-5.98	-5.32	-4.09	-2.59	-2.06	45.22	2.26	2.93	4.41	6.1	8.03
20y30y	-6.42	-5.83	-4.66	-2.93	-2.32	42.96	2.69	3.55	5.31	7.19	9.2

Table 5.3: *Swaption normal volatilities. Volatilities and strikes are in basis points.*

Chapter 6

Tests

In this chapter we describe the stripping procedure in detail, SABR calibration and we show some issues and tests about the Crank-Nicholson scheme.

6.1 Stripping caplet volatilities

This procedure arises from the nature of the SABR model. The SABR model is designed to get an implied volatility for the most basic instruments in markets which are vanilla options. A cap/floor is not a straightforward option, in fact, they are a basket of vanilla options (caplets/floorlets). In this way, the problem comes when we are going to look in markets those caplet/floorlet volatilities and it turns out that they are not quoted, instead cap/floor volatilities are quoted. In light of the foregoing, a procedure to extract those caplet/floorlet volatilities is needed. That procedure is the so-called **stripping caplet volatilities**.

In the previous section we introduced a key concept: the forward start volatiltiy. The stripping procedure is based on computing the forward start volatilities from forward start premiums.[1] It works as follows:

1. Order the caps in ascending order of maturity: $T_0 < T_1 < \cdots < T_n$.

2. For a given strike K, compute the sequence of price differences:

$$V_{cap}^{MKT}(t, T_j, K, \sigma_j^{cap}) - V_{cap}^{MKT}(t, T_{j-1}, K, \sigma_{j-1}^{cap}) \quad j = 1, \dots n, \tag{6.1}$$

with $V_{cap}^{MKT}(t, T_0, K, \sigma_0^{cap}) = 0$ and σ_j^{cap} is the cap's volatility for the maturity T_j.

[1] In literature there are many ways to extract the caplet volatilities from cap volatitilies as shown in [33]. However, the presented procedure is the most common and simplest. The described procedure in this Thesis is inspired in (for example) [34] among many other references.

3. Every price difference is mapped to the caplets between the maturities T_{j-1} and T_j. For instance, let us suppose a 6 month Euribor cap with maturity 1 year (1Y), hence the corresponding mapping would be

$$V_{cap}^{MKT}(t,1Y,K,\sigma_{1Y}^{cap}) = V_{caplet}(t,T_0,6M,K,\sigma_0^{caplet})+V_{caplet}(t,6M,1Y,K,\sigma_{6M}^{caplet}), \tag{6.2}$$

where the value $V_{caplet}(t,T_0,6M,K,\sigma_0^{caplet})$ must not be taken into account since it is the value of a caplet with reset date T_0 ($T_0 = t$) and thus its value is already known and there is not point in that price being part of the cap's premium. Hence, it must be

$$V_{cap}^{MKT}(t,1Y,K,\sigma_{1Y}^{cap}) = V_{caplet}(t,6M,1Y,K,\sigma_{6M}^{caplet}), \tag{6.3}$$

where σ_{6M}^{caplet} is the caplet's volatility (to be more precise it is the Libor's volatility since the Libor rate is the random variable) between the valuation date until the reset date (6M) because when the reset date comes the Libor rate is no longer random. Notice that V_{caplet} accounts for the caplet's value in the Bachelier model or the Black model. Hence, in this case, the unknown caplet volatility to be found is σ_{6M}^{caplet} throughout a 1 dimensional root finder like Newton Raphson.

In the next step, considering the 6 month Euribor cap with maturity 2 years and the same strike, the equation would be

$$\begin{aligned} &V_{cap}^{MKT}(t,2Y,K,\sigma_{2Y}^{cap}) - V_{cap}^{MKT}(t,1Y,K,\sigma_{1Y}^{cap}) \\ &= V_{caplet}(t,1Y,18M,K,\sigma_{1Y}^{caplet}) + V_{caplet}(t,18M,2Y,K,\sigma_{18M}^{caplet}), \end{aligned} \tag{6.4}$$

and the crucial hypothesis is that $\sigma_{1Y}^{caplet} = \sigma_{18M}^{caplet}$ and then the unknown caplet's volatility to be found is σ_{18M}^{caplet}. Notice that the left hand side of (6.4) corresponds with the forward start cap introduced in Section 5.2. In this case, (6.4) is the forward start cap that starts within a year with maturity 1 year length (or 2 years from the valuation date t) and following the notation introduced in Section 5.2 it would be

$$V_{cap}^{Bachelier}(t,1Y,2Y,K,\bar{\sigma}^{1Y\times 2Y}). \tag{6.5}$$

Thus, forward start caps fully define the caplet volatility structure. In such a way, we can easly pass from the premium dataset in Table 5.2 to cap's (floor's) volatilities dataset in Table 5.1.

Some remarks can be taken into account. The Cap/Floor volatilities in Table 5.1 can be understood as caplets volatilities because they are the flat volatilities that we introduced in formula (3.4) in order to get the cap price.

As commented in [34], the process does not work for the ATM strike. The reason is by definition in itself, namely, the ATM strike of a cap is different for every maturity. The procedure to find the caplet's volatility for the ATM strike is a little bit trickier and the interested reader is referred to [34] for more detailed information.

6.2 Calibration in practice

Once the differences between caplets volatilities and caps volatilities have been explained in detail, it is easy to understand the calibrating procedure. Although, it was already explained in Section 3.4.4, we will make some clarifications.

As discussed in Section 4.1, we allowed the paramater β to be calibrated. In addition, according to the notation introduced in Section 5.2 for forward start volatilities, the calibration formula (3.23) can be rewrite as

$$(\hat{\nu}, \hat{\rho}, \hat{\alpha}, \hat{\beta}) = \underset{\nu,\rho,\alpha,\beta}{\arg\min} \sum_{K \in \mathcal{S}} \left(\sigma_{caplet}^{T_{Start} \times T_{Mat}}(T_{ex}, K) - \sigma_N(T_{ex}, K, f)\right)^2, \tag{6.6}$$

where T_{start} is the beginning and T_{Mat} is the maturity of the corresponding cap. T_{ex} stands for the expiration date of an underlying caplet between dates T_{Start} and T_{Mat}. For example, for the caplet volatility smile calibrated in Section 4.1 (Figure 4.2), the crucial dates are $T_{Start} = 10Y$, $T_{Mat} = 15Y$ and we have chosen $T_{ex} = 10Y$.[2]

6.3 Crank-Nicholson testing

6.3.1 Caps

In Section 4.1 a market caplet volatility smile with expiration date $T_{ex} = 10$ years and initial forward $f = 0.01291$ was fitted with the implied normal volatility and the calibrated paramaters were

$$\hat{\alpha} = 0.0063, \quad \hat{\beta} = 0.0384, \quad \hat{\rho} = 0.4118, \quad \hat{\nu} = 0.1819. \tag{6.7}$$

[2]All underlying caplets between dates $10Y$ and $15Y$, have the same volatility (by hypothesis) and then we could have chosen any expiration date from the set $\{10Y, 10.5Y, 11Y, \ldots, 14.5Y\}$ with its corresponding initial forward. In our case, the initial forward f was computed for the caplet's prevailing period $[10Y, 10.5Y]$.

In view of the above, we solve the PDE throughout the equations system (4.32) using the calibrated parameters (6.7) in order to check how the probability density function looks like (see Figure 6.1).

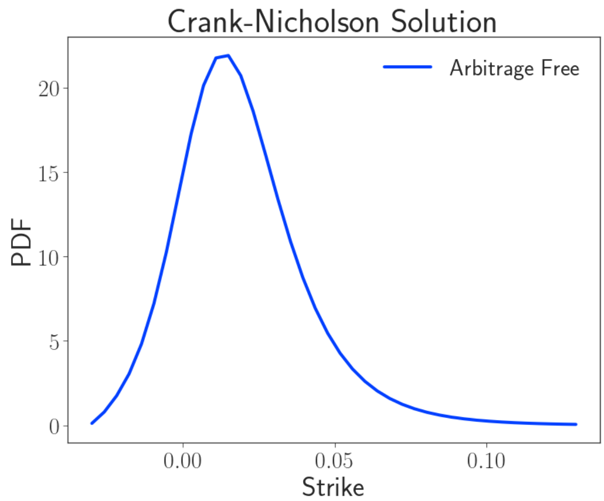

Figure 6.1: *Probability density function of the forward rate* $F(T_{ex})$*, based on the calibrated parameters (6.7).*

As we can see in Figure 6.1, the density function draw a smooth curve even for negative strikes. This makes sense because if the density function was negative we would have found in the market an arbitrage opportunity as explained in Section 4.1. The formula (6.7) has been used in this Thesis.[3]

6.3.2 Swaptions

In this section swaption's market volatilities 1×5 are calibrated to market data. The swaption volatility smile is shown in Figure 6.2.

In order to calibrate normal swaption volatilities, the implied normal volatility is used. In Figure 6.3 we can observe the probability density function obtained throughout Crank-Nicholson with the calibrated parameters:

$$\hat{\beta} = 0.1536, \hat{\alpha} = 0.0055, \hat{\rho} = 0.3880, \hat{\nu} = 1. \tag{6.8}$$

In the following point we introduce some important disadvantages of using the Crank-Nicholson scheme to solve the PDE (4.25).

[3] For all calculations Python 3.7 has been used, and for the optimization of (6.6) the dual annealing optimization has been used. For more details about this procedure, the reader can find further information in [35].

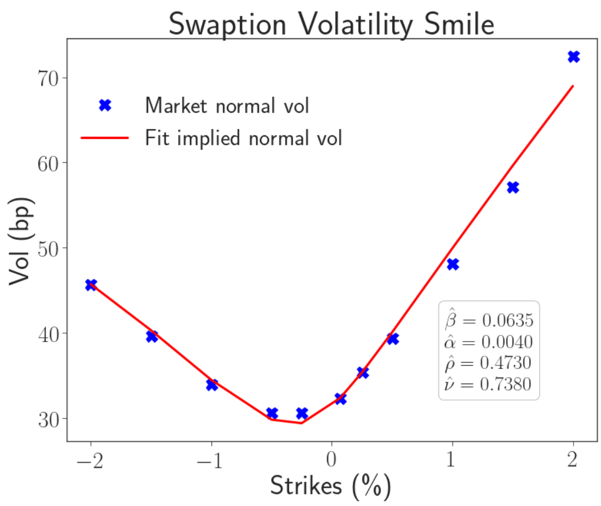

Figure 6.2: *Market swaption normal volatilities* $1Y \times 5Y$ *and its fit implied normal volatility.*

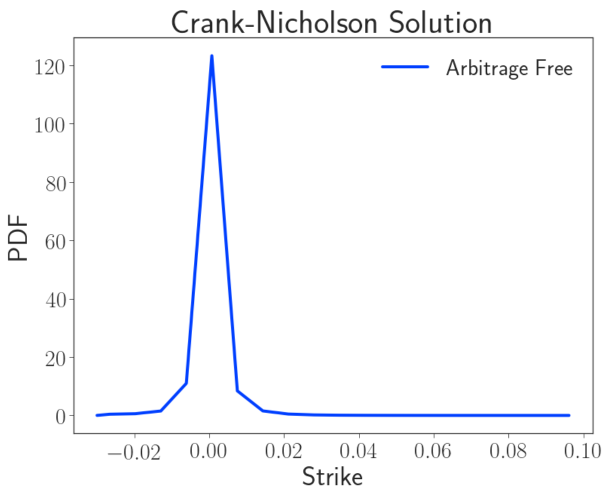

Figure 6.3: *Probability density function of the forward swap rate* $S_{1Y}^{1Y,5Y}$ *with the calibrated parameters (6.8).*

6.3.3 Grid discretisation problem

As indicated in [30], Hagan et al. recommend to use 200 to 500 points for the grid space and 30 to 100 timesteps. It is well known that the Crank-Nicholson approach is not always a good choice to solve the PDE (4.25) as commented in [36].

In the literature, most partial differential equations are approximated with numerical methods which present different kinds of stability.[4] In line with this point,

[4]The concept of stability accounts for a good approximation to the exact solution.

the Crank-Nicholson is not always stable.[5] The paper [36] examines how this instability affects the arbitrage-free approach introduced by Hagan et al. (2014) [30].

Below, we present the instability for a specific grid discretisation. The chosen parameters which cause this instability are shown in Table 6.1.

F_{min}	F_{max}	T_{ex}	f	α	ρ	β	ν	J
0.001	0.1	0.5	0.05	0.01	-0.8	0	0.1	500

Table 6.1: Input paramters for Crank-Nicholson solution shown in Figure 6.4.

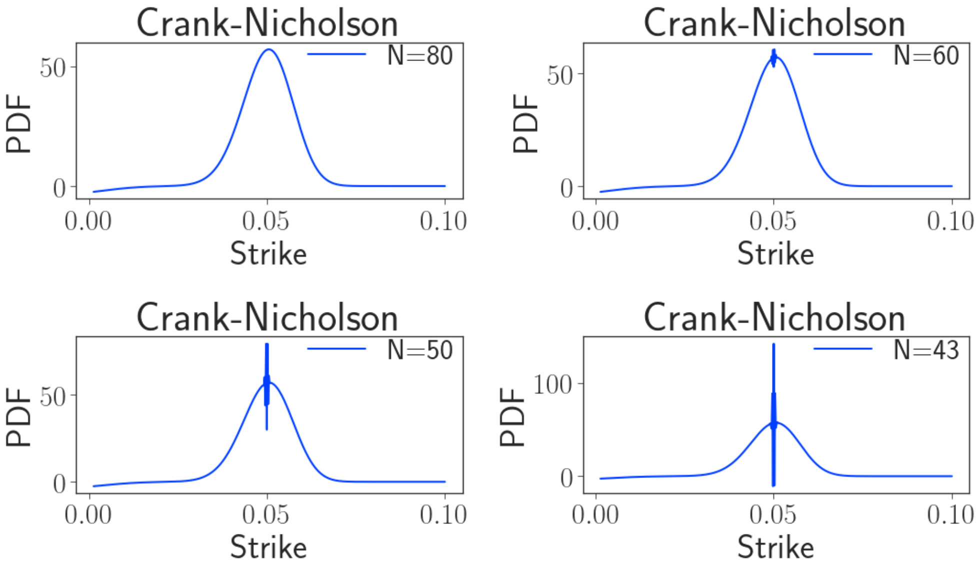

Figure 6.4: Crank-Nicholson instability with the input parameters shown in Table 6.1.

As we can observe in Figure 6.4, the problem arises just around the initial forward. Moreover, as N (time step) decreases the instability is more evident until the probability density function becomes negative ($N = 43$). Hence, it seems to be that Crank-Nicholson instability also affects the AF-SABR approach and this happens when $\frac{\Delta t}{h^2}$ is enough large as can be seen in Figure 6.4. This means that when N decreases, the Crank-Nicholson scheme also has problems with arbitrage opportunities.

This problem is well known in the literature and there have been numerous efforts to overcome this problem. For example, in [36] 8 different finite difference schemes are mentioned to address the problem of Crank-Nicholson instability. As concluded

[5]More specifically, the Crank-Nicholson is A-stable and not L-stable as discussed in [36]. These concepts are very technical and they are beyond the scope of this Thesis.

in [36], the Lawson-Swayne scheme seems to be one of the most efficient schemes, even when the timestep decreases and for long maturities.

Conclusion

All concepts related to the SABR model and its predecessors models have been reviewed in depth throughout this Thesis. Specifically, the problem of the SABR model for the valuation of caps, floors and swaptions in negative interest rate environments has been studied in detail.

The context of negative rates seems to be maintained throughout 2019 and as mentioned in this Thesis, given this context, the revision of the so-called SABR model has been necessary to perform a correct valuation of interest rate derivatives.

The density function negativity problem in an environment of near zero or even negative rates is a crucial aspect in the valuation of caps, floors and swaptions under the SABR model. One of the solutions proposed in the literature is that of Hagan et al. (2014), which consists in the resolution of a partial differential equation by means of Crank-Nicholson's finite difference scheme, this is the Arbitrage-Free SABR (AF-SABR) model. This approach has been tested in this Thesis with market data from caps and swaptions.

The instability of Crank-Nicholson for specific grid values has also been tested and a possible solution has been discussed and is shown mainly in [36].

As a possible future line of research there are many relevant points to deal with. For instance, a portfolio of swaptions or caps could be valued under the AF-SABR model and compare it with the valuation under the SABR model. In the same way, we can perform an analytical study of the scheme proposed by [36], and perform the valuation to compare the information obtained with the two previous approaches: the SABR model and the AF-SABR (Crank-Nicholson). In addition, we can also numerically test sensitivities with market data. Finally, finding a real arbitrage opportunity in the market, showing the negativiy problem and its corresponding correction by means of the AF-SABR approach is a turning point of the extensions just described.

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Appendix A

PDE Remarks

A.1 Joint conditional density function

In this section we adress the issue about how a conditional density function can be approximated as a conditional distribution probability function for a special case.

We will follow [38] where this topic is explained in general (a simpler case) and then we are going to apply to our case. In this way, we start with a simple case to ease the comprehension.

Let $f_{X|Y}(x|y)$ be the conditional density of X given Y. Let $f(x,y)$ and $f(y)$ be the joint density of the random vector (X,Y) and the marginal density function of Y, respectively. We will prove as it was done in [38] that the conditional distribution probability $\mathbb{P}(x < X < x + dx \mid Y = y)$, where $dx > 0$ is small enough, can be approximated as follows

$$\mathbb{P}(x < X < x + dx \mid Y = y) \approx dx \cdot f_{X|Y}(x|y). \tag{A.1}$$

Conditioning on the event $\{Y = y\}$ is almost the same as conditioning on the event $\{y \leq Y \leq y + dy\}$ for a small $dy > 0$. Therefore, for $f(y) > 0$ we then have

$$\begin{aligned}
\mathbb{P}(x < X < x + dx \mid Y = y) &\approx \mathbb{P}(x < X < x + dx \mid y \leq Y \leq y + dy) \\
&= \frac{\mathbb{P}(x < X < x + dx,\ y \leq Y \leq y + dy)}{\mathbb{P}(y \leq Y \leq y + dy)} \\
&= \frac{\int_x^{x+dx} \int_y^{y+dy} f(x,y)\ dy\ dx}{\int_y^{y+dy} f(y)\ dy}
\end{aligned}$$

$$
\begin{aligned}
&\approx \frac{dx \cdot dy \cdot f(x,y)}{dy \cdot f(y)} \qquad \text{(A.2)}\\
&= dx \cdot f_{X|Y}(x|y),
\end{aligned}
$$

as we wanted to prove. We are now going to apply the argument done in (A.2) to the conditional (joint) probability density $p(T, F, A; t, f, \alpha)$ as defined in (4.8). For the sake of clarity, the conditional probability density is exactly given by

$$p(T, F, A; t, f, \alpha) \equiv p(F(T), \sigma(T) \mid F(t) = f, \sigma(t) = \alpha). \qquad \text{(A.3)}$$

Notice the dependence of the temporary indexes T and t on the left hand side definition in (A.3). In such a way, if we denote $G = \{F' < F(T) < F' + dF\} \cap \{A < \sigma(T) < A + dA\}$, then replicating the previous argument for a small $\varepsilon > 0$ and $\delta > 0$, we then obtain that

$$
\begin{aligned}
&\mathbb{P}(G \mid F(t) = f,\ \sigma(t) = \alpha) \approx \mathbb{P}(G \mid f < F(t) < f + \varepsilon,\ \alpha < \sigma(t) < \alpha + \delta)\\
&= \frac{\mathbb{P}(G,\ f < F(t) < f + \varepsilon,\ \alpha < \sigma(t) < \alpha + \delta)}{\mathbb{P}(f < F(t) < f + \varepsilon,\ \alpha < \sigma(t) < \alpha + \delta)}\\
&\approx \frac{dF \cdot dA \cdot \varepsilon \cdot \delta \cdot h(G, f < F(t) < f + \varepsilon,\ \alpha < \sigma(t) < \alpha + \delta)}{\varepsilon \cdot \delta \cdot g(f < F(t) < f + \varepsilon,\ \alpha < \sigma(t) < \alpha + \delta)} \qquad \text{(A.4)}\\
&\approx dF \cdot dA \cdot p(G \mid F(t) = f\ \sigma(t) = \alpha)\\
&= dF \cdot dA \cdot p(T, F, A; t, f, \alpha),
\end{aligned}
$$

where the $h(\ \cdot\ ,\ \cdot\ ,\ \cdot\ ,\ \cdot\ )$ and $g(\ \cdot\ ,\ \cdot\ )$ accounts for the joint density function of $(F(T),\, \sigma(T), F(t), \sigma(t))$ and $(F(t), \sigma(t))$, respectively. In short, we have the relation

$$p(T, F, A; t, f, \alpha)\, dF\, dA \approx \mathbb{P}(F < F(T) < F{+}dF,\ A < \sigma(T) < A{+}dA \mid F(t) = f,\ \sigma(t) = \alpha) \qquad \text{(A.5)}$$

as it is defined in [30]. However, note that the authors of [30] define the relation (A.5) with an equality and it is actually an approximation.

Appendix B

Finite Difference Schemes

We can find this theory in any numerical methods book. However, we will include in this appendix a brief look about the main finite difference derivatives approximations and the Crank-Nicholson scheme to solve a partial differential equation in order to ease the reading of this project.

B.1 Finite difference derivatives approximations

Let $f(x)$ be a smooth enough function. Then, by the Taylor's theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f^{(3)}(x) + \cdots \tag{B.1}$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f^{(3)}(x) + \cdots \tag{B.2}$$

By substracting (B.1) minus (B.2), dividing by $2h$ and neglecting the terms of order h^3, we obtain the **central difference approximation** for the first derivative:

$$\boxed{f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}} \tag{B.3}$$

Neglecting terms of order h^2 in (B.1) and (B.2), we obtain the **forward difference approximation**:

$$\boxed{f'(x) \approx \frac{f(x+h) - f(x)}{h}} \tag{B.4}$$

and the **backward difference approximation**:

$$\boxed{f'(x) \approx \frac{f(x) - f(x-h)}{h}} \tag{B.5}$$

Adding (B.1) and (B.2), we get the **central difference approximation** for the second derivative:

$$\boxed{f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}} \tag{B.6}$$

B.2 Crank-Nicholson

Let $f(t,x)$ be a smooth function. There are two ways of solving the PDE

$$f_t = f_{xx}. \tag{B.7}$$

One way is approximating f_t throughout forward difference approximation (B.4) and f_{xx} with the central difference approximation (B.3). This way of solving (B.7) is called classical scheme. Another way, is to approximate f_{xx} by the average of the central differences of the row j-th and the row $j+1$-th and f_t via forward difference approximation, namely,

$$\frac{f_{n+1,j} - f_{n,j}}{\Delta t} = \frac{1}{2}\left(\frac{f_{n+1,j+1} - 2f_{n+1,j} + f_{n+1,j-1}}{h^2} + \frac{f_{n,j+1} - 2f_{n,j} + f_{n,j-1}}{h^2}\right), \tag{B.8}$$

where $f_{n,j} \equiv (t_n, x_j)$ is the approximation of f evaluated at the grid point (t_n, x_j), Δt is the time step, h is the space step, with $t_n = n\Delta t$, $n = 0, 1\ldots, N$ and $x_j = jh$, $j = 0, \ldots, J$. This way of approximating is called the **Crank-Nicholson scheme** and in Figure B.1 we can see the generated grid.

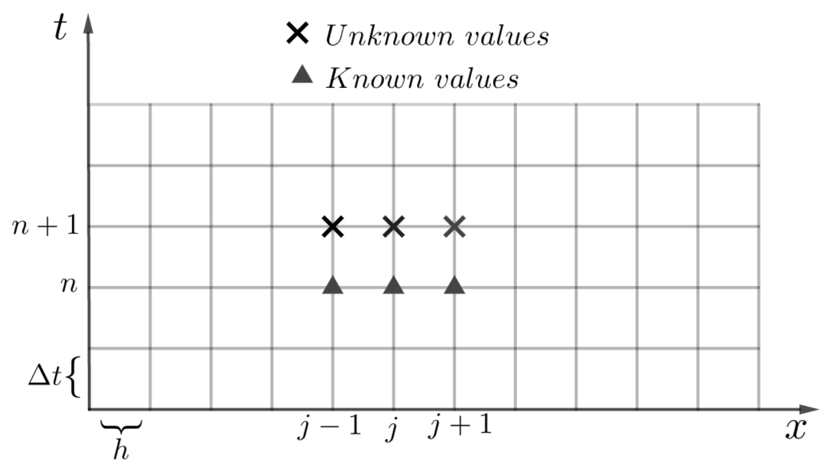

Figure B.1: Crank-Nicholson grid.

For every n time step a linear equation system must be solved. The coefficient's matrix of the linear system is a tridiagonal matrix and thus it can be easily solved.

The main advantage is that this scheme is stable[1] for any $\frac{\Delta t}{h^2} > 0$ as opposed to the classical scheme. Despite this, it is convenient that Δt and h are small. The main disadvantage is that at every time step a linear equations system must be solved.

[1]The concept "stable" refers to a good approximation to the exact solution for any value of $\frac{\Delta t}{h^2} > 0$.

Appendix C

Discount factors data

We present in this appendix the discount factors data associated with the graphs shown in Figures 5.1 and 5.2.

Discount curve	
Maturities	**Discount Factor**
28-may.-19	1
29-may.-19	1.00001025
06-jun.-19	1.000090888
13-jun.-19	1.000162265
28-jun.-19	1.000315224
30-ago.-19	1.000959086
29-nov.-19	1.001930227
28-feb.-20	1.002951591
29-may.-20	1.004000463
31-ago.-20	1.005090623
30-nov.-20	1.006129533
26-feb.-21	1.007075608
31-may.-21	1.008059528
30-may.-22	1.010972383
30-may.-23	1.012467092
30-may.-24	1.011957269
30-may.-25	1.009363047
29-may.-26	1.004747519
31-may.-27	0.998138897
30-may.-28	0.989734795
30-may.-29	0.979768344
30-may.-31	0.956719771
30-may.-34	0.918168238
30-may.-39	0.858342228
30-may.-44	0.8104248
31-may.-49	0.770964684
30-may.-59	0.707436487
30-may.-69	0.653878968
30-may.-79	0.604820421

Table C.1: *Discount curve data at date 28 May 2019.*

Estimation Curve for Forwarding	
Maturities	**Discount Factor**
28-may.-19	1
28-jun.-19	1.000203376
29-nov.-19	1.001244818
30-dic.-19	1.001490013
29-may.-20	1.002653992
30-nov.-20	1.004036774
31-may.-21	1.005142216
30-may.-22	1.006383829
30-may.-23	1.005938512
30-may.-24	1.00355645
30-may.-25	0.999155488
29-may.-26	0.992863776
31-may.-27	0.984600695
30-may.-28	0.974591871
30-may.-29	0.963357034
30-may.-31	0.937918497
30-may.-34	0.896934629
30-may.-39	0.834060395
30-may.-44	0.783834422
31-may.-49	0.742775636
30-may.-59	0.676420501
30-may.-69	0.621704817
30-may.-79	0.57236501

Table C.2: *Estimated discount curve at date 28 May 2019.*