EXCHANGE RATE, THAT GREAT FORGOTTEN

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Trabajo de investigación 21/002

Master en Banca y Finanzas Cuantitativas

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TRABAJO FIN DE MÁSTER

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Abstract

Hedging feature of gold in equity portfolios during turmoil periods is well known. Furthermore, there are evidences of US dollar appreciation in stock market distressed scenarios, a fact that also has an impact on gold price. The safe-haven property of gold has been widely studied in the literature, although, to our knowledge, none author has so far analyse how the exchange rate movements could affect the hedging strategy of non-US investors.

The contribution of this study is to analyse the role of exchange rate in the construction of international stock portfolios in Europe, Japan, Brazil and UK. Moreover, we compute the cost of not taking into account the exchange rate risk in the portfolio construction.

To model the interrelationships between gold, equity index and exchange rates we rely on copula methodology, using a DCC-GARCH approach to capture the time-varying dependence. We evaluate the impact of the exchange rate in the portfolio performance built under different criteria, i.e. equally-weighted, minimum-variance and minimum-Expected Shortfall, in terms of several risk measures.

According to our findings, the exchange rate plays a leading role to hedge stock losses and even in Japanese and British portfolios outshines the gold position. Indeed, including an extra-investment in exchange rate in any portfolio enhances the hedging strategies and reduces the tail risk in a significant quantity. In addition, an out-of-sample exercise serves as a robustness check of these results.

The implications of this study go beyond risk management decisions. It also could be useful for regulatory and supervisory authorities as a powerful stress test approach to evaluate the performance of domestic stock markets under distress scenarios for the exchange rate.

Keywords: *turmoil periods, exchange rate risk, copula models, risk measures, portfolio optimization.*

Contents

1 Introduction

Gold positions have shown a hedging feature against tail events in stock markets. On the one hand, gold acts as a hegde asset when disjointed or opposite movements in gold and stock or exchange rate markets occur only on average (Reboredo and Rivera-Castro, 2014a). On the other hand, gold is a safe haven when it co-moves on opposite direction to benchmark markets in distress scenarios.

Throughout the successive crisis of recent decades, like oil crisis, dot-com bubble or Great Recession, stock markets have experienced sharp decreases. The high uncertainty in stock markets during these periods encourages investors' herd behaviour to hold haven assets like gold, and consequently its price surges. Safe-haven property of gold has been widely studied by the literature, especially since the begining of the Great Recession (Gürgün and Ünalmis, 2014; Kristoufek and Vosvrda, 2016; Nguyen et. al, 2020), and it is related to the thought that gold is able to limit investment losses during black swan events. In fact, this is not only a thought, but there are empirical evidences on the portfolio diversification benefits of commodities, especially of gold, and low performance of stocks and bonds.

There is a general consensus in the literature on the key role of gold as a safe haven against movements in stock markets, which comes from a negative correlation between gold and these markets, both in developed and emerging economies, and especially in the short term (Iqbal, 2017; Junttila et al., 2018). Similar results have been found in exchange rate markets: gold acts as hedge and as a weak safe haven against extreme exchange rate movements, with a decreasing hedging property for longer periods (Wang, 2013; Reboredo and Rivera-Castro, 2014b; Qureshi et al., 2018). Also, there are evidences of dependence on average and tail dependence between stock indices and exchange rates (Ojea-Ferreiro and Reboredo, 2021).

The interrelationships between exchange rate, stock and gold markets make that shocks in one of them cannot be seen in an isolated way. For instance, a depreciation of the US dollar against a domestic currency turns gold investment more attractive for investors whose portfolio is not denominated in US dollars. Furthermore, this depreciation is equivalent to an appreciation of a domestic currency, which discourages the foreing investment in the local equity index, causing a downward movement in this market. The evolution of the exchange rate, in turn, is conditional on the investment flows in international markets. Indeed, on the one hand, the increase of foreing investment in equity would push the demand of the local currency and, hence, its appreciation against the foreign currency (USD). On the other hand, the investment in gold market reduces the demand of the local currency, which envolves its depreciation against the US dollar.

A non-US investor, looking for a hedging strategy for his stock portfolio would face an exposure to the exchange rate risk because the US dollar is the main currency to trade gold. This exchange rate risk could lead to a loss in the portfolio when the investor wanted to close the position in gold depending on the movements in currency markets. To our knowledge, the literature which has examined the usefulness of gold as hedge or safe haven asset, either in stock or in exchange rate markets, focus on the relationships between only two of these assets. In addition, authors usually transform gold price into a local currency to perform their analysis, overlooking that in float exchange rate systems this implies the introduction of a new stochastic component in the portfolio dynamics and ignoring the potential profit of including the exchange rate as an investment asset in the portfolio. The goal of our study is to analyse the importance of the exchange rate risk when a non-US investor creates a hedging strategy with gold. The questions we endeavor to answer in this study are:

1. Which role plays the exchange rate in the hedge of market risk when we use gold as a hedging strategy?
2. What is the cost of ignoring the exchange rate risk in terms of Value at Risk or Expected Shortfall?
3. How does the hedging strategy change depending on the currency in which is denominated the portfolio?
4. How does the tail dependence affect the above questions?

Thus, the contribution of this study to the literature is, firstly, highlighting the importance for a non-US investor of considering exchange rate risk when investing in an asset denominated in dollars, like gold. Secondly, we consider the possibility of using the exchange rate as a third asset in a portfolio, and we analyse its additional role in a hedging strategy considering the portfolio performance in terms of different risk measures. Finally, we carry out an out-of-sample exercise in order to make a robustness check of the results obtained in-sample.

Regarding the methodological approach, we rely on a complex technique that combines the marginal features with the dependence structure of the data. To model the marginal distribution of each asset we use autoregresive models for the conditional mean and GARCH-type models for the conditional variance. We also consider innovations that take into account asymmetric behaviours and heavy tails, i.e. Skewed Student t distribution. Concerning the dependence between variables, we use copula theory to obtain flexible dependence structures. In particular, we employ elliptical copula models, using a DCC-GARCH approach to obtain the dynamic correlations between assets (Joy, 2011; Baruník et al., 2016).

We analyse the role of the exchange rate in international stock portfolios of non-US investors within a twenty-year sample for a data set including financial returns of gold, exchange rate and local stock index. Our results are based on the performance of stock portfolios built following different criteria of risk management, i.e. equally-weighted, minimum-variance and minimum-Expected Shortfall. These criteria have been widely employed in the literature (Reboredo, 2013; Reboredo and Rivera-Castro 2014b). We propose the construction of these portfolios considering the exchange rate as an instrument to invest in gold and considering it as another asset in the portfolio, so that the investment in this asset can be divided in a part intended to acquire gold and another intended to improve the hedging strategy. We show, by means of different risk measures, the effect of considering the stochastic dynamic introduced by exchange rate risk in portfolios of non-US investors.

To finish, we must also consider that the effects of the exchange rate in an investment depends on in which currency is denominated the portfolio (Beckers and Soenen, 1984; Reboredo and Rivera-Castro, 2014a). In order to determine this effect, the aforementioned methodology will be repeated in four economies, in particular Europe, Japan, Brazil and United Kingdom.

Our findings evidence the leading role of exchange rate in stock portfolios, even outshining the gold position in Japanese and British cases. By comparing the performance of portfolios in terms of risk measures, we verify that an extra-investment in exchange rate enhances the hedging strategies, reducing the tail losses and the systemic risk of the international portfolios.

Understanding the evolution of co-movements between different asset classes or markets and estimating properly the dependence between them is crucial in many portfolio and risk management assignments. This study has straightforward implications for international investors in terms of portfolio optimization and risk management decisions. In addition, the analysis of the relationship between economic and financial markets is useful for policy makers and the evaluation of the performance of domestic stock markets under distress scenarios for the exchange rate might constitute a stress test approach for regulatory and supervisory authorities.

The rest of the paper is organized as follows. Section 2 contains a review of the literature on the subject. Section 3 describes the methodology, while Section 4 introduces the data employed in the empirical exercise from Section 5. Section 6 closes this study with the main conclusions.

2 Literature review

The literature has provided an exhaustive study of the hedging properties of gold and its dependence with other markets. Baur and McDermott (2010), Reboredo (2013) or Gürgün and Ünalmis (2014), among others, point out the difference between the role of gold as hedge or safe haven asset. A hedge asset exhibits null (weak hedge) or negative (strong hegde) correlation on average with another market or variable of interest, as exchange rates, stock or inflation, while a safe haven asset exhibits null (weak safe haven) or negative (strong safe haven) correlation with another market or variable when extreme returns occurs in it.

Reboredo (2013), Iqbal (2017), Junttila et al. (2018) or Qureshi et al. (2018) have studied the role of gold as hedging or safe haven asset in crisis times in which abrupt falls occur in equity or exchange rate markets, using diverse methods to measure their dependence, from linear regression to copula models. On the one hand, Baur and McDermott (2010), Gürgün and Ünalmis (2014) and Junttila et al. (2018) based their studies in the hedging properties of gold in stock markets.

Firstly, Baur and McDermott (2010) carried out linear regression of gold on stock to value the capacity of the former to act as a safe haven depending on the value obtained in the estimation, i.e. negative sign of the beta estimate implies a safe haven characteristic in the gold asset. The dataset encompasses daily prices of equity indices (denominated in local currency) of 13 countries, including developed and emerging economies, and the World Index over a thirty-year period, from March 1979 to March 2009. Their findings indicate that a) the price of gold and the level of the World Index co-move in tandem or in opposite directions depending on the period, b) a common currency denomination of both stock indices and gold generally increases the co-movement in all market conditions, reducing the safe-haven property of gold, and c) negative correlations exist between gold and the regional indices for extreme market shocks.

Secondly, Gürgün and Ünalmis (2014) studied the relation between gold and emerging stock indices expresed in US dollars from 1979 to 2009, using the same methodology as Baur and McDermott (2010) and including dummies which represent extreme events in stock markets. They noticed that gold performs as hedge and safe haven asset in most of the financial markets for domestic investors. Furthermore, gold is a safe haven for foreign investors in only a few markets, although in distress periods it acts as safe haven in more countries.

Thirdly, Junttila et al. (2018) studied a hedge strategy based on gold and oil against price risk of a portfolio formed by S&P500 index. Using daily prices from September 1989 to September 2016, they analysed the dynamic correlation between these assets based on a VAR model with DCC-GARCH errors. After computing a dynamic hegde ratio with the commodities, they found a negative relationship between those assets in the short run, while in the long run the effect of safe haven decreases.

On the other hand, Reboredo (2013), Reboredo and Rivera-Castro (2014a), Reboredo and Rivera-Castro (2014b), Qureshi et al. (2018) and Nguyen et al. (2020) focused on gold and exchange rate markets.

Reboredo (2013) used a sample between January 2000 and September 2012 to identify the relationships between gold and a set of exchange rates against US dollar. He captured the dependence between assets by the use of copulas. Their results showed a positive dependence on average between gold and USD depreciation. In addition, the tail dependence is symmetric, which means that gold has safe-haven properties against extreme movements of USD, both downward and upward.

Reboredo and Rivera-Castro (2014a) replicates Reboredo (2013) changing the methodological part. They design a likelihood ratio test that draws a distinction between hedging and safe-haven characteristics on the basis of the conditional dependence structure under different market conditions. The findings are in line with the results in Reboredo (2013), concluding that gold can hedge dollar depreciation but it provides a weak safe haven against extreme US dollar movements.

Similarly, Reboredo and Rivera-Castro (2014b) employed the denominated wavelet analysis[1] to explore whether gold provides hedging and downside risk benefits against currency movements at different investment horizons, meeting the different maturity goals of investors. The sample encompasses daily returns from January 2000 to March 2013 of gold and a set of exchange rates of USD against local currencies of some countries, including among others Canada, UK or Japan. They evidenced a positive relationship between gold and the aforementioned exchange rates at some time frequencies, i.e., gold can serve as hedge asset for a portfolio formed with currencies. Moreover, thanks to the creation of four portfolios, namely, currency-only

[1] The wavelet approach decomposes a serie into different frequency components: lower time scales capture higher frequency time series components which occur over very short periods of time, whilst higher time scales capture lower frequency components occurring over very long periods of time. Thus, it is useful to study the properties of gold at differents time horizons.

(benchmark), minimum-variance, maximum-hedge ratio and equally-weighted, they verified a significant improvement of the portfolios in terms of variance, Value at Risk (VaR) and Expected Shortfall (ES).

Qureshi et al. (2018) applied the same approach than Reboredo and Rivera-Castro (2014b) on a sample of daily returns of gold (denominated in PKR per ounce) and exchange rates of rupee against a set of four currencies. The results indicated that gold performs as a temporary hedge and as safe haven against depreciation of de rupee against some of the currencies.

Nguyen et al. (2020), in line with Reboredo (2013), used a copula approach to analyse the relation between gold and the depreciation of USD, EUR and JPY on average and in turmoil periods, i.e. sub-prime crisis. With a sample from January 2000 to March 2018 they noticed that gold serves as a strong hedge and as a weak safe-haven asset against exchange rate devaluation, and strong safe haven during crisis periods.

Iqbal (2017), in turn, analysed the correlation between gold and stock, exchange rate and inflation, respectively. This paper showed the gold hegding properties against each one of these variables in three countries, namely India, Pakistan and USA. He used a sample of 22 years, from January 1991 to November 2013, of daily and monthly returns of the assets. Using a similar methodology than Gürgün and Ünalmis (2014), he concluded that a) gold is a safe haven against exchange rate risk in Pakistan and India; b) gold acts as hegde in USA on average and in bearish conditions of gold market; and c) gold is a safe haven in Pakistan.

Finally, Lin (2011), Wang et al. (2013), Reboredo et al. (2016) and Ojea-Ferreiro and Reboredo (2021) used copulas to measure the dependence between stock and exchange rate markets.

Lin (2011) studied stock indices and the exchange rates against USD of five asiatic economies, namely Hong Kong, Indonesia, South Korea, Singapore and Taiwan. Using daily data from 1987 or 1997 (depending on the country) to June 2010, Lin (2011) applied different copula models, concluding that Hong Kong and Singapur do not present tail dependence, whereas for Indonesia and South Korea the tail dependence is asymmetic, and Taiwan exhibits symmetric tail dependence. Furthermore, he evidenced that the linear correlation coefficient provides an unaccurate measure of dependence between non-Normal distributed variables.

Wang et al. (2013) used a switching-copula model to get the joint distribution between stock markets and exchange rates in Germany, UK, France, Italy, Japan and Canada over a twenty-year period, from 1990 to 2010. This model captured the dependence between the assets in four states, i.e. bullish stock-appreciating currency, bearish stock-depreciating currency, bullish stock-depreciating currency and bearish stock-appreciating currency. They found that dependence and tail dependence are asymmetric for most countries in a negative correlation regime but are symmetric for all countries in a positive correlation regime.

Reboredo et al. (2016), in turn, use a sample from April 2001 to November 2014 of weekly prices of local stock indices and exchange rates of EUR and USD against local currency of eight emerging economies to applicate the copula approach, computing also the VaR and Conditional VaR as risk measures. This study found that exists a positive relation between stock prices and currency values in emerging economies with respect to the dollar and the euro, and that there is asymmetric tail risk.

To conclude, Ojea-Ferreiro and Reboredo (2021) based their study in the role played by exchange rates in the transmission of shocks across equity markets. Covering a sample from 2002 to 2020, they used vine copulas to analyse the dependence between the stock index of USA and Europe, the stock index of an emerging economy, namely Chile, Argentina, Mexico and Brazil, and the respectives exchange rates (unit of USD and EUR against each Latin American currencies) and, from a copula framework, they obtain the conditional expectation and measure the exchange rate contribution to shock propagation between those equity markets. Their findings showed a relation between these different assets, indicating that the contribution of the exchange rates to the transmission of shocks in international equity markets is time-varying and asymmetric.

In short, many papers have examined the usefulness of gold as hedge asset or safe haven, either in stock or in exchange rate markets. Moreover, several authors have used copula methodology to model the dependence between them. However, according to our research, all the studies focus on the relationships between only two types of assets. We instead propose analysing the relationship between gold, stock and exchange rate by means of copulas in order to determine the role of the exchange rate risk in portfolios of non-US investors.

3 Methodology

This section is laid out in three stages. Firstly, we analyse the marginal structure to take into account autocorrelation, heterocedasticity and higher moments of the financial returns. Secondly, we study the dependence structure by means of a copula approach, which would be enhanced by the use of dynamics for the evolution of the dependence parameters over time. Finally, the mathematical expression of the investment portfolio and the risk measures that relied on this methodology close this section.

3.1 Marginal density

For the marginal distributions we consider an ARMA(p,q) model for the mean and a GARCH(m,s) model for the variance of the distribution.

ARMA(p,q) models are very useful to describe the dynamic of a variable. These processes can be decomposed in autoregressive (AR) and moving average (MA) parts. The former relates the return at time t, r_t, with its last p values, while the moving average term tries to explain this variable as a weighted sum of the last q values of the innovations. Given these definitions, the dynamic of an asset returns can be written as:

$$r_t = \phi_0 + \sum_{i=1}^{p} \phi_i \cdot r_{t-i} + \sum_{j=1}^{q} \theta_i \cdot a_{t-j} + a_t \tag{3.1.1}$$

where p and q are non-negative integers and ϕ and θ are AR and MA parameters, respectively. Moreover, $a_t = \sigma_t \cdot \varepsilon_t$, being ε_t a random variable with mean zero and unit variance (white noise).

Changes in the variance (σ_t^2) are very important to understand the financial markets, while the investors demand higher expected returns as compensation for taking on more risks in their investments (Hamilton, 1994). We compare several models to take into account heterocedasticity, which some of them are nested in other ones, to fit in the most accurate way the dynamics of the volatility. We introduced them hereinbelow.

1. GARCH (*Generalized AutoRegressive Conditional Heteroscedasticity*) models are used to adjust the variance of the serie, and they also contain an autoregressive and a moving average term:

$$\begin{aligned} r_t &= \mu_t + a_t = \mu_t + \sigma_t \cdot \varepsilon_t \\ \sigma_t^2 &= \omega + \alpha \cdot a_{t-1}^2 + \beta \cdot \sigma_{t-1}^2 \end{aligned} \tag{3.1.2}$$

where ε_t is a white noise which follows a certain distribution with null mean and unit variance. To fulfill the stacionarity conditions, it should happen that $\omega \geq 0$, $\alpha,\ \beta > 0$ and $\alpha + \beta < 1$.

This model has some limitations such as not having into account the skewness of the residuals or the leverage effect. To overcome these limitations, some extensions of standard GARCH model appear.

2. GJR-GARCH model (Glosten et al., 1993) is one of the most used extensions because it includes the mentioned leverage effect. GJR-GARCH(1,1) model has the following expression:

$$\sigma_t^2 = \omega + \left(\alpha + \gamma \cdot \mathbf{1}_{\{\varepsilon_{t-1}<0\}}\right) \cdot \varepsilon_{t-1}^2 + \beta \cdot \sigma_{t-1}^2 \tag{3.1.3}$$

Asymmetry is captured by γ. When γ is positive, it means that negative shocks introduce more volatility than positive shocks of the same size (Huang et al., 2009). Because of that, is expected that in GJR model $\gamma > 0$. Other conditions for the stacionarity of the model are $\omega \geq 0$, $\alpha, \beta > 0$ and $\alpha + \beta + \gamma \cdot \mathbf{1}_{\{\varepsilon_{t-1}<0\}} < 1$

3. EWMA (*Equaly-Weighted Moving Average*) model is a particular case of GARCH where $\omega = 0$ and $\alpha = 1 - \beta$, i.e. the parameters which accompained act as weights:

$$\sigma_t^2 = (1 - \lambda) \cdot a_{t-1}^2 + \lambda \cdot \sigma_{t-1}^2 \tag{3.1.4}$$

This model does not present mean reversion, i.e., it has not a term for the average variance in the long term.

4. The APARCH(1,1) (*Asymmetric Power ARCH*) model is defined as follows:

$$\sigma_t^\delta = \omega + \alpha \cdot (|\varepsilon_{t-1}| + \gamma \cdot \varepsilon_{t-1})^\delta + \beta \cdot \sigma_{t-1}^\delta \tag{3.1.5}$$

This model will be stationary only if δ is positive. When $\delta = 2$ and $\gamma = 0$, APARCH is reduced to the GARCH model. Furthermore, when $\delta = 2$, for any value of γ, APARCH becomes GJR-GARCH model.

5. GARCH-M model tries to capture the fact that in finance the return of an asset depends on its volatility, so greater volatility of the asset should imply a greater expected return. GARCH(1,1)-M model does not include a variation respect to the standard GARCH in the variance equation, but in the mean equation:

$$r_t = \mu_t + c \cdot \sigma_t^2 + a_t = \mu_t + c \cdot \sigma_t^2 + \sigma_t \cdot \varepsilon_t \tag{3.1.6}$$

where $\mu_t = \phi_0 + \sum_{i=1}^p \phi_i \cdot r_{t-i} + \sum_{j=1}^q \theta_i \cdot a_{t-j}$ and c represents the risk premium. A positive value of this parameter means that the more volatility, the more return we will obtain with the asset. There are not additional conditions for the parameter c respect to the standard GARCH to be a stacionary model.

To select the marginal model which captures better the behaviour of each serie, we compute GARCH(1,1), GJR(1,1), EWMA, APARCH(1,1) and GARCH(1,1)-M models for each one of the series under different distribution assumptions, i.e. Normal, Student t or Hansen's Skewed Student t distribution (Hansen, 1994).

Under **Gaussian distributed assumption**, the density function we work with is:

$$f(\varepsilon_t|\Omega_{t-1}) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \cdot e^{-\frac{\varepsilon_t^2}{2}} \tag{3.1.7}$$

where, σ_t^2 is adjusted by the GARCH(m,s)-type model specificated and the standardized residual is defined as $\varepsilon_t = \frac{a_t}{\sigma_t} = \frac{r_t - \mu_t}{\sigma_t}$, where μ_t summarizes the ARMA(p,q) model.

Student t distribution is widely applied in studies which are based on data with outliers or heavy tails (Ding, 2016). If we assume innovations follow this distribution, we have the following density function:

$$f(\varepsilon_t|\Omega_{t-1}) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\sigma_t^2\pi(\nu-2)}} \cdot \left(1 + \frac{\varepsilon_t^2}{\nu-2}\right)^{-\frac{\nu+1}{2}} \tag{3.1.8}$$

Finally, Hansen's Skewed Student t is a flexible distribution accomodating the skewness and excess kurtosis often present in financial data (Theodossiou, 1998). The density function of the **Skewed Student t distribution** is given by:

$$f(\varepsilon_t|\nu,\lambda) = \begin{cases} bc\left(1 + \frac{1}{\nu-2}\left(\frac{a+b\varepsilon_t}{1-\lambda}\right)^2\right)^{-\frac{\nu+1}{2}}, & \varepsilon_t < -\frac{a}{b} \\ bc\left(1 + \frac{1}{\nu-2}\left(\frac{a+b\varepsilon_t}{1+\lambda}\right)^2\right)^{-\frac{\nu+1}{2}}, & \varepsilon_t \geq -\frac{a}{b} \end{cases} \tag{3.1.9}$$

where $2 < \nu < \infty$ and $-1 < \lambda < 1$. The parameter ν controls the tails of the density, and the skewness parameter, λ, represents the rate of descent of the density around $\varepsilon = 0$. The constants a, b and c are given by:

$$a = 4\lambda c\frac{\nu-2}{\nu-1}, \quad b^2 = 1 + 3\lambda^2 - a^2, \quad c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)}$$

Remark that this distribution becomes Student t distribution when $\lambda = 0$, i.e., when there is not asymmetry, and it becomes Normal distribution if also $\nu \to \infty$.

3.2 Dependence structure

The linear correlation coefficient of Pearson has traditionally been used as a measure of dependency, but it has some limitations. Firstly, it only measures linear relations between variables and it is not able to handle the problem of heavy tails of the distributions. Secondly, it is a static coefficient, i.e., it measures the correlation of two series by a unique value. Therefore an alternative dependence measure emerges, namely, copula, which allows to study the evolution over the time of the dependence on average and in tail events between two or more variables. Copula approach has certain advantages over other traditional measures: it allows modeling no-linear relationships, it does not require assumptions about marginal distributions, and it is useful to model extreme events. Its greater flexibility explains the increasing attention that has received in the latest years to analyse of dependence between markets (Cherubini and Luciano, 2001; Sun et al., 2008; Reboredo and Rivera-Castro, 2014a; Bolance et al., 2015; Berger, 2016).

The problem when we want to adjust a copula model to a data series is that the marginal distributions of variables are usually unknown. However, this fact does not prevent us from using copulas. We use meta-distributions to model the dependence between the assets. A meta-distribution consists in using a certain copula model with whatever marginal distributions. We can obtain Uniform(0,1) variables from the standardized residuals of an ARMA-GARCH model. The use of meta-distributions allows us:

a. To estimate probabilities when the returns of a portfolio (copula distribution) is not the same that individual asset's distributions (marginal distributions).

b. To generate series with marginal distributions differents from those derivated from the copula.

It is important to remark that a misspecification in the marginal distribution could imply a misspecified copula model (Patton, 2006). Hence, it is extremely relevant the model selection for the marginal model, to ensure that the Probability Integral Transforms (PIT), i.e., $u_1 = F(x_1)$, $u_2 = F(x_2)$, are Uniform(0,1).

3.2.1 Copula functions

The copula approach allows us to separate a joint distribution between two or more variables in the marginal distributions of each variable and the dependence structure between them. This methodology gives us the possibility to combine marginal models with a variety of dependence models, so it provides a more flexible key to capture complex joint distributions. Likewise, this approach allows to reflect asymmetries in the dependences and tail dependence.

By Sklar's Theorem, from d marginal distributions and a copula model that adjust the dependence structure between them, it is possible to obtain the joint distribution of the d variables:

$$F(x_1, ..., x_d) = C\left(F_1(x_1), ..., F_d(x_d)\right) \tag{3.2.1}$$

where $F_i(x_i) = u_i \in [0, 1]$ can be any distribution, known or unknown. If all the margins are continuous, then the copula is unique.

Citing Patton (2001), given any two marginal distributions and any copula we have a joint distribution, and from any joint distribution we can extract the implied marginal distributions and copula.

There exist two families of copulas, namely elliptical and Archimedean. In this study we employ the former, based on known distributions. Gaussian and Student t copulas are elliptic, allowing for symmetric dependence, which could be positive or negative. While Gaussian copula does not reflect tail dependence, Student t could exhibit tail dependence for low values of the numbers of degrees of freedom and non-zero correlation. Since they have not an explicit close form, they are also known as implicit copulas. Annex A contains the expression of the distribution and density functions of elliptical copulas used in this study. Furthermore, Table 6 in this Annex summarizes their principal features.

The estimation of the optimal parameters of the copula can be done in one step, namely canonical maximum likelihood methodology, or, as is the case here, in a two-steps procedure named Inference Functions for Margins (IFM). In the first step of this procedure we estimate the optimal parameters of each marginal distribution individually, and in the second step these optimal parameters are replaced in the copula expression to obtain later the optimal parameters of the copula.

The main drawback of the IFM approach is that in its second step copula parameters are estimated from the pseudo-observations obtained from the estimated marginals. It means that the optimal parameters are conditional on the estimates of parameters of the marginal distributions. Therefore, although the parameters estimated using the IFM approach are consistent and asymptotically Normal, these approach is less efficient than canonical maximum likelihood one. To overcome this limitation, following Joe (2014) and Ojea-Ferreiro and Reboredo (2021), we use a Monte Carlo procedure to simulate and re-estimate W times the model in order to obtain the distribution of the estimated parameters.

Copula convolutions

In the study of the dependence structure between the returns of gold denominated in a local currency and the returns of the stock market, we assume the former variable is a convolution between the returns of the gold in US dollars and the exchange rate of euros, yens, Brazilian reals or pounds against this currency. This means changes in gold returns denominated in a local currency could be derived from a variation in the supply and demand of the metal or from the appreciation or depreciation of the currency.

The concept of convolution appears when our interest relies on a variable defined as the sum of two stochastic processes. In this case, the financial variable of gold denominated in a local currency other than USD (r_{ge}) is the result of the convolution of two dependent stochastic processes, namely exchange rate (r_e) and gold returns in USD (r_g), which is the main currency for gold trade. Ojea-Ferreiro (2020) measured the relation between stock, commodities and exchange rates markets, in particular between EUROSTOXX, EUR/USD and oil, assuming that the price of oil in euros is a convolution between its price in US dollars and the exchange rate. Then, he used copula models to obtain the jointly distribution between these assets.

Assume there is a variable r_{ge} defined as $r_{ge} = r_e + r_g$, where r_e and r_g are not independents. The copula-convolution or C-convolution function (Cherubini et al., 2016) is:

$$F_{ge}(r_{ge}) = \int_0^1 C_{g|e}\left(F_g(r_{ge} - F_e^{-1}(u))|u\right) du \tag{3.2.2}$$

Moreover, its density function is given by:

$$f_{ge}(r_{ge}) = \int_0^1 c_{g|e}\left(u, F_g(r_{ge} - F_e^{-1}(u)) \cdot f_g(r_{ge} - F_e^{-1}(u))\right) du \tag{3.2.3}$$

3.2.2 Dynamics

Previously we have used GARCH-type models to capture the dynamic of the conditional volatility of the marginals. In this section, we use the Dynamic Conditional Correlation (DCC) model with the aim to capture the time-varying features of the dependence structure.

The **Dynamic Conditional Correlation (DCC) approach**, introduced by Engle (2002), has the aim to capture the dynamic correlation between pairs of variables. Assuming that innovations follow a Normal or Student t distribution, the DCC model output, $\rho_{ij,t}$, acts as a Gaussian or Student t copula model varying parameter, respectively. We must highlight that for the Student t case we assume that the number of degrees of freedom is constant and only the correlation parameter is time-varying. Other authors have used this methodology, like Huang et al. (2009), Joy (2011), Ciner et al. (2013), Baruník et al. (2016) and Ascorbebeitia et al. (2021).

The DCC-GARCH model can be estimated in two steps (Ciner et al., 2013). To obtain the evolution of the correlation between assets we use the auxiliar variables $q_{ij,t}$:

$$q_{ij,t} = \overline{\rho}_{ij} + \alpha \cdot (\varepsilon_{i,t-1} \cdot \varepsilon_{j,t-1} - \overline{\rho}_{ij}) + \beta \cdot (q_{ij,t-1} - \overline{\rho}_{ij}) \tag{3.2.4}$$

where $\overline{\rho}_{ij}$ represents the correlation between i and j in the long term and $\varepsilon_{i,t}$ is the standardized innovation of asset i at time t. Then, the dynamic correlations are calculated as:

$$\rho_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{i,t} \cdot q_{j,t}}} \tag{3.2.5}$$

Huang et al. (2009) used Kendall's tau ρ_τ transform method to estimate the parameter ρ in Gaussian copula: $\rho_\tau = \frac{2}{\pi} arcsin(\rho)$. However, in this study the parameters of the correlation model are estimated using a maximum likelihood methodology.

3.3 Portfolio construction

We propose the construction of international investment portfolios as a next step in this study (Huang et al., 2009; Reboredo, 2013; Reboredo and Rivera-Castro, 2014b; Jin and Lehnert, 2018; Ojea-Ferreiro and Reboredo, 2021). The analysis of the portfolio performance built under different assumptions can shed light on determine the role of the exchange rate in a stock portfolio.

We consider, on the one hand, an international stock portfolio and gold positions where the portfolio is denominated in a different currency that gold and, on the other hand, the same stock portfolio with gold positions where the exchange rate between the local currency and the gold currency, i.e. US dollars, is taking into account in the investment strategy. The latter option is suitable when the exchange rate acts as "shock absorber", that is, when an appreciation of the local currency due to the turn back of the investment in foreing markets reduces the impact of the spillover in the equity market (Wang et al., 2013).

In this section we present the expressions of the distribution function of a portfolio in these two cases in terms of copulas, which would be the cornerstone to build risk measures on the performance of the portfolio.

3.3.1 Distribution function of a portfolio in terms of copulas and marginal distributions

Firstly, under the assumption that the portfolio is made up of stock and gold denominated in a certain currency other than US dollar, the portfolio amount can be expressed as a percentage $(1-\omega)\%$ invested in the stock market (r_s) and a percentage $\omega\%$ invested in a haven asset than could hedge the left heavy tail from the stock returns, i.e., gold (r_g). We must keep in mind that for investors which have their portfolio denominated in a currency different from US dollar the latter investment implies an exposure to the corresponding exchange rate (r_e) in the same proportion than in gold, so the returns of the portfolio is:

$$r_{p,t} = (1-\omega) \cdot r_{s,t} + \omega \cdot (r_{g,t} + r_{e,t}) = (1-\omega) \cdot r_s s, t + \omega \cdot r_{ge,t} \tag{3.3.1}$$

where $r_{ge,t}$ denotes the gold return denominated in the local currency at time t. From here we ignore the subscript t for the sake of brevity.

The cumulative distribution function of this portfolio can be expressed in terms of copulas as follows:

$$F(r_p) = \int_0^1 \int_0^1 C_{g|e,s} \left(F_g \left(\frac{r_p - (1-\omega) F_s^{-1}(u_s) - \omega F_e^{-1}(u_e)}{\omega} \right) \middle| u_e, u_s; \mathbf{R} \right) du_e du_s \tag{3.3.2}$$

where $\mathbf{R}$ is the correlation matrix.

In the particular case of $\omega = 0$, we only invest in the stock market, so $F(r_p) = F(r_s)$. On the contrary, $\omega = 1$ implies that we are only interested in invest in gold, and in consequence in exchange rate for non-USD investors. Then, the cdf of the portfolio in terms of copulas is:

$$F(r_p) = \int_0^1 C_{e|g} \left(F_e \left(r_p - F_g^{-1}(u_g) \right) \middle| u_g; \rho_{g,e} \right) \, du_g$$

Secondly, when the exchange rate acts as a shock absorber in a distress event in stock markets, we can do an extra-investment in this asset, improving the hedging strategy. Considering this approach, Eq. (3.3.1) is modified to include an extra-weight in exchange rate, resulting in:

$$r_p = \omega_1 \cdot r_g + (\omega_1 + \omega_2) \cdot r_e + (1 - \omega_1 - \omega_2) \cdot r_s \tag{3.3.3}$$

Hence, the cumulative distribution function of the portfolio must be rewritten:

$$F(r_p) = \begin{cases} \int_0^1 \int_0^1 C_{g|e,s}\left(F_g\left(\frac{r_p-(1-\omega_1-\omega_2)F_s^{-1}(u_s)-(\omega_1+\omega_2)F_e^{-1}(u_e)}{\omega_1}\right)\Big|u_e,u_s;\mathbf{R}\right)du_e du_s & \text{if } 0<\omega_i<1; \quad i=1,2 \\ \int_0^1 C_{s|e}\left(F_s\left(\frac{r_p-\omega_2 F_e^{-1}(u_e)}{1-\omega_2}\right)\Big|u_e;\rho_{e,s}\right)du_e & \text{if } \omega_1=0\ ,\ 0<\omega_2\leq 1 \\ \int_0^1 C_{e|g}\left(F_e\left(r_p-F_g^{-1}(u_g)\right)\big|u_g;\rho_{g,e}\right)\ du_g & \text{if } \omega_1=1,\ \omega_2=0 \\ F(r_s) & \text{if } \omega_i=0; \quad i=1,2 \end{cases} \quad (3.3.4)$$

Equations (3.3.2) and (3.3.4) could be easily expressed by means of conditional copulas for elliptic forms of dependence. The advantage of multivariate elliptical models when we express the cdf of the portfolio as a conditional copula is that the order of the variables does not matter[2]. Because of that, we can rewrite the previous distribution functions as a copula in which the conditional variable is not gold, but exchange rate or stock. This fact provides greater flexibility in integrating and in calculating all risk measures desired.

On the one hand, we know that the conditional copula for a bivariate Gaussian dependence structure is:

$$C_{s|ge}\left(u_s|u_{ge};\rho_{s,ge}\right) = \Phi\left(\frac{\Phi^{-1}\left(u_s\right)-\rho_{s,ge}\Phi^{-1}\left(u_{ge}\right)}{\sqrt{1-\rho_{s,ge}^2}}\right) \quad (3.3.5)$$

The trivariate case of the conditional copula can be obtained straightforward by using the Cholesky decomposition:

$$C_{g|e,s}\left(u_g|u_e,u_s;\mathbf{R}\right) = \Phi\left(\frac{\Phi^{-1}\left(u_g\right)-\rho_{g,e}\Phi^{-1}\left(u_e\right)-\frac{\rho_{s,g}-\rho_{g,e}\rho_{e,s}}{\sqrt{1-\rho_{s,e}^2}}\Phi^{-1}\left(u_s\right)}{\sqrt{1-\rho_{g,e}^2-\frac{\left(\rho_{s,g}-\rho_{g,e}\rho_{e,s}\right)^2}{1-\rho_{s,e}^2}}}\right) \quad (3.3.6)$$

On the other hand, the conditional Student t bivariate copula is defined as:

$$C_{s|ge}\left(u_s|u_{ge};\nu,\rho_{s,ge}\right) = T_{\nu+1}\left(\sqrt{\frac{\nu+1}{\nu+T_\nu^{-1}(u_s)^2}}\cdot\frac{T_{\nu+1}^{-1}\left(u_{ge}\right)-\rho_{s,ge}T_{\nu+1}^{-1}\left(u_s\right)}{\sqrt{1-\rho_{s,ge}^2}}\right) \quad (3.3.7)$$

The trivariate Student t copula expression is:

$$C_{g|s,e}\left(u_g|u_s,u_e;\nu,\mathbf{R}\right) =$$
$$= T_{\nu+2}\left(\sqrt{\frac{\nu+2}{\nu+T_{\nu+2}(u_s)^2+T_{\nu+2}(u_e)^2}}\cdot\frac{T_{\nu+2}^{-1}\left(u_g\right)-\rho_{g,s}T_{\nu+2}^{-1}\left(u_s\right)-\frac{\rho_{g,e}-\rho_{g,s}\rho_{e,s}}{\sqrt{1-\rho_{g,e}^2}}T_{\nu+2}^{-1}\left(u_e\right)}{\sqrt{1-\rho_{g,s}^2-\frac{\left(\rho_{g,e}-\rho_{g,s}\rho_{e,s}\right)^2}{1-\rho_{e,s}^2}}}\right) \quad (3.3.8)$$

Annex B explains in more detail the process of obtaining these formulas.

3.3.2 Risk measures

The use of risk measures is useful in order to evaluate the impact of the exchange rate in the international portfolio performance built under different construction criteria, i.e. equally-weighted, minimum-variance and minimum-Expected Shortfall. We recall that these portfolios will be formed under two assumptions, i.e., including and without including an extra-investment in exchange rate.

The most common tools to measure the potential losses incurred by a portfolio in an extreme event are the Value at Risk (VaR) and the Expected Shortfall (ES). The α%-VaR is the maximum loss of a portfolio at

[2]In other applications of copula theory, as vine copula structures, the order of the variables in each pair-copula is crucial to define the dependence structure (Aas et al., 2009; Cooke et al., 2011; Aas and Berg, 2011).

α% confidence level (Adrian and Brunnermeier, 2016), and the ES is defined as the expected loss obtained by a portfolio when there is an excedance of the VaR. The expressions of both measures are, respectively:

$$P\left(r_p \leq VaR_p\left(\alpha\right)\right) = \alpha \Leftrightarrow VaR_p(\alpha) = F_p^{-1}(\alpha) \tag{3.3.9}$$

$$ES_p(\alpha) = E\left(r_p \middle| \, r_p \leq VaR_p\left(\alpha\right)\right) = \frac{1}{\alpha}\int_0^{\alpha} F_p^{-1}\left(q\right)\, dq = \frac{1}{\alpha}\int_0^{\alpha} VaR_p\left(q\right)\, dq \tag{3.3.10}$$

By comparing the VaR obtained in both portfolios, we can determine which one has a higher (inconditional) risk, so we can observe the effect of adding the exchange rate as an asset on the risk of the investment. Furthermore, calculating the Marginal ES (MES) (Cai et al., 2015; Acharya et al., 2017), which decomposes the value of the ES of a portfolio in the ES of its components, we can obtain the percentage of the ES of the portfolio which is because of the exchange rate investment. The MES of the component i of the portfolio is:

$$MES_i\left(\alpha\right) = E\left(\, r_i \middle| \, r_p < VaR_p(\alpha)\right) \tag{3.3.11}$$

The Conditional VaR (CoVaR) could be more suitable for analysing the tail risk, as it is based on calculating the β-quantile of the lower tail of the returns distribution, conditional on another variable is in its α-quantile:

$$P\left(r_p < CoVaR_p(\beta)|r_i < VaR_i\left(\alpha\right)\right) = \beta \tag{3.3.12}$$

That means CoVaR is a stressed risk measure because it does not show the perform of a portfolio in an isolated way, but in a distress scenario of one (or more) of its components. As an example, we could calculate the CoVaR of a portfolio conditional on gold, exchange rate or both were in the tail of their respective distributions. This can give us an idea of the effect separetly of the gold returns and exchange rate variations on a portfolio. In addition, the Conditional ES (CoES) measures the expected losses of a portfolio when it is under its β%-CoVaR:

$$CoES_{p|e}(\beta) = E\left(r_p \middle| r_p \leq CoVaR_{p|e}\left(\alpha, \beta\right)\right) = \frac{1}{\beta}\int_0^{\beta} F_{p|e}^{-1}\left(u_p\right) du_p = \frac{1}{\beta}\int_0^{\beta} CoVaR_{p|e}\left(\alpha, q\right) dq \tag{3.3.13}$$

ΔCoVaR, in turn, can be understood as a proxy of the systemic risk of the portfolios when we make an extra-investment in exchange rate and when we do not make it, because it measures how much the VaR(β) of a portfolio changes when returns of an asset moves from the quantile α to under the median, i.e.

$$\Delta CoVaR_{p|e} = CoVaR_{p|e}(\alpha, \beta) - CoVaR_{p|e}(0.5, \beta) \tag{3.3.14}$$

There exist other interesting risk measures, as Conditional Diversification Benefits (CBD), which captures the evolution of the benefits of portfolio diversification while the dependence between assets is changing over time. This measure, introduced by Christoffersen et al. (2012), uses upper ($\overline{ES}_p(\alpha)$) and lower ($\underline{ES}_p(\alpha)$) bounds of the ES of the portfolio:

$$CDB_p(\alpha) = \frac{\overline{ES}_p(\alpha) - ES_p(\alpha)}{\overline{ES}_p(\alpha) - \underline{ES}_p(\alpha)} \tag{3.3.15}$$

where $ES_p(\alpha)$ denotes the ES of the portfolio at hand.

In addition, the conditional variance is useful to measure the variance of an asset when the portfolio is in its VaR. The conditional variance of the component i of the portfolio is defined as

$$\sigma^2_{i|r_p<VaR_p(\alpha)} = E(r_i^2|r_p < VaR_p(\alpha)) - E\left(r_i|r_p < VaR_p(\alpha)\right)^2 \tag{3.3.16}$$

Using the previous cdfs we could compute all of these risk measures for any portfolio. Annex C presents in more detail these measures and their expressions them in terms of copulas. Those expressions are valid both for Gaussian and Student t copulas.

4 Data

We empirically investigate the role of exchange rate in a stock portfolio of a non-US investor using data of the prices of gold (in USD per ounce) and the stock index and exchange rate (unit of the currency against USD) of each region studied (see Figures 30 to 33, Annex D). Then, our data set is composed of gold, EUROSTOXX50 and EUR/USD for Europe, Nikkei225 and JPY/USD for Japan, BOVESPA and BRL/USD for Brazil, and FTSE100 and GBP/USD for UK. An increase (decrease) of the exchange rate implies a depreciation (appreciation) of the domestic currency. We take the data from Datastream.

The sample period extends from 29 December 2000 to 1 October 2020, including the global financial crisis, European debt crisis from 2008 and the begin of the Coronavirus crisis in March 2020. After calculating the (log) changes in the daily prices of each asset, the logaritmic returns data is aggregated to a weekly frequency (Friday-Friday). The sample between 29 December 2000 and 30 December 2019 is reserved for the study of the dependence structure between assets, while the remaining observations are used for an out-of-sample exercise. The use of weekly data is more appropiated for characterizing the dependence between assets because daily data may be affected by drifts and noise that could mask the relationships and complicate modelling the marginal distributions through non-stationary variances or sudden jumps (Reboredo, 2013).

Table 1 provides summary statistics of the distributions analysed. We appreciate a lower annualized average return, always positive except in the cases of EUROSTOXX50 and GBP/USD. BOVESPA returns present the widest range of values, as shown by the interquartile rank and the difference between the maximum and minimum. It implies this asset reachs more extreme returns, both positive and negative, than any other, and in consecuence it is also the most volatile. Respect to higher moments, all assets present asymmetric and leptokurtic features. On the one hand, the skewness parameter is positive for the exchange rate of yen and Brazilian real against USD and for the gold, and negative for the rest. On the other hand, all assets have a leptokurtic distribution, being EUR/USD, GBP/USD and gold those with a lower value for the kurtosis statistic. The relevance of higher moments in the marginal distributions leads to the rejection of the null hypothesis in the Jarque Bera test for any significance level. Histogram is a very useful graphical tool to appreciate these features of the distributions (Figures 34 to 38, Annex D).

Table 1: Descriptive statistics of returns distributions for the complete set of variables.

	EUR/USD	EUROSTOXX50	JPY/USD	Nikkei225	BRL/USD	BOVESPA	GBP/USD	FTSE100	Gold
Mean	0.0018	-0.0027	0.0009	0.0054	0.0071	0.0098	-0.0011	0.0019	0.0169
Standard Deviation	0.1061	0.2224	0.1110	0.2588	0.1924	0.4265	0.1030	0.2041	0.1924
Maximum	0.0752	0.1668	0.0777	0.1788	0.1784	0.2468	0.0475	0.1113	0.1167
Min imum	-0.0593	-0.2296	-0.0648	-0.3078	-0.1024	-0.4734	-0.0646	-0.1543	-0.0890
Quantile 1%	-0.0393	-0.1094	-0.0394	-0.0924	-0.0641	-0.1605	-0.0387	-0.1042	-0.0713
Quantile 25%	-0.0082	-0.0127	-0.0080	-0.0171	-0.0125	-0.0279	-0.0081	-0.0105	-0.0107
Median	0.0006	0.0014	-0.0006	0.0031	-0.0011	0.0035	0.0001	0.0023	0.0016
Quantile 75%	0.0086	0.0149	0.0072	0.0210	0.0133	0.0314	0.0081	0.0138	0.0159
Quantile 99%	0.0364	0.0701	0.0464	0.0910	0.0843	0.1479	0.0391	0.0717	0.0797
Skewness	-0.0099	-1.0885	0.4297	-0.8166	0.9045	-0.8579	-0.2589	-1.1244	0.0430
Kurtosis	4.9113	10.9370	5.6827	11.4907	8.3891	10.3377	4.7906	9.2743	4.7205
Jarque Bera (p-value)	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Lastly but not least, we compute the Kendall's tau correlation coefficient between the three assets of each region. Table 2 shows the great difference in the relation between the stock market and the exchange rate in developed and emerging countries. Furthermore, the lower asociation between stock and gold in any region is probably because we have measured the correlation on average and not only in distress periods.

Table 2: Correlation matrix for the assets used in each region.

	EUR/USD	**EUROSTOXX50**	**Gold**		**JPY/USD**	**Nikkei225**	**Gold**
EUR/USD	1			**JPY/USD**	1		
EUROSTOXX50	0.0492	1		**Nikkei225**	-0.1381	1	
Gold	-0.3198	-0.0518	1	**Gold**	0.1844	-0.0342	1
	BRL/USD	**BOVESPA**	**Gold**		**GBP/USD**	**FTSE100**	**Gold**
BRL/USD	1			**GBP/USD**	1		
BOVESPA	-0.5115	1		**FTSE100**	-0.0247	1	
Gold	-0.1319	0.0656	1	**Gold**	0.2394	-0.0023	1

Note: the correlation is measured by Kendall's tau coefficient.

5 Empirical results

Analysing the behaviour of a serie must consider a) selecting the marginal model which describes better its features, and b) modelling the dependence between this and the other assets corresponding to its region. Then we propose built portfolios under diverse criteria, i.e. equally-weighted, minimum-variance and minimum-Expected Shortfall, including and without including an extra-investment in exchange rate. By comparing their performance we will be able to see the role of exchange rate in a hedge strategy. Finally, we close this section with an out-of-sample analysis (Berger, 2016; Ojea-Ferreiro, 2019).

5.1 Marginal model selection

We characterize the marginal densities of the assets with different ARMA and GARCH models to capture the behaviour of the mean and the heteroscedasticity of the serie. In particular we fit GARCH(1,1), GARCH(1,1)-M, EWMA, APARCH(1,1) and GJR(1,1). Furthermore, we assume three different hypothesis about the distribution of the residuals of each model, namely Gaussian, Student t and Skewed Student t. In total we test 15 different models for each serie. Figures 48 to 56 (Annex E) represent the annualized volatility resulting of applying these models.

To select the best GARCH model we use diverse methods. Firstly, we calculate the maximum loglikelihood of each model. Secondly, Akaike and Swarchtz (or Bayesian) information criteria are useful estimators of prediction error, i.e., given a set of models, these criteria estimate the quality of each model, relative to each of the other models of the collection. Thirdly, we obtain the parametric VaR of the assets assuming its variance is adjusted by each one of the GARCH models selected. The parametric VaR is defined as: $VaR_{i,t}(\alpha) = F_i^{-1}(\alpha) \cdot \sigma_{i,t} + \mu_{i,t}$, where $F^{-1}(\alpha)$ is the inverse of the cdf of the distribution used. Once we have calculated the VaR, we apply two backtesting test, namely Kupiec and Christoffersen. The objective of both of them is to verify if a model is suitable to calculate the VaR of an asset or portfolio, so the null hypothesis is that the number of excedances of the VaR, i.e., the observations in which the returns of the asset are under the VaR, is statistically equal to the level of significance of the test.

The evidence reported in Annex E indicates that the best marginal model is ARMA(p,q)-APARCH(1,1). In any case the residuals of the models follow a Skewed Student t distribution, capturing higher moments such as skewness and kurtosis.

We must remark that for the Brazilian stock index an ARMA(1,1)-APARCH(1,1) results the best model following loglikelihood and information criteria. However, with the VaR backtesting we obtain better results for an ARMA(1,1)-GJR(1,1) model. The differences between the two models in terms of the information criteria are not very high, so we decide to apply this latter GARCH extension to this serie. In fact we must keep in mind that GJR is a particular case of APARCH model in which $\delta = 2$ (see Section 3.1). In the same sense, for FTSE100 distribution an ARMA(1,2)-GJR(1,1) provides the best fit to the serie, but ARMA(1,2)-APARCH(1,1) performs better for Kupiec and Christoffersen tests, so we will use this model to the rest of the analysis. Table 3 summarizes these best marginal models.

Table 3: Optimal marginal model selected for each returns serie.

	AR(p)	MA(q)	GARCH(m,s)	Distribution
EUR/USD	1	12	APARCH(1,1)	Skewed Student t
EUROSTOXX50	1	12	APARCH(1,1)	Skewed Student t
JPY/USD	1	1	APARCH(1,1)	Skewed Student t
Nikkei225	1	1	APARCH(1,1)	Skewed Student t
BRL/USD	1	1	APARCH(1,1)	Skewed Student t
BOVESPA	1	1	GJR(1,1)	Skewed Student t
GBP/USD	12	1	APARCH(1,1)	Skewed Student t
FTSE100	1	2	APARCH(1,1)	Skewed Student t
Gold	1	12	APARCH(1,1)	Skewed Student t

The time structure of the returns are fitted by the porposed model as shown the Figures 57 to 60 (Annex E), where the acf and pacf are represented. Moreover, we use two test to check that the model is actually suitable. Firstly, we use the Ljung-Box test to corroborate if the lags of the acf and pacf of the residuals are not significative and so the model is well specificated, i.e., the residuals are white noise. The alternative hypothesis is that some autocorrelation coefficient is non-zero. Secondly, the ARCH test of Engle assesses the null hypothesis that a series of residuals exhibits no conditional heteroscedasticity, against the alternative that an ARCH(1) model. The impossibility of rejecting the null hypothesis of the tests indicates that the model is adecuate and there is not any ARCH component in the residuals.

We also test the distribution of the model. On the one hand, Kolmogorov-Smirnov test verifies if independent random samples, X_1 and X_2, are drawn from the same underlying continuous population. X_1 represents the residuals of the model, and X_2 represents a simulated drawn of the same length obtained from the theorical distribution, in particular from a Skewed Student t. On the other hand, the Anderson-Darling is a statistical test of whether a sample of data is drawn from a given probability distribution, i.e. Skewed Student t. Table 4 contains the optimal parameters and all the mentioned tests for the optimal marginal models.

Table 4: Estimates of optimal marginal models and backtesting results.

	EUR/USD	**EUROSTOXX**	**JPY/USD**	**Nikkei**	**BRL/USD**	**BOVESPA**	**GBP/USD**	**FTSE**	**Gold**
Mean									
ϕ_0	0.0001	-0.0003	0.0000	0.0016	0.0013	0.0019	0.0001	0.0012	0.0029
	(0.0003)	(0.0008)	(0.0005)	(0.0014)	(0.0011)	(0.0015)	(0.0005)	(0.0009)	(0.0011)
ϕ_1	0.0093	-0.0822	-0.0173	-0.1733	-0.0574	0.0094	0.0693	-0.0909	-0.0302
	(0.0291)	(0.0516)	(0.0334)	(0.2245)	(0.0549)	(0.1947)	(0.0342)	(0.0372)	(0.0402)
θ_1	0.0957	0.0111	-0.0030	0.1319	0.0009	-0.0330	0.0189	-0.1410	0.1352
	(0.0593)	(0.0521)	(0.0070)	(0.2268)	(0.0418)	(0.1881)	(0.0425)	(0.0384)	(0.0355)
Variance									
ω	0.0013	0.0002	0.0002	0.0057	0.0000	0.0015	0.0000	0.0000	0.0000
	(0.0000)	(0.0048)	(0.0018)	(0.0090)	(0.0005)	(0.1473)	(0.0003)	(0.0012)	(0.0006)
α	0.0810	0.1825	0.0922	0.1287	0.1559	0.0602	0.0571	0.1816	0.0277
	(0.0128)	(0.0381)	(0.0493)	(0.0368)	(0.1198)	(0.0194)	(0.0390)	(0.0558)	(0.0587)
β	0.9072	0.8175	0.8783	0.8101	0.8441	0.8897	0.8334	0.7854	0.7615
	(0.0288)	(0.0386)	(0.0552)	(0.0810)	(0.1549)	(0.3827)	(0.0887)	(0.1014)	(0.2010)
γ	0.0073	-0.0006	0.0056	0.5641	0.0040	0.9896	-0.0014	-0.0016	-0.0050
	(0.0015)	(0.0081)	(0.0118)	(0.1593)	(0.0231)	(0.2973)	(0.0256)	(0.0096)	(0.0977)
δ	0.7394	1.5517	1.3938	0.8377	2.4445	2.0000	3.4713	2.0887	5.0269
	(1.2510)	(0.4114)	(1.4746)	(0.5367)	(12.9134)	(4.6878)	(4.3516)	(4.6995)	(7.9205)
Tail	7.6543	4.8704	0.0760	5.5312	3.2536	4.8892	7.1671	4.5427	4.3262
	(0.0322)	(0.0388)	(0.0451)	(0.0497)	(0.0623)	(0.0589)	(0.0516)	(0.0509)	(0.0640)
Skewness	-0.0634	-0.1604	3.5083	-0.1540	0.1440	-0.0745	-0.0531	-0.2638	0.0398
	(2.8762)	(0.6792)	(0.7413)	(1.6235)	(0.4114)	(0.9024)	(8.5539)	(0.7744)	(0.8136)
Loglikelihood	2127.74	1699.13	2103.23	1507.94	1756.29	1142.02	2155.19	1768.19	1697.76
LBQ test	0.6933	0.9819	0.8487	0.9631	0.3685	0.9806	0.9118	0.8257	0.8482
ARCH test	0.4872	0.8813	0.5686	0.8601	0.9575	0.5337	0.9702	0.291	0.6191
KS test	0.4693	0.7223	0.7021	0.6142	0.2502	0.4469	0.7453	0.373	0.9332
AD test	0.5188	0.6266	0.6060	0.6495	0.4908	0.302	0.6193	0.4964	0.5992
Percentage of excedances(1)									
VaR(1%)	1.2129	1.0782	0.6739	0.6739	0.0000**	0.0000**	0.0000**	0.0000**	0.9434
VaR(2.5%)	2.9650	3.3693	2.4259	2.4259	1.4825	1.8868	1.3477*	0.0000**	3.5040
VaR(5%)	5.7951	5.3908	5.5256	4.7170	5.5256	3.5040*	4.4474	2.6954**	6.4690
VaR(10%)	10.6469	11.4555	10.5121	10.3774	11.4555	7.9515	11.1860	9.4340	9.9730
Kupiec									
VaR(1%)	0.0001	0.0029	0.0064	0.0064	0.0029	0.0064	0.0064	0.0064	0.0064
VaR(2.5%)	0.2637	0.0550	0.0278	0.0000	0.2661	0.2637	0.0351	0.0000	0.0550
VaR(5%)	0.8523	0.1567	0.2167	0.0000	0.4135	0.0487	0.4819	0.0017	0.6293
VaR(10%)	0.7333	0.3074	0.8263	0.0966	0.1231	0.0546	0.2897	0.6042	0.9769
Christoffersen									
VaR(1%)	0.9858	0.9578	0.9858	0.9858	0.9578	0.9858	0.9858	0.9858	0.9858
VaR(2.5%)	0.4624	0.1332	0.6007	0.9169	0.4621	0.4624	0.8348	0.9858	0.5645
VaR(5%)	0.5192	0.1135	0.1492	0.4308	0.0244	0.1687	0.6672	0.5635	0.5669
VaR(10%)	0.9836	0.5934	0.3971	0.6450	0.0141	0.1418	0.0216	0.3069	0.5568

	VaR(1%)	**VaR**(2.5%)	**VaR**(5%)	**VaR**(10%)
(1) Lower bound (in percentage) of exceedances for not reject the null for $\alpha = 5\%$	0.4043	1.4825	3.6388	7.9515
Upper bound (in percentage) of exceedances for not reject the null for $\alpha = 5\%$	1.7520	3.6388	6.6038	12.1294
Lower bound (in percentage) of exceedances for not reject the null for $\alpha = 1\%$	0.2695	1.2129	3.0997	7.2776
Upper bound (in percentage) of exceedances for not reject the null for $\alpha = 1\%$	2.0216	4.0431	7.1429	12.8032

Notes: the estimates of the optimal parameters are obtained from maximum likelihood. The first Panel present the optimal parameters of each ARMA(p,q)-APARCH(1,1) model assuming Skewed Student t distribution for the innovations. The standard deviation of the optimal parameters is calculated by Monte Carlo simulation (see Section 3.2.1). Then are presented the results of Ljung-Box (LBQ), ARCH, Kolmogorov-Smirnov (KS) and Anderson-Darling (AD) tests for the godness of fit of thoose models. Finally we show the results of Kupiec and Christoffersen tests for the Value at Risk backtesting to verify the model adjusted to the variables dynamic is appropiated to calculate risk measures. The values with (*)/(**) represent values for the VaR(α%) which are under the lower bound or above the upper bound of exceedances for a level of significance of $\alpha = 1\%$ / $\alpha = 5\%$ (we will reject the null hypothesis).

5.2 DCC-GARCH model

To compute dynamic elliptical copulas we use DCC-GARCH methodology, obtaining the evolution of the correlation between each pair of assets, i.e., exchange rate-stock, exchange rate-gold and stock-gold. We consider the degrees of freedom of the Student t copula (DCC-GARCH-T) time-invariant. As a result we find that the evolution of the correlations in Gaussian and Student t models are very similar, except in UK, where the estimation of the dynamic correlations with a Student t copula provides a less smoothed serie.

We can think in the logical relationships between the assets. Firstly, the correlation between the exchange rate and the index takes positive values when a depreciation of the local currency attracts foreing capital and the equity index increases its value. On the contrary, when there is a shock in stock markets, the supply of the domestic currency is higher so it will be depreciated, and the correlation between these assets is negative. Secondly, we should expect that the correlation between the depreciation of a local currency against USD and gold is negative, because of an appreciation of the local currency, or equivalently a depreciation of the dollar, will attract investors to those markets traded in USD, like gold market. Finally, due to the safe haven properties of gold, we should expect a negative correlation between its price and the value of the local stock index. Furthermore, when the depreciation (appreciation) of the local currency against dollar makes the investment in this stock index more (less) attractive, the investment in gold will fall (grow).

Summarizing the results shown in Figures 1 to 4, we find that in Europe the expected relationships explained above are fulfilled. We can see in Figure 1 a sharp movement in correlations, more pronounced when we use a Student t copula, in the first months of 2015. In this period there is a depreciation of the euro originated by a strategy of the ECB to inject liquidity to the European economies. This depreciation causes a great increase of EUROSTOXX50 value, and also an increase of the correlation between this exchange rate and gold value, which was having a bearish trend in these period.

In the Japanese case (see Figure 2) we find a positive correlation between JPY/USD and gold in the whole sample, while the correlation between this exchange rate and the stock index is negative, except in certain moments in which the correlation becomes positive but takes low values. The correlation between Nikkei225 and gold alternates between negative (until 2006 and since 2010) and positive values (between 2006-2010). We can better understand this evolution with bottom Panel of Figure 31 (Annex D).

For its part, Brazil is an emerging country. Because of that, on the one hand, the investment in BOVESPA increases a lot in the early years of the 21st century (see Figure 32, Annex D). Due to this great rise while gold price was also increasing we found a positive correlation between these assets in Figure 3. On the other hand, we must keep in mind that the investment in emerging countries implies a higher expected profit than in developed markets, but also a greater risk. Because of that, when there is uncertainty in the markets, investors flee to safer markets, causing a great depreciation of the Brazilian real. The consequence is a negative correlation between the depreciation of this currency and both stock index and gold. It is worth noting the very narrow range in which the three conditional correlations move.

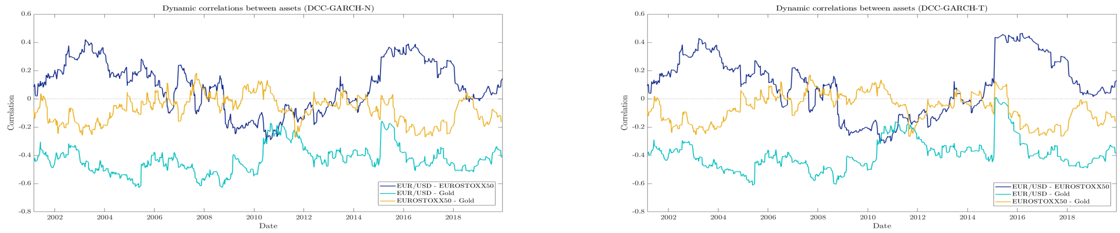
Finally, Figure 4 shows that in UK the only correlation which takes always positive values is that between the pound depreciation against dollars and gold, while the other two correlations alterns positive and negative values, depending of the trend of their prices (see Figure 33, Annex D).

Table 5: Optimal parameters of Gaussian and Student t copula, obtained with DCC-GARCH(1,1) model.

		Europe	Japan	Brazil	UK
DCC-GARCH(1,1)-N	α	0.0241	0.0145	0.0121	0.0117
		(0.0106)	(0.0099)	(0.0131)	(0.0107)
	β	0.9540	0.9815	0.9285	0.9777
		(0.1246)	(0.1541)	(0.1389)	(0.0770)
DCC-GARCH(1,1)-T	α	0.0213	0.0141	0.0145	0.0331
		(0.0124)	(0.0094)	(0.0122)	(0.0092)
	β	0.9668	0.9809	0.9203	0.9116
		(0.1264)	(0.1624)	(0.2193)	(0.0830)
	ν	5.2142	4.3206	6.7872	5.7772
		(7.8536)	(2.5068)	(6.7040)	(3.4551)

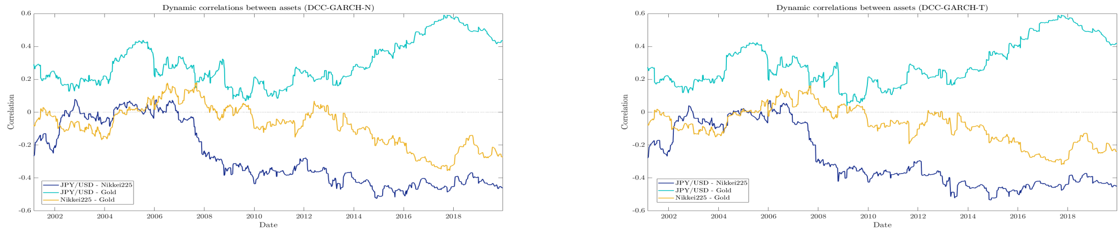
Notes: *Europe* refers to the following set of variables: EUR/USD, EUROSTOXX50 and gold; *Japan* includes JPY/USD, Nikkei225 and gold; *Brazil* refers to the variables BRL/USD, BOVESPA and gold; *UK* refers to GBP/USD, FTSE100 and gold. The estimates of the parameters are obtained from maximum likelihood, and their standard deviations are obtained from Monte Carlo simulation.

Figure 1: Dynamic pairwise correlation between assets for Europe using DCC-GARCH(1,1) approach.



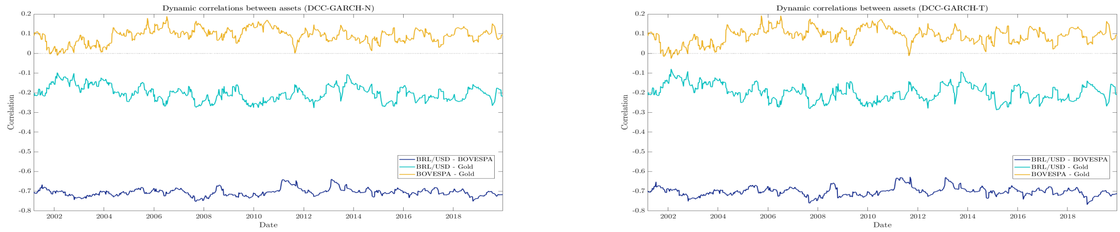
Notes: Left Panel shows the evolution of the correlation parameters of assets under a Gaussian copula approach, while in the right Panel we present the results of a Student t copula case. The optimal parameters are estimated by maximum likelihood.

Figure 2: Dynamic pairwise correlation between assets for Japan using DCC-GARCH(1,1) approach.



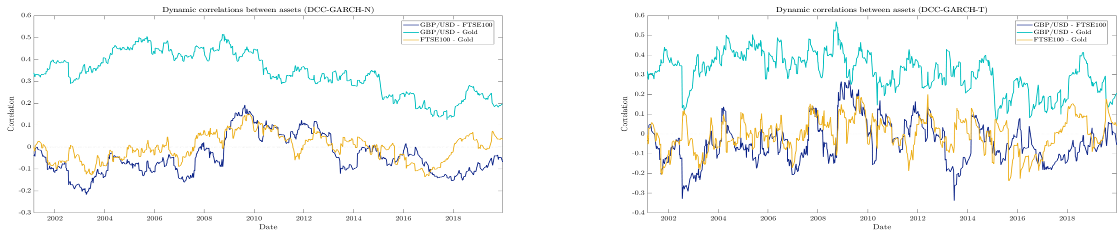
Notes: Left Panel shows the evolution of the correlation parameters of assets under a Gaussian copula approach, while in the right Panel we present the results of a Student t copula case. The optimal parameters are estimated by maximum likelihood.

Figure 3: Dynamic pairwise correlation between assets for Brazil using DCC-GARCH(1,1) approach.



Notes: Left Panel shows the evolution of the correlation parameters of assets under a Gaussian copula approach, while in the right Panel we present the results of a Student t copula case. The optimal parameters are estimated by maximum likelihood.

Figure 4: Dynamic pairwise correlation between assets for UK using DCC-GARCH(1,1) approach.



Notes: Left Panel shows the evolution of the correlation parameters of assets under a Gaussian copula approach, while in the right Panel we present the results of a Student t copula case. The optimal parameters are estimated by maximum likelihood.

5.3 Portfolio construction

The construction of portfolios is a key step in this study. Top Panels in Figures 77 to 80 (Annex G) and in Figures 5 to 8 present the evolution of the weights of gold denominated in the local currency and stock index of each region in the minimum-variance and minimum-ES portfolio, respectively. The results show, on the one hand, that when we only consider exchange rate as an instrument to invest in gold, the leading position in all portfolios is gold in all the sample, although when we assume a Student t copula to model the dependence between assets the prominence of gold is reduced. Moreover, we notice that in turbulent periods the investment in gold is usually greater. However, despite of increasing the investment in the safe haven, the risk of the portfolios reachs signficantly higher values than in the rest of the sample, indicating a strong systemic risk.

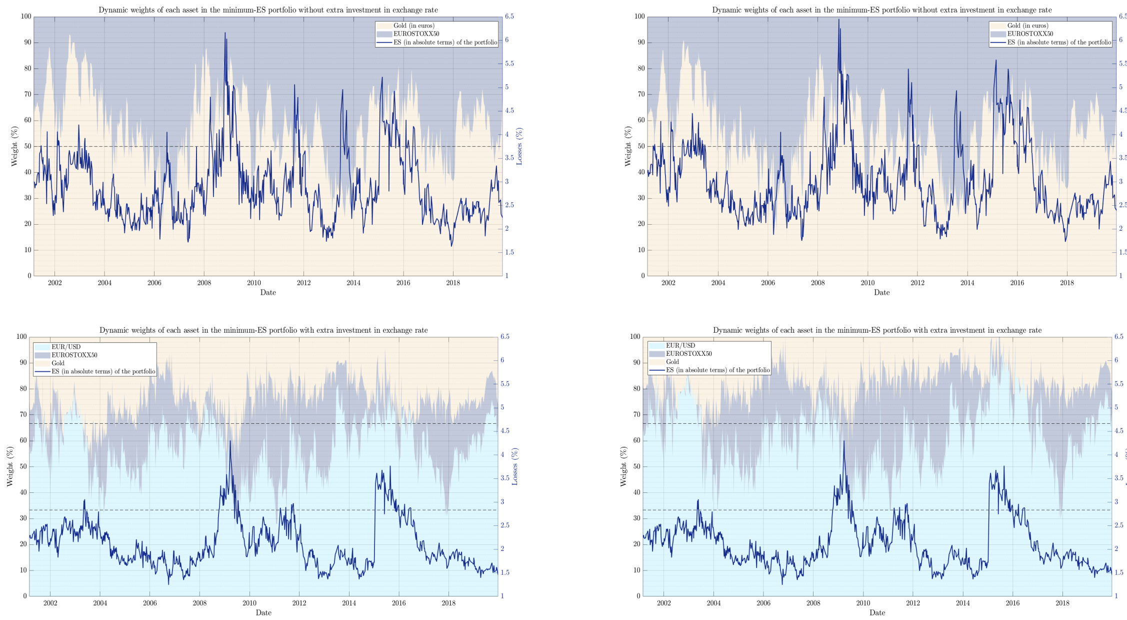
Bottom Panels of Figures 77 to 80 (Annex G) and Figures 5 to 8 analyse the situation in which the investment in exchange rate may be divided into a part intended for acquire gold and another part intended to hedge the portfolio. The blue area represents this extra-investment in exchange rate. Here two distinct patterns of behaviour can be observed. Firstly, in the European and Brazilian cases the weight of the exchange rate is majority in the portfolios, i.e., exchange rate is better hedging an equity position than gold, but all assets have positive weights in them. Secondly, in Japan and UK we can advise that the exchange rate outshines the gold position, having gold a non-zero weight in only briefly periods. Under the Student t copula assumption, the presence of gold decreases respect to the previous copula model in Europe and Brazil and increases in the other two regions. The latter implies that, when we consider extreme events and tail dependence, gold strengthens its role as hedge asset.

In Europe (Brazil) the correlation between the depreciation of the euro (Brazilian real) and the stock index is positive (negative) in almost the whole sample, and between gold and the index, negative (positive) (see Figure 1 (3)). This means that in the event of a spillover in the stock market, it occurs a depreciation (appreciation) of the currency and an increase (decrease) of the gold price. Since the correlation between the depreciation of both the euro and the Brazilian reals and gold price is negative, it could be suitable including both assets in the portfolio to hedge a position in equity. In fact, on the one hand, in European portfolios investing in the exchange rate reduces the expected tail risk in 91.64 bp in the case of the Gaussian copula and in 91.96 bp when the dependence structure is given by a Student t copula. On the other hand, for Brazil the presence of the exchange rate in the portfolio reduces the tail risk in 129.65 bp when we use a Gaussian copula to model the dependence between the assets and in 133.74 bp when the copula model is Student t one.

In the Japanese (British) case, the depreciation of both the yen (pound) and the price of gold have, in general terms, a negative correlation with the Nikkei225 (FTSE100). This fact means that a spillover in stock markets will be accompained by an appreciation of the local currency and an increase in the price of gold. But we must to take into account two facts. Firstly, the correlation between the depreciation of the currency and gold is positive, and it is, in absolute terms, higher than the correlation between the exchange rate or gold and the stock index, respectively. Because of it a potencial profit of the investment in gold caused by a rise in its price will be accompained by a depreciation of the domestic currency. Secondly, the negative correlation between the exchange rate and stock index is, in absolute terms, greater than the correlation between the precious metal and this index. Knowing this, and if we observe Figures 78 and 6 for Japan and Figures 80 and 8 for UK we can notice that the best strategy for hedge a stock portfolio is to acquire exchange rate, without buying gold (in minimum-variance portfolios) or investing in gold in only a few certain moments (in minimum-ES portfolios). In Japan, when the dependence structure is defined by a Gaussian copula, including the exchange rate in the portfolio reduces its tail risk 156.87 bp and in 141.95 in the Student t case. For British portfolios, when we introduce the exchange rate in them the tail risk is reduced in 137.56 or 121.11 bp when we assume a Gaussian or Student t copula, respectively, for the dependence structure.

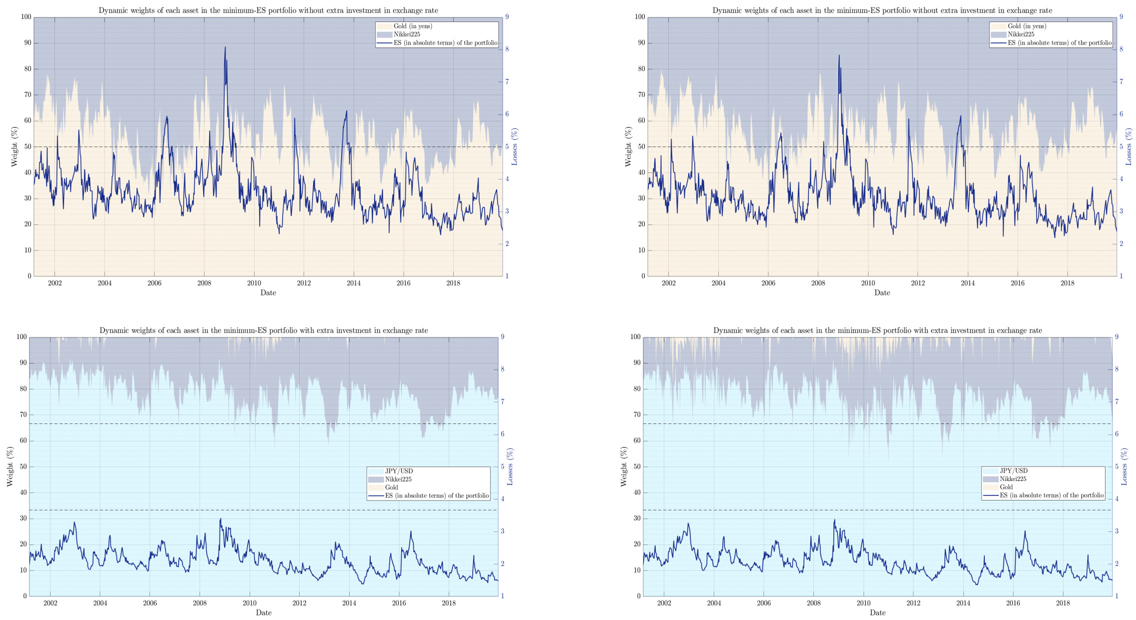
It is relevant to notice that in each one of the uncertainty periods in which the variance or the ES of the portfolio takes greater values the evolution of the weights of gold and exchange rate in the porfolios is different. In some cases, due to the stress in stock markets the capital allocated in the index, and then in the domestic currency, refugees in gold market, causing a decrease on the investment in exchange rate. On the contrary, in other periods a depreciation of the local currency makes its investment more attractive and suitable to reduce the risk of the portfolio, so its weight in the portfolios becomes higher.

Figure 5: Evolution of the weights of each asset in minimum-Expected Shortfall portfolios for Europe.



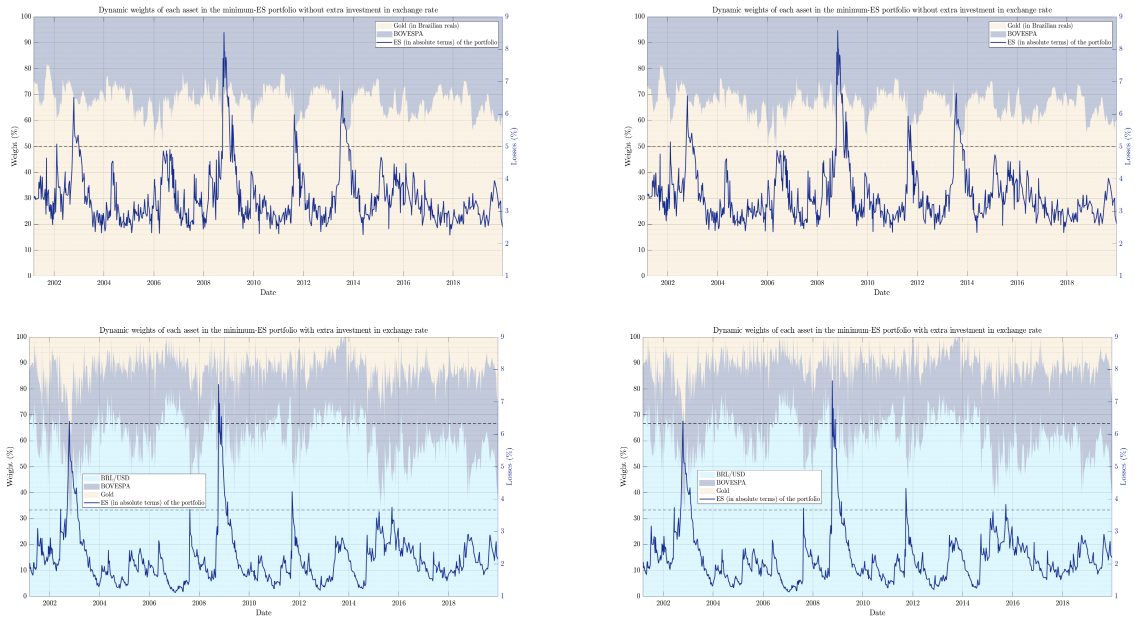
Note: Figures in upper (lower) row show the portfolios without (with) extra investment in the exchange rate assuming Gaussian (left Panel) and Student t (right Panel) copula approach.

Figure 6: Evolution of the weights of each asset in minimum-Expected Shortfall portfolios for Japan.



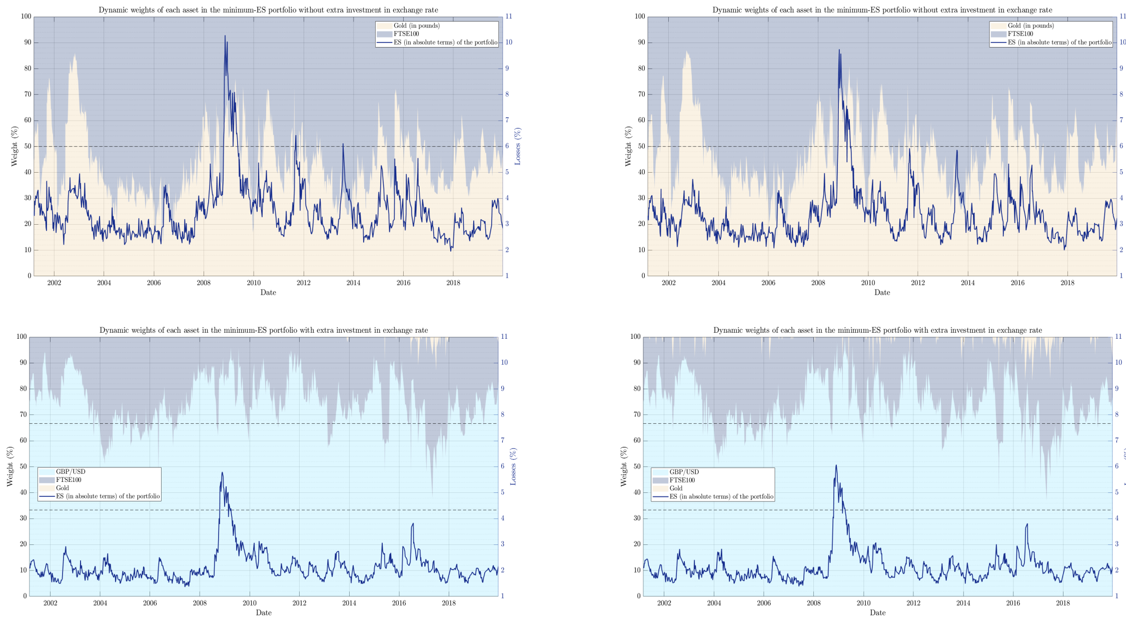
Note: Figures in upper (lower) row show the portfolios without (with) extra investment in the exchange rate assuming Gaussian (left Panel) and Student t (right Panel) copula approach.

Figure 7: Evolution of the weights of each asset in minimum-Expected Shortfall portfolios for Brazil.



Note: Figures in upper (lower) row show the portfolios without (with) extra investment in the exchange rate assuming Gaussian (left Panel) and Student t (right Panel) copula approach.

Figure 8: Evolution of the weights of each asset in minimum-Expected Shortfall portfolios for UK.

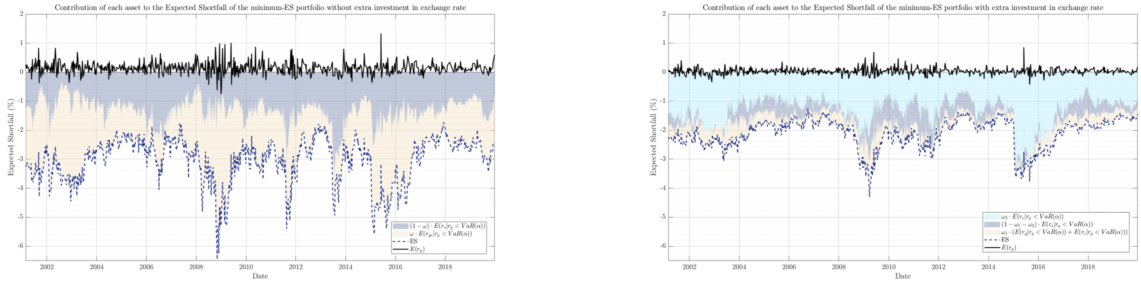


Note: Figures in upper (lower) row show the portfolios without (with) extra investment in the exchange rate assuming Gaussian (left Panel) and Student t (right Panel) copula approach.

Figures 9 to 12 show the contribution of each asset to the Expected Shortfall of the portfolio, computed as the Marginal Expected Shortfall (MES) of each asset weighted by their respective weight in the portfolio (see Annex C). They perfectly reflect the presence of the assets in the international portfolios under Student t copula approach[3].

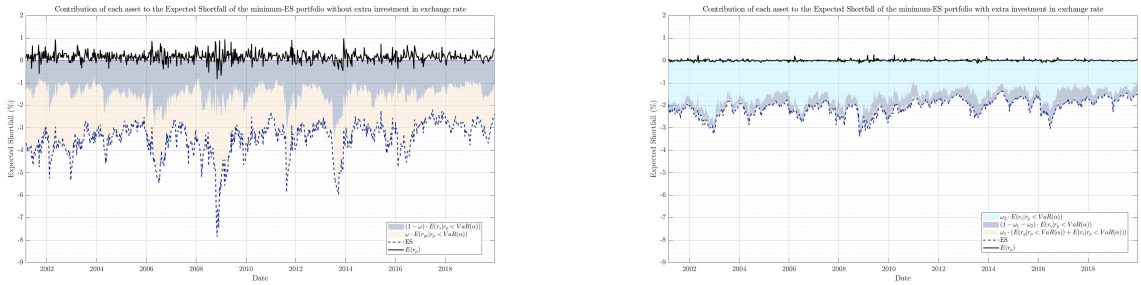
[3]From here figures related with risk measures of portfolios under Gaussian copula assumption appear in Annex G.

Figure 9: Contribution of each asset to the Expected Shortfall of the minimum-ES portfolio for Europe (assuming Student t copula model).



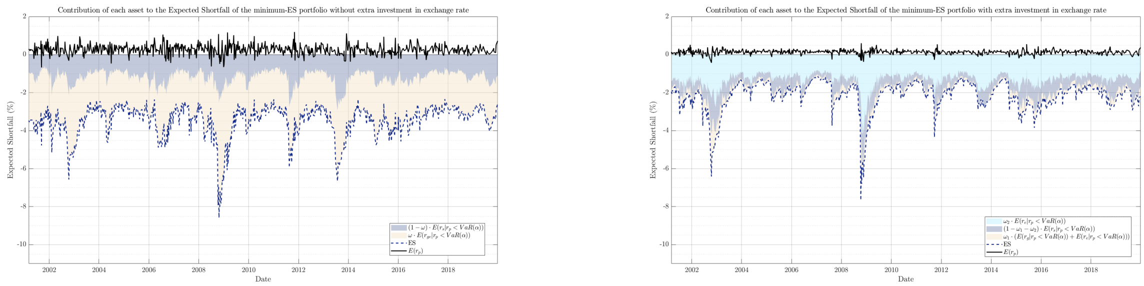
Notes: left (right) Panel performs the contribution of each asset to the ES without (with) extra-investment in exchange rate.

Figure 10: Contribution of each asset to the Expected Shortfall of the minimum-ES portfolio for Japan (assuming Student t copula model).



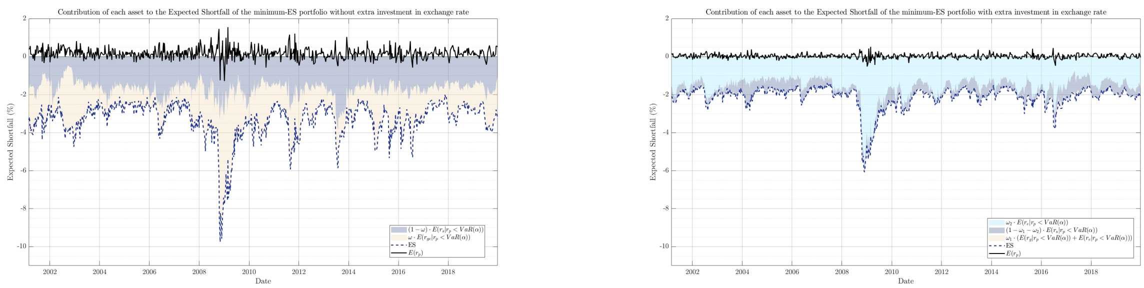
Notes: left (right) Panel performs the contribution of each asset to the ES without (with) extra-investment in exchange rate.

Figure 11: Contribution of each asset to the Expected Shortfall of the minimum-ES portfolio for Brazil (assuming Student t copula model).



Notes: left (right) Panel performs the contribution of each asset to the ES without (with) extra-investment in exchange rate.

Figure 12: Contribution of each asset to the Expected Shortfall of the minimum-ES portfolio for UK (assuming Student t copula model).



Notes: left (right) Panel performs the contribution of each asset to the ES without (with) extra-investment in exchange rate.

In addition to the MES, other several risk measures are useful to analyse the role of the exchange rate in stock portfolios (see Annex C). The most common tools to measure the potential losses incurred by a portfolio in an extreme event are the VaR and the ES, while the CoVaR is more useful to measure the tail dependence between assets in a portfolio. Figures 13 to 16 perform VaR, ES and CoVaR of all portfolios conditional on one of the assets is under or above its VaR when a Student t copula defines the dependence structure.

Firstly, the graphical evidence indicates that VaR, ES and CoVaR values of the portfolios have a similar trend in all regions, although there are differences in the magnitude of the risk across them. We notice the evident impact of the global financial crisis, reflected by an abrupt increase of the risk measures.

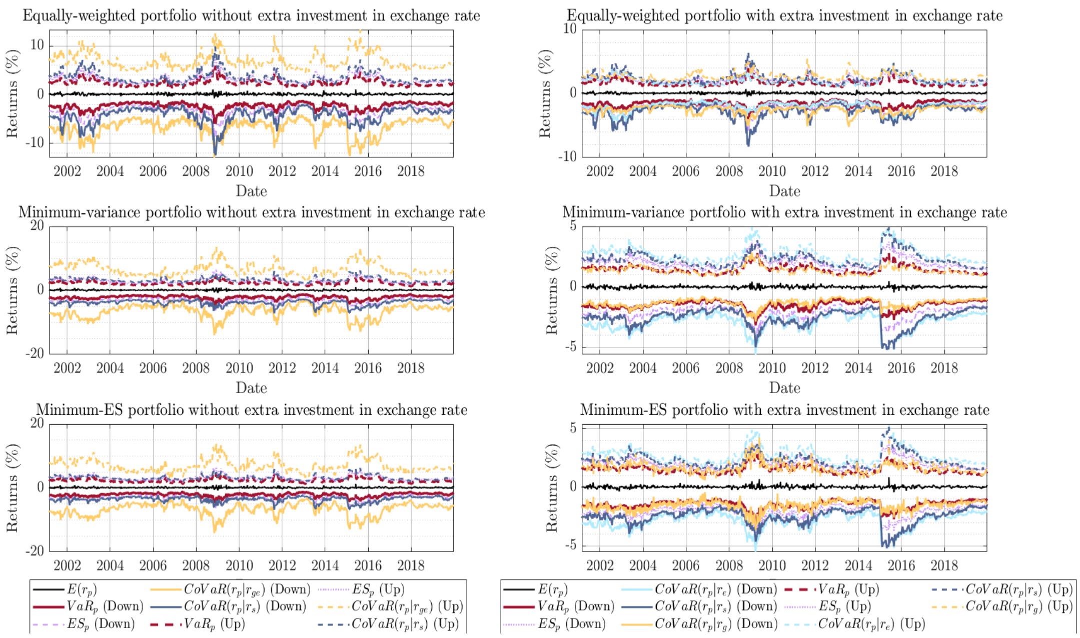
Secondly, broadly speaking, we see that the tail risk, both downside and upside, is higher in equally-weighted portfolios (respect to other construction criteria) and in those portfolios which do not make an extra-investment in exchange rate (respect those which do make it). Hence, in these portfolios a bearish scenario in either exchange rate, stock or gold market has a major impact on portfolio risk.

For instance, when the local stock index suffers a sharply fall in its price, the CoVaR of the portfolios which do not include an extra-weight in exchange rate is greater. In other words, the inclusion of the exchange rate in the portfolios reduces the impact of a spillover in stock markets, improving the hedging strategy.

Thirdly, the greater conditional tail risk of a portfolio composed of stock and gold denominated in a local currency occurs when the latter suffers a sharpe decrease. On the contrary, the maximum loss incurred by the minimum-variance and minimum-ES portfolios composed of exchange rate, stock and gold is produced when the former variable is under its VaR, while gold is the asset which provides less conditional tail risk. However, if we gave equal importance to all assets, i.e. in the equally-weighted portfolio, we can see that in fact the exchange rate provides less tail risk to the portfolio.

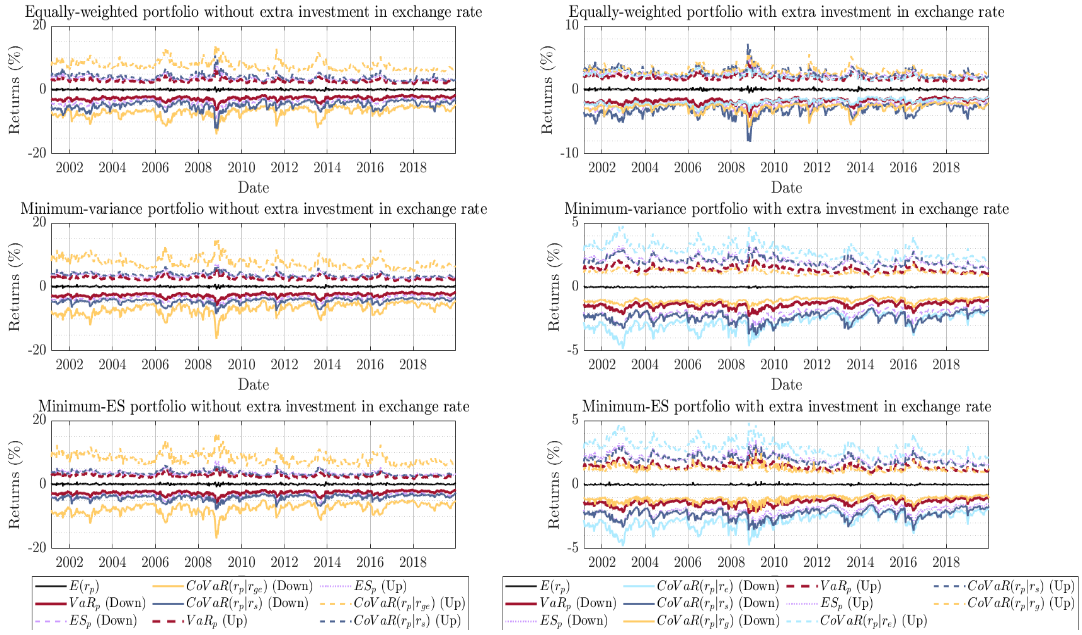
Finally, considering the downside risk, we observe that bearish CoVaR values are systematically below the VaR threshole in any portfolio. It indicates that extreme downwards movements in the price of any asset have a spillover effect in the portfolio. It is also a sign of co-movements in the lower tail of assets returns. In the same line, when the upside CoVaR values are greater than the upside VaR, we can conclude that an extreme rise of one of the assets has an impact on the upside risk of the portfolio returns.

Figure 13: CoVaR of European portfolios conditional on only one of the assets is under or above its VaR (assuming Student t copula model).



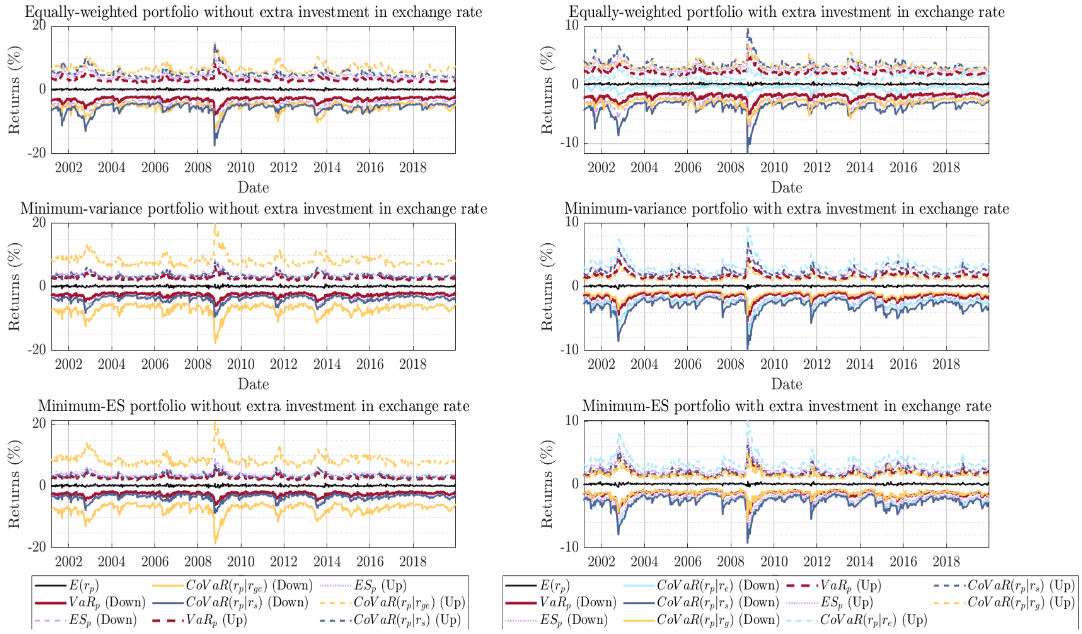
Notes: left Panels consider the situation in which there is not extra investment in exchange rate, while right Panels present the situation in which portfolios are composed of exchange rate, stock and gold.

Figure 14: CoVaR of Japanese portfolios conditional on only one of the assets is under or above its VaR (assuming Student t copula model).



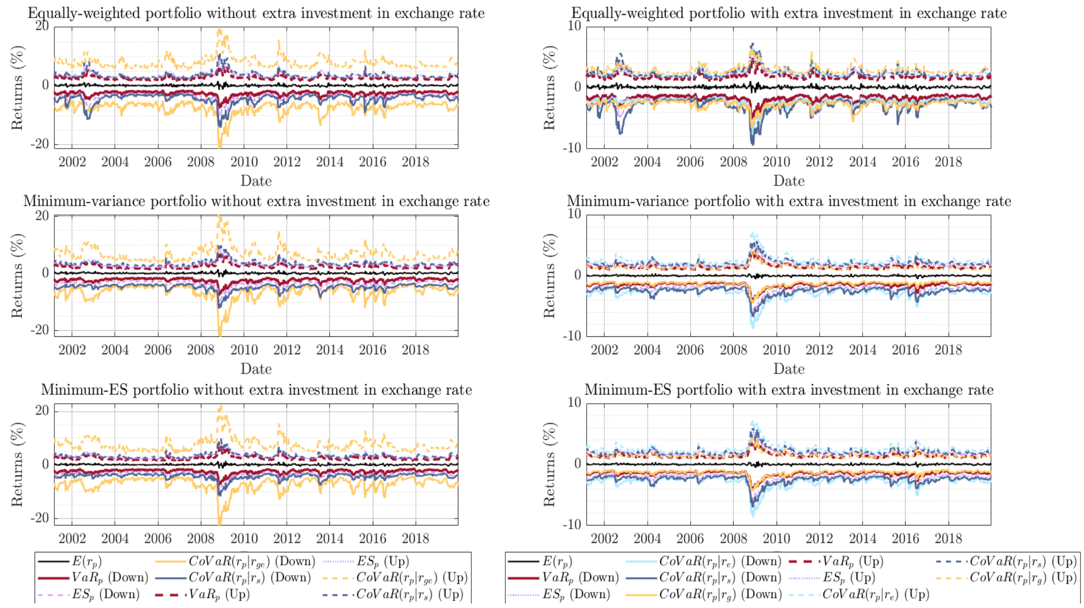
Notes: left Panels consider the situation in which there is not extra investment in exchange rate, while right Panels present the situation in which portfolios are composed of exchange rate, stock and gold.

Figure 15: CoVaR of Brazilian portfolios conditional on only one of the assets is under or above its VaR (assuming Student t copula model).



Notes: left Panels consider the situation in which there is not extra investment in exchange rate, while right Panels present the situation in which portfolios are composed of exchange rate, stock and gold.

Figure 16: CoVaR of British portfolios conditional on only one of the assets is under or above its VaR (assuming Student t copula model).

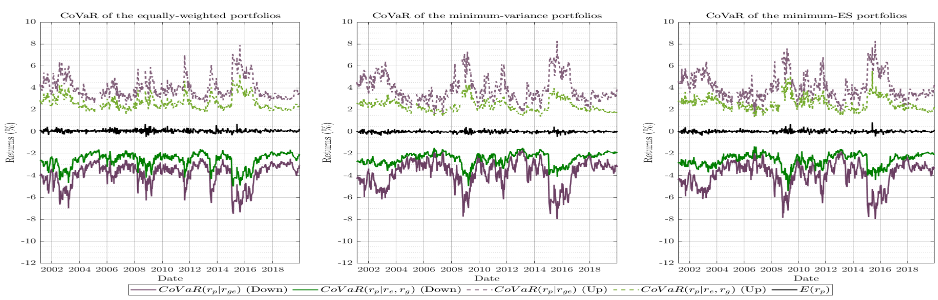


Notes: left Panels consider the situation in which there is not extra investment in exchange rate, while right Panels present the situation in which portfolios are composed of exchange rate, stock and gold.

To our purpose, i.e. analyse the role of exchange rate in a non-US stock portfolio, it is important to compare the tail risk of the portfolio conditional on the jointly scenario for gold and this exchange rate, when including it as a third asset and when we only use it to invest in gold. The probabilities of a bearish scenario in both portfolios are $P(r_p < CoVaR_p(\beta)|r_{ge} < VaR_{ge}(\alpha)) = \beta$ and $P(r_p < CoVaR_p(\beta)|r_e < VaR_e(\alpha), r_g < VaR_g(\alpha)) = \beta$, where r_{ge} denotes the returns of gold in a local currency. In Figures 17 to 20 we find again that including the exchange rate in all portfolios actually reduces their tail risk.

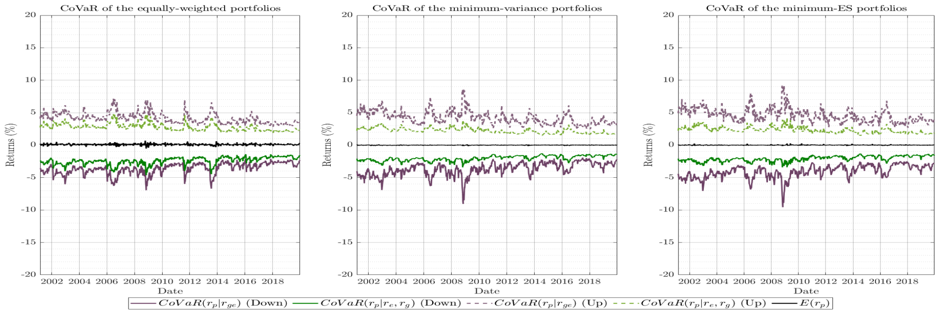
Moreover, it also reduces their systemic risk in terms of ΔCoVaR, which measures how much the VaR of a portfolio changes when returns of an asset moves from the tail to the median (see Eq. (3.3.14)). In Europe the systemic risk is reduced in 50.61 bp assuming a Gaussian copula to model the dependence, and in 37.85 bp when the Student t copula represents the dependence structure. In Japan, the ΔCoVaR is reduced in 159.94 bp and 67 bp under Gaussian and Student t copula, respectively. Brazilian portfolios, in turn, reduce their systemic risk in 18.74 bp Gaussian copula assumptions and in 77.70 bp when the copula model assumed is the Student t one. Finally, in the UK the value of the measure falls in 144.38 bp under Gaussian copula and in 63.32 bp in the Student t case.

Figure 17: CoVaR of European portfolios conditional on the situation of exchange rate and gold (assuming Student t copula model).



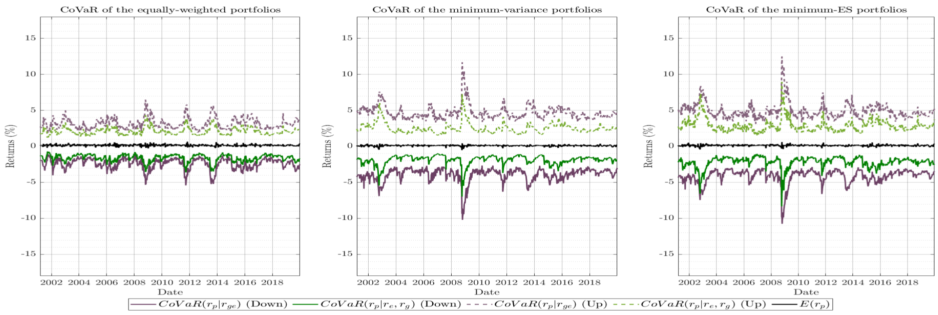
Note: the figure shows the downside and upside CoVaR of portfolios, when gold denominated in euros (violet lines) or gold and exchange rate (green lines) are under or above its VaR, respectively.

Figure 18: CoVaR of Japanese portfolios conditional on the situation of exchange rate and gold (assuming Student t copula model).



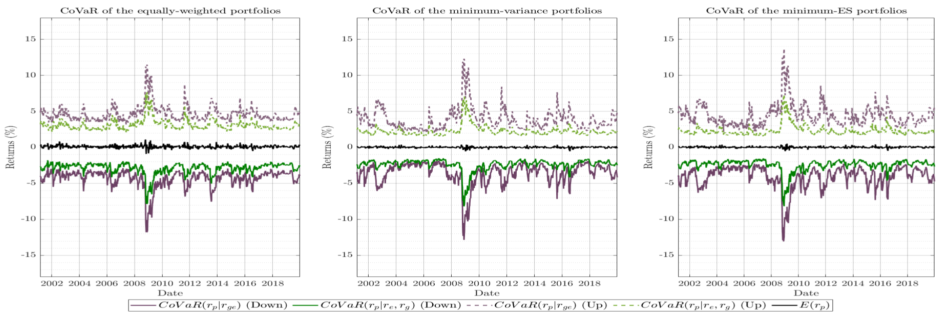
Note: the figure shows the downside and upside CoVaR of portfolios, when gold denominated in euros (violet lines) or gold and exchange rate (green lines) are under or above its VaR, respectively.

Figure 19: CoVaR of Brazilian portfolios conditional on the situation of exchange rate and gold (assuming Student t copula model).



Note: the figure shows the downside and upside CoVaR of portfolios, when gold denominated in euros (violet lines) or gold and exchange rate (green lines) are under or above its VaR, respectively.

Figure 20: CoVaR of British portfolios conditional on the situation of exchange rate and gold (assuming Student t copula model).



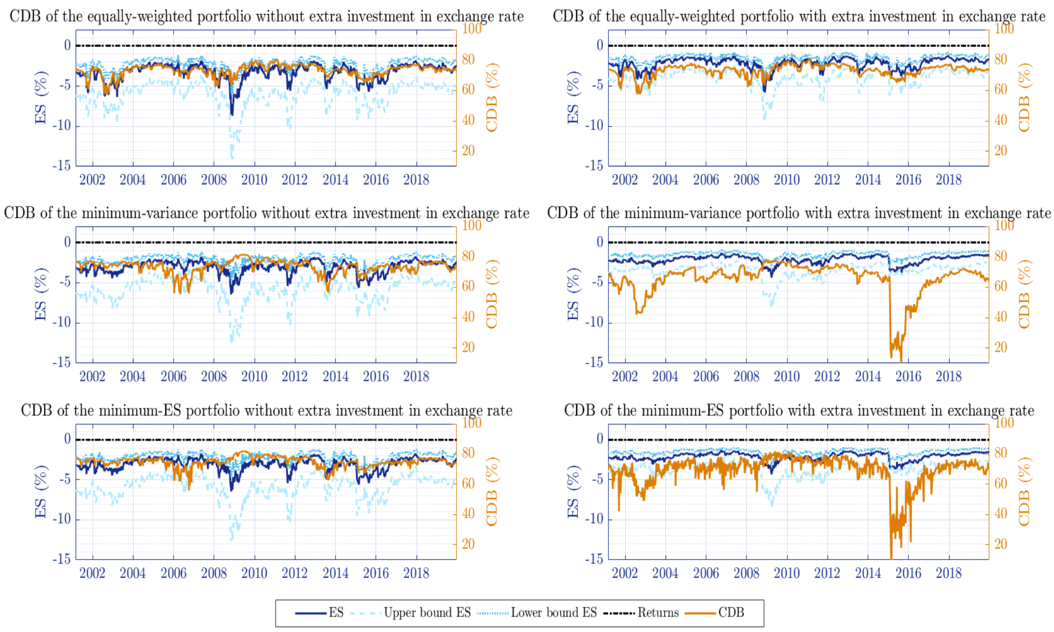
Note: the figure shows the downside and upside CoVaR of portfolios, when gold denominated in euros (violet lines) or gold and exchange rate (green lines) are under or above its VaR, respectively.

The same results are found when we compute the CoVaR of all portfolios in the best and in the worst cases[4] (Figure 95), as well as in the most probable scenarios in each region (Figures 96 to 99). We obtain the probability of all of these scenarios in Annex F.

Remark that in the case of Europe the probabilities of both best and worst scenarios are very low, and in Japan these probabilities decrease sharply since the financial crisis (see Figures 61 and 65, respectively). A more extreme case is given in Brazil. In this country, the best and the worst scenarios are so improbable (see Figure 69) that measuring the CoVaR in these scenarios is not relevant. For this reason, the CoVaR of the European, Japanese and Brazilian portfolios conditional on all the assets are under their $VaR(\alpha)$ or above their $VaR(1-\alpha)$ are not consider as a relevant measure of the risk of the mentioned portfolios, and consequently these figures will not be presented.

To conclude, Figures 21 to 24 present the Conditional Diversification Benefits (CDB) measure, as well as the ES of the portfolio, its upper and its lower bounds. Contrary to what we might expect, in general the CDB index is lower in portfolios which include the exchange rate as a third asset. However, we must take into account that portfolios in left and right Panels of these figures are not comparable since their ES are very different. By including this extra-investment in the exchange rate, the ES, its lower bound, and specially its upper bound are reduced. Considering the expression of this risk measure, given by Eq. (3.3.15), we notice that in this situation the distance between the ES ($ES_p(\alpha)$) and its upper bound ($\overline{ES}_p(\alpha)$) is reduced, so the CDB will be lower. Nevertheless this does not imply that the inclusion of the exchange rate in the portfolio reduces its diversification degree.

Figure 21: Conditional Diversification Benefits of European portfolios (assuming Student t copula model).



[4] The best scenario possible is that in which occurs a great depreciation of the currency and a rise in stock index and gold prices. In the worst scenario there is an appreciation of the local currency, a bearish stock market and a decrease of gold prices.

Figure 22: Conditional Diversification Benefits of Japanese portfolios (assuming Student t copula model).

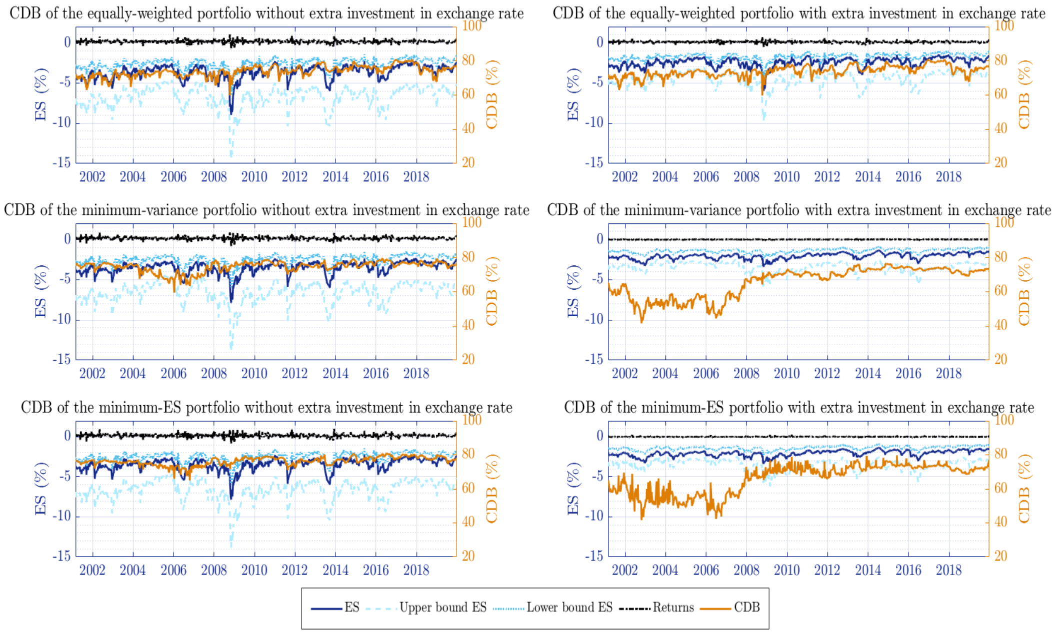


Figure 23: Conditional Diversification Benefits of Brazilian portfolios (assuming Student t copula model).

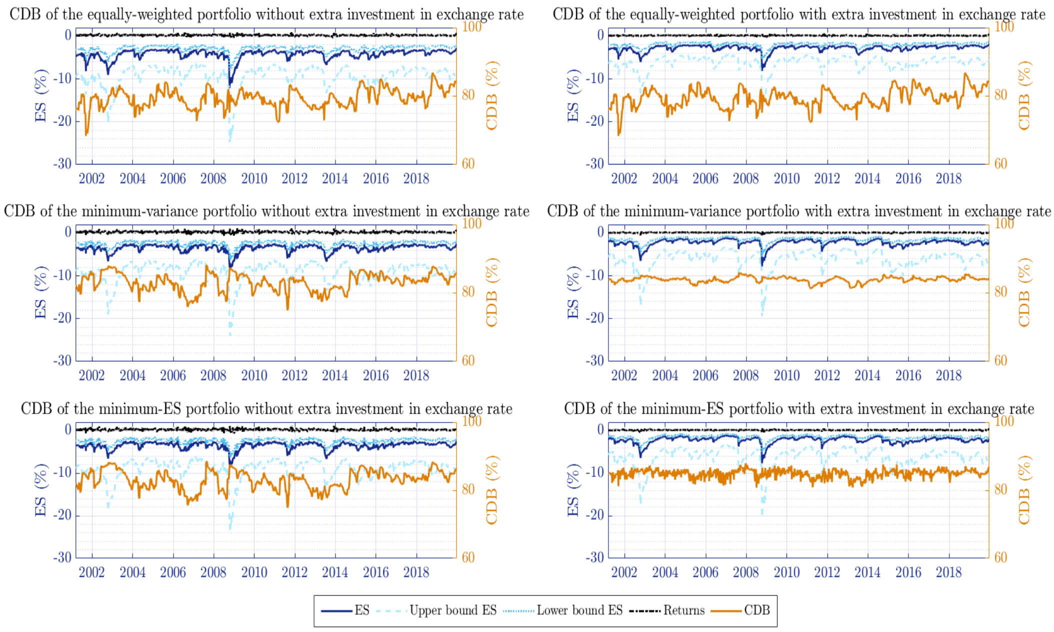
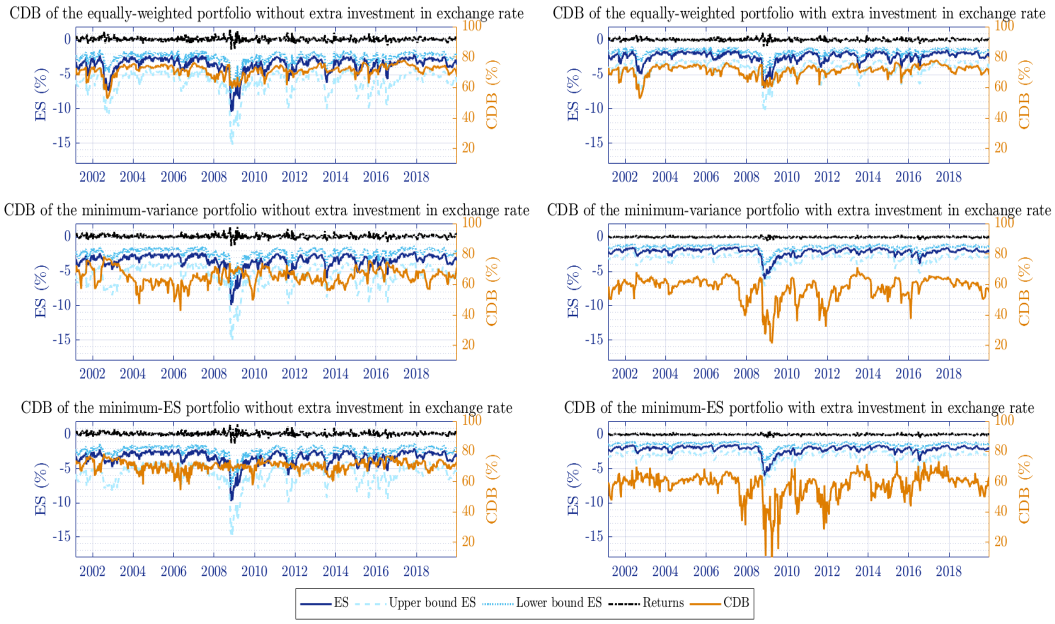


Figure 24: Conditional Diversification Benefits of British portfolios (assuming Student t copula model).



5.4 Out-of-sample portfolio performance

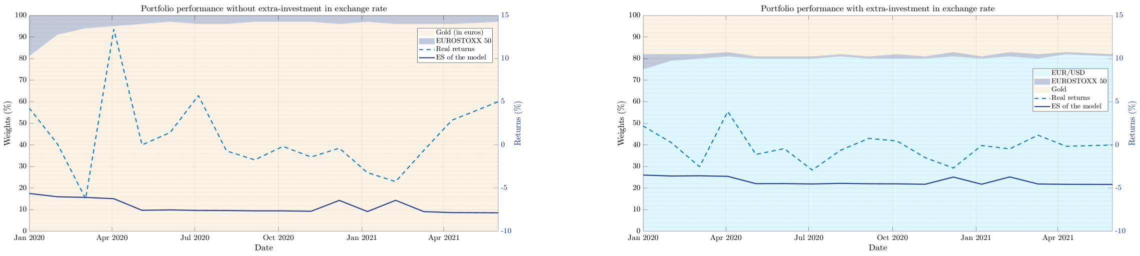
We study the use of multivariate probability models incorporating asymmetries and tail dependence within the marginal distributions and symmetric tail dependence in the dependence structure in portfolio returns forecast. This analysis is performed using a sample from 1 January to 1 June 2021.

After simulating logaritmic returns which follow an ARMA(p,q)-APARCH(1,1) model with Hansen's Skewed Student t innovations under elliptical copula models and transforming them into arithmetic returns, we built two minimum-ES portfolios. On the one hand, we build a portfolio composed of a stock index and gold denominated in a domestic currency. On the other hand, we build an additional portfolio where we take into account the position in exchange rate. We explore their performance out-of-sample in relation to real returns observed in these months.

Figures 25 to 28 show the return of an initial hypothetical investment of 100 CU in both minimum-ES strategies implemented in the four regions, assuming a monthly rebalancing of the portfolio, as well as the dynamic composition of the portfolios and the ES given by the model, when the portfolio is built under a Student t copula approach.

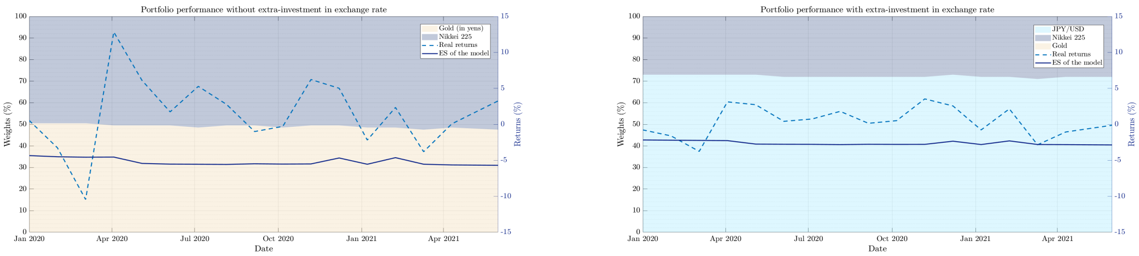
In line with the in-sample analysis, we see that an extra-investment in exchange rate reduces the tail risk, and this asset becomes the leading position in all portfolios, outshining gold position in Japanese and British cases. The figures also show the portfolio returns, obtained with the real returns of the three assets in the out-of-sample period. We notice that considering the exchange rate as a third asset to build the portfolio the losses at the begining of the Coronavirus crisis are lower, although in the following months the potential profit of this portfolio is also lower than that of portfolios which not include it.

Figure 25: Out-of-sample performance of minimum-ES portfolio in Europe (assuming Student t copula model).



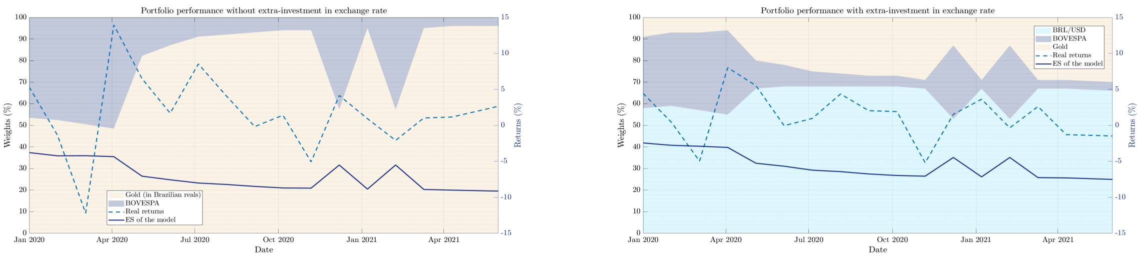
Note: left (right) Panel shows the portfolios without (with) extra investment in the exchange rate.

Figure 26: Out-of-sample performance of minimum-ES portfolio in Japan (assuming Student t copula model).



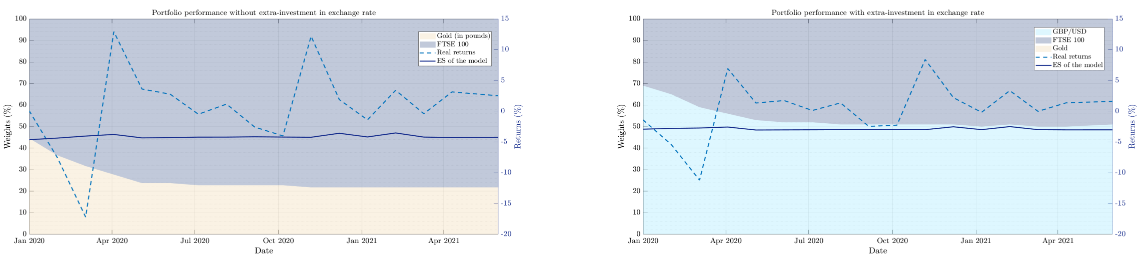
Note: left (right) Panel shows the portfolios without (with) extra investment in the exchange rate.

Figure 27: Out-of-sample performance of minimum-ES portfolio in Brazil (assuming Student t copula model).



Note: left (right) Panel shows the portfolios without (with) extra investment in the exchange rate.

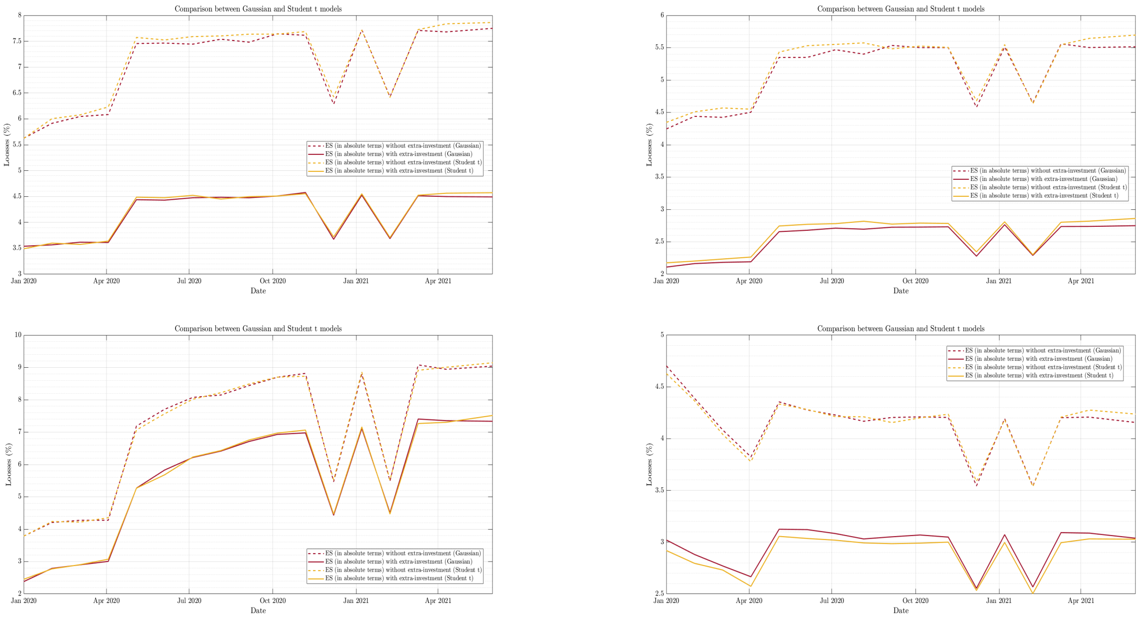
Figure 28: Out-of-sample performance of minimum-ES portfolio in UK (assuming Student t copula model).



Note: left (right) Panel shows the portfolios without (with) extra investment in the exchange rate.

Finally, we compare in Figure 29 the losses of portfolios including and without including the exchange rate as a third asset, built under Gaussian or Student t copula approach. Firstly, we notice that the differences between the ES resulting when using the Student t copula to model the dependence structure and that obtained with a Gaussian copula are of little relevance. Secondly, the reduction of the tail risk when we make an extra-investmen in exchange rate becomes evident in any region.

Figure 29: Comparison between the ES of the portfolio out-of-sample with Gaussian and Student t copula model.



Notes: the figure present the ES of portfolios without and with extra investment in the corresponding exchange rate for Europe (top left Panel), Japan (top right Panel), Brazil (bottom left Panel) and UK (bottom right Panel). top left Panel shows the ES of European portfolios.

6 Conclusions

Gold positions in equity portfolios constitute an effective hedging strategy against downward movements in stock markets. The behaviour of gold in stock portfolios has been widely studied in the literature, although, to our knowledge, none study has so far analysed how the exchange rate movements could affect the hedging strategy of non-USD investors. In addition, the possibility of including the exchange rate like a third asset in the portfolio to improve the hedging strategy has hitherto been overlooked.

Using a twenty-year sample of weekly data of float exchange rate between US dollar, stock and gold prices, we analyse the role of exchange rate in the construction of stock portfolios for the cases of Europe, Japan, Brazil and UK. On the one hand, the marginal features of the series, i.e. autocorrelation and heterocedasticity, are captured by an ARMA-APARCH model, assuming Skewed Student t errors to take into account higher moments. On the other hand, the time-varying dependence structure and tail dependence are reflected by elliptical copula models, using a DCC-GARCH approach. We evaluate the impact of the exchange rate in the portfolio performance built under different construction criteria, namely equally-weighted, minimum-variance and minimum-Expected Shortfall. The use of risk measures allows us to analyse and compare the characteristic of the different portfolios.

In light of our results, using the exchange rate as an additional asset in international stock portfolios enhances the hedging strategies both in terms of volatility and tail losses. On the one hand, according to our findings, the exchange rate plays a leading role in all portfolios to hedge stock losses and even for the Japanese and British cases outshines the gold position. On the other hand, when we compute the cost in terms of risk of not-taking into account the exchange rate as an investment asset in the portfolios, we find that bearish (bullish) CoVaR values are systematically greater in absolute terms that bearish (bullish) VaR, i.e. extreme downwards (upwards) movements in the price of one or more assets have an impact on the downside (upside) risk of the portfolios. In fact, an extra-investment in this asset can reduce the tail risk in a significant quantity that goes from 91.64 bp in Europe to 156.87 bp in Japan.

With an out-of-sample exercise we corroborate that, from the point of view of a risk-averse investor, including the exchange rate in his portfolio reduces the potential losses in turmoil periods. Furthermore, it is also verified the goodness of the model to predict the tail of the joint distribution.

This study has straightforward implications for international investors in terms of portfolio optimization and risk management decisions for non-US investors, although the relevance of our findings goes beyond investment strategies. It also provides important information for the policy makers to understand the interdependence between economic and financial variables. Likewise, regulatory and supervisory authorities might find in this study a powerful stress test approach to evaluate the performance of domestic stock markets under distress scenarios for the exchange rate.

The use of elliptical copula models may have certain shortcomings when the upper and lower tails of the marginals behave differently. Other authors, like Aas et al. (2009), Yew-Low et al. (2013) or Ojea-Ferreiro (2020), opt instead for a vine copula approach, which brings greater flexibility to the analysis. Thus, the use of vine copulas methodology offers plenty of scope for further research and could be helpful to analyse more complex dependence structures where asymmetric tail dependence might appear.

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A Set of copula and conditional copula models

In this Annex we present the principal features of the copula models used in this study. The expression of their distribution and density functions, as well as their tail dependence coefficients and the relation between their dependence parameter and the rank correlation coefficients, i.e., Spearman's rho and Kendall's tau are summarized in Table 6[5].

Gaussian copula

The distribution function of the Gaussian copula is:

$$C(u_1, u_2; \rho) = \Phi_2\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right)$$

where Φ_2 is the cumulative distribution function (cdf) of a bivariate Gaussian distribution, and Φ^{-1} is the inverse cdf of a (univarite) Gaussian distribution.
As elliptical copulas have not a close form, they can be expressed as an integral:

$$C(u_1, u_2; \rho) = \int_0^{\Phi^{-1}(u_1)=z_1} \int_0^{\Phi^{-1}(u_2)=z_2} \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot exp\left(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2 \cdot (1-\rho^2)}\right) dx_1\ dx_2 \tag{A.1}$$

where $x_j = \Phi^{-1}(u_j)$ for $j = 1, 2$ and Φ^{-1} is the inverse cumulative Gaussian distribution.
Derivating copula distribution function with respect to all marginal distributions we obtain the copula density:

$$c(u_1, u_2) = \frac{\partial^2 F(x_1, x_2)}{\partial X_1 \partial X_2} = \frac{\partial^2 C\left(F_1(x_1), F_2(x_2)\right)}{\partial F_1(x_1) \partial F_2(x_2)} \cdot \frac{\partial F_1(x_1)}{\partial X_1} \cdot \frac{\partial F_2(x_2)}{\partial X_2} \tag{A.2}$$

The density of the Gaussian copula is defined as:

$$c(u_1, u_2; \rho) = \frac{1}{\sqrt{1-\rho^2}} \cdot exp\left\{-\frac{\rho^2 x_1^2 - 2\rho x_1 x_2 + \rho^2 x_2^2}{2(1-\rho^2)}\right\} \tag{A.3}$$

This copula model does not allow tail dependence.

Student copula

Student t copula has the following distribution function:

$$C(u_1, u_2; \rho, \nu) = T_\nu\left(T_{\nu_1}^{-1}(u_1), T_{\nu_2}^{-1}(u_2)\right)$$

$$C(u_1, u_2; \rho, \nu) = \int_0^{t_{\nu_1}^{-1}(u_1)=z_1} \int_0^{t_{\nu_2}^{-1}(u_2)=z_2} \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot exp\left(1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{\nu \cdot (1-\rho^2)}\right)^{-\frac{\nu+2}{2}} dx_1\ dx_2 \tag{A.4}$$

where $x_j = T^{-1}(u_j)$ for $j = 1, 2$ and T_ν^{-1} is the inverse cumulative Student t distribution.
The density function, in turn, is:

$$c(u_1, u_2; \rho, \nu) = K \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \left[1 + \frac{\rho^2 x_1^2 - 2\rho x_1 x_2 + \rho^2 x_2^2}{\nu(1-\rho^2)}\right]^{-\frac{\nu+2}{2}} \cdot \left[(1+\nu^{-1}x_1^2)(1+\nu^{-1}x_2^2)\right]^{\frac{\nu+1}{2}} \tag{A.5}$$

with $K = \Gamma\left(\frac{\nu}{2}\right) \cdot \Gamma\left(\frac{\nu+1}{2}\right)^{-2} \cdot \Gamma\left(\frac{\nu+2}{2}\right)$
This copula does allow tail dependence, although it must be symmetric.

[5] From here we will present equations for a bivariate copula model, but they can be easily extrapolated to the multivariate case.

Conditional copulas

In probability theory and statistics, given two jointly distributed random variables X and Y, the conditional probability distribution of Y given X is the probability distribution of Y when X is known to be a particular value, x. This concept can be applied to copula theory since any copula can be obtained from the joint distribution of the d variables and their marginal distributions (inverse Sklar's Theorem).

By Sklar's Theorem for continuous conditional distributions, there exists a unique conditional copula such as

$$H(x, y|w) = C(F(x|w), G(y|w)|w) \tag{A.6}$$

where $x = F^{-1}(u_1)$, $y = F^{-1}(u_2)$, and F and G are the conditional distribution of $X|W$ and $Y|W$, respectively (with W scalar or vector). Remark that the conditioning variable(s), W, must be the same for both marginal distributions and the copula (Patton, 2006).

The conditional copula comes from the derivation of the copula function from one input variable, e.g.

$$C_{2|1}(u_2|u_1) = \frac{\partial C(u_1, u_2)}{\partial u_1} \tag{A.7}$$

where $C_{2|1}(u_2|u_1)$ indicates the distribution of u_2 given the realization of u_1.

Table 6: Principal features of the elliptical copulas.

	Gaussian	**Student t**
Distribution function	$C(u, v; \rho) = \Phi_2\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right)$	$C(u, v; \rho, \nu) = T_\nu\left(T_\nu^{-1}(u_1), T_\nu^{-1}(u_2)\right)$
Density function	$c(u, v; \rho) = \frac{1}{\sqrt{1-\rho^2}} \cdot exp\left\{-\frac{\rho^2\xi_1^2 - 2\rho\xi_1\xi_2 + \rho^2\xi_2^2}{2(1-\rho^2)}\right\}$	$c(u, v; \rho, \nu) = K \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \left[1 + \frac{\rho^2\xi_1^2 - 2\rho\xi_1\xi_2 + \rho^2\xi_2^2}{\nu(1-\rho^2)}\right]^{-\frac{\nu+2}{2}} \cdot$ $\cdot \left[(1+\nu^{-1}\xi_1^2)(1+\nu^{-1}\xi_2^2)\right]^{\frac{\nu+1}{2}}$ $K = \Gamma\left(\frac{\nu}{2}\right) \cdot \Gamma\left(\frac{\nu+1}{2}\right)^{-2} \cdot \Gamma\left(\frac{\nu+2}{2}\right)$
Conditional copula	$C_{2\|1}(u_2\|u_1; \rho) = \Phi\left(\frac{\Phi^{-1}(u_2) - \rho \cdot \Phi^{-1}(u_1)}{\sqrt{1-\rho^2}}\right)$	$C_{2\|1}(u_2\|u_1; \rho, \nu) = T_{\nu+1}\left(\sqrt{\frac{\nu+1}{\nu + \left[T_\nu^{-1}(u_1)\right]^2}} \cdot \frac{T_\nu^{-1}(u_2) - \rho \cdot T_\nu^{-1}(u_1)}{\sqrt{1-\rho^2}}\right)$
Tail dependence	$\lambda_U = \lambda_L = 0$	$\lambda_U = \lambda_L = 2 \cdot t_{\nu+1}\left(-\frac{\sqrt{\nu+1} \cdot \sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$
Rank correlation	$\rho = sin\left(\frac{\pi}{2} \cdot \tau\right)$ $\rho = 2 \cdot sin\left(\frac{\pi}{6} \cdot \rho_s\right)$	$\rho = sin\left(\frac{\pi}{2} \cdot \tau\right)$ $\rho = 2 \cdot sin\left(\frac{\pi}{6} \cdot \rho_s\right)$

Notes: All distribution, density and conditional copula functions are expressed for the particular case of bivariate copula models, but they can be easily extrapolated to the multivariate case. *Tail dependence* refers to the two indicators of the tail dependence of the copulas, namely upper-tail (λ_U) and lower-tail (λ_L) dependence indicators. Both are defined as the value of the copula in the limit of the value of the marginals, i.e., the limit when the uniform variables tend to 1 or to 0, respectively. *Rank correlation* presents the relationship between the Spearman's (ρ_s) or Kendall's (τ) correlation coefficients and the dependence parameter of each copula (ρ).

B Conditional elliptical copula expressions

It is easy to introduce dependence into a set of independent variables applying Cholesky matrix:

$$X = L \cdot Z$$

Where Z are independent Gaussian variables, i.e., $Z \overset{iid}{\sim} N(0,1)$, X is a set of dependent variables and L is the Cholesky matrix.

Cholesky matrix is a triangular matrix such as a matrix R can be decomposed as $R = LL'$. In our case, R refers to the correlation matrix, so, for the bivariate case, the Cholesky decomposition is:

$$\begin{pmatrix} L_{11} & 0 \\ L_{12} & L_{22} \end{pmatrix} \cdot \begin{pmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{pmatrix} = \begin{pmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{pmatrix}$$

$$\Rightarrow L_{11} = 1; \quad L_{12} = \rho_{12}; \quad L_{22} = \sqrt{1-\rho_{12}^2}$$

So

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

$$X_1 = Z_1$$

$$X_2 = \rho_{12} Z_1 + \sqrt{1-\rho_{12}^2} Z_2 \Rightarrow Z_2 = \frac{X_2 - \rho_{12} X_1}{\sqrt{1-\rho_{12}^2}}$$

In terms of copulas we will have:

$$C_{2|1}\left(u_2|u_1;\mathbf{R}\right) = \Phi\left(\frac{\Phi^{-1}\left(u_2\right) - \rho_{12}\Phi^{-1}\left(u_1\right)}{\sqrt{1-\rho_{12}^2}}\right) \tag{B.1}$$

If now we consider the trivariate case, the Cholesky decomposition follows the same procedure:

$$\begin{pmatrix} L_{11} & 0 & 0 \\ L_{12} & L_{22} & 0 \\ L_{13} & L_{23} & L_{33} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ 0 & L_{22} & L_{23} \\ 0 & 0 & L_{33} \end{pmatrix} = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}$$

$$L_{11} = 1; \quad L_{12} = \rho_{12}; \quad L_{22} = \sqrt{1-\rho_{12}^2};$$

$$L_{13} = \rho_{13}; \quad L_{23} = \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}; \quad L_{33} = \sqrt{1-\rho_{13}^2 - \frac{\left(\rho_{23}-\rho_{12}\rho_{13}\right)^2}{1-\rho_{12}^2}}$$

Applying this dependence matrix, the dependent variables resulting have the following expression:

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & 0 \\ \rho_{13} & \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} & \sqrt{1-\rho_{13}^2 - \frac{\left(\rho_{23}-\rho_{12}\rho_{13}\right)^2}{1-\rho_{12}^2}} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

$$\Rightarrow Z_3 = \frac{X_3 - \rho_{13}X_1 - \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} X_2}{\sqrt{1-\rho_{13}^2 - \frac{\left(\rho_{23}-\rho_{12}\rho_{13}\right)^2}{1-\rho_{12}^2}}}$$

The trivariate conditional copula resulting is:

$$C_{3|1,2}\left(u_3|u_1,u_2;\mathbf{R}\right) = \Phi\left(\frac{\Phi^{-1}\left(u_3\right) - \rho_{13}\Phi^{-1}\left(u_1\right) - \frac{\rho_{23}-\rho_{13}\rho_{12}}{\sqrt{1-\rho_{23}^2}}\Phi^{-1}\left(u_2\right)}{\sqrt{1-\rho_{13}^2 - \frac{\left(\rho_{23}-\rho_{13}\rho_{12}\right)^2}{1-\rho_{12}^2}}}\right) \tag{B.2}$$

We repeat these steps for the Student t copula. The problem is that we need the Student t conditional distribution function expression to compute the copula. Demarta and McNeil (2005) and Chan and Li (2008) notice that a multivariate t-distribution $t(\mu, \Sigma, \nu)$ can be expressed as the distribution of the following multivariate Normal mixture:

$$(X_1, ..., X_n) = (\mu_1, ..., \mu_n) + \sqrt{S}(Z_1, ..., Z_n)$$

where $(\mu_1, ..., \mu_n)$ is a vector of component means and the vector $(Z_1, ..., Z_n)$ has a joint Normal distribution $N(0, \Sigma)$. The scale variable $\sqrt{S}$ is independent of the vector of Gaussian variables $(Z_1, ..., Z_n)$ and follows an inverse Gamma distribution $IG(\frac{\nu}{2}, \frac{\nu}{2})$.
Ding (2016) concludes the conditional distribution of a random variable X_2 given X_1 is:

$$X_2|X_1 \sim t_{p_2}\left(\mu_{2|1}, \frac{\nu + d_1}{\nu + p_1}\Sigma_{22|1}, \nu + p_1\right) \Rightarrow \sqrt{\frac{\nu + p_1}{\nu + d_1}}X_2|X_1 \sim t_{p_2}\left(\mu_{2|1}, \Sigma_{22|1}, \nu + p_1\right)$$

where p_1 and p_2 are the dimensionality of X_1 and X_2, respectively, $\nu + p_1$ represents the degrees of freedom of the conditional distribution function, and $d_1 = (X_1 - \mu_1)'\Sigma_{11}^{-1}(X_1 - \mu_1)$ is the squared Mahalanobis distance of X_1 from μ_1 with scale matrix Σ_{11}. Moreover, we have that $\mu_{2|1} = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(X_1 - \mu_1)$ is the linear regression of X_2 on X_1 and $\Sigma_{22|1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$. This conditional distribution differs from the multivariate Normal distribution only by the scaling factor $\frac{\nu+d_1}{\nu+p_1}$, which inflates the conditional scale matrix, $\Sigma_{22|1}$.
In the bivarate case we have that X_1 and X_2 are scalars, so $p_1 = 1$ and $d_1 = X_1^2$ assuming $\mu_1 = 0$ and $\Sigma_{11} = 1$.
Torrent-Gironella and Fortiana (2011) compute an algorithm for the Student t copula simulation given a correlation matrix $\mathbf{R}$. In this procedure they obtain the dependent Gaussian variable $Y = LZ$ applying Cholesky decomposition (where $Z \overset{iid}{\sim} N(0, 1)$ and L is the Cholesky matrix). Finally they obtain dependent variables following a Student t distribution adding a scaling factor.
With all this information, the Cholesky decomposition in the case of a bivariate conditional Student t distribution results in

$$Z_2 = \sqrt{\frac{\nu + 1}{\nu + X_1^2}} \cdot \frac{X_2 - \rho_{12}X_1}{\sqrt{1 - \rho_{12}^2}}$$

Thus, the conditional Student t bivariate copula can be expressed as:

$$C_{2|1}\left(u_2|u_1; \nu, \mathbf{R}\right) = T_{\nu+1}\left(\sqrt{\frac{\nu + 1}{\nu + T_{\nu+1}^{-1}(u_1)^2}} \cdot \frac{T_{\nu+1}^{-1}\left(u_2\right) - \rho_{12}T_{\nu+1}^{-1}\left(u_1\right)}{\sqrt{1 - \rho_{12}^2}}\right) \tag{B.3}$$

In the case in which we have two conditioning variables, X_1^* is a vector with these variables. Then $p_{1^*} = 2$ and

$$d_{1^*} = \begin{pmatrix} X_1 - \mu_1 & X_2 - \mu_2 \end{pmatrix} \Sigma_{11}^{-1} \begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = X_1^2 + X_2^2$$

$$\Rightarrow Z_3 = \sqrt{\frac{\nu + 2}{\nu + X_1^2 + X_2^2}} \cdot \frac{X_3 - \rho_{13}X_1 - \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1 - \rho_{12}^2}}X_2}{\sqrt{1 - \rho_{13}^2 - \frac{(\rho_{23} - \rho_{12}\rho_{13})^2}{1 - \rho_{12}^2}}}$$

In consequence the trivariate conditional Student t copula has the following expression:

$$C_{3|1,2}\left(u_3|u_1, u_2; \nu, \mathbf{R}\right) = T_{\nu+2}\left(\sqrt{\frac{\nu + 2}{\nu + T_{\nu+2}(u_1)^2 + T_{\nu+2}(u_2)^2}} \cdot \frac{T_{\nu+2}^{-1}\left(u_3\right) - \rho_{13}T_{\nu+2}^{-1}\left(u_1\right) - \frac{\rho_{23} - \rho_{13}\rho_{12}}{\sqrt{1 - \rho_{23}^2}}T_{\nu+2}^{-1}\left(u_2\right)}{\sqrt{1 - \rho_{13}^2 - \frac{(\rho_{23} - \rho_{13}\rho_{12})^2}{1 - \rho_{12}^2}}}\right) \tag{B.4}$$

C Risk measures for conditional trivariate copula

Case 1

In this section we calculate the risk measures of international portfolio described in Eq. (3.3.1) using the expression for its cdf (given by Eq. (3.3.2)).

The **Value at Risk (VaR)** is used frequently to quantify the market risk of an asset or a portfolio (Hotta et al., 2008). It measures the maximum loss expected in a certain horizon with a confidence level of $\alpha\%$. In other words, this means that we are $(1-\alpha)\%$ confident that the portfolio loss in a certain period will not be larger than the VaR threshole. Mathematically, this is equivalent to:

$$P\left(r_p < VaR_p\left(\alpha\right)\right) = \alpha \tag{C.1}$$

where r_p denotes the return of the portfolio.

VaR is also defined as the α-quantile of a distribution: $VaR_p(\alpha) = F_p^{-1}(\alpha)$

The **Conditional VaR (CoVaR)** represents the β-quantile of the returns of a portfolio conditional on the return of its component i is under its VaR, i.e., $r_i \leq VaR_i(\alpha)$ (for $i = e, s, g$):

$$P\left(r_p < CoVaR_p(\beta) | r_i < VaR_i\left(\alpha\right)\right) = \beta \tag{C.2}$$

It is exacty the definition of tail dependence that give Aas et al. (2009) and Aas and Berg (2011): "tail dependence in a bivariate distribution can be represented by the probability that the first variable exceeds its q-quantile, given that the other exceeds its own q-quantile".

Assume that the exchange rate is in its α-lower quantile. Another definition of CoVaR (Karimalis and Nomikos, 2018) is the following:

$$CoVaR_{p|e}(\beta) = P\left(r_p < F_p^{-1}(\beta) | r_e < VaR_e\left(\alpha\right)\right) \tag{C.3}$$

It is possible to calculate the CoVaR of the portfolio conditional on exchange rate, stock or gold returns are lower than its VaR.

By Bayes Theorem, a conditional probability can be expresed as the ratio of the joint probability of seeing both scenarios to the probability of observing the conditioning scenario:

$$P\left(r_p < CoVaR_p(\beta) | r_e < VaR_e\left(\alpha\right)\right) = \frac{P\left(r_p < CoVaR_p(\beta), r_e < VaR_e\left(\alpha\right)\right)}{P\left(r_e < VaR_e\left(\alpha\right)\right)} = \beta$$

where $P\left(r_e < VaR_e\left(\alpha\right)\right) = \alpha$ and $P\left(r_p < CoVaR_p(\beta), r_e < VaR_e\left(\alpha\right)\right)$ can be understood in terms of distribution function of the portfolio, i.e.,

$$P\left(r_p < CoVaR_p(\beta), r_e < VaR_e\ \left(\alpha\right)\right) = F_p(r_p < CoVaR_p(\beta), r_e < VaR_e\left(\alpha\right))$$

Moreover, in Section 3.3 we have seen that the cdf of a portfolio can be written in terms of copulas, so

$$P\left(r_p < CoVaR_p(\beta), r_e < VaR_e\left(\alpha\right)\right) = C(F_p(CoVaR_p(\beta)), \alpha) = C(u_p, u_e)$$

Now, the problem is that $u_p = F_p(CoVaR_p(\beta))$ is unknowed. To solve it we can use conditional copula definition, by which $C(u_p, u_e) = C_{g|e,s}(u_g|u_e, u_s)$

Remind that the portfolio return, given by Eq. (3.3.1), is $r_p = (1-\omega)r_s + \omega(r_e + r_g)$, so $r_p < CoVaR_p(\beta)$ is equivalent to

$$r_g < \frac{CoVaR_p(\beta) - (1-\omega)\, r_s - \omega r_e}{\omega} \Rightarrow u_g < F_g\left(\frac{CoVaR_p(\beta) - (1-\omega)\, F_s^{-1}\left(u_s\right) - \omega F_e^{-1}\left(u_e\right)}{\omega}\right)$$

Then

$$\begin{aligned} &P\left(r_p < CoVaR_p(\beta) | r_e < VaR_e\left(\alpha\right)\right) = \\ &= \frac{1}{\alpha}\int_0^{\alpha}\int_0^1 C_{g|e,s}\left(F_g\left(\frac{CoVaR_p(\beta) - (1-\omega)\, F_s^{-1}\left(u_s\right) - \omega F_e^{-1}\left(u_e\right)}{\omega}\right)\middle| u_e, u_s\right) du_s du_e = \beta \end{aligned} \tag{C.4}$$

In the same line, we can compute the CoVaR of the portfolio conditional on the other components of the portfolio, namely, stock and gold, are under their corresponding VaR threshole:

$$P\left(r_p < CoVaR_p(\beta)|r_s < VaR_s\left(\alpha\right)\right) = \frac{P\left(r_p < CoVaR_p(\beta), r_s < VaR_s\left(\alpha\right)\right)}{P\left(r_s < VaR_s\left(\alpha\right)\right)} = \\ = \frac{1}{\alpha}\int_0^1\int_0^\alpha C_{g|e,s}\left(F_g\left(\frac{CoVaR_p(\beta)-(1-\omega)F_s^{-1}\left(u_s\right)-\omega F_e^{-1}\left(u_e\right)}{\omega}\right)\middle|u_e,u_s\right)du_s du_e = \beta \tag{C.5}$$

$$P\left(r_p < CoVaR_p(\beta)|r_g < VaR_g\left(\alpha\right)\right) = \frac{P\left(r_p < CoVaR_p(\beta), r_g < VaR_g\left(\alpha\right)\right)}{P\left(r_g < VaR_g\left(\alpha\right)\right)} = \\ = \frac{1}{\alpha}\int_0^1\int_0^\alpha C_{s|e,g}\left(F_s\left(\frac{CoVaR_p(\beta)-\omega\left[F_g^{-1}\left(u_g\right)+F_e^{-1}\left(u_e\right)\right]}{(1-\omega)}\right)\middle|u_e,u_g\right)du_g du_e = \beta \tag{C.6}$$

It could be interesting to consider the probability that portfolio returns being under its CoVaR at β-level while the component i is in its α-upper quantile. This is equivalent to $P\left(r_p < CoVaR_p(\beta)|r_i > VaR_i\left(\alpha\right)\right)$
Assuming the asset i is the exchange rate, we have:

$$P\left(r_p < CoVaR_p(\beta)|r_e > VaR_e\left(1-\alpha\right)\right) = \frac{P\left(r_p < CoVaR_p(\beta), r_e > VaR_e\left(1-\alpha\right)\right)}{P\left(r_e > VaR_e\left(1-\alpha\right)\right)} = \beta \tag{C.7}$$

where $P\left(r_e > VaR_e\left(1-\alpha\right)\right) = \alpha$
Thus, the bullish CoVaR of the portfolio is obtained from:

$$P\left(r_p < CoVaR_p(\beta)|r_e > VaR_e\left(1-\alpha\right)\right) = \\ = \frac{1}{\alpha}\int_0^1\int_{1-\alpha}^1 C_{g|e,s}\left(F_g\left(\frac{CoVaR_p(\beta)-(1-\omega)F_s^{-1}\left(u_s\right)-\omega F_e^{-1}\left(u_e\right)}{\omega}\right)\middle|u_e,u_s\right)du_e du_s = \beta \tag{C.8}$$

When the other assets are in their corresponding α-higher quantile, this measure can be computed as:

$$P\left(r_p < CoVaR_p(\beta)|r_s > VaR_s\left(1-\alpha\right)\right) = \\ = \frac{1}{\alpha}\cdot\int_0^1\int_{1-\alpha}^1 C_{g|s,e}\left(F_g\left(\frac{CoVaR_p(\beta)-(1-\omega)F_s^{-1}\left(u_s\right)-\omega F_e^{-1}\left(u_e\right)}{\omega}\right)\middle|u_s,u_e\right)du_s du_e = \beta \tag{C.9}$$

$$P\left(r_p < CoVaR_p(\beta)|r_g > VaR_g\left(1-\alpha\right)\right) = \\ = \frac{1}{\alpha}\cdot\int_0^1\int_{1-\alpha}^1 C_{s|g,e}\left(F_s\left(\frac{CoVaR_p(\beta)-\omega\left(F_g^{-1}\left(u_g\right)+F_e^{-1}\left(u_e\right)\right)}{1-\omega}\right)\middle|u_g,u_e\right)du_g du_e = \beta \tag{C.10}$$

When building a portfolio, portfolio managers must to take into account the aggregated behaviour of part or even all of its components. In other words, it is important to consider the probability of the return of the portfolio is under its CoVaR conditional on more than one of its components are under its VaR level. This probability is:

$$P\left(r_p < CoVaR_p(\gamma)|r_i < VaR_i\left(\alpha\right),\ r_j < VaR_j(\beta)\right) = \gamma \tag{C.11}$$

Assume stock and gold prices suffer a sharp drop. We can obtain this measure as follows:

$$P\left(r_p < CoVaR_p(\gamma)|r_g < VaR_g\left(\alpha\right), r_s < VaR_s\left(\beta\right)\right) = \frac{P\left(r_p < CoVaR_p(\gamma), r_g < VaR_g\left(\alpha\right), r_s < VaR_s\left(\beta\right)\right)}{P\left(r_g < VaR_g\left(\alpha\right), r_s < VaR_s\left(\beta\right)\right)}$$

In this case, the denominator is not a parameter like in previous cases, but is a joint probability, which can also be captured by a copula:

$$P\left(r_g < VaR_g\left(\alpha\right), r_s < VaR_s\left(\beta\right)\right) = C_{g,s}\left(\alpha,\beta\right)$$

Then the CoVaR of the portfolio is:

$$P\left(r_p < CoVaR_p(\gamma)|r_g < VaR_g\left(\alpha\right), r_s < VaR_s\left(\beta\right)\right) = \\ = \frac{1}{C_{g,s}\left(\alpha,\beta\right)}\int_0^\beta\int_0^\alpha C_{e|s,g}\left(F_e\left(\frac{CoVaR_p(\gamma)-(1-\omega)F_s^{-1}\left(u_s\right)-\omega F_g^{-1}\left(u_g\right)}{\omega}\right)\middle|u_s,u_g\right)du_g du_s = \gamma \tag{C.12}$$

Like before, we can also think in that scenario in which the returns of the portfolio components are in their upper tails. The probability of the portfolio return being under its CoVaR(γ) conditional on the aforementioned situation is:

$$P\left(r_p < CoVaR_p(\gamma)|r_g > VaR_g\left(1-\alpha\right),\ r_s > VaR_s(1-\beta)\right) = \gamma \tag{C.13}$$

Using again Bayes Theorem,

$$P\left(r_p < CoVaR_p(\gamma)|r_g > VaR_g\left(1-\alpha\right),\ r_s > VaR_s(1-\beta)\right) =$$

$$= \frac{P\left(r_p < CoVaR_p(\gamma), r_g > VaR_g\left(1-\alpha\right),\ r_s > VaR_s(1-\beta)\right)}{P\left(r_g > VaR_g\left(1-\alpha\right),\ r_s > VaR_s(1-\beta)\right)}$$

The denominator can be capture by a copula as follows:

$$P\left(r_g > VaR_g\left(1-\alpha\right),\ r_s > VaR_s(1-\beta)\right) = P\left(r_g > VaR_g\left(1-\alpha\right)\right) - P\left(r_g > VaR_g\left(1-\alpha\right),\ r_s < VaR_s(1-\beta)\right) =$$

$$= \alpha - \left[P\left(r_s < VaR_s(1-\beta)\right) - P\left(r_g < VaR_g\left(1-\alpha\right),\ r_s < VaR_s(1-\beta)\right)\right] = \alpha - (1-\beta) + C_{g,s}(1-\alpha, 1-\beta)$$

So the CoVaR in this case is:

$$\begin{gathered} P\left(r_p < CoVaR_p(\gamma)|r_g > VaR_g\left(1-\alpha\right),\ r_s > VaR_s(1-\beta)\right) = \frac{1}{\alpha+\beta-1+C_{g,s}(1-\alpha,1-\beta)} \\ \int_{1-\beta}^{1}\int_{1-\alpha}^{1} C_{e|s,g}\left(F_e\left(\frac{CoVaR_p(\gamma)-(1-\omega)\,F_s^{-1}\left(u_s\right)-\omega F_g^{-1}\left(u_g\right)}{\omega}\right)\middle|\, u_s, u_g\right) du_g du_s = \gamma \end{gathered} \tag{C.14}$$

Adrian and Brunnermeier (2011) propose $\Delta CoVaR$ as a systemic risk measure. **ΔCoVaR** captures the change in CoVaR as one shifts the conditioning event from the median return of the component i to the adverse $VaR_i(\alpha)$. The definition of this systemic risk of the portfolio given by the authors is:

$$\Delta CoVaR_\beta^{p|i} = CoVaR_\beta^{r_p|r_i=VaR_i(\alpha)} - CoVaR_\beta^{r_p|r_i=VaR_i(0.5)}$$

We instead use the alternative definition of ΔCoVaR proposed by Girardi and Ergün (2013), i.e.

$$\Delta CoVaR_\beta^{p|i} = CoVaR_\beta^{r_p|r_i\leq VaR_i(\alpha)} - CoVaR_\beta^{r_p|r_i\leq VaR_i(0.5)} \tag{C.15}$$

Calculating this risk measure of the components of the portfolio, it measures the tail dependency between an asset and the portfolio returns because it quantify how much the VaR of a portfolio changes when returns of the asset i moves from the median to quantile α (Adrian and Brunnermeier, 2011).

VaR and CoVaR are not sub-additive measures. Instead, the Expected Shortfall (ES) and the Conditional Expected Shortfall (CoES), respectively, overcome this limitation, i.e. they are coherent risk measures which complement the information provided by VaR and CoVaR (Ojea-Ferreiro, 2019). The sub-additivity property of the ES (CoES) of a portfolio implies that it is not higher than the ES (CoES) weighted of its components, i.e., $ES_p(\alpha) \leq \sum_{i=1}^{N}\omega_i \cdot ES_i(\alpha)$. The **Expected Shortfall (ES)** at α-level is the expected return on the portfolio in the worst α% of cases, that is, when the portfolio returns is under its $VaR(\alpha)$. Then, the ES can be computed as:

$$ES_p(\alpha) = E\left(r_p|\ r_p \leq VaR_p\left(\alpha\right)\right) = \frac{1}{\alpha}\int_0^\alpha F_p^{-1}\left(q\right)\,dq = \frac{1}{\alpha}\int_0^\alpha VaR_p\left(q\right)\,dq \tag{C.16}$$

With this expression we can understand the ES as a mean of VaRs at certain levels in $[0,\alpha]$. This mean is normalized by the factor $\frac{1}{\alpha}$ because it is a conditional measure.

In the same sense, we can obtain the **Conditional Expected Shortfall (CoES)**. It measures the expected loss when a portfolio obtain a return under its CoVaR at β level:

$$CoES_{p|e}(\alpha,\beta) = E\left(r_p \middle| r_p \leq CoVaR_{p|e}(\alpha,\beta)\right) = \frac{1}{\beta}\int_0^\beta F_{p|e}^{-1}(q)\,dq = \frac{1}{\beta}\int_0^\beta CoVaR_{p|e}(\alpha,q)\,dq \tag{C.17}$$

The conditional inverse cumulative distribution function $F_{p|e}^{-1}(q)$ is such that $F_{p|e}(r_p) = q$, where the cdf of the portfolio conditional on the scenario for the exchange rate, i.e. $F_{p|e}(r_p)$, is:

$$\int_0^1\int_0^\alpha C_{g|e,s}\left(F_g\left(\frac{CoVaR_p(\beta) - (1-\omega)F_s^{-1}(u_s) - \omega F_e^{-1}(u_e)}{\omega}\right)\middle| u_e, u_s\right)du_e du_s$$

The CoES can be interpreted as a mean of CoVaR at level q, $\forall q \in [0,\beta]$.

The **Marginal Expected Shortfall (MES)** represents the contribution of each asset to portfolio risk, i.e., is defined as the marginal impact of a certain asset in the ES of a portfolio, so it is useful as a measure of systemic risk (Cai et al., 2015; Acharya et al., 2017).
Assume the portfolio return is given by $r_p = \sum_N \omega_i r_i$. Then, its ES can be rewriten as:

$$ES_p(\alpha) = E\left(\sum_N \omega_i r_i |\; r_p < VaR_p(\alpha)\right)$$

Assuming that ω_i is determinist (or at least known at time $t-1$),

$$ES_p(\alpha) = \sum_N \omega_i E(r_i|\; r_p < VaR_p(\alpha))$$

The MES of asset i is:

$$MES_i(\alpha) = E(r_i|\; r_p < VaR_p(\alpha)) \tag{C.18}$$

If we multiply the MES of the component i of the portfolio by its weigth, ω_i, we will obtain the contibution of the asset to the ES of the portfolio.
For instance, the MES of the stock component of the portfolio is:

$$MES_s(\alpha) = E(\;r_s|\; r_p < VaR_p(\alpha)) = \int_{-\infty}^{\infty} r_s \cdot f(\;r_s|\; r_p < VaR_p(\alpha))\,dr_s = \int_{-\infty}^{\infty} r_s \cdot \frac{f(r_s, r_p < VaR_p(\alpha))}{P(r_p < VaR_p(\alpha))}dr_s =$$
$$= \frac{1}{\alpha}\int_{-\infty}^{\infty} r_s \cdot f\left(r_s, r_p < r_p^*\right)dr_s$$

where

$$f\left(r_s, r_p < r_p^*\right) = f\left(r_p < r_p^* \middle| r_s\right)\cdot f(r_s)$$

$$\Rightarrow E(\;r_s|\; r_p < VaR_p(\alpha)) = \frac{1}{\alpha}\int_{-\infty}^{\infty} r_s \cdot P\left(r_p < r_p^*|r_s\right)\cdot f(r_s)\,dr_s \tag{C.19}$$

Moreover, $f(r_s)\,dr_s = dF(r_s) = du_s; r_s = F^{-1}(u_s); F(-\infty) = 0; F(\infty) = 1$
So we can rewrite Eq. (C.19) as follows:

$$E(r_s|\; r_p < VaR_p(\alpha)) = \frac{1}{\alpha}\int_0^1 F_s^{-1}(u_s)\;C_{p|s}\left(u_p^*|u_s\right)du_s \tag{C.20}$$

In addition, $P(r_p < VaR_p(\alpha)) = F_p(VaR_p(\alpha))$, where the formula for F_p is provided in Eq. (3.3.2)
Then:

$$E(r_s|\; r_p < VaR_p(\alpha)) = \frac{1}{\alpha}\int_0^1\int_0^1 F_s^{-1}(u_s)\;C_{g|e,s}\left(F_g\left(\frac{VaR_p(\alpha) - (1-\omega)F_s^{-1}(u_s) - \omega F_e^{-1}(u_e)}{\omega}\right)\middle| u_e, u_s\right)du_e du_s \tag{C.21}$$

The MES of exchange rate and of the gold are given, respectively, by the following equations:

$$E\left(r_e|\ r_p < VaR_p(\alpha)\right) = \frac{1}{\alpha}\int_0^1\int_0^1 F_e^{-1}\left(u_e\right)\ C_{g|e,s}\left(F_g\left(\frac{VaR_p\left(\alpha\right) - \left(1-\omega\right)F_s^{-1}\left(u_s\right) - \omega F_e^{-1}\left(u_e\right)}{\omega}\right)\Bigg|u_e, u_s\right)du_e du_s \tag{C.22}$$

$$E\left(r_g|\ r_p < VaR_p(\alpha)\right) = \frac{1}{\alpha}\int_0^1\int_0^1 F_g^{-1}\left(u_g\right)\ C_{s|e,g}\left(F_s\left(\frac{VaR_p\left(\alpha\right) - \omega F_g^{-1}\left(u_g\right) - \omega F_e^{-1}\left(u_e\right)}{\left(1-\omega\right)}\right)\Bigg|u_e, u_g\right)du_e du_g \tag{C.23}$$

The **conditional variance** of the component i of the portfolio is defined as

$$\sigma^2_{i|r_p<VaR_p(\alpha)} = E(r_i^2|r_p < VaR_p(\alpha)) - E\left(r_i|r_p < VaR_p(\alpha)\right)^2 \tag{C.24}$$

For instance, the elements to compute the conditional variance of the exchange rate are:

$$E\left(r_e^2|\ r_p < VaR_p(\alpha)\right) = \int_{-\infty}^{\infty} r_e^2 \cdot f\left(\ r_e|\ r_p < VaR_p(\alpha)\right)dr_e =$$

$$= \frac{1}{\alpha}\int_0^1\int_0^1 F_e^{-1}\left(u_e\right)^2 C_{g|e,s}\left(F_g\left(\frac{VaR_p\left(\alpha\right) - \left(1-\omega\right)F_s^{-1}\left(u_s\right) - \omega F_e^{-1}\left(u_e\right)}{\omega}\right)\Bigg|u_e, u_s\right)du_e du_s$$

$$E\left(\ r_e|\ r_p < VaR_p(\alpha)\right)^2 =$$

$$= \left(\frac{1}{\alpha}\int_0^1\int_0^1 F_e^{-1}\left(u_e\right)\ C_{g|e,s}\left(F_g\left(\frac{VaR_p\left(\alpha\right) - \left(1-\omega\right)F_s^{-1}\left(u_s\right) - \omega F_e^{-1}\left(u_e\right)}{\omega}\right)\Bigg|u_e, u_s\right)du_e du_s\right)^2$$

If the level of dependence is changing over time, then the benefits of portfolio diversification are likely changing as well. Christoffersen et al. (2012) develop a dynamic measure, the **Conditional Diversification Benefits (CDB)**, which has into account the sub-additivity property of ES to obtain its lower and upper bounds. Due to this property, the upper bound of the ES of a portfolio is $\overline{ES}_p(\alpha) = \sum_{i=1}^N \omega_i \cdot ES_i(\alpha)$, which represents the particular case of no diversification. Moreover, in the case where the portfolio never losses more than its α-quantile the lower bound of its ES is $\underline{ES}_p(\alpha) = -F_p^{-1}(\alpha) = VaR_p(\alpha)$
With this elements, the CDB is defined as:

$$CDB_p(\alpha) = \frac{\overline{ES}_p(\alpha) - ES_p(\alpha)}{\overline{ES}_p(\alpha) - \underline{ES}_p(\alpha)} \tag{C.25}$$

where $ES_p(\alpha)$ denotes the ES of the portfolio at hand.

Case 2

It is also possible to calculate the risk measures of portfolio described in Eq. (3.3.3) using the expression of its cdf (Eq. (3.3.4)). This cdf is represented by $C_{g|e,s}(u_g|u_e, u_s)$, or well by $C_{g|s,e}(u_g|u_s, u_e)$ or $C_{e|g,s}(u_e|u_g, u_s)$, i.e. the order of the variables does not matter when we use elliptical conditional copulas.
The former gives us this definition of the cdf of the portfolio:

$$r_g = \frac{r_p - (1-\omega_1-\omega_2)\cdot r_s - (\omega_1+\omega_2)\cdot r_e}{\omega_1}$$

$$\Rightarrow F_p(r_p) = \int_0^1\int_0^1 C_{g|e,s}\left(F_g\left(\frac{r_p - (1-\omega_1-\omega_2)\cdot F_s^{-1}(u_s) - (\omega_1+\omega_2)\cdot F_e^{-1}(u_e)}{\omega_1}\right)\Bigg|u_e u_s\right)du_e du_s$$

This cdf can be also expressed using gold as conditioning and exchange rate or stock as conditional variable:

$$r_e = \frac{r_p - (1-\omega_1-\omega_2)\cdot r_s - \omega_1 \cdot r_g}{\omega_1+\omega_2}$$

$$\Rightarrow F_p(r_p) = \int_0^1 \int_0^1 C_{e|g,s}\left(F_e \left(\frac{r_p - (1-\omega_1-\omega_2)\cdot F_s^{-1}(u_s) - \omega_1 \cdot F_g^{-1}(u_g)}{\omega_1+\omega_2} \right) \middle| u_g u_s \right) du_g du_s$$

Or

$$r_s = \frac{r_p - \omega_1 \cdot r_g - (\omega_1+\omega_2)\cdot r_e}{1-\omega_1-\omega_2}$$

$$\Rightarrow F_p(r_p) = \int_0^1 \int_0^1 C_{s|e,g}\left(F_s \left(\frac{r_p - \omega_1 \cdot F_g^{-1}(u_g) - (\omega_1+\omega_2)\cdot F_e^{-1}(u_e)}{1-\omega_1-\omega_2} \right) \middle| u_e u_g \right) du_e du_g$$

From here onwards the calculation of the risk measures is identical to that presented in Case 1.

D Data analysis

Figures 30 to 33 present the prices of all assets including in the data set in the sample between 29 December 2000 and 30 December 2019.

Figure 30: Prices of EUR/USD, EUROSTOXX50 and gold between 2001-2019.

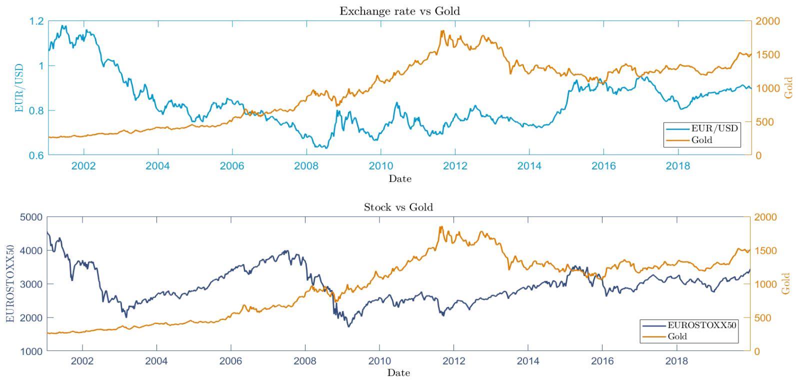


Figure 31: Prices of JPY/USD, Nikkei225 and gold between 2001-2019.

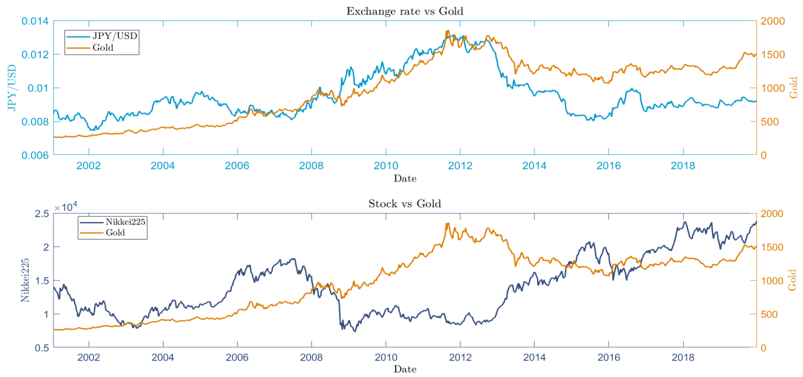


Figure 32: Prices of BRL/USD, BOVESPA and gold between 2001-2019.

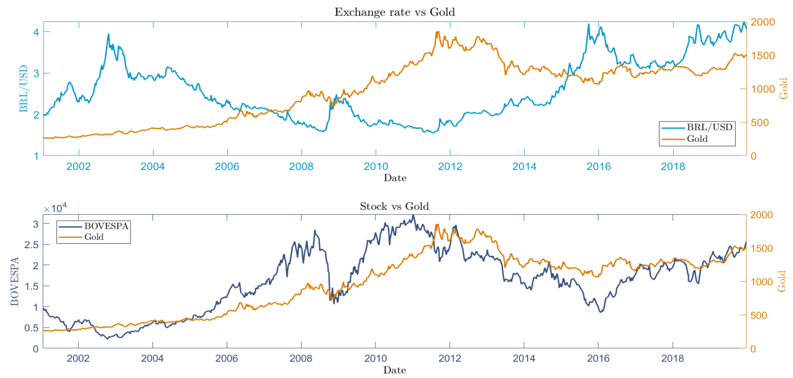
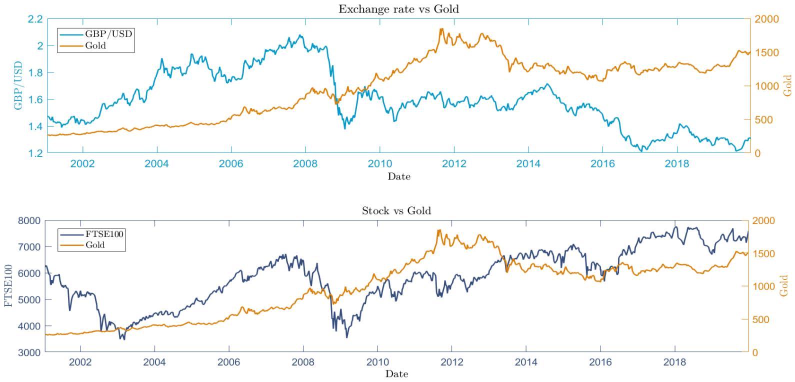
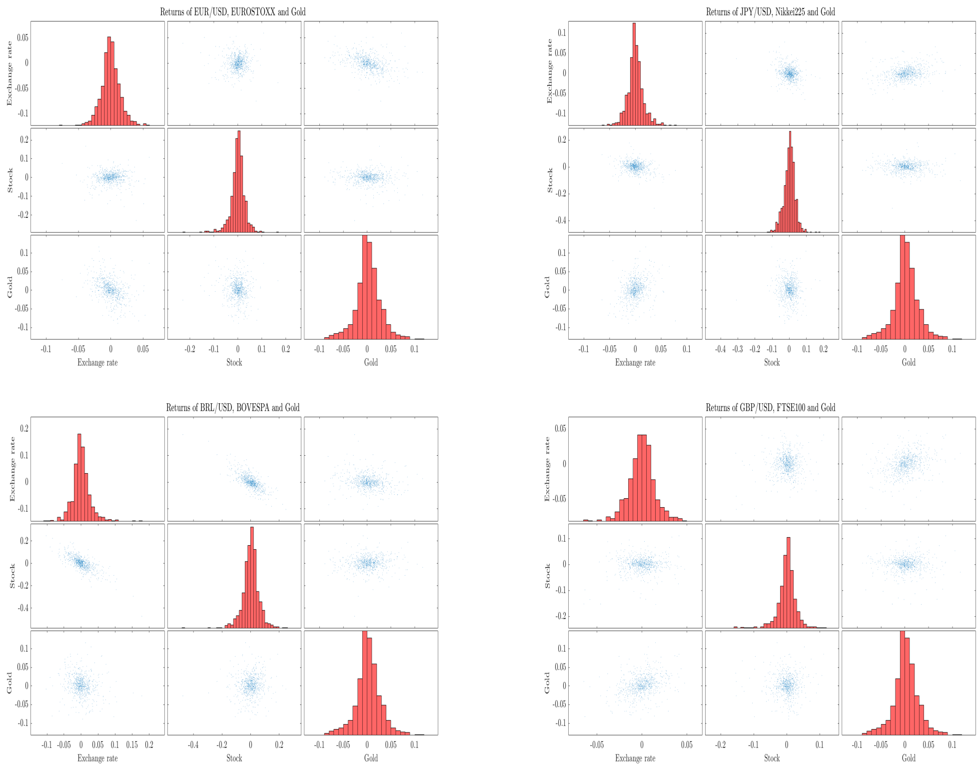


Figure 33: Prices of GBP/USD, FTSE100 and gold between 2001-2019.



Histogram is a very useful graphical tool to appreciate the features of the asset distributions (see Table 1), especially those related to higher moments. In Figure 34 we can see the histograms of the distributions of the returns of the nine assets analyzed, as well as scatter plots of the returns of assets per pair.

Figure 34: Histograms and scatter plots for the bivariate relationship between assets.



Notes: in top left Panel appear the relation between the returns of European assets, i.e., EUR/USD exchange rate, EUROSTOXX50 and gold. Top right Panel shows the relation between the group of assets of Japan, i.e., JPY/USD, Nikkei225 and gold. In bottom left Panel we can see graphics to explain the relationship between BRL/USD, BOVESPA and gold. Finally, bottom right Panel contains the histogram and scatter plot for GBP/USD, FTSE100 and gold. We can appreciate the high kurtosis and skewness (frequently negative) in all distributions, and the scatter plots show a certain dependence between the returns of the assets.

In addition, Figures 35 to 38 show the same histograms of the logaritmic returns of each serie with a Gaussian fit. Thus we can appreciate better the non-Normality of the distributions.

Figure 35: Histogram of European assets returns

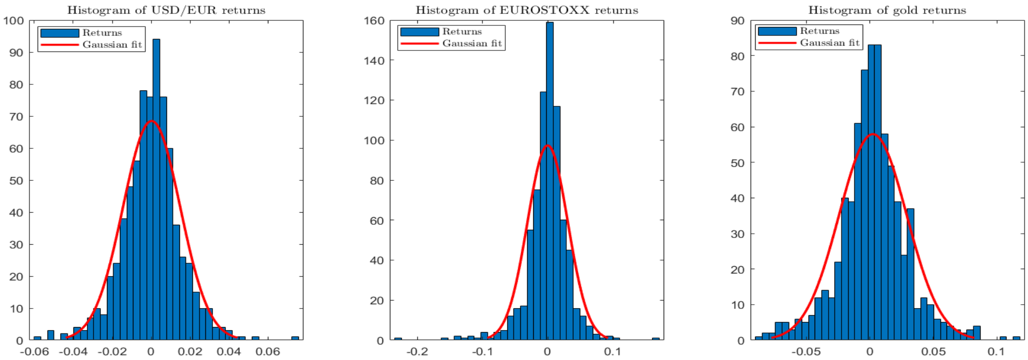


Figure 36: Histogram of Japanese assets returns

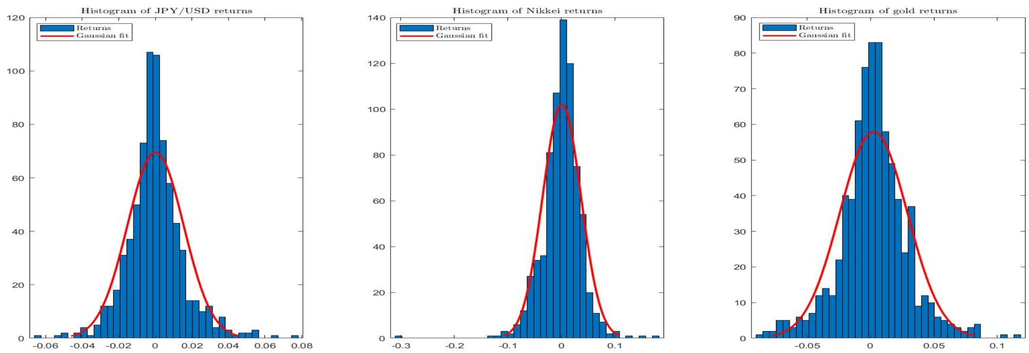


Figure 37: Histogram of Brazilian assets returns

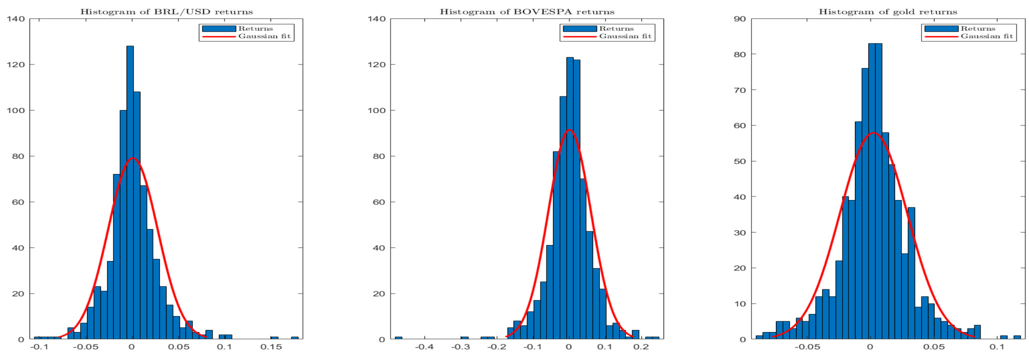
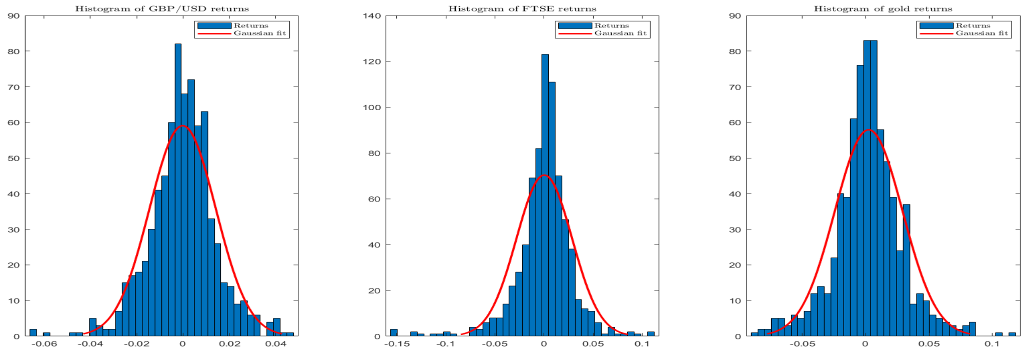


Figure 38: Histogram of British assets returns



Finally, we compute the correlation between the three assets corresponding to each region. Table 7 presents the total correlation of each asset respect to the others variables of each group. For example, the total correlation of EUROSTOXX50 is the sum of the correlation between this index and EUR/USD and between the index and gold; the total correlation of JPY/USD has into acount the correlation between the exchange rate and both stock index (Nikkei225) and gold; etc.

We can see that the exchange rate of any region has the highest total correlation. This result could be logical if we think that any investor must acquire exchange rate if he wants to form a portfolio which includes any asset traded in a currency other than his own, like gold. Moreover, a shock occurred in a market would have consequences as a fall of the stock index or an increase of the prices of gold, and these facts always impact on the exchange rate. Moreover, assume it is produced a shock in European, Japanese, Brazilian or British market, which causes a fall in the respective stock index. In this situation foreing investors withdraw their investments in these markets, which produces a depreciation of the euro, yen, Brazilian real or pound, respectively, and an appreciation of their domestic currency. This remarks that it is not necessary to have an investment in gold or in other asset denominated in a non-local currency for movements in stock markets have and impact on exchange rate markets.

Table 7: Total correlation of each asset of the data set.

	Pearson	Spearman	Kendall
Total correlation EUR/USD	0.4037	0.5126	**0.3665**
Total correlation EUROSTOXX50	0.0381	0.1345	0.0965
Total correlation gold	0.4132	0.5047	0.3591
Total correlation JPY/USD	0.5460	0.4629	**0.3225**
Total correlation Nikkei225	0.3286	0.2459	0.1723
Total correlation gold	0.3287	0.3090	0.2186
Total correlation BRL/USD	0.9231	0.8833	**0.6434**
Total correlation BOVESPA	0.8323	0.7879	0.5771
Total correlation gold	0.2589	0.2834	0.1975
Total correlation GBP/USD	0.4138	0.3742	**0.2640**
Total correlation FTSE100	0.0966	0.0344	0.0269
Total correlation gold	0.3350	0.3423	0.2416

Note: this table presents the sum of the Pearson, Spearman and Kendall correlation coefficients, respectively, between each asset and the other assets of its group, divided by the currency in which each stock index is denominated.

E Model selection

Gold

Table 8: Optimal parameters, information criteria and maximum loglikelihood of each tested model for gold.

GARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	Degrees of Freedom	Skewness
Normal	0.0028	-0.0464	0.1385	0.00007	0.1046	0.8007	-	-
Student's t	0.0024	-0.0263	0.1378	0.00005	0.0910	0.8484	4.3356	-
Skewed Student's t	0.0028	-0.0272	0.1366	0.00005	0.0931	0.8474	4.3431	0.0367

GJR(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	Degrees of Freedom	Skewness
Normal	0.0028	-0.0520	0.0220	0.0001	0.0997	0.8105	0.0000	-	-
Student's t	0.0024	0.0023	-0.0217	0.0235	0.0000	0.0895	0.8594	0.0000	4.3709
Skewed Student's t	0.0027	-0.0230	0.0229	0.0000	0.0918	0.8578	0.0000	4.3817	0.0409

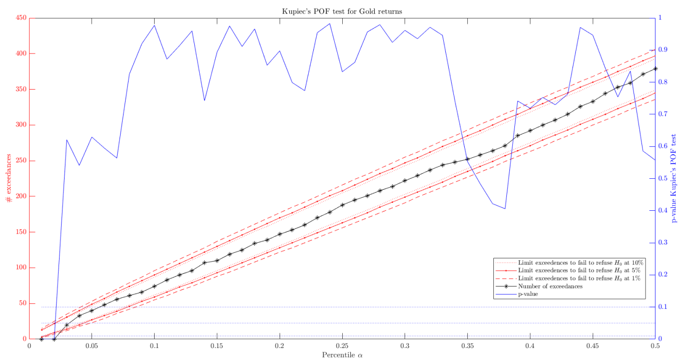
EWMA	ARMA Constant	AR(1)	MA(12)	λ	Degrees of Freedom	Skewness
Normal	0.0029	-0.0471	0.1446	0.9472	-	-
Student's t	0.0023	0.0022	-0.0254	0.1356	0.9511	4.7238
Skewed Student's t	0.0025	-0.0263	0.1348	0.9491	4.7766	0.0340

APARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	δ	Degrees of Freedom	Skewness
Normal	0.0029	-0.0420	0.1363	0.00000	0.0471	0.7654	-0.0506	3.9286	-	-
Student's t	0.0025	-0.0285	0.1367	0.00000	0.0271	0.7618	-0.0014	5.0126	4.3305	-
Skewed Student's t	0.0029	-0.0302	0.1352	0.00000	0.0277	0.7615	-0.0050	5.0269	4.3262	0.0398

GARCH-M(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	c	Degrees of Freedom	Skewness
Normal	0.0028	-0.0493	0.0223	0.00006	0.1012	0.8103	0.0025	-	-
Student's t	0.0023	-0.0210	0.0235	0.00005	0.0930	0.8559	0.0099	4.3436	-
Skewed Student's t	0.0027	-0.0223	0.0229	0.00005	0.0953	0.8543	0.0066	4.3559	0.0413

	Akaike	Schwartz	Loglikelihood
GARCH (Normal)	3361.07	3388.75	1674.53
GJR (Normal)	3344.36	3376.66	1665.18
EWMA (Normal)	3317.09	3335.54	1654.54
APARCH (Normal)	3367.93	3404.84	1675.96
GARCH-M (Normal)	3345.56	3377.85	1665.78
GARCH (Student t)	3405.31	3437.61	1695.66
GJR (Student t)	3388.42	3425.33	1686.21
EWMA (Student t)	3383.75	3406.82	1686.88
APARCH (Student t)	3412.79	3454.31	1697.40
GARCH-M (Student t)	3389.23	3426.14	1686.61
GARCH (Skewed Student t)	3407.93	3444.84	1695.96
GJR (Skewed Student t)	3391.16	3432.68	1686.58
EWMA (Skewed Student t)	3386.39	3414.07	1687.20
APARCH (Skewed Student t)	**3415.51**	**3461.65**	**1697.76**
GARCH-M (Skewed Student t)	3391.98	3433.50	1686.99

Figure 39: Result of Kupiec test for optimal model for gold.



Notes: we calculate the VaR for a certain percentile $\alpha \in [0,\ 0.5]$; the p-value of the Kupiec test done for each VaR is represented by the blue line. Considering the p-value, we will reject the null (the model is suitable) for VaR(α) with $\alpha \in [0,\ 0.10]$ with a level of confidence of 95% and 90%. The number of exceedances (black line) is between the upper and lower bounds of number of exceedances of the VaR(α) for VaR(α) with $\alpha \geq 0.025$ for a level of significance of $\tau = 1\%,\ 5\%,\ 10\%$.

EUR/USD

Table 9: Optimal parameters, information criteria and maximum loglikelihood of each tested model for EUR/USD.

GARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	Degrees of Freedom	Skewness
Normal	0.0005	0.0013	0.1210	0.0000	0.0603	0.9173	-	-
Student's t	0.0005	-0.0032	0.0908	0.0000	0.0664	0.9133	7.2795	-
Skewed Student's t	0.0004	0.0087	0.0915	0.0000	0.0670	0.9126	7.2847	-0.0255

GJR(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	Degrees of Freedom	Skewness
Normal	0.0005	-0.0025	-0.0977	0.0000	0.0615	0.9151	-0.0027	-	-
Student's t	0.0005	-0.0074	-0.0645	0.0000	0.0663	0.9113	-0.0070	7.3299	-
Skewed Student's t	0.0005	-0.0075	-0.0637	0.0000	0.0663	0.9113	0.0159	7.3105	-0.0153

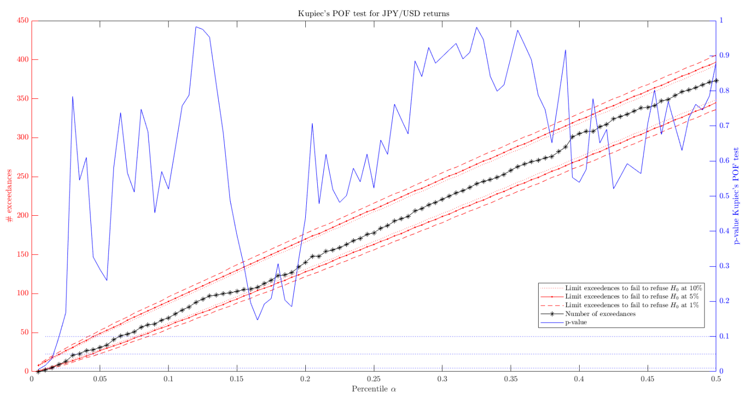
EWMA	ARMA Constant	AR(1)	MA(12)	λ	Degrees of Freedom	Skewness
Normal	0.0006	0.0007	0.1224	0.94710	-	-
Student's t	0.0005	-0.0045	0.0969	0.9487	7.3774	-
Skewed Student's t	0.0005	-0.0048	0.0969	0.9485	7.3580	-0.0221

APARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	δ	Degrees of Freedom	Skewness
Normal	0.0010	0.0348	0.1400	0.0074	0.0642	0.9110	-0.0020	0.3690	-	-
Student's t	0.0004	0.0134	0.0987	0.0010	0.0820	0.9069	0.0063	0.8090	7.8791	-
Skewed Student's t	0.0001	0.0093	0.0957	0.0013	0.0810	0.9072	0.0073	0.7394	7.6543	-0.0634

GARCH-M(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	c	Degrees of Freedom	Skewness
Normal	0.0004	0.0034	-0.0944	0.0000	0.0262	0.9251	0.0459	-	-
Student's t	0.0004	0.0013	-0.0625	0.0000	0.0329	0.9173	0.0495	7.4072	-
Skewed Student's t	0.0004	0.0045	-0.0622	0.0000	0.0331	0.9172	0.0498	7.3852	-0.0099

	Akaike	Schwartz	Loglikelihood
GARCH (Normal)	4404.89	4432.78	2196.45
GJR (Normal)	4404.77	4437.30	2195.39
EWMA (Normal)	4386.56	4405.15	2189.28
APARCH (Normal)	4416.81	4453.99	2200.40
GARCH-M (Normal)	4399.00	4431.53	2192.50
GARCH (Student t)	4429.45	4461.99	2207.73
GJR (Student t)	4427.21	4464.39	2205.60
EWMA (Student t)	4418.71	4441.94	2204.35
APARCH (Student t)	4435.32	4477.15	2208.66
GARCH-M (Student t)	4422.88	4460.07	2203.44
GARCH (Skewed Student t)	4431.61	4468.79	2207.80
GJR (Skewed Student t)	4429.21	4471.04	2205.61
EWMA (Skewed Student t)	4420.93	4448.82	2204.47
APARCH (Skewed Student t)	**4437.71**	**4484.19**	**2208.85**
GARCH-M (Skewed Student t)	4424.99	4466.82	2203.49

Figure 40: Result of Kupiec test for optimal model for EUR/USD.



Notes: we calculate the VaR for a certain percentile $\alpha \in [0,\ 0.5]$; the p-value of the Kupiec test done for each VaR is represented by the blue line. Considering the p-value, we will reject the null (the model is suitable) for VaR(α) with $\alpha \in [0,\ 0.10]$ with a level of confidence of 95% and 90%. The number of exceedances (black line) is always between the upper and lower bounds of number of exceedances of the VaR(α) for a level of significance of $\tau = 1\%,\ 5\%,\ 10\%$.

EUROSTOXX50

Table 10: Optimal parameters, information criteria and maximum loglikelihood of each tested model for EUROSTOXX50.

GARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	Degrees of Freedom	Skewness
Normal	0.0020	-0.0346	0.0263	0.00006	0.3253	0.6747	-	-
Student's t	0.0024	-0.0766	-0.0137	0.00004	0.1964	0.7902	3.7739	-
Skewed Student's t	0.0013	-0.0967	-0.0076	0.00005	0.1969	0.7751	4.1141	-0.1589

GJR(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	Degrees of Freedom	Skewness
Normal	0.0016	-0.0356	0.0394	0.00005	0.3022	0.6971	-0.0120	-	-
Student's t	0.0025	-0.0797	-0.0233	0.00004	0.1989	0.7906	0.0396	3.7360	-
Skewed Student's t	0.0014	-0.0981	-0.0291	0.00005	0.2064	0.7683	0.1962	4.0765	-0.1577

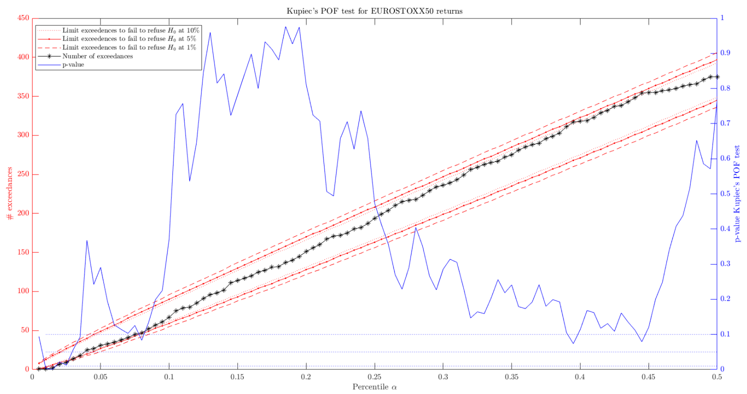
EWMA	ARMA Constant	AR(1)	MA(12)	λ	Degrees of Freedom	Skewness
Normal	0.0010	-0.0223	-0.0243	0.8808	-	-
Student's t	0.0024	-0.0898	0.0003	0.9125	4.3854	-
Skewed Student's t	0.0018	-0.1110	0.0073	0.9128	4.5853	-0.1271

APARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	δ	Degrees of Freedom	Skewness
Normal	0.0024	-0.0076	0.0310	0.0172	0.2684	0.6991	-0.0099	0.4781	-	-
Student's t	0.0022	-0.0683	0.0332	0.0024	0.2110	0.7890	-0.0043	0.9228	3.9691	-
Skewed Student's t	-0.0003	-0.0822	0.0111	0.0002	0.1825	0.8175	-0.0006	1.5517	4.8704	-0.1604

GARCH-M(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	c	Degrees of Freedom	Skewness
Normal	-0.0007	-0.0303	0.0433	0.00004	0.2613	0.7351	0.0071	-	-
Student's t	0.0024	-0.0813	-0.0218	0.00004	0.1843	0.8061	0.0000	3.6949	-
Skewed Student's t	0.0014	-0.1010	-0.0236	0.00004	0.1797	0.7967	0.0000	4.0249	-0.1557

	Akaike	Schwartz	Loglikelihood
GARCH (Normal)	3368.82	3396.70	1678.41
GJR (Normal)	3353.03	3385.56	1669.51
EWMA (Normal)	3288.74	3307.33	1640.37
APARCH (Normal)	3393.59	3430.78	1688.80
GARCH-M (Normal)	3366.36	3398.89	1676.18
GARCH (Student t)	3487.95	3520.48	1736.97
GJR (Student t)	3490.46	3527.64	1737.23
EWMA (Student t)	3458.39	3481.63	1724.20
APARCH (Student t)	3493.56	3535.39	1737.78
GARCH-M (Student t)	3491.57	3528.75	1737.78
GARCH (Skewed Student t)	3500.81	3537.99	1742.41
GJR (Skewed Student t)	3502.76	3544.59	1742.38
EWMA (Skewed Student t)	3469.84	3497.72	1728.92
APARCH (Skewed Student t)	3502.88	**3549.36**	**1743.34**
GARCH-M (Skewed Student t)	**3504.68**	3546.51	1741.44

Figure 41: Result of Kupiec test for optimal model for EUROSTOXX50.



Notes: we calculate the VaR for a certain percentile $\alpha \in [0,\ 0.5]$; the p-value of the Kupiec test done for each VaR is represented by the blue line. Considering the p-value, we will reject the null (the model is suitable) for VaR(α) with $\alpha \in [0,\ 0.10]$ with a level of confidence of 95% and 90%. The number of exceedances (black line) is always between the upper and lower bounds of number of exceedances of the VaR(α) for a level of significance of $\tau = 1\%,\ 5\%,\ 10\%$.

JPY/USD

Table 11: Optimal parameters, information criteria and maximum loglikelihood of each tested model for JPY/USD.

GARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	Degrees of Freedom	Skewness
Normal	0.0001	0.2263	-0.2696	0.00002	0.0665	0.8564	-	-
Student's t	-0.0005	-0.0197	0.0036	0.00001	0.0678	0.9014	3.4063	-
Skewed Student's t	0.0000	-0.0269	0.0093	0.00001	0.0721	0.8904	0.0770	3.4816

GJR(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	Degrees of Freedom	Skewness
Normal	0.0001	-0.0411	-0.0210	0.00002	0.0663	0.8570	0.0000	-	-
Student's t	-0.0005	-0.0172	-0.0246	0.00001	0.0680	0.9011	0.0000	3.4079	-
Skewed Student's t	0.0000	-0.0190	-0.0220	0.00001	0.0724	0.8903	0.0000	0.0736	3.4780

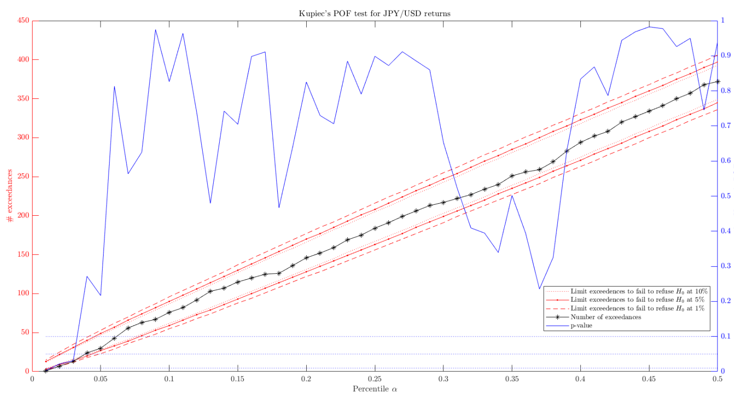
EWMA	ARMA Constant	AR(1)	MA(12)	λ	Degrees of Freedom	Skewness
Normal	0.0001	0.2564	-0.3032	0.9718	-	-
Student's t	-0.0004	0.3446	-0.3695	0.9627	3.9628	-
Skewed Student's t	-0.0003	-0.0198	-0.0001	0.9626	0.0554	4.0144

APARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	δ	Degrees of Freedom	Skewness
Normal	0.0001	-0.0450	0.0030	0.00052	0.0890	0.8424	0.0027	1.2365	-	-
Student's t	-0.0005	-0.0268	0.0085	0.00019	0.0895	0.8869	0.0075	1.3692	3.4297	-
Skewed Student's t	0.0000	-0.0173	-0.0030	0.00019	0.0922	0.8783	0.0056	1.3938	0.0760	3.5083

GARCH-M(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	c	Degrees of Freedom	Skewness
Normal	0.0001	-0.0399	0.0022	0.00002	0.0672	0.8550	0.0033	-	-
Student's t	-0.0005	-0.0172	-0.0248	0.00001	0.0679	0.9007	-1.9958	3.4131	-
Skewed Student's t	0.0000	-0.0190	-0.0220	0.00001	0.0723	0.8903	0.0616	0.0737	3.4789

	Akaike	Schwartz	Loglikelihood
GARCH (Normal)	4129.46	4157.14	2058.73
GJR (Normal)	4125.46	4157.76	2055.73
EWMA (Normal)	4102.86	4121.31	2047.43
APARCH (Normal)	4135.33	4172.24	2059.66
GARCH-M (Normal)	4125.10	4157.39	2055.55
GARCH (Student t)	4217.28	4249.58	2101.64
GJR (Student t)	4213.64	4250.55	2098.82
EWMA (Student t)	4203.41	4226.48	2096.71
APARCH (Student t)	*4221.74*	*4263.26*	*2101.87*
GARCH-M (Student t)	4213.65	4250.56	2098.82
GARCH (Skewed Student t)	4222.02	4258.93	2103.01
GJR (Skewed Student t)	4218.19	4259.71	2100.10
EWMA (Skewed Student t)	4206.82	4234.50	2097.41
APARCH (Skewed Student t)	**4226.47**	**4272.60**	**2103.23**
GARCH-M (Skewed Student t)	4218.19	4259.71	2100.10

Figure 42: Result of Kupiec test for optimal model for JPY/USD.



Notes: we calculate the VaR for a certain percentile $\alpha \in [0,\ 0.5]$; the p-value of the Kupiec test done for each VaR is represented by the blue line. Considering the p-value, we will reject the null (the model is suitable) for VaR(α) with $\alpha \in [0,\ 0.10]$ with a level of confidence of 95% and 90%. The number of exceedances (black line) is always between the upper and lower bounds of number of exceedances of the VaR(α) for a level of significance of $\tau = 1\%,\ 5\%,\ 10\%$.

Nikkei225

Table 12: Optimal parameters, information criteria and maximum loglikelihood of each tested model for Nikkei225.

GARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	Degrees of Freedom	Skewness
Normal	0.0030	-0.0094	-0.0009	0.00009	0.2015	0.7506	-	-
Student's t	0.0032	-0.0436	0.0053	0.00004	0.1075	0.8687	5.2058	-
Skewed Student's t	0.0020	-0.0536	0.0024	0.00004	0.1050	0.8661	5.3891	-0.1348

GJR(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	Degrees of Freedom	Skewness
Normal	0.0017	-0.0147	-0.0077	0.0002	0.0352	0.6835	0.2460	-	-
Student's t	0.0031	-0.0345	0.0018	0.0001	0.0659	0.8196	0.1037	5.4034	-
Skewed Student's t	0.0018	-0.0412	0.0004	0.0001	0.0645	0.8079	0.1119	5.6107	-0.1409

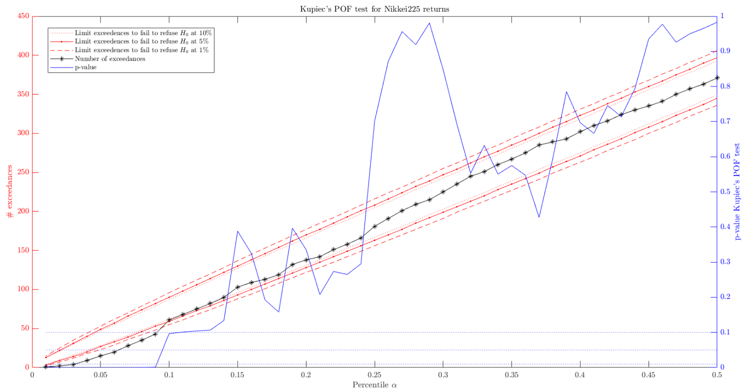
EWMA	ARMA Constant	AR(1)	MA(12)	λ	Degrees of Freedom	Skewness
Normal	0.0020	-0.0054	0.0005	0.9295	-	-
Student's t	0.0033	-0.0379	-0.0016	0.9237	5.6639	-
Skewed Student's t	0.0026	-0.0497	-0.0014	0.9259	5.8280	-0.1044

APARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	δ	Degrees of Freedom	Skewness
Normal	0.0001	0.9209	-0.9102	0.0032	0.1641	0.7325	0.6809	1.1248	-	-
Student's t	0.0034	-0.2427	0.2026	0.0033	0.1288	0.8254	0.4597	0.9365	5.3890	-
Skewed Student's t	0.0016	-0.1733	0.1319	0.0057	0.1287	0.8101	0.5641	0.8377	5.5312	-0.1540

GARCH-M(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	c	Degrees of Freedom	Skewness
Normal	0.0021	-0.0095	-0.0123	0.00017	0.1707	0.6751	8.5197	-	-
Student's t	0.0032	-0.0390	-0.0062	0.00004	0.1071	0.8685	0.0801	5.2490	-
Skewed Student's t	0.0020	-0.0535	-0.0110	0.00004	0.1036	0.8670	-0.0966	5.4426	-0.1376

	Akaike	Schwartz	Loglikelihood
GARCH (Normal)	2940.06	2967.74	1464.03
GJR (Normal)	2962.01	2994.30	1474.00
EWMA (Normal)	2902.25	2920.70	1447.13
APARCH (Normal)	2974.18	3011.08	1479.09
GARCH-M (Normal)	2954.73	2987.03	1470.37
GARCH (Student t)	3011.32	3043.61	1498.66
GJR (Student t)	3011.73	3048.64	1497.86
EWMA (Student t)	2994.44	3017.51	1492.22
APARCH (Student t)	3025.23	3066.75	1503.62
GARCH-M (Student t)	3008.61	3045.52	1496.31
GARCH (Skewed Student t)	3020.33	3057.23	1502.16
GJR (Skewed Student t)	3021.41	3062.93	1501.71
EWMA (Skewed Student t)	3001.66	3029.34	1494.83
APARCH (Skewed Student t)	**3035.88**	**3082.01**	**1507.94**
GARCH-M (Skewed Student t)	3017.78	3059.30	1499.89

Figure 43: Result of Kupiec test for optimal model for Nikkei225.



Notes: we calculate the VaR for a certain percentile $\alpha \in [0,\ 0.5]$; the p-value of the Kupiec test done for each VaR is represented by the blue line. Considering the p-value, we will reject the null (the model is suitable) for VaR(α) with $\alpha \in [0,\ 0.10]$ with a level of confidence of 95% and 90%. The number of exceedances (black line) is between the upper and lower bounds of number of exceedances of the VaR(α) for $\alpha \geq 0.10$ for a level of significance of $\tau = 1\%,\ 5\%,\ 10\%$.

BRL/USD

Table 13: Optimal parameters, information criteria and maximum loglikelihood of each tested model for BRL/USD.

GARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	Degrees of Freedom	Skewness
Normal	-0.0008	-0.0071	-0.0024	0.00006	0.2029	0.7355	-	-
Student's t	-0.0010	-0.0176	-0.0017	0.00002	0.1626	0.8374	3.4887	-
Skewed Student's t	0.0006	0.0033	-0.0065	0.00002	0.1466	0.8534	3.5546	0.1245

GJR(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	Degrees of Freedom	Skewness
Normal	-0.0008	-0.0080	0.0225	0.00006	0.2032	0.7355	0.0000	-	-
Student's t	-0.0011	-0.0190	0.0047	0.00002	0.1641	0.8359	0.0000	3.4643	-
Skewed Student's t	-0.0004	-0.0783	-0.0040	0.00002	0.1569	0.8430	0.0002	3.4886	0.0868

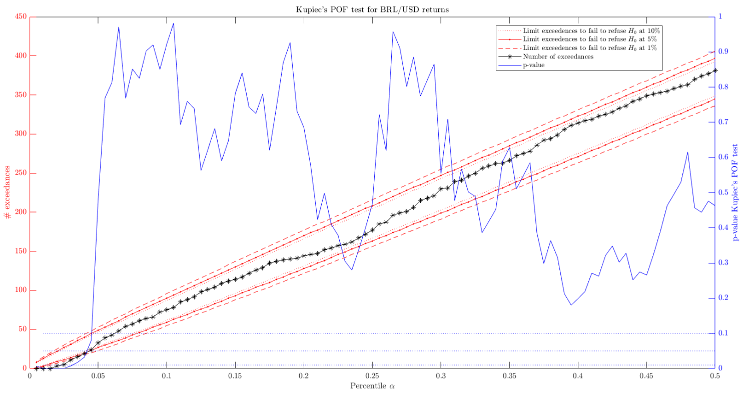
EWMA	ARMA Constant	AR(1)	MA(12)	λ	Degrees of Freedom	Skewness
Normal	0.0010	-0.0222	-0.0009	0.9085	-	-
Student's t	-0.0010	-0.0160	-0.0009	0.8943	4.0545	-
Skewed Student's t	-0.0004	-0.0266	-0.0018	0.8988	4.1035	0.0934

APARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	δ	Degrees of Freedom	Skewness
Normal	-0.0015	-0.0317	-0.0024	0.00123	0.2142	0.7518	-0.0133	1.1868	-	-
Student's t	-0.0007	-0.0059	-0.0006	0.00011	0.1686	0.8314	0.0068	1.6343	3.5707	-
Skewed Student's t	0.0013	-0.0574	0.0009	0.00000	0.1559	0.8441	0.0040	2.4445	3.2536	0.1440

GARCH-M(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	c	Degrees of Freedom	Skewness
Normal	-0.0009	-0.0098	0.0013	0.00006	0.2047	0.7338	0.0163	-	-
Student's t	-0.0010	-0.0267	0.0047	0.00002	0.1592	0.8362	0.0118	3.4990	-
Skewed Student's t	-0.0001	-0.0446	-0.0024	0.00002	0.1529	0.8471	0.0033	3.4734	0.1064

	Akaike	Schwartz	Loglikelihood
GARCH (Normal)	3408.44	3436.12	1698.22
GJR (Normal)	3407.19	3439.48	1696.59
EWMA (Normal)	3315.17	3333.62	1653.58
APARCH (Normal)	3417.96	3454.87	1700.98
GARCH-M (Normal)	3406.62	3438.92	1696.31
GARCH (Student t)	3524.70	3556.99	1755.35
GJR (Student t)	3524.24	3561.15	1754.12
EWMA (Student t)	3502.41	3525.48	1746.21
APARCH (Student t)	3527.10	3568.62	1754.55
GARCH-M (Student t)	3523.73	3560.64	1753.87
GARCH (Skewed Student t)	3530.25	3567.16	**1757.13**
GJR (Skewed Student t)	3528.45	3569.97	1755.23
EWMA (Skewed Student t)	3509.40	3537.08	1748.70
APARCH (Skewed Student t)	**3532.57**	**3578.70**	*1756.29*
GARCH-M (Skewed Student t)	3530.39	3571.91	1756.20

Figure 44: Result of Kupiec test for optimal model for BRL/USD.



Notes: we calculate the VaR for a certain percentile $\alpha \in [0,\ 0.5]$; the p-value of the Kupiec test done for each VaR is represented by the blue line. Considering the p-value, we will reject the null (the model is suitable) for VaR(α) with $\alpha \in [0,\ 0.04]$ with a level of confidence of 95%, and we will reject the null for VaR(α) with $\alpha \in [0,\ 0.05]$ with a level of confidence of 90%. The number of exceedances (black line) is always between the upper and lower bounds of number of exceedances of the VaR(α) for $\alpha \geq 0.05$ for a level of significance of $\tau = 1\%,\ 5\%,\ 10\%$.

BOVESPA

Table 14: Optimal parameters, information criteria and maximum loglikelihood of each tested model for BOVESPA.

GARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	Degrees of Freedom	Skewness
Normal	0.0049	-0.0642	0.0021	0.0003	0.1315	0.8003	-	-
Student's t	0.0036	-0.0249	-0.0001	0.0002	0.0718	0.8868	4.1101	-
Skewed Student's t	0.0021	-0.0332	0.0001	0.0001	0.0698	0.8930	4.2115	-0.0805

GJR(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	Degrees of Freedom	Skewness
Normal	0.0025	-0.0539	0.0010	0.0003	0.0000	0.8147	0.1808	-	-
Student's t	0.0032	-0.0196	0.0171	0.0002	0.0000	0.8826	0.1084	4.4401	-
Skewed Student's t	0.0018	-0.0241	0.0193	0.0002	0.0000	0.8873	0.1079	4.5764	-0.0828

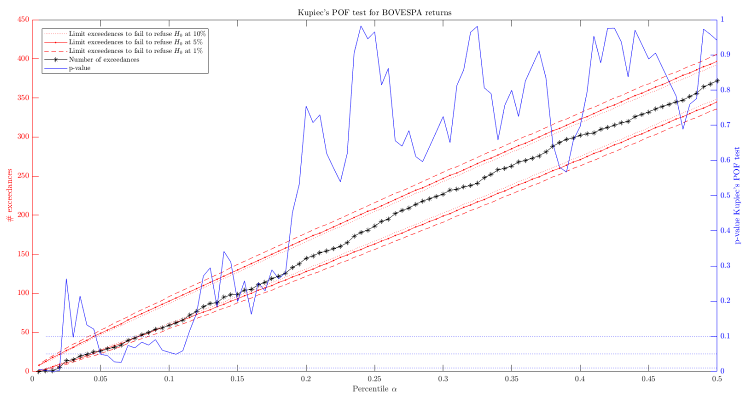
EWMA	ARMA Constant	AR(1)	MA(12)	λ	Degrees of Freedom	Skewness
Normal	0.0027	-0.0419	-0.0004	0.9514	-	-
Student's t	0.0035	-0.0288	-0.0001	0.9478	4.5039	-
Skewed Student's t	0.0022	-0.0380	-0.0009	0.9469	4.6473	-0.0864

APARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	δ	Degrees of Freedom	Skewness
Normal	0.0061	-0.1823	0.0866	0.0572	0.0949	0.8305	0.2431	0.1310	-	-
Student's t	0.0002	0.9044	-0.8917	0.0076	0.0859	0.8879	0.8140	0.6822	4.7204	-
Skewed Student's t	0.0019	0.0094	-0.0330	0.0015	0.0602	0.8897	0.9896	1.2646	4.8892	-0.0745

GARCH-M(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	c	Degrees of Freedom	Skewness
Normal	0.0052	-0.0640	-0.0262	0.0003	0.1390	0.7938	0.0469	-	-
Student's t	0.0037	-0.0239	0.0106	0.0002	0.0725	0.8864	-0.0541	4.0366	-
Skewed Student's t	0.0022	-0.0318	0.0115	0.0001	0.0706	0.8921	-0.0269	4.1350	-0.0779

	Akaike	Schwartz	Loglikelihood
GARCH (Normal)	2173.65	2201.33	1080.83
GJR (Normal)	2217.10	2249.39	1101.55
EWMA (Normal)	2128.78	2147.23	1060.39
APARCH (Normal)	2195.83	2232.74	1089.92
GARCH-M (Normal)	2174.90	2207.19	1080.45
GARCH (Student t)	2282.97	2315.26	1134.48
GJR (Student t)	2297.46	2334.37	1140.73
EWMA (Student t)	2262.22	2285.29	1126.11
APARCH (Student t)	*2300.29*	*2341.81*	*1141.14*
GARCH-M (Student t)	2284.53	2321.44	1134.27
GARCH (Skewed Student t)	2287.41	2324.32	1135.71
GJR (Skewed Student t)	*2302.04*	*2343.56*	*1142.02*
EWMA (Skewed Student t)	2267.76	2295.44	1127.88
APARCH (Skewed Student t)	**2306.58**	**2352.71**	**1143.29**
GARCH-M (Skewed Student t)	2288.86	2330.38	1135.43

Figure 45: Result of Kupiec test for optimal model for BOVESPA.



Notes: we calculate the VaR for a certain percentile $\alpha \in [0,\ 0.5]$; the p-value of the Kupiec test done for each VaR is represented by the blue line. Considering the p-value, we will reject the null (the model is suitable) for VaR(α) with $\alpha \in [0,\ 0.02]U[0.05,\ 0.6]$ with a level of confidence of 95%, and we will reject the null for VaR(α) with $\alpha \in [0,\ 0.025]U[0.05,\ 0.11]$ with a level of confidence of 90%. The number of exceedances (black line) is between the upper and lower bounds of number of exceedances of the VaR(α) for $\alpha = 0.02$ for a level of significance of $\tau = 1\%,\ 5\%,\ 10\%$.

GBP/USD

Table 15: Optimal parameters, information criteria and maximum loglikelihood of each tested model for GBP/USD.

GARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	Degrees of Freedom	Skewness
Normal	0.0002	0.0711	0.0134	0.00001	0.0931	0.8359	-	-
Student's t	0.0003	0.0649	-0.0019	0.00001	0.0947	0.8577	7.1049	-
Skewed Student's t	0.0001	0.0651	-0.0079	0.00001	0.0923	0.8627	7.0161	-0.0536

GJR(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	Degrees of Freedom	Skewness
Normal	0.0002	0.0672	0.0427	0.00001	0.0939	0.8347	0.0000	-	-
Student's t	0.0000	0.0571	0.0314	0.00001	0.0245	0.8913	0.0807	7.6599	-
Skewed Student's t	0.0002	0.0629	0.0217	0.00001	0.0929	0.8609	0.0000	7.3473	-0.0551

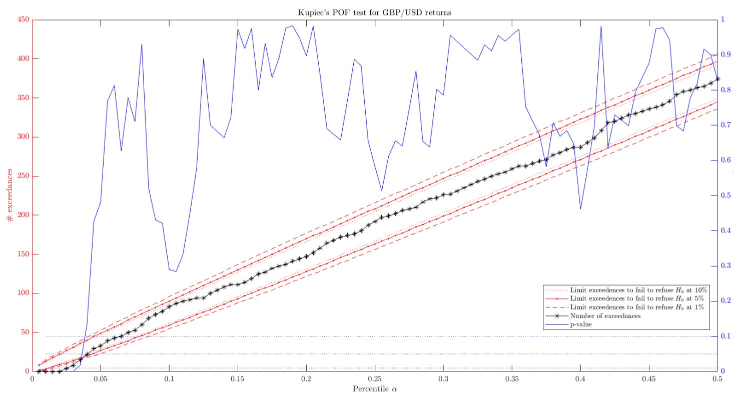
EWMA	ARMA Constant	AR(1)	MA(12)	λ	Degrees of Freedom	Skewness
Normal	0.0002	0.0747	0.0075	0.9464	-	-
Student's t	0.0003	0.0689	-0.0087	0.9409	7.1972	-
Skewed Student's t	0.0002	0.0687	-0.0169	0.9423	7.1905	-0.0647

APARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	δ	Degrees of Freedom	Skewness
Normal	0.0002	0.0779	0.0147	0.0000	0.0535	0.8099	-0.0025	3.5098	-	-
Student's t	0.0006	0.0650	-0.0053	0.0000	0.0496	0.8284	0.0229	3.7785	7.0624	-
Skewed Student's t	0.0001	0.0693	0.0189	0.0000	0.0571	0.8334	-0.0014	3.4713	7.1671	-0.0531

GARCH-M(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	c	Degrees of Freedom	Skewness
Normal	0.0002	0.0662	0.0433	0.00001	0.0935	0.8354	0.0057	-	-
Student's t	0.0003	0.0619	0.0234	0.00001	0.0952	0.8574	0.0061	7.1932	-
Skewed Student's t	0.0001	0.0622	0.0220	0.00001	0.0934	0.8615	0.0121	7.1097	-0.0574

	Akaike	Schwartz	Loglikelihood
GARCH (Normal)	4302.20	4329.88	2145.10
GJR (Normal)	4230.80	4263.10	2108.40
EWMA (Normal)	4271.18	4289.64	2131.59
APARCH (Normal)	4308.19	4345.09	2146.09
GARCH-M (Normal)	4229.12	4261.41	2107.56
GARCH (Student t)	4321.87	4354.16	2153.93
GJR (Student t)	4255.78	4292.69	2119.89
EWMA (Student t)	4305.97	4329.04	2147.99
APARCH (Student t)	4329.18	4370.70	**2155.59**
GARCH-M (Student t)	4247.91	4284.82	2115.96
GARCH (Skewed Student t)	4324.94	4361.85	2154.47
GJR (Skewed Student t)	4252.28	4293.80	2117.14
EWMA (Skewed Student t)	4309.91	4337.59	2148.96
APARCH (Skewed Student t)	**4330.38**	**4376.51**	*2155.19*
GARCH-M (Skewed Student t)	4251.17	4292.69	2116.58

Figure 46: Result of Kupiec test for optimal model for GBP/USD.



Notes: we calculate the VaR for a certain percentile $\alpha \in [0,\ 0.5]$; the p-value of the Kupiec test done for each VaR is represented by the blue line. Considering the p-value, we will reject the null (the model is suitable) for VaR(α) with $\alpha \in [0,\ 0.035]$ with a level of confidence of 95%, and we will reject the null for VaR(α) with $\alpha \in [0,\ 0.04]$ with a level of confidence of 90%. The number of exceedances (black line) is between the upper and lower bounds of number of exceedances of the VaR(α) for $\alpha = 0.04$ for a level of significance of $\tau = 1\%,\ 5\%,\ 10\%$.

FTSE100

Table 16: Optimal parameters, information criteria and maximum loglikelihood of each tested model for FTSE100.

GARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	Degrees of Freedom	Skewness
Normal	0.0017	-0.0686	-0.1429	0.00003	0.1571	0.8140	-	-
Student's t	0.0026	-0.0673	-0.1000	0.00003	0.1873	0.7939	3.8864	-
Skewed Student's t	0.0012	-0.0917	-0.1405	0.00004	0.1827	0.7893	4.5515	-0.2648

GJR(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	Degrees of Freedom	Skewness
Normal	0.0015	-0.0575	0.0127	0.00004	0.1685	0.7975	0.0000	-	-
Student's t	0.0026	-0.0587	-0.0051	0.00004	0.1967	0.7893	0.0000	3.7854	-
Skewed Student's t	0.0003	-0.0389	0.0022	0.00006	0.0004	0.7472	0.2859	4.7333	-0.2704

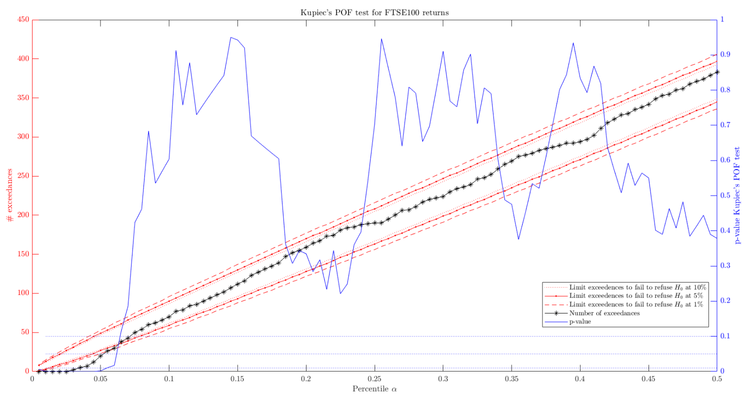
EWMA	ARMA Constant	AR(1)	MA(12)	λ	Degrees of Freedom	Skewness
Normal	0.0005	-0.0597	-0.1688	0.9190	-	-
Student's t	0.0026	-0.0671	-0.0964	0.8994	4.8519	-
Skewed Student's t	0.0019	-0.0895	-0.1313	0.8976	5.4083	-0.1973

APARCH(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	γ	δ	Degrees of Freedom	Skewness
Normal	0.0013	-0.0449	-0.0990	0.00108	0.1629	0.8371	-0.0027	1.0285	-	-
Student's t	0.0020	-0.0490	-0.0837	0.00125	0.1167	0.8453	1.0000	1.0557	4.3039	-
Skewed Student's t	0.0012	-0.0909	-0.1410	0.00003	0.1816	0.7854	-0.0016	2.0887	4.5427	-0.2638

GARCH-M(1,1)	ARMA Constant	AR(1)	MA(12)	ω	α	β	c	Degrees of Freedom	Skewness
Normal	0.0014	-0.0556	0.0126	0.00004	0.1669	0.7986	-0.0855	-	-
Student's t	0.0026	-0.0587	-0.0051	0.00004	0.1953	0.7906	-0.0063	3.7874	-
Skewed Student's t	0.0011	-0.0775	-0.0227	0.00004	0.2103	0.7673	-0.0052	4.3581	-0.2306

	Akaike	Schwartz	Loglikelihood
GARCH (Normal)	3413.51	3441.19	1700.76
GJR (Normal)	3399.94	3432.24	1692.97
EWMA (Normal)	3361.04	3379.50	1676.52
APARCH (Normal)	3418.50	3455.41	1701.25
GARCH-M (Normal)	3399.65	3431.95	1692.83
GARCH (Student t)	3523.92	3556.21	1754.96
GJR (Student t)	3515.29	3552.20	1749.64
EWMA (Student t)	3492.49	3515.55	1741.24
APARCH (Student t)	3554.18	3595.70	1768.09
GARCH-M (Student t)	3515.10	3552.01	1749.55
GARCH (Skewed Student t)	3552.50	3589.41	1768.25
GJR (Skewed Student t)	**3565.38**	**3606.90**	**1773.69**
EWMA (Skewed Student t)	3514.99	3542.67	1751.50
APARCH (Skewed Student t)	*3556.38*	*3602.51*	1768.19
GARCH-M (Skewed Student t)	3537.81	3579.33	1759.91

Figure 47: Result of Kupiec test for optimal model for FTSE100.



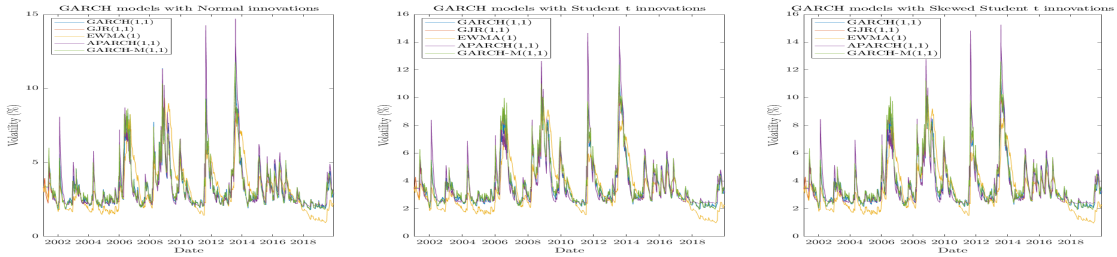
Notes: we calculate the VaR for a certain percentile $\alpha \in [0,\ 0.5]$; the p-value of the Kupiec test done for each VaR is represented by the blue line. Considering the p-value, we will reject the null (the model is suitable) for VaR(α) with $\alpha \in [0,\ 0.06]$ with a level of confidence of 95%, and we will reject the null for VaR(α) with $\alpha \in [0,\ 0.07]$ with a level of confidence of 90%. The number of exceedances (black line) is always between the upper and lower bounds of number of exceedances of the VaR(α) for $\alpha = 0.07$ for a level of significance of $\tau = 1\%,\ 5\%,\ 10\%$.

The next figures represent the annualized volatility resulting of the different GARCH models apply to the returns' series. For each asset are shown three graphics, depending on the distribution of the innovations, with the five GARCH models used. The annualized volatility is not calculated as $T\sigma^2$ because the returns of the assets en each moment are not independent, but have an ARMA structure. Then, the temporal accumulation of variances must also consider the covariances between r_{t+i} and r_{t+j}. As a result we have:

$$Std(r_{ht}) = \sqrt{\sigma^2_{\varepsilon,t} \cdot \left(h + 2\frac{\phi}{(1-\phi)^2}\left[(h-1)(1-\phi) - \phi(1-\phi^{h-1})\right] \right)}$$

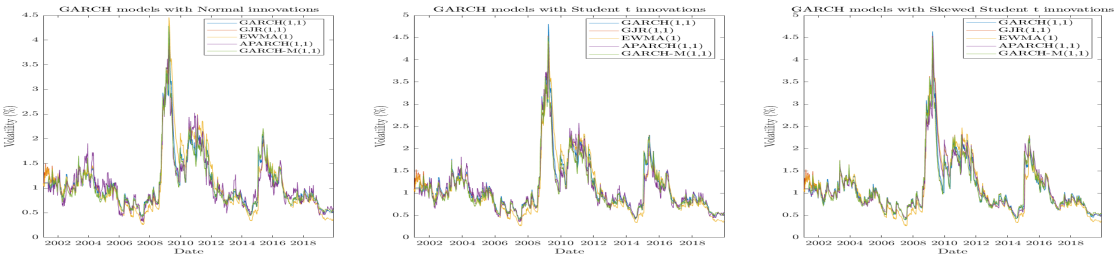
where $r_{ht} = \sum_{i=0}^{h-1} r_{t+i}$, $h = 52$ (weeks per year) and ϕ is the autorregresive parameter of the model.

Figure 48: GARCH models for gold serie.



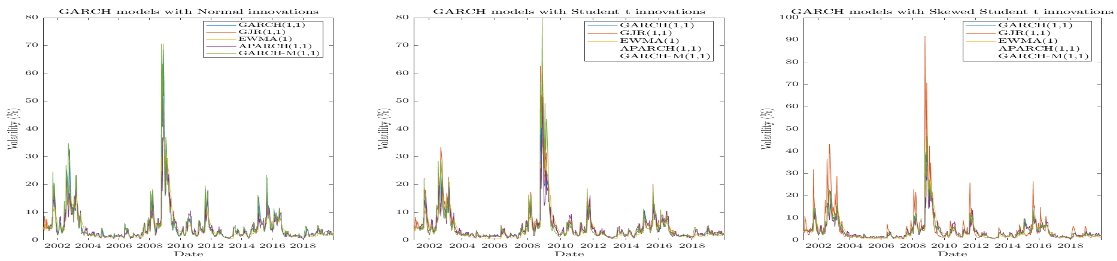
Notes: the figure shows the value of the standard deviation extrapolated 52 periods. We consider innovations follow a Normal, Student t or Skewed Student t distribution.

Figure 49: GARCH models for EUR/USD serie.



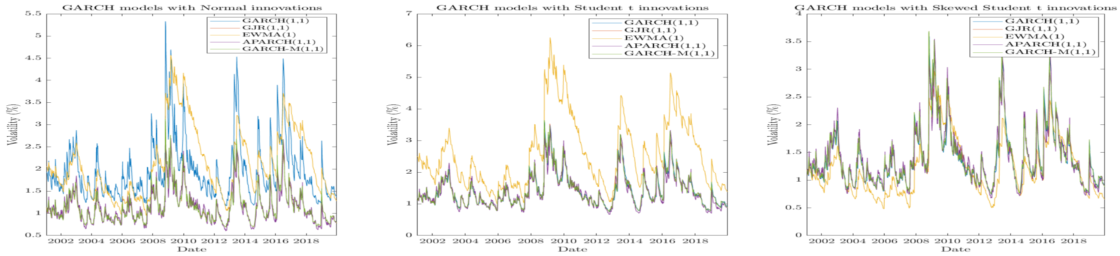
Notes: the figure shows the value of the standard deviation extrapolated 52 periods. We consider innovations follow a Normal, Student t or Skewed Student t distribution.

Figure 50: GARCH models for EUROSTOXX50 serie.



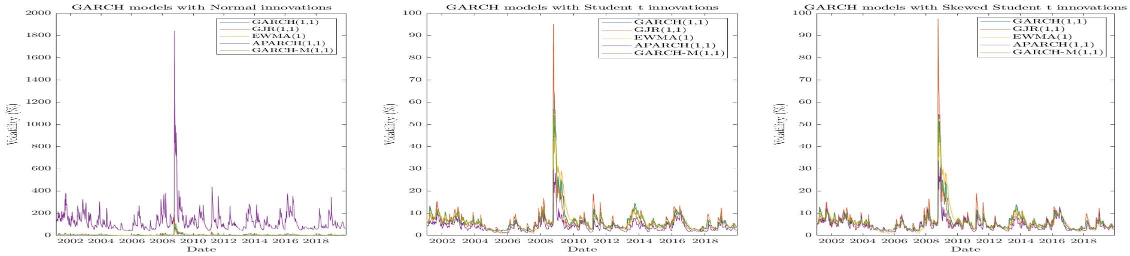
Notes: the figure shows the value of the standard deviation extrapolated 52 periods. We consider innovations follow a Normal, Student t or Skewed Student t distribution.

Figure 51: GARCH models for JPY/USD serie.



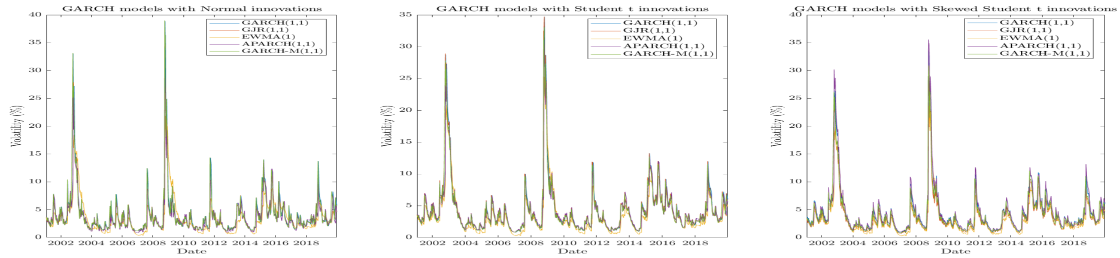
Notes: the figure shows the value of the standard deviation extrapolated 52 periods. We consider innovations follow a Normal, Student t or Skewed Student t distribution.

Figure 52: GARCH models for Nikkei225 serie.



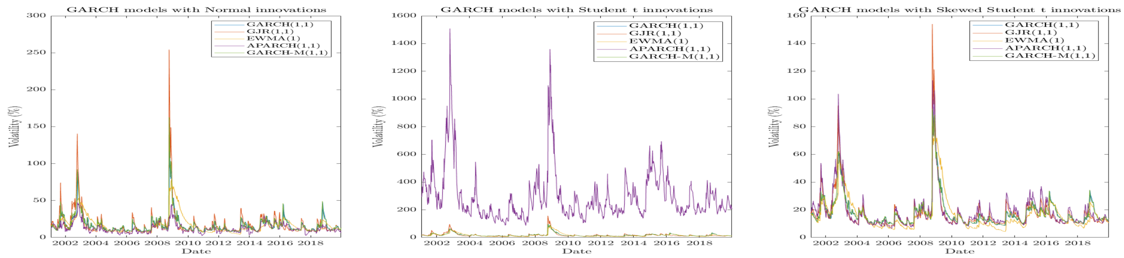
Notes: the figure shows the value of the standard deviation extrapolated 52 periods. We consider innovations follow a Normal, Student t or Skewed Student t distribution.

Figure 53: GARCH models for BRL/USD serie.



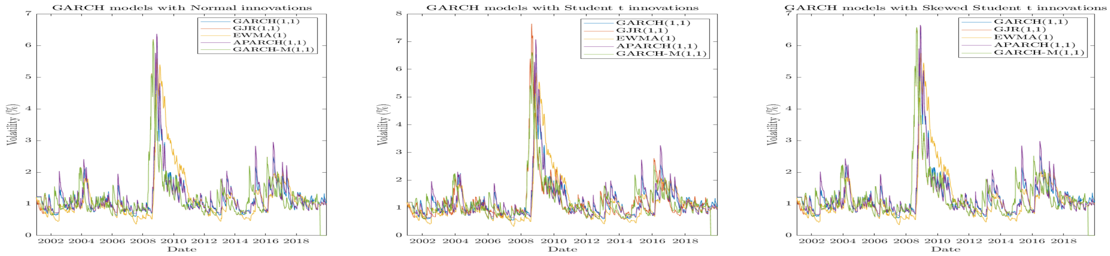
Notes: the figure shows the value of the standard deviation extrapolated 52 periods. We consider innovations follow a Normal, Student t or Skewed Student t distribution.

Figure 54: GARCH models for BOVESPA serie.



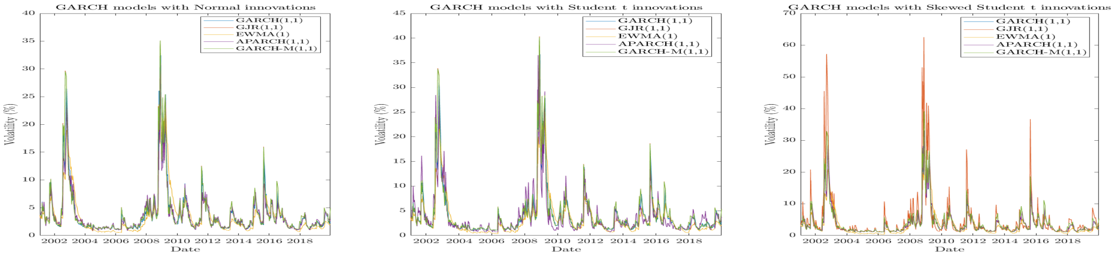
Notes: the figure shows the value of the standard deviation extrapolated 52 periods. We consider innovations follow a Normal, Student t or Skewed Student t distribution.

Figure 55: GARCH models for GBP/USD serie.



Notes: the figure shows the value of the standard deviation extrapolated 52 periods. We consider innovations follow a Normal, Student t or Skewed Student t distribution.

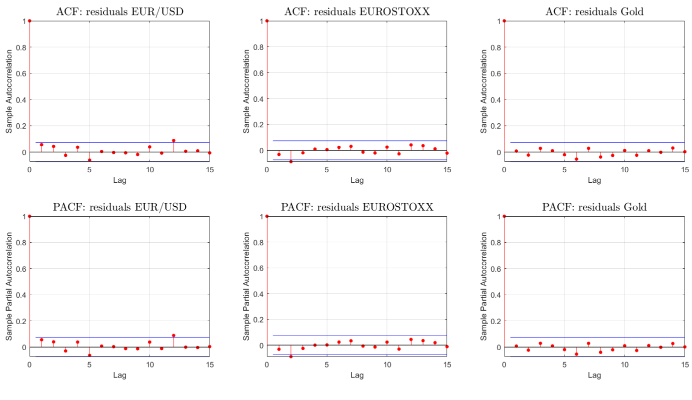
Figure 56: GARCH models for FTSE100 serie.



Notes: the figure shows the value of the standard deviation extrapolated 52 periods. We consider innovations follow a Normal, Student t or Skewed Student t distribution.

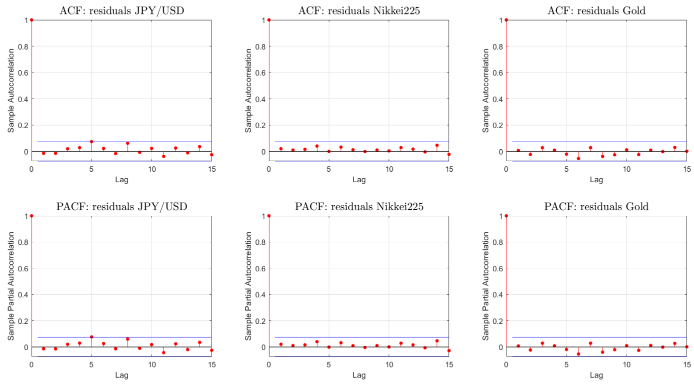
We can observe in the auto-correlation (acf) and partial auto-correlation (pacf) functions of the residuals of the three returns in each region (Figures 57 to 60) that these innovations are actually white noise.

Figure 57: Auto-correlation and partial auto-correlation of the residuals for EUR/USD, EUROSTOXX50 and gold.



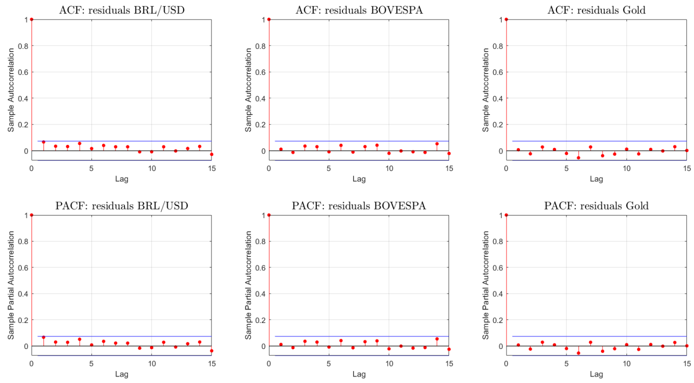
Notes: returns adjusted by an ARMA(1,12)-APARCH(1,1) model for the three cases and assuming innovations are Skewed Student t. The parameters are estimated using maximum loglikelihood.

Figure 58: Auto-correlation and partial auto-correlation of the residuals for JPY/USD, Nikkei225 and gold.



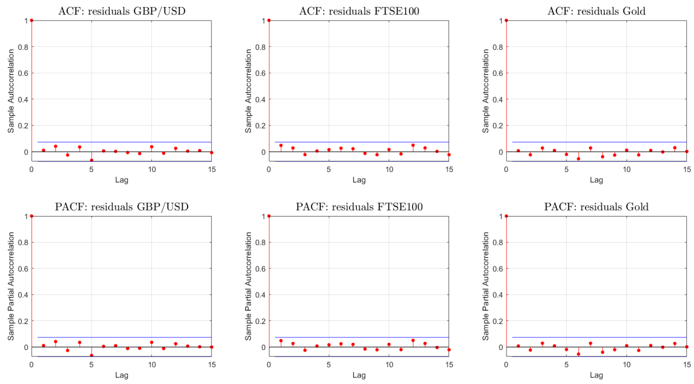
Notes: returns adjusted by an ARMA(1,12)-APARCH(1,1) model for the three cases and assuming innovations are Skewed Student t. The parameters are estimated using maximum loglikelihood.

Figure 59: Auto-correlation and partial auto-correlation of the residuals for BRL/USD, BOVESPA and gold.



Notes: returns adjusted by an ARMA(1,12)-APARCH(1,1) model for the three cases and assuming innovations are Skewed Student t. The parameters are estimated using maximum loglikelihood.

Figure 60: Auto-correlation and partial auto-correlation of the residuals for GBP/USD, FTSE100 and gold.



Notes: returns adjusted by an ARMA(1,12)-APARCH(1,1) model for the three cases and assuming innovations are Skewed Student t. The parameters are estimated using maximum loglikelihood.

F Probabilities of the different scenarios

In this study we talk about the role of exchange rate and gold as safe haven against adverse movements in stock markets. If first we do not consider the importance of the exchange rate in portfolios, we have two possibilities: first, stock and gold could move in the same direction, i.e., both prices could arise and so the potencial profit of our portfolio increases, or well both prices fall and in consecuence the losses of the portfolio are aggravated. A second possibility is that stock and gold move in opposite directions. In this case, gold will act as a safe haven so the impact of the stock market fall will be cushioned, but if stock market goes up the profit of our portfolio will be reduced. However we must also consider the role of the exchange rate risk, which is the principal object of this thesis. The exchange rate can minimize or aggravate the losses of the portfolio if the price of gold falls, as well it can enhance or reduce its profits in the case the price of gold increases. So now we do not have four possible scenarios, but eight:

- Gold and stock move in the same direction:
  - **Scenario 1:** Bullish stock-bullish gold-bullish exchange rate. This is the best scenario possible, because we will obtain a profit from the investments in stock and gold, and the depreciation of the currency increases this result. The depreciation of the investor's domestic currency implies that he needs less dollars to buy an unit of that currency, or, in an equivalent form, with one US dollar he obtains more units of his domestic currency.

    The probability of this scenario is $P(r_e > x, r_s > y, r_g > z)$; where r_e, r_s and r_g refer to the returns of the three assets, i.e., exchange rate, stock and gold, respectively. On the other hand, x, y and z represent the value given by the inverse cdf of the distribution of each asset in a certain quantile. This means these values can be expressed as: $x = VaR_e(\alpha)$, $y = VaR_s(\beta)$ and $z = VaR_g(\gamma)$. Since α, β and γ represents the lowest quantiles of the distributions, we have that the probability $P(r_e > x, r_s > y, r_g > z)$ is $P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma))$.

    We can calculate this probability as follows:

$$\begin{aligned} &P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = \\ &= P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta)) - P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g < VaR_g(1-\gamma)) \end{aligned} \tag{F.1}$$

    where

$$P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta)) = P(r_e > VaR_e(1-\alpha)) - P(r_e > VaR_e(1-\alpha), r_s < VaR_s(1-\beta)) \tag{F.2}$$

    Here the minuend $P(r_e > VaR_e(1-\alpha))$ is $1 - P(r_e < VaR_e(1-\alpha))$ and the subtrahend, $P(r_e > VaR_e(1-\alpha), r_s < VaR_s(1-\beta))$, is equal to $P(r_s < VaR_s(1-\beta)) - P(r_e < VaR_e(1-\alpha), r_s < VaR_s(1-\beta))$. Substituting these expressions in (F.2) we can rewrite it as

$$\begin{aligned} &P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta)) = \\ &= 1 - P(r_e < VaR_e(1-\alpha)) - P(r_s < VaR_s(1-\beta)) + P(r_e < VaR_e(1-\alpha), r_s < VaR_s(1-\beta)) \end{aligned} \tag{F.3}$$

    The subtrahend of (F.1) can be also rewritten:

$$\begin{aligned} &P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g < VaR_g(1-\gamma)) = \\ &= P(r_g < VaR_g(1-\gamma)) - P(r_e < VaR_e(1-\alpha), r_g < VaR_g(1-\gamma)) \\ &- P(r_s < VaR_s(1-\beta), r_g < VaR_g(1-\gamma)) + P(r_e < VaR_e(1-\alpha), r_s < VaR_s(1-\beta), r_g < VaR_g(1-\gamma)) \end{aligned} \tag{F.4}$$

    Hence, using (F.3) and (F.4) in Eq. (F.1) we have the following expression:

$$\begin{aligned} &P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = 1 - P(r_e < VaR_e(1-\alpha)) \\ &-P(r_s < VaR_s(1-\beta)) + P(r_e < VaR_e(1-\alpha), r_s < VaR_s(1-\beta)) - P(r_g < VaR_g(1-\gamma)) \\ &+P(r_e < VaR_e(1-\alpha), r_g < VaR_g(1-\gamma)) + P(r_s < VaR_s(1-\beta), r_g < VaR_g(1-\gamma)) \\ &-P(r_e < VaR_e(1-\alpha), r_s < VaR_s(1-\beta), r_g < VaR_g(1-\gamma)) \end{aligned}$$

$$
\begin{aligned}
\Rightarrow & P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = \\
& = 1 - P(r_e < VaR_e(1-\alpha)) - P(r_s < VaR_s(1-\beta)) - P(r_g < VaR_g(1-\gamma)) \\
& \quad + P(r_e < VaR_e(1-\alpha), r_s < VaR_s(1-\beta)) + P(r_e < VaR_e(1-\alpha), r_g < VaR_g(1-\gamma)) \\
& \quad + P(r_s < VaR_s(1-\beta), r_g < VaR_g(1-\gamma)) \\
& \quad - P(r_e < VaR_e(1-\alpha), r_s < VaR_s(1-\beta), r_g < VaR_g(1-\gamma))
\end{aligned} \tag{F.5}
$$

In terms of copulas, Eq. (F.5) can be rewritten as:
$P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = 1 - (1-\alpha) - (1-\beta) - (1-\gamma)$
$+C_{e,s}(1-\alpha, 1-\beta) + C_{e,g}(1-\alpha, 1-\gamma) + C_{s,g}(1-\beta, 1-\gamma) - C_{e,s,g}(1-\alpha, 1-\beta, 1-\gamma)$

$$
\begin{aligned}
\Rightarrow & P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = \alpha + \beta + \gamma - 2 \\
& + C_{e,s}(1-\alpha, 1-\beta) + C_{e,g}(1-\alpha, 1-\gamma) + C_{s,g}(1-\beta, 1-\gamma) - C_{e,s,g}(1-\alpha, 1-\beta, 1-\gamma)
\end{aligned} \tag{F.6}
$$

- **Scenario 2:** Bullish stock-bullish gold-bearish exchange rate. In this case the return of the exchange rate reduces the profit obtained with the invest in stock and gold. Depending on the magnitude of the variations in the prices of the three assets the benefit would be positive, zero or even negative.
  This probability is:

$$
\begin{aligned}
& P(r_e < VaR_e(\alpha), r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = \\
& = P(r_e < VaR_e(\alpha), r_s > VaR_s(1-\beta)) - P(r_e < VaR_e(\alpha), r_s > VaR_s(1-\beta), r_g < VaR_g(1-\gamma))
\end{aligned} \tag{F.7}
$$

  On the one hand, the first term of this equation is:

$$
P(r_e < VaR_e(\alpha), r_s > VaR_s(1-\beta)) = P(r_e < VaR_e(\alpha)) - P(r_e < VaR_e(\alpha), r_s < VaR_s(1-\beta)) \tag{F.8}
$$

  On the other hand, the subtrahend of Eq. (F.7), is:

$$
\begin{aligned}
& P(r_e < VaR_e(\alpha), r_s > VaR_s(1-\beta), r_g < VaR_g(1-\gamma)) = \\
& = P(r_e < VaR_e(\alpha), r_g < VaR_g(1-\gamma)) - P(r_e < VaR_e(\alpha), r_s < VaR_s(1-\beta), r_g < VaR_g(1-\gamma))
\end{aligned} \tag{F.9}
$$

  Given this expression and the previous definition of $P(r_e < VaR_e(\alpha), r_s > VaR_s(1-\beta))$, we can rewrite Eq. (F.7) as:

$$
\begin{aligned}
& P(r_e < VaR_e(\alpha), r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = \\
& = P(r_e < VaR_e(\alpha)) - P(r_e < VaR_e(\alpha), r_s < VaR_s(1-\beta)) \\
& - P(r_e < VaR_e(\alpha), r_g < VaR_g(1-\gamma)) + P(r_e < VaR_e(\alpha), r_s < VaR_s(1-\beta), r_g < VaR_g(1-\gamma))
\end{aligned} \tag{F.10}
$$

  Expressing the latter equation in terms of copulas we finally obtain:

$$
\begin{aligned}
& P(r_e < VaR_e(\alpha), r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = \\
& = \alpha - C_{e,s}(\alpha, 1-\beta) - C_{e,g}(\alpha, 1-\gamma) + C_{e,s,g}(\alpha, 1-\beta, 1-\gamma)
\end{aligned} \tag{F.11}
$$

- **Scenario 3:** Bearish stock-bearish gold-bullish exchange rate. Here including the exchange rate in the strategy favors us because it reduces the losses caused by the drop in the gold market. Like in the previous case, depending on the magnitude of the variations in the prices of the three assets the result may remain negative, but it is possible also to obtain no-losses or even a profit.
  This probability can be expressed as:

$$
\begin{aligned}
& P(r_e > VaR_e(1-\alpha), r_s < VaR_s(\beta), r_g < VaR_g(\gamma)) = \\
& = P(r_s < VaR_s(\beta), r_g < VaR_g(\gamma)) - P(r_e < VaR_e(1-\alpha), r_s < VaR_s(\beta), r_g < VaR_g(\gamma))
\end{aligned} \tag{F.12}
$$

Expressing Eq. (F.12) in terms of copulas:

$$P(r_e > VaR_e(1-\alpha), r_s < VaR_s(\beta), r_g < VaR_g(\gamma)) = C_{s,g}(\beta,\gamma) - C_{e,s,g}(1-\alpha,\beta,\gamma) \quad \text{(F.13)}$$

- **Scenario 4:** Bearish stock-bearish gold-bearish exchange rate. This is the worst scenario possible because of the appreciation of the local currency amplifies the bad performance of the portfolio. The probability of this scenario can be expressed in terms of copulas directly:

$$P(r_e < VaR_e(\alpha), r_s < VaR_s(\beta), r_g < VaR_g(\gamma)) = C_{e,s,g}(\alpha,\beta,\gamma) \quad \text{(F.14)}$$

The probabilities of such situations occurring in Europe appear in Figure 61.

- Gold and stock move in opposite directions:
  - **Scenario 5:** Bullish stock-bearish gold-bullish exchange rate. This depreciation minimizes the impact of the fall in gold prices on the return of the portfolio. The sign of the result of this portfolio in unknowed, it will depend on the magnitude of the variations on the prices of all assets.
    This situation can be represented by the following probability:

$$P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g < VaR_g(\gamma)) =$$
$$= P(r_e > VaR_e(1-\alpha), r_g < VaR_g(\gamma)) - P(r_e > VaR_e(1-\alpha), r_s < VaR_s(1-\beta), r_g < VaR_g(\gamma)) \quad \text{(F.15)}$$

where

$$P(r_e > VaR_e(1-\alpha), r_g < VaR_g(\gamma)) = P(r_g < VaR_g(\gamma)) - P(r_e < VaR_e(1-\alpha), r_g < VaR_g(\gamma)) \quad \text{(F.16)}$$

and

$$P(r_e > VaR_e(1-\alpha), r_s < VaR_s(1-\beta), r_g < VaR_g(\gamma)) =$$
$$= P(r_s < VaR_s(1-\beta), r_g < VaR_g(\gamma)) - P(r_e < VaR_e(1-\alpha), r_s < VaR_s(1-\beta), r_g < VaR_g(\gamma)) \quad \text{(F.17)}$$

Sustituing (F.16) and (F.17) in equation (F.15) we obtain:

$$P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g < VaR_g(\gamma)) =$$
$$= P(r_g < VaR_g(\gamma)) - P(r_e < VaR_e(1-\alpha), r_g < VaR_g(\gamma))$$
$$- P(r_s < VaR_s(1-\beta), r_g < VaR_g(\gamma)) + P(r_e < VaR_e(1-\alpha), r_s < VaR_s(1-\beta), r_g < VaR_g(\gamma)) \quad \text{(F.18)}$$

We can express this probability in terms of copulas:

$$\begin{aligned} &P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g < VaR_g(\gamma)) = \\ &= \gamma - C_{e,g}(1-\alpha,\gamma) - C_{s,g}(1-\beta,\gamma) + C_{e,s,g}(1-\alpha,1-\beta,\gamma) \end{aligned} \quad \text{(F.19)}$$

  - **Scenario 6:** Bullish stock-bearish gold-bearish exchange rate. The appreciation of the local currency which increases the losses caused by the invest in gold. Only if the positive returns of the stock are higher than the negative returns of exchange rate and gold the portfolio will provide a profit.
    This probability can be computed as follows:

$$P(r_e < VaR_e(\alpha), r_s > VaR_s(1-\beta), r_g < VaR_g(\gamma)) =$$
$$= P(r_e < VaR_e(\alpha), r_g < VaR_g(\gamma)) - P(r_e < VaR_e(\alpha), r_s < VaR_s(1-\beta), r_g < VaR_g(\gamma)) \quad \text{(F.20)}$$

We can express Eq. (F.20) in terms of copulas as:

$$P(r_e < VaR_e(\alpha), r_s > VaR_s(1-\beta), r_g < VaR_g(\gamma)) = C_{e,g}(\alpha,\gamma) - C_{e,s,g}(\alpha,1-\beta,\gamma) \quad \text{(F.21)}$$

- **Scenario 7:** Bearish stock-bullish gold-bullish exchange rate. In this scenario gold acts as a safe haven, helping to mitigate losses produced in stock markets, and the depreciation of the local currency amplifies the profits in gold investment. Thanks to it, the profit of the portfolio increases, or at least its losses are lower.

  We can obtain the probability of this scenario as follows:

$$P(r_e > VaR_e(1-\alpha), r_s < VaR_s(\beta), r_g > VaR_g(1-\gamma)) = \\ = P(r_e > VaR_e(1-\alpha), r_s < VaR_s(\beta)) - P(r_e > VaR_e(1-\alpha), r_s < VaR_s(\beta), r_g < VaR_g(1-\gamma)) \tag{F.22}$$

  The minuend of Eq. (F.22) is:

$$P(r_e > VaR_e(1-\alpha), r_s < VaR_s(\beta)) = P(r_s < VaR_s(\beta)) - P(r_e < VaR_e(1-\alpha), r_s < VaR_s(\beta)) \tag{F.23}$$

  And the subtrahend:

$$P(r_e > VaR_e(1-\alpha), r_s < VaR_s(\beta), r_g < VaR_g(1-\gamma)) = \\ = P(r_s < VaR_s(\beta), r_g < VaR_g(1-\gamma)) - P(r_e < VaR_e(1-\alpha), r_s < VaR_s(\beta), r_g < VaR_g(1-\gamma)) \tag{F.24}$$

  If we rewrite equation (F.22) sustituing the minuend and sustraend by expressions in (F.23) and (F.24), respectively, we obtain the following equation:

$$P(r_e > VaR_e(1-\alpha), r_s < VaR_s(\beta), r_g > VaR_g(1-\gamma)) = \\ = P(r_s < VaR_s(\beta)) - P(r_e < VaR_e(1-\alpha), r_s < VaR_s(\beta)) \\ - P(r_s < VaR_s(\beta), r_g < VaR_g(1-\gamma)) + P(r_e < VaR_e(1-\alpha), r_s < VaR_s(\beta), r_g < VaR_g(1-\gamma)) \tag{F.25}$$

  The result of expressing the probability in equation (F.25) in terms of copulas is:

$$P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g < VaR_g(\gamma)) = \\ = \beta - C_{e,s}(1-\alpha, \beta) - C_{s,g}(\beta, 1-\gamma) + C_{e,s,g}(1-\alpha, \beta, 1-\gamma) \tag{F.26}$$

- **Scenario 8:** Bearish stock-bullish gold-bearish exchange rate. Also in this case gold acts as a safe haven, but the fall in the exchange rate reduces the potencial benefits of including gold in the portfolio.

  The probability of this scenario can be expressed as follows:

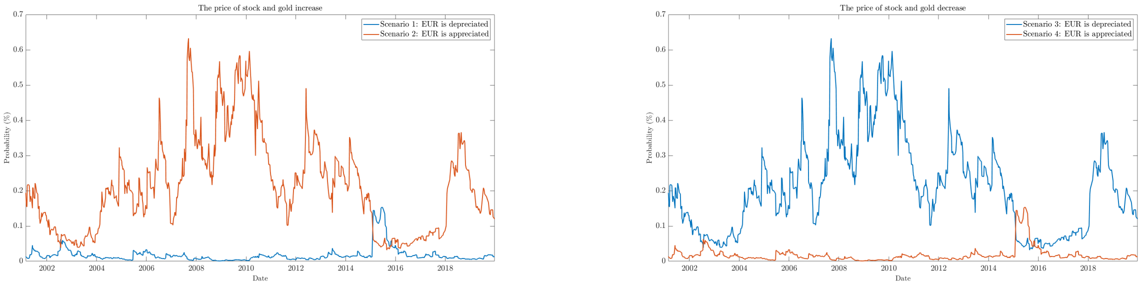
$$P(r_e < VaR_e(\alpha), r_s < VaR_s(\beta), r_g > VaR_g(1-\gamma)) = \\ = P(r_e < VaR_e(\alpha), r_s < VaR_s(\beta)) - P(r_e < VaR_e(\alpha), r_s < VaR_s(\beta), r_g < VaR_g(1-\gamma)) \tag{F.27}$$

  By means of copulas, this equation can be rewritten as follows:

$$P(r_e < VaR_e(\alpha), r_s < VaR_s(\beta), r_g > VaR_g(1-\gamma)) = C_{e,s}(\alpha, \beta) - C_{e,s,g}(\alpha, \beta, 1-\gamma) \tag{F.28}$$

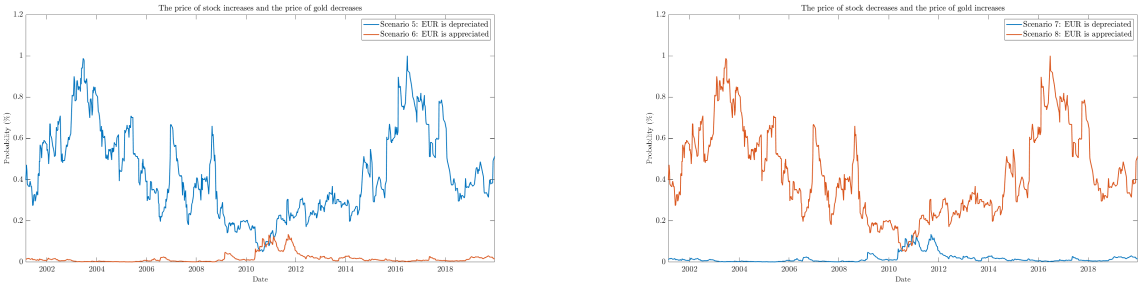
The probabilities of these four situations for the case of Europe appear in Figure 62.

Figure 61: Probability of scenarios in which stock index and gold move in the same direction in Europe.



Notes: left Panel shows bullish scenarios for EUROSTOXX50 and gold market. In the first possible situation (blue line) the local currency is depreciated, so the profit of the portfolio is greater. In the second situation (orange line) there is an appreciation of the euro, and consecuently the results of the portfolio are lower, zero or even negative. In right Panel appear the probability bearish scenarios for EUROSTOXX50 and gold market. In the first possible situation (blue line) the euro is depreciated, so the impact of the markets' falls are deadened. The second scenario (orange line) shows an appreciation of the euro, which accentuates the losses of the portfolio.

Figure 62: Probability of scenarios in which stock index and gold move in opposite directions in Europe.



Notes: left Panel shows a bullish scenario for EUROSTOXX50 and a bearish scenario for gold market. In the first possible situation (blue line) there is a depreciation of the euro, and this fact reduces the negative return of the gold investment. In the second situation (orange line) there is an appreciation of this currency, which accentuates the losses in this investment. In both cases the final result of the portfolio is uncertain. In right Panel appears the probability of a bearish scenario for EUROSTOXX50 jointly a bullish scenario for gold market. In the first possible situation (blue line) the euro suffers a depreciation, so this asset also helps to minimize the impact of the stock loss in the portfolio. In the second situation (orange line) there is an appreciation of the domestic currency, which reduces the safe haven role of gold in the portfolio. In both cases the final result of the portfolio is uncertain.

Figures 61 and 62 present "reflected probabilities", in the sense we can observate the probability of Scenarios 1 and 4, Scenarios 2 and 3, Scenarios 5 and 8, and Scenarios 6 and 7, respectively, are statistically identical. In terms of probabilities that means that:

$$P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = P(r_e < VaR_e(\alpha), r_s < VaR_s(\beta), r_g < VaR_g(\gamma))$$

$$P(r_e < VaR_e(\alpha), r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = P(r_e > VaR_e(1-\alpha), r_s < VaR_s(\beta), r_g < VaR_g(\gamma))$$

$$P(r_e > VaR_e(1-\alpha), r_s > VaR_s(1-\beta), r_g < VaR_g(\gamma)) = P(r_e < VaR_e(\alpha), r_s < VaR_s(\beta), r_g > VaR_g(1-\gamma))$$

and

$$P(r_e < VaR_e(\alpha), r_s > VaR_s(1-\beta), r_g < VaR_g(\gamma)) = P(r_e > VaR_e(1-\alpha), r_s < VaR_s(\beta), r_g > VaR_g(1-\gamma))$$

It shows that the distribution of the Gaussian copula is symmetric, as we can supose given the characteristics of Normal distribution.

In Figure 61 we can see that extreme situations in which all assets are above its τ-highest quantile or well under its τ-lowest quantile are not very probable. The probability of being in both rarely is over 0.1%. We can notice in left Panel of this Figure that the probability of a depreciation of the euro takes its highest values in 2015, when occurs a great injection of liquidity due to a policy of ECB. Because of the symmetry of Normal distribution, the opposite scenario, i.e. stock, gold and exchange rate crashing markets, has the same probability. In addition, in the period from 2007 to 2010, EUROSTOXX50 value falls and the euro was depreciated in consecuence, so the probability of an scenario in which these two facts occurs, i.e. Scenarios 2 and 3, can be elevated.

A similar situation is presented in Figure 62. The probabilities of Scenarios 6 and 7 are very low, but reach their highest values between 2010 and 2012. If we pay attention to Scenario 7, is logical to think that the probability of a sharp fall on EUROSTOXX50 price and due to the greater offer of euros this currency is depreciated and the price of gold incresases will be higher.

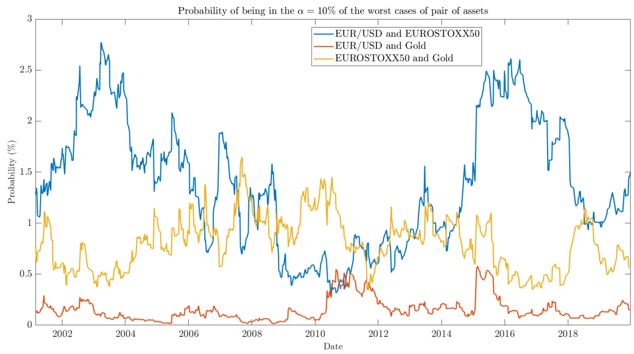
Those more probables scenarios are 5 and 8, both with the same probability. In the former we have explained that the depreciation of the euro minimizes the impact of the fall in gold prices on the return of the portfolio. In this situation an extra-investment in the exchange rate could help the investor reducing his losses or increasing his profits (the sign of the result of the portfolio depends also on the profit in the stock market). However, in Scenario 8 stock prices decrease, gold prices go up and it occurs an appreciation of the euro which reduces the potencial benefits of including gold in the portfolio, so in this case a possible extra-investment in the exchange rate would be detrimental to the investor.

We can also obtain the probability of two of the assets are in their τ-worst cases, whatever is the situation of the remain asset and express it in terms of copulas:

- $P(r_e < VaR_e(\alpha), r_s < VaR_s(\beta)) = C_{e,s}(\alpha, \beta)$
- $P(r_e < VaR_e(\alpha), r_g < VaR_g(\gamma)) = C_{e,g}(\alpha, \gamma)$
- $P(r_s < VaR_s(\beta), r_g < VaR_g(\gamma)) = C_{s,g}(\beta, \gamma)$

These probabilities appear in Figure 63.

Figure 63: Worst scenario for two of the variables, EUR/USD, EUROSTOXX50 or gold.



Note: the figure presents the probability of scenarios in which two of the three assets, EUR/USD and EUROSTOXX50, EUR/USD and gold, or EUROSTOXX50 and gold, respectively, are in their τ% lower quantile.

In the same sense, we can obtain the probability of two of the assets are in their τ-best cases, or well the probability of one of the assets is in its upper tail while other is in the opposite tail. In other words, it is useful to study not only the probability of that scenario in which two assets are under their corresponding VaR threshole, but the probabilities of all scenarios which envolves both assets. For example, it can be useful to obtain the four possible scenarios existing combining different situations for stock and gold, ignoring the behaviour of exchange rate:

- Bullish stock-bullish gold. It is equivalent to say that both assets are in their β or γ-highest quantile, respectively. This probability is:

  $$P(r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = P(r_s > VaR_s(1-\beta)) - P(r_s > VaR_s(1-\beta), r_g < VaR_g(1-\gamma)) \quad \text{(F.29)}$$

  The subtrahend of Eq. (F.29) is:

  $$P(r_s > VaR_s(1-\beta), r_g < VaR_g(1-\gamma)) = P(r_g < VaR_g(1-\gamma)) - P(r_s < VaR_s(1-\beta), r_g < VaR_g(1-\gamma)) \quad \text{(F.30)}$$

Given that $P(r_s > VaR_s(1-\beta)) = 1 - P(r_s < VaR_s(1-\beta))$ and the previous definition of $P(r_s > VaR_s(1-\beta), r_g < VaR_g(1-\gamma))$, we have:

$$P(r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = \\ = 1 - P(r_s < VaR_s(1-\beta)) - P(r_g < VaR_g(1-\gamma)) + P(r_s < VaR_s(1-\beta), r_g < VaR_g(1-\gamma)) \tag{F.31}$$

Expressed in terms of copulas, Eq. (F.31) is equal to:

$$P(r_s > VaR_s(1-\beta), r_g > VaR_g(1-\gamma)) = \\ = 1 - (1-\beta) - (1-\gamma) + C_{s,g}(1-\beta, 1-\gamma) = \beta + \gamma - 1 + C_{s,g}(1-\beta, 1-\gamma) \tag{F.32}$$

- Bearish stock-bearish gold. In this occasion both assets are in their β or γ-lowest quantile, respectively, i.e., are under its VaR at β or γ level. In terms of copulas, its probability is:

$$P(r_s < VaR_s(\beta), r_g < VaR_g(\gamma)) = C_{s,g}(\beta, \gamma) \tag{F.33}$$

- Bullish stock-bearish gold. The probability of this scenario is:

$$P(r_s > VaR_s(1-\beta), r_g < VaR_g(\gamma)) = P(r_g < VaR_g(\gamma)) - P(r_s < VaR_s(1-\beta), r_g < VaR_g(\gamma)) \tag{F.34}$$

  If we rewrite this equation using copulas we have:

$$P(r_s > VaR_s(1-\beta), r_g < VaR_g(\gamma)) = \gamma - C_{s,g}(1-\beta, \gamma) \tag{F.35}$$

- Bearish stock-bullish gold. The probability of this scenario can be expressed as:

$$P(r_s < VaR_s(\beta), r_g > VaR_g(1-\gamma)) = P(r_s < VaR_s(\beta)) - P(r_s < VaR_s(\beta), r_g < VaR_g(1-\gamma)) \tag{F.36}$$

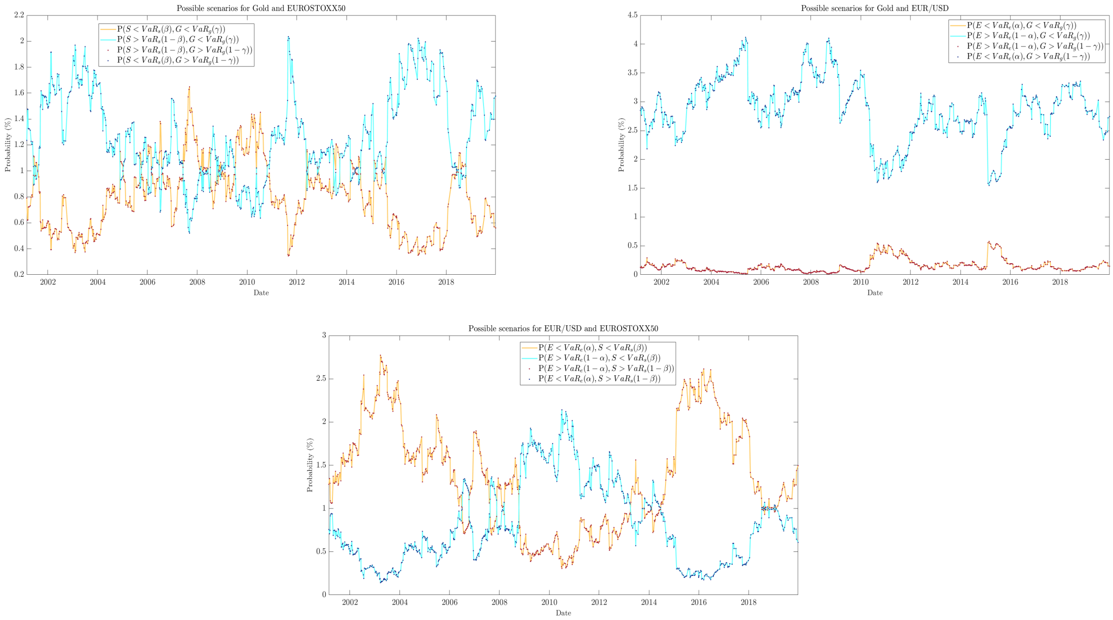
  And this equation can be expressed in terms of copulas as follows:

$$P(r_s < VaR_s(\beta), r_g > VaR_g(1-\gamma)) = \beta - C_{s,g}(\beta, 1-\gamma) \tag{F.37}$$

The probabilities of EUROSTOXX50 and gold being in these four situations are contained in top left Panel of Figure 64.

Apart form the analysis of the jointly behaviour of stock index and gold prices, the study of the relationships between exchange rate and gold can be useful to analyse the role of exchange rate in a portfolio which includes gold. They appear in top right Panel of Figure 64 for the case of Europe. Furthermore, bottom Panel of Figure 64 presents the probability of those possible scenarios for the exchange rate, EUR/USD, and the stock index.

Figure 64: Possible scenarios for pair of assets in Europe.



Notes: top left Panel presents the different possible scenarios for gold and EUROSTOXX50. Top right Panel contains the probabilities of all possible scenarios for gold and EUR/USD. Finally, bottom Panel shows the different possible scenarios for EUR/USD and EUROSTOXX50.

We notice in top left Panel of Figure 64 that the probability of both assets being in the same situation (both increase or both decrease) and the probabilities of both assets being in the opposite situation (one increases and the other decreases) are almost symmetric.

Top right Panel of Figure 64 evidences that the probability of gold and EUR/USD being in the same situation, i.e., both on its τ-highest or on its τ-lowest quantile, is much lower than the probability of they are in opposite situations. It implies that when gold price grows the appreciation of the euro will reduce the profit of the investment in gold, but when the price of this asset decreases, the depreciation of the euro will reduce the impact of this downward movement.

We can do the same scenario analysis for the rest of regions.

In Figures 65 and 66 appear the eight possible scenarios if we consider the different situations for JPY/USD, Nikkei225 and gold. Figure 67 shows the probability of the worst scenario of each pair of assets; i.e., the probability of JPY/USD and gold, Nikkei225 and gold, and JPY/USD and Nikkei225 are respectively under their VaR($\alpha = 10\%$). Figure 68 contains all the possible situations when we the behaviour of two of the assets.

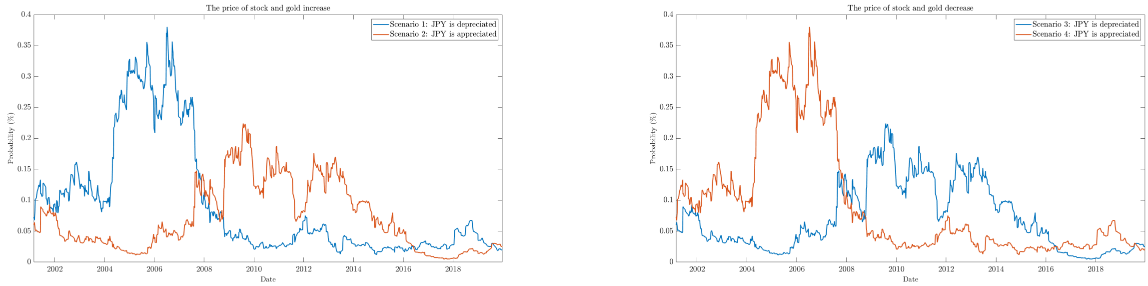
We can appreciate in Figures 65 and 66 that the begin of the financial global crisis produces a change in the probabilities of all scenarios. On the one hand, in the latter Figure we see the probability of Scenarios 6 and 7 is always higher than probability of Scenarios 5 and 8, but since 2008 the probability of the former scenarios increases while Scenarios 5 and 8 become even less probable.

On the other hand, in Figure 65 this change is more clear. We will comment the case of left Panel of this figure, and later we can extrapolate the results and comments to the right Panel because of the symmetry of Gaussian distribution. Until 2008 the probability of a bullish scenario in the three markets is higher than that situation in which exchange rate is appreciated while stock index and gold prices are increasing. The explanation of this fact will be found in Figure 31, where we can see that between 2003 and 2008 the three assets moves in the same direction, but since the begining of the global financial crisis Nikkei225 starts a bearish trend, unlike exchange rate and gold. Because of that the probability of Scenarios 1 and 4 becomes much lower. However, since 2016 these Scenarios turn again more probable than Scenarios 2 and 3 because from that year to the end of the sample the all three assets prices increase.

This fact also explains the probabilities in Figure 67. Here we can see that from 2008 the probability of

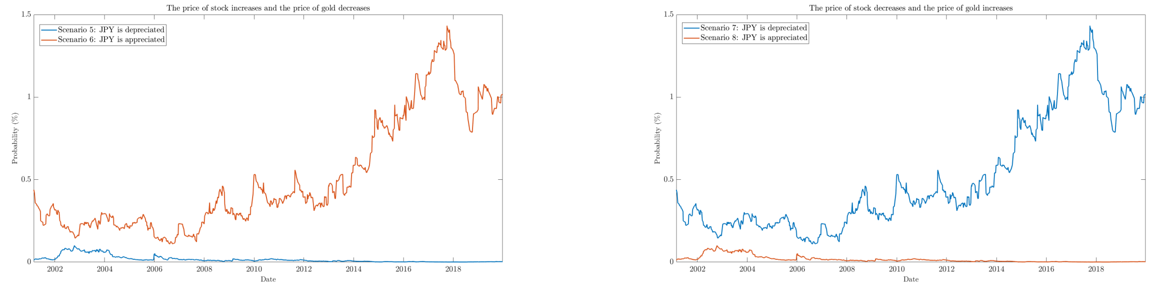
JPY/USD exchange rate and gold are under their VaR(10%) increases, while the probability of this situation for Nikkei225 and gold and JPY/USD and Nikkei225 decreases since stock index takes an opposite trend than to other two assets.

Figure 65: Probability of scenarios in which stock index and gold move in the same direction in Japan.



Notes: left Panel shows bullish scenarios for Nikkei225 and gold market. In the first possible situation (blue line) the local currency is depreciated, so the profit of the portfolio is greater. In the second situation (orange line) there is an appreciation of the yen, and consecuently the results of the portfolio are lower, zero or even negative. In right Panel appear the probability bearish scenarios for Nikkei225 and gold market. In the first possible situation (blue line) the yen is depreciated, so the impact of the markets' falls are deadened. The second scenario (orange line) shows an appreciation of the yen, which accentuates the losses of the portfolio.

Figure 66: Probability of scenarios in which stock index and gold move in opposite directions in Japan.

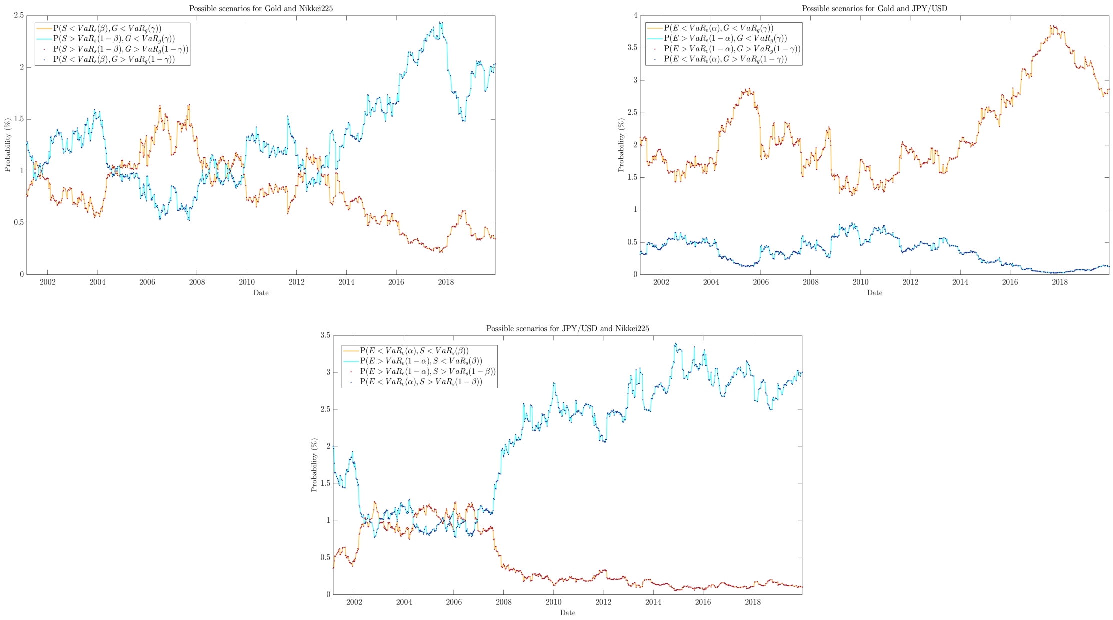


Notes: left Panel shows a bullish scenario for Nikkei225 and a bearish scenario for gold market. In the first possible situation (blue line) there is a depreciation of the yen, and this fact reduces the negative return of the gold investment. In the second situation (orange line) there is an appreciation of this currency, which accentuates the losses in this investment. In both cases the final result of the portfolio is uncertain. In right Panel appears the probability of a bearish scenario for Nikkei225 jointly a bullish scenario for gold market. In the first possible situation (blue line) the yen suffers a depreciation, so this asset also helps to minimize the impact of the stock loss in the portfolio. In the second situation (orange line) there is an appreciation of the domestic currency, which reduces the safe haven role of gold in the portfolio. In both cases the final result of the portfolio is uncertain.

Figure 67: Worst scenario for two of the variables, JPY/USD, Nikkei225 or gold.

Note: the figure presents the probability of scenarios in which two of the three assets, JPY/USD and Nikkei225, JPY/USD and gold, or Nikkei225 and gold, respectively, are in their $\tau\%$ lower quantile.

Figure 68: Possible scenarios for pair of assets in Japan.



Notes: top left Panel presents the different possible scenarios for gold and Nikkei225. Top right Panel contains the probabilities of all possible scenarios for gold and JPY/USD. Finally, bottom Panel shows the different possible scenarios for JPY/USD and Nikkei225.

In Figure 68 we can see that the scenarios more probables, especially since the financial crisis, are, respectively, those in which gold and Nikkei225 prices take opposite directions, in which gold and depreciation of the yen have positive correlation, and in which the appreciation of yen is accompanied by a sharp rise of Nikkei225, or viceversa. Summarizing, the most probable scenario occurs when JPY/USD and Nikkei225 move in the same direction while gold price is in the opposite situation. This is exactly what we can observate also in Figure 66 (Scenarios 6 and 7).

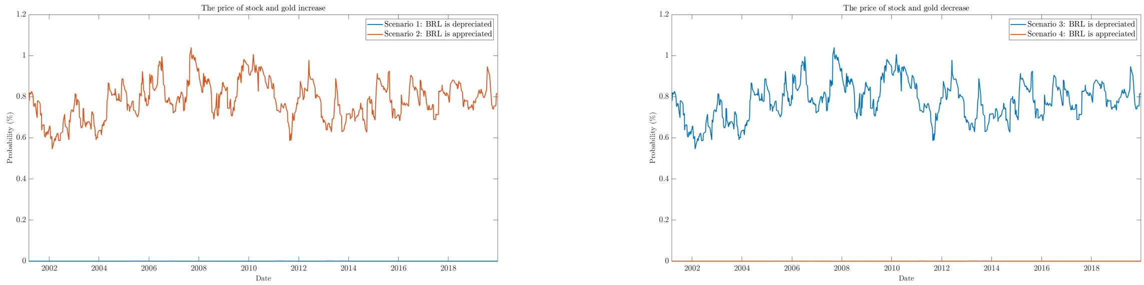
Figures 69 and 70 present the probability of the eight possible scenarios in Brazil. Figure 71 shows the probability of the worst scenario of each pair of assets, i.e., the probability of BRL/USD and gold, BOVESPA and gold, and BRL/USD and BOVESPA are respectively under their VaR(10%). Finally, Figure 72 contains all possible situations when we study the behaviour of two of the assets.

In Figures 69 and 70, on the one hand, we found the most probable scenario ocurring when there is bullish stock and gold markets and the Brazilian real is appreciated (Scenario 2), and when the stock and gold markets are crashing and the Brazilian real is depreciated (Scenario 3), both with the same probability. If Scenario 2 happens, the portfolio will obtain profits from gold and BOVESPA investment, but in this occasion the profit obtained with gold will be reduced because of the appreciation of Brazilian real. On the contrary, if Scenario 3 occurs it would be suitable to increase the investment in this exchange rate.

On the other hand, we notice that scenarios in which the stock index value grows and the currency suffers a depreciation, or well the index value falls and the currency is appreciated, take probabilities almost meaningless. These findings appear to be deducible in view of the results in Section 5.2. (and Figure 3), where we have advice a strong negative correlation between the evolution of the index and the depreciation of the Brazilian real.

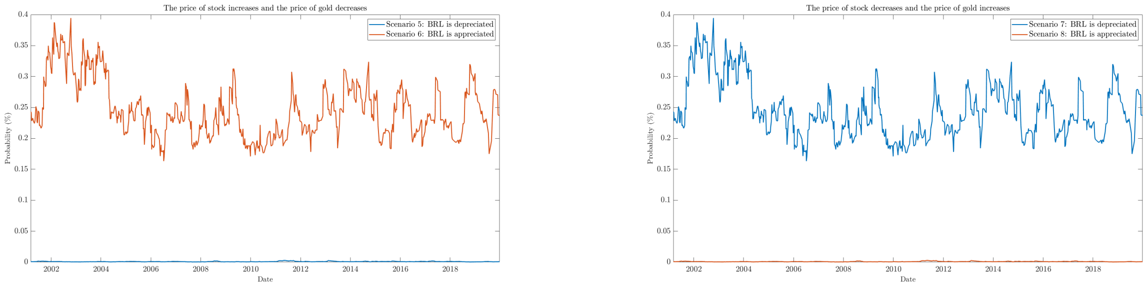
The same results can be sawn in Figure 71, in which appears the probability of BOVESPA is under its VaR and an extremely depreciation of local currency occurs is almost zero in all the sample, and in bottom Panel of Figure 72, in which the difference between the first and third and the second and fourth scenarios is remarkable.

Figure 69: Probability of scenarios in which stock index and gold move in the same direction in Brazil.



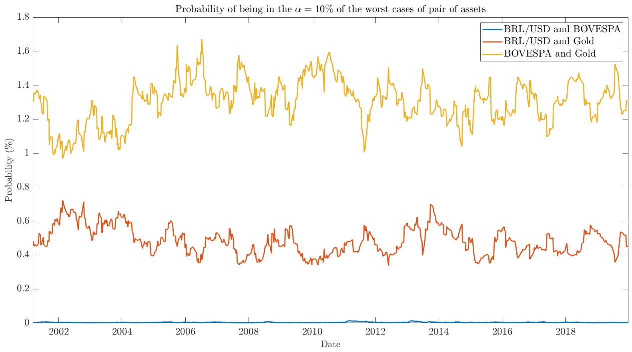
Notes: left Panel shows bullish scenarios for BOVESPA and gold market. In the first possible situation (blue line) the local currency is depreciated, so the profit of the portfolio is greater. In the second situation (orange line) there is an appreciation of the Brazilian real, and consecuently the results of the portfolio are lower, zero or even negative. In right Panel appear the probability bearish scenarios for BOVESPA and gold market. In the first possible situation (blue line) the Brazilian real is depreciated, so the impact of the markets' falls are deadened. The second scenario (orange line) shows an appreciation of the Brazilian real, which accentuates the losses of the portfolio.

Figure 70: Probability of scenarios in which stock index and gold move in opposite directions in Brazil.



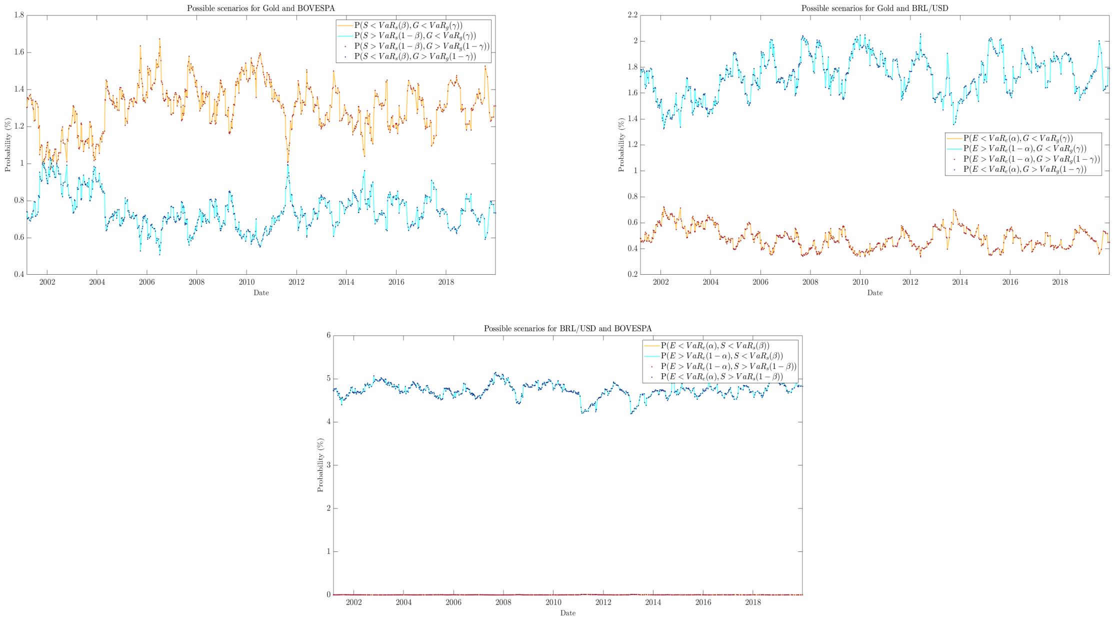
Notes: left Panel shows a bullish scenario for BOVESPA and a bearish scenario for gold market. In the first possible situation (blue line) there is a depreciation of the Brazilian real, and this fact reduces the negative return of the gold investment. In the second situation (orange line) there is an appreciation of this currency, which accentuates the losses in this investment. In both cases the final result of the portfolio is uncertain. In right Panel appears the probability of a bearish scenario for BOVESPA jointly a bullish scenario for gold market. In the first possible situation (blue line) the Brazilian real suffers a depreciation, so this asset also helps to minimize the impact of the stock loss in the portfolio. In the second situation (orange line) there is an appreciation of the domestic currency, which reduces the safe haven role of gold in the portfolio. In both cases the final result of the portfolio is uncertain.

Figure 71: Worst scenario of two of the variables, BRL/USD, BOVESPA or gold.



Note: the figure presents the probability of scenarios in which two of the three assets, BRL/USD and BOVESPA225, BRL/USD and gold, or BOVESPA and gold, respectively, are in their τ% lower quantile.

Figure 72: Possible scenarios for pair of assets in Brazil.



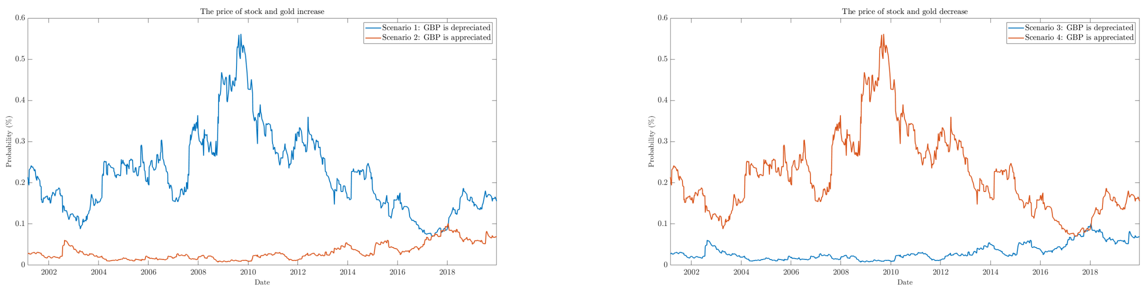
Notes: top left Panel presents the different possible scenarios for gold and BOVESPA. Top right Panel contains the probabilities of all possible scenarios for gold and BRL/USD. Finally, bottom Panel shows the different scenarios for BRL/USD and BOVESPA.

In Figures 73 and 74 appear the eight possible scenarios combining different behaviours of GBP/USD, FTSE100 and gold, Figure 75 shows the probability of the worst scenario of each pair of assets, and Figure 76 contains the four possible situations when we study the behaviour of two of the assets.

In both Figures 73 and 74 we can see, on the one hand, that the scenarios less probables are those in which an increase in the price of gold is accompained by an appreciation of the pound, or well a decrease in the price of this metal is accompained by a depreciation of the currency. The same result can be found in top right Panel of Figure 76, in which we can see it is much more probable that GBP/USD and gold are both in their τ-highest or lowest quantile.

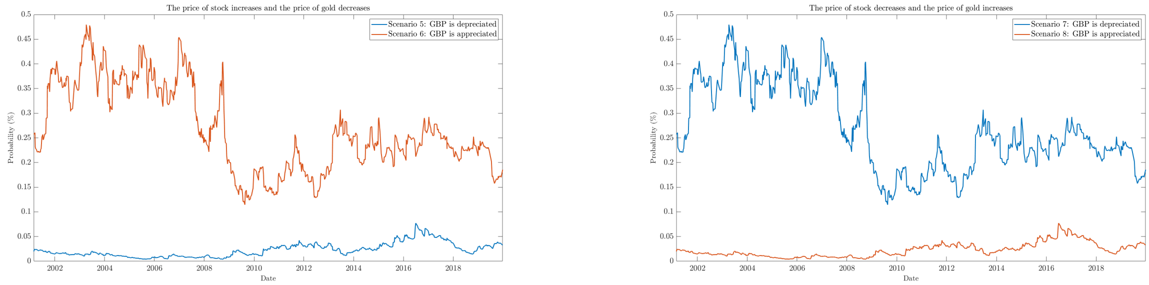
On the other hand, Scenarios 1 and 4 have the same probability, and the highest of the eight possible scenarios. Moreover, this probability increases a lot around 2009-2010. In all the sample gold has a more or less estable bullish trend, and FTSE100, after recovering from distress period with the financial crisis, in 2009 starts also a bullish trend, so the probability of that index is increasing joint to the gold is very high. We can see in top left Panel of Figure 76 that in these years the probability of being both assets under their VaR(10%) or over their VaR(90%) is higher than the other two possible scenarios, unlike the most of the rest of the sample. It is also more probable in that period that a great depreciation of the pound was accompained by a shock in FTSE100, or viceversa (see bottom Panel of Figure 76).

Figure 73: Probability of scenarios in which stock index and gold move in the same direction in UK.



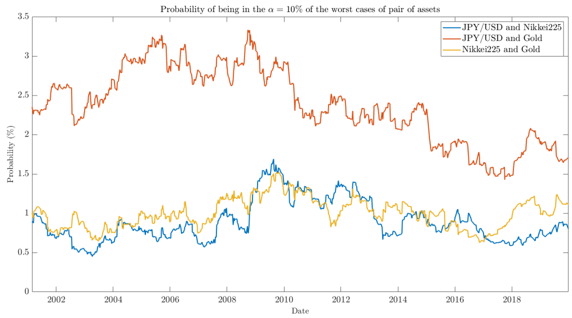
Notes: left Panel shows bullish scenarios for FTSE100 and gold market. In the first possible situation (blue line) the local currency is depreciated, so the profit of the portfolio is greater. In the second situation (orange line) there is an appreciation of the pound, and consecuently the results of the portfolio are lower, zero or even negative. In right Panel appear the probability bearish scenarios for FTSE100 and gold market. In the first possible situation (blue line) the pound is depreciated, so the impact of the markets' falls are deadened. The second scenario (orange line) shows an appreciation of the pound, which accentuates the losses of the portfolio.

Figure 74: Probability of scenarios in which stock index and gold move in opposite directions in UK.



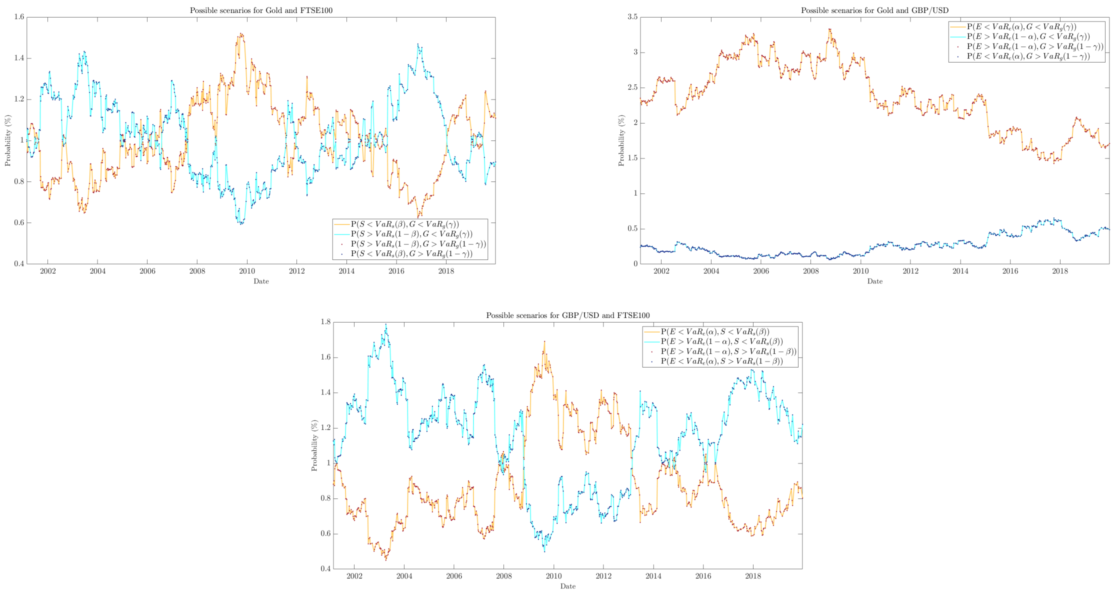
Notes: left Panel shows a bullish scenario for FTSE100 and a bearish scenario for gold market. In the first possible situation (blue line) there is a depreciation of the pound, and this fact reduces the negative return of the gold investment. In the second situation (orange line) there is an appreciation of this currency, which accentuates the losses in this investment. In both cases the final result of the portfolio is uncertain. In right Panel appears the probability of a bearish scenario for FTSE100 jointly a bullish scenario for gold market. In the first possible situation (blue line) the pound suffers a depreciation, so this asset also helps to minimize the impact of the stock loss in the portfolio. In the second situation (orange line) there is an appreciation of the domestic currency, which reduces the safe haven role of gold in the portfolio. In both cases the final result of the portfolio is uncertain.

Figure 75: Worst scenario for two of the variables, GBP/USD, FTSE100 or gold.



Note: the figure presents the probability of scenarios in which two of the three assets, GBP/USD and FTSE100, GBP/USD and gold, or FTSE100 and gold, respectively, are in their τ%-lower quantile.

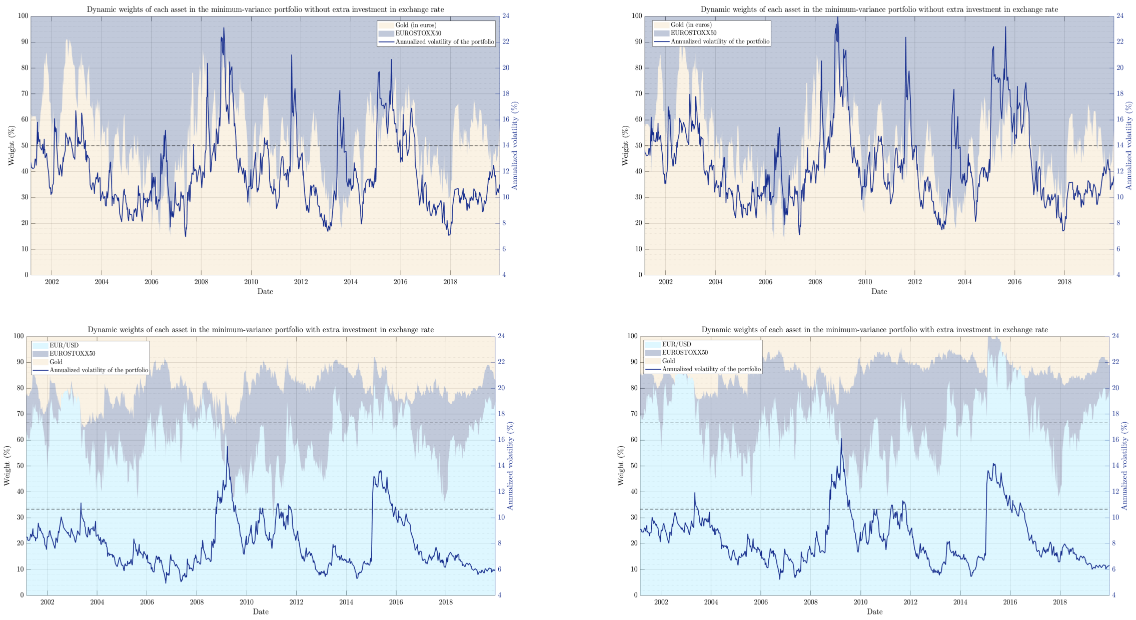
Figure 76: Possible scenarios for pair of assets in UK.



Notes: top left Panel presents the different possible scenarios for gold and FTSE100. Top right Panel contains the probabilities of all possible scenarios for gold and GBP/USD. Finally, bottom Panel shows the possible scenarios for GBP/USD and FTSE100.

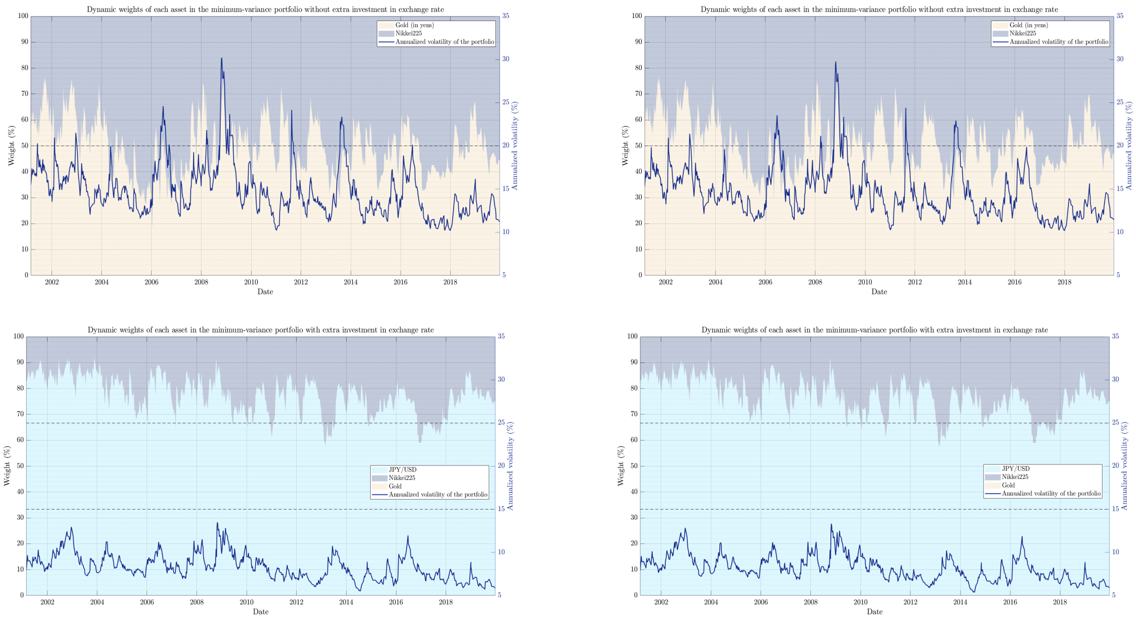
G Figures

Figure 77: Evolution of the weights of each asset in minimum-variance portfolios for Europe.



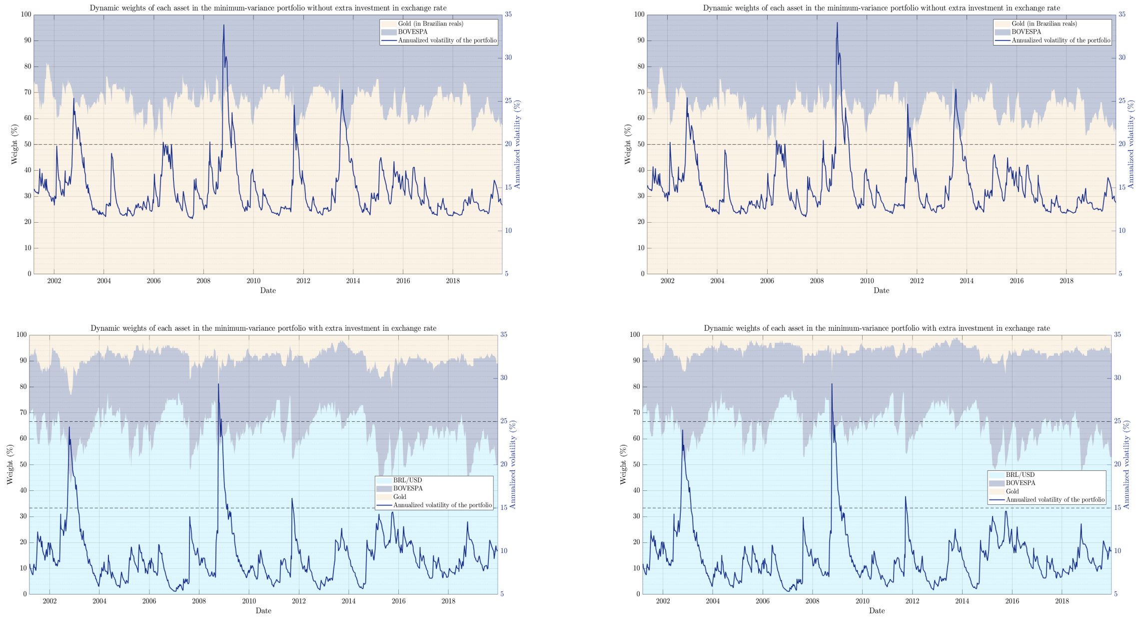
Note: Figures in upper (lower) row show the portfolios without (with) extra investment in the exchange rate assuming Gaussian (left Panel) and Student t (right Panel) copula approach.

Figure 78: Evolution of the weights of each asset in minimum-variance portfolios for Japan.



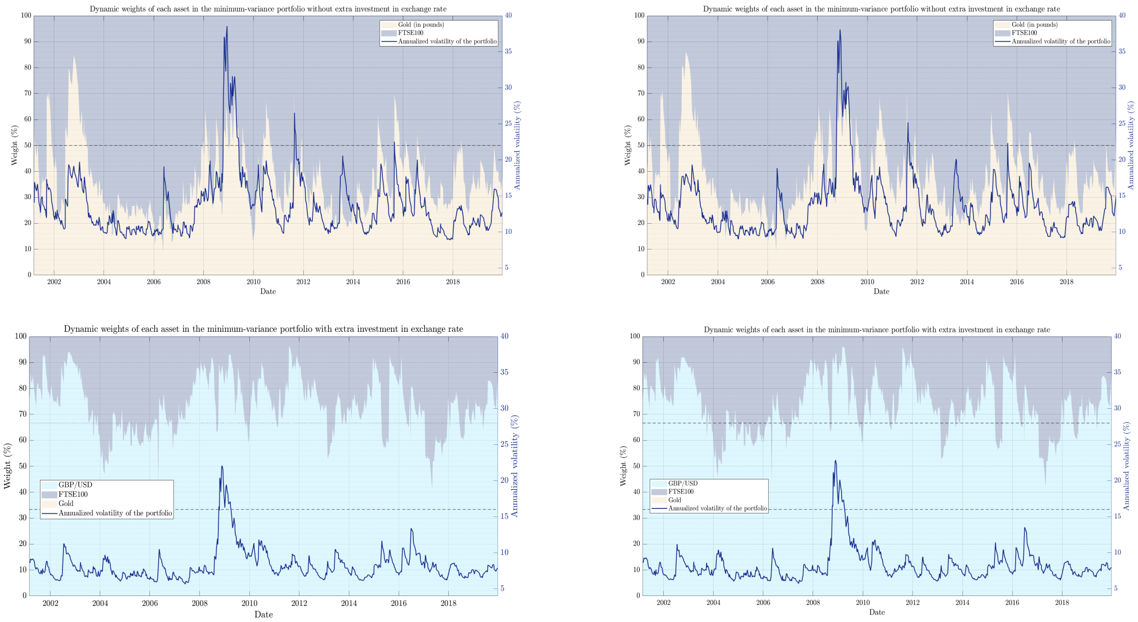
Note: Figures in upper (lower) row show the portfolios without (with) extra investment in the exchange rate assuming Gaussian (left Panel) and Student t (right Panel) copula approach.

Figure 79: Evolution of the weights of each asset in minimum-variance portfolios for Brazil.



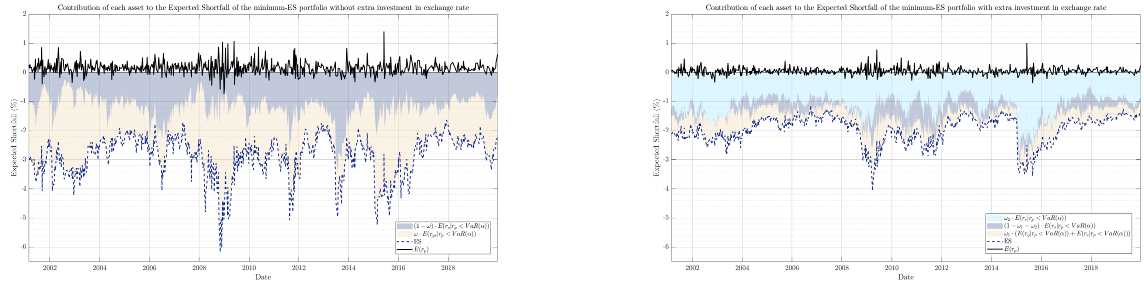
Note: Figures in upper (lower) row show the portfolios without (with) extra investment in the exchange rate assuming Gaussian (left Panel) and Student t (right Panel) copula approach.

Figure 80: Evolution of the weights of each asset in minimum-variance portfolios for UK.



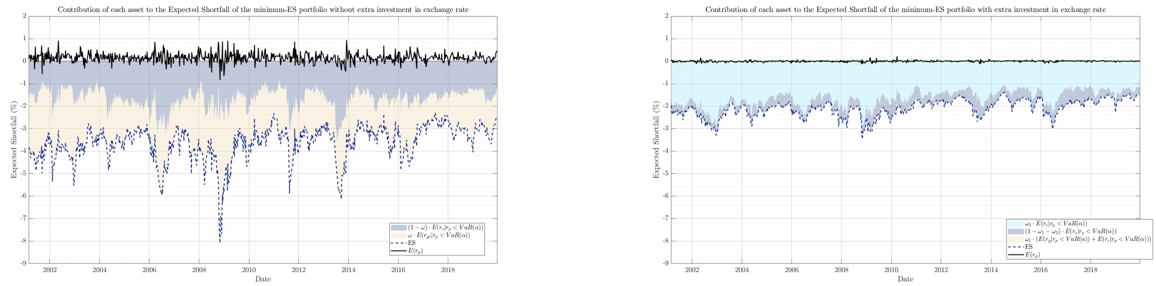
Note: Figures in upper (lower) row show the portfolios without (with) extra investment in the exchange rate assuming Gaussian (left Panel) and Student t (right Panel) copula approach.

Figure 81: Contribution of each asset to the Expected Shortfall of the minimum-ES portfolio for Europe (assuming Gaussian copula model).



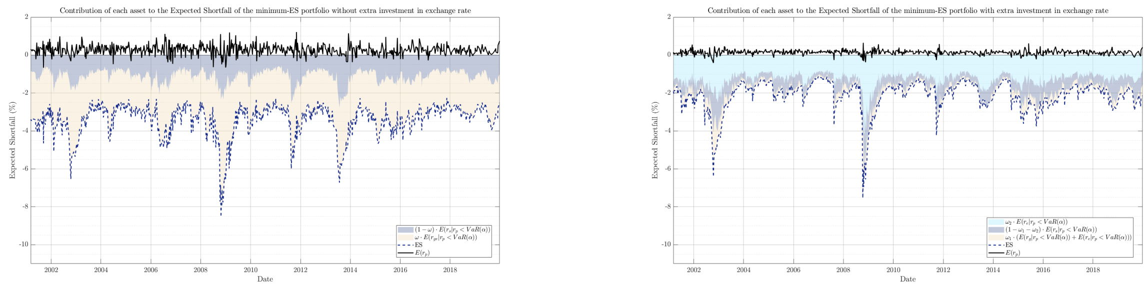
Notes: left (right) Panel performs the contribution of each asset to the ES without (with) considering an extra-investment in exchange rate.

Figure 82: Contribution of each asset to the Expected Shortfall of the minimum-ES portfolio for Japan (assuming Gaussian copula model).



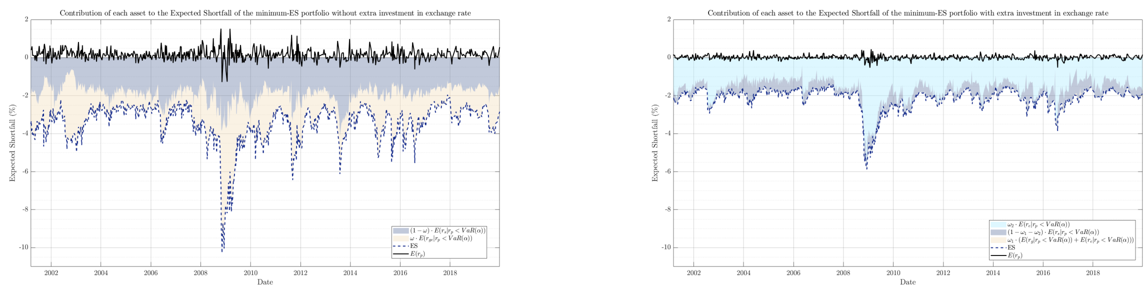
Notes: left (right) Panel performs the contribution of each asset to the ES without (with) extra-investment in exchange rate.

Figure 83: Contribution of each asset to the Expected Shortfall of the minimum-ES portfolio for Brazil (assuming Gaussian copula model).



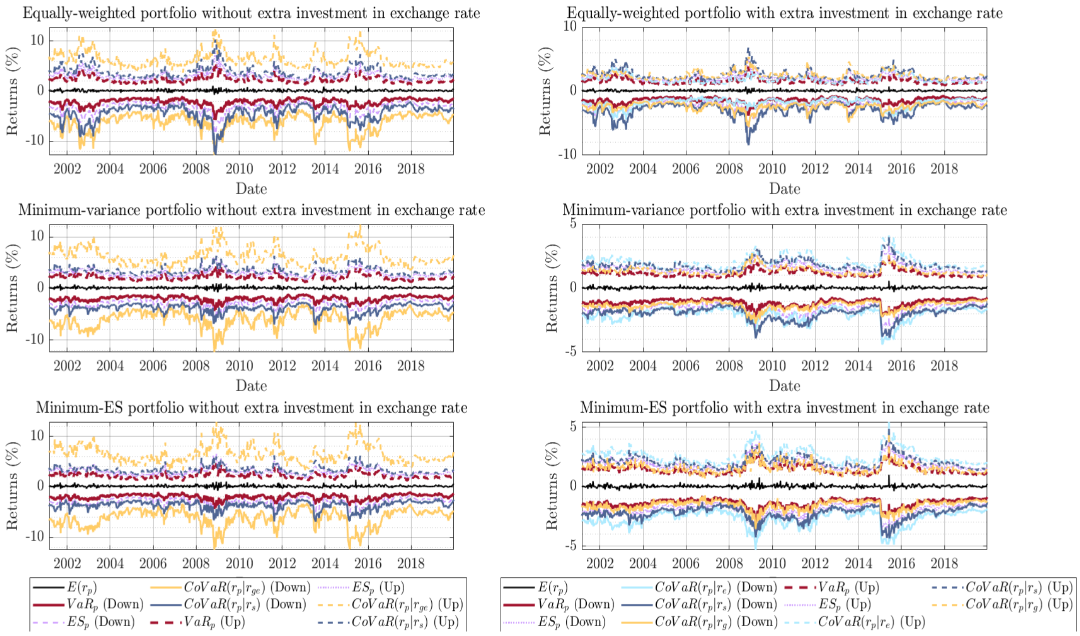
Notes: left (right) Panel performs the contribution of each asset to the ES without (with) extra-investment in exchange rate.

Figure 84: Contribution of each asset to the Expected Shortfall of the minimum-ES portfolio for UK (assuming Gaussian copula model).



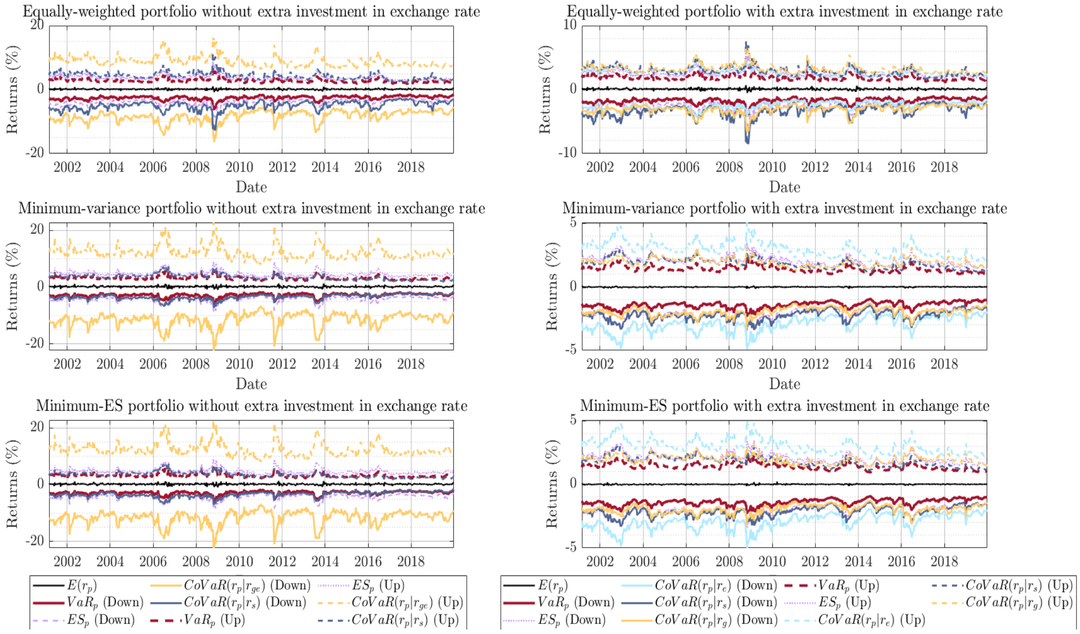
Notes: left (right) Panel performs the contribution of each asset to the ES without (with) extra-investment in exchange rate.

Figure 85: CoVaR of European portfolios conditional on only one of the assets is under or above its VaR (assuming Gaussian copula model).



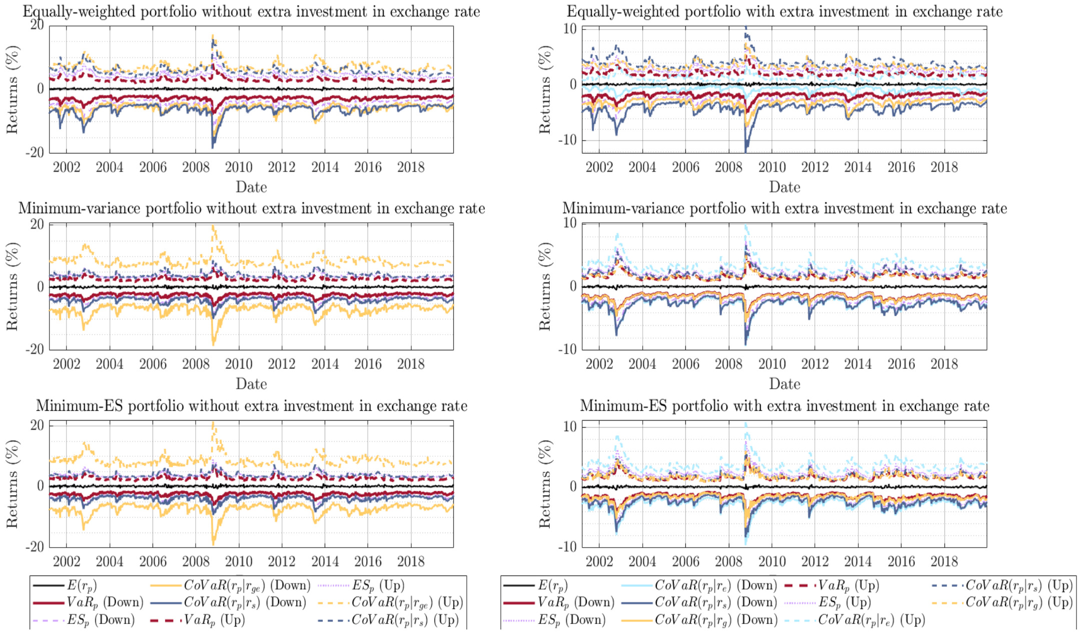
Notes: left Panels consider the situation in which there is not extra investment in exchange rate, while right Panels present the situation in which portfolios are composed of exchange rate, stock and gold.

Figure 86: CoVaR of Japanese portfolios conditional on only one of the assets is under or above its VaR (assuming Gaussian copula model).



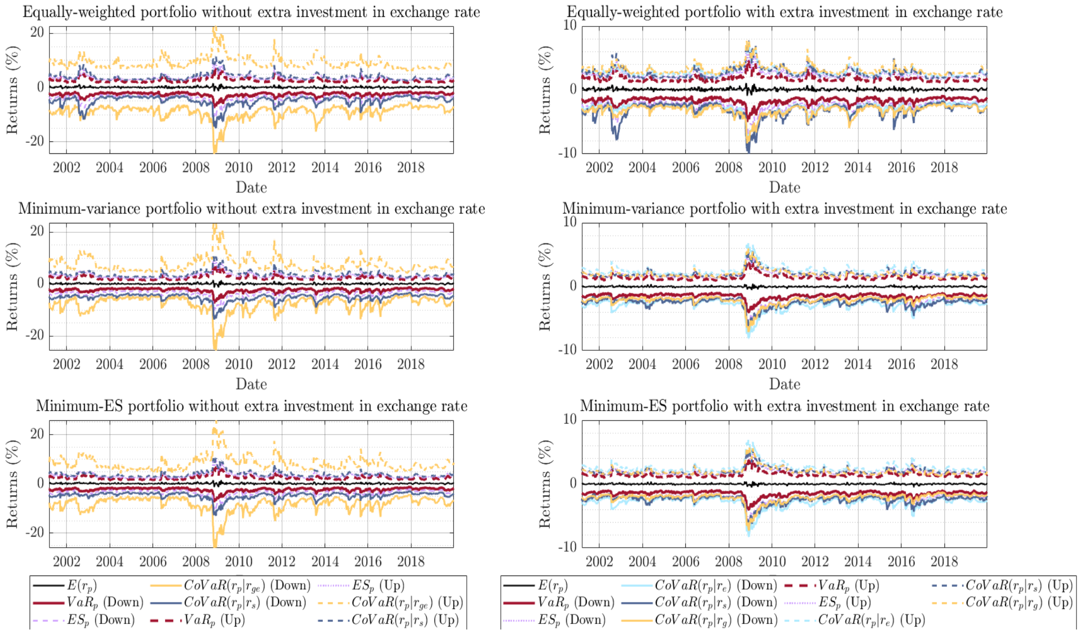
Notes: left Panels consider the situation in which there is not extra investment in exchange rate, while right Panels present the situation in which portfolios are composed of exchange rate, stock and gold.

Figure 87: CoVaR of Brazilian portfolios conditional on only one of the assets is under or above its VaR (assuming Gaussian copula model).



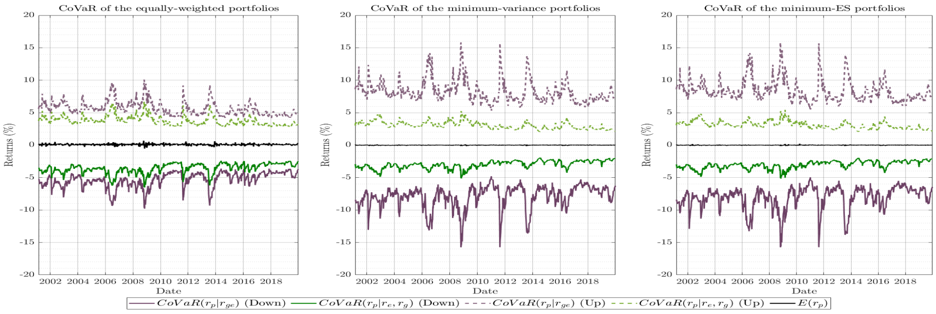
Notes: left Panels consider the situation in which there is not extra investment in exchange rate, while right Panels present the situation in which portfolios are composed of exchange rate, stock and gold.

Figure 88: CoVaR of British portfolios conditional on only one of the assets is under or above its VaR (assuming Gaussian copula model).



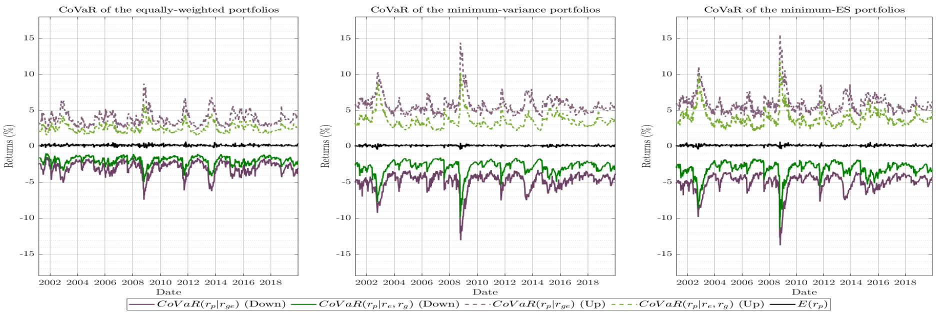
Notes: left Panels consider the situation in which there is not extra investment in exchange rate, while right Panels present the situation in which portfolios are composed of exchange rate, stock and gold.

Figure 89: CoVaR of Japanese portfolios conditional on the situation of exchange rate and gold (assuming Gaussian copula model).



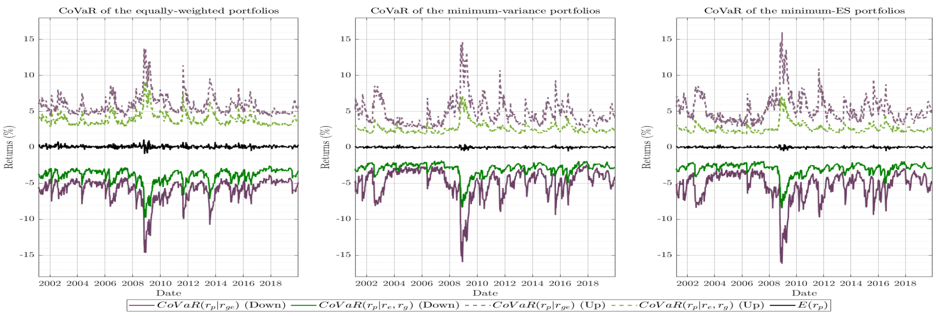
Notes: the figure shows the downside and upside CoVaR of portfolios, when gold denominated in euros (violet lines) or gold and exchange rate (green lines) are under or above its VaR, respectively.

Figure 90: CoVaR of Brazilian portfolios conditional on the situation of exchange rate and gold (assuming Gaussian copula model).



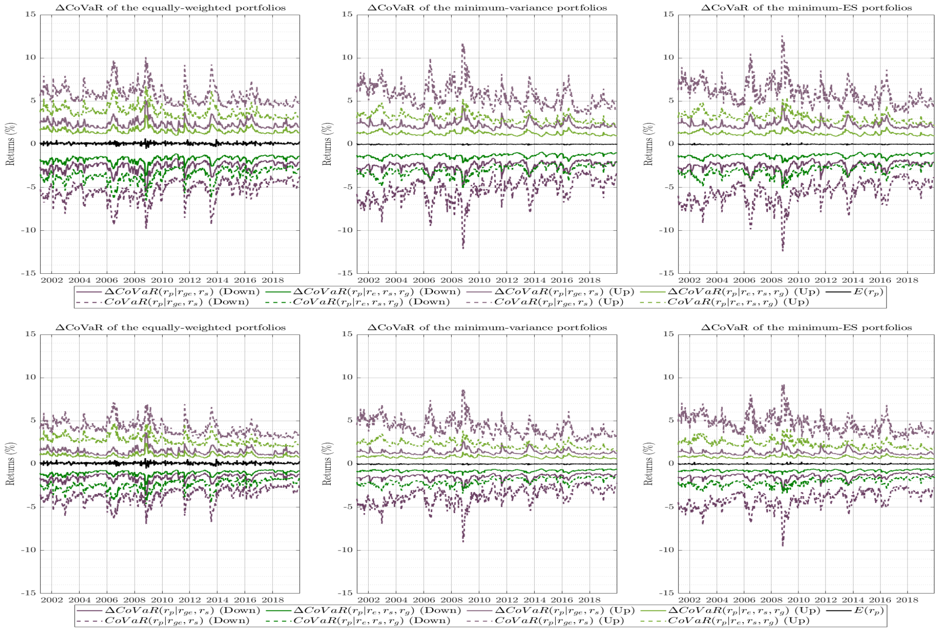
Notes: the figure shows the downside and upside CoVaR of portfolios, when gold denominated in euros (violet lines) or gold and exchange rate (green lines) are under or above its VaR, respectively.

Figure 91: CoVaR of British portfolios conditional on the situation of exchange rate and gold (assuming Gaussian copula model).



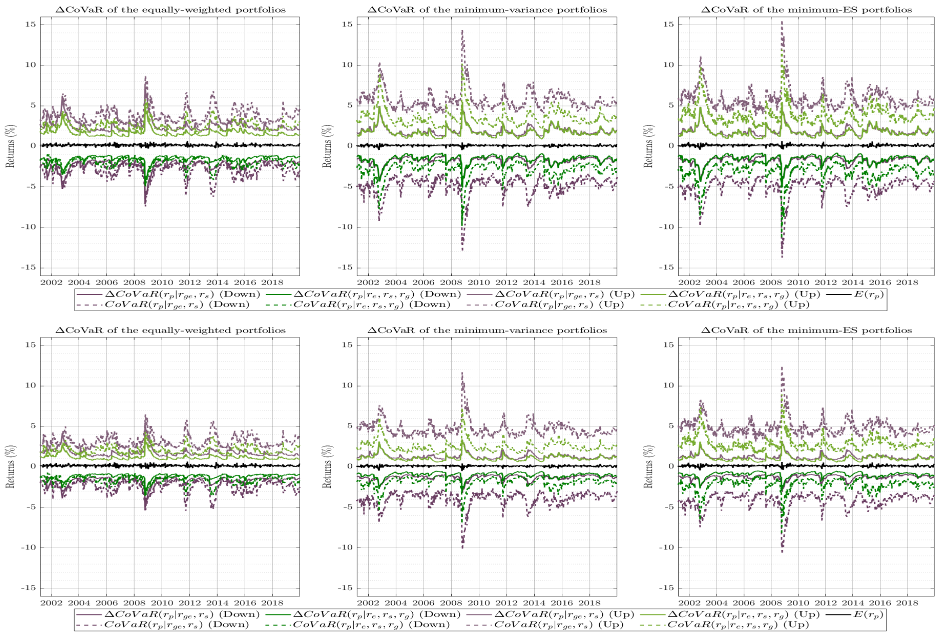
Notes: the figure shows the downside and upside CoVaR of portfolios, when gold denominated in euros (violet lines) or gold and exchange rate (green lines) are under or above its VaR, respectively.

Figure 92: ΔCoVaR of Japanese portfolios conditional on the situation of exchange rate and gold.



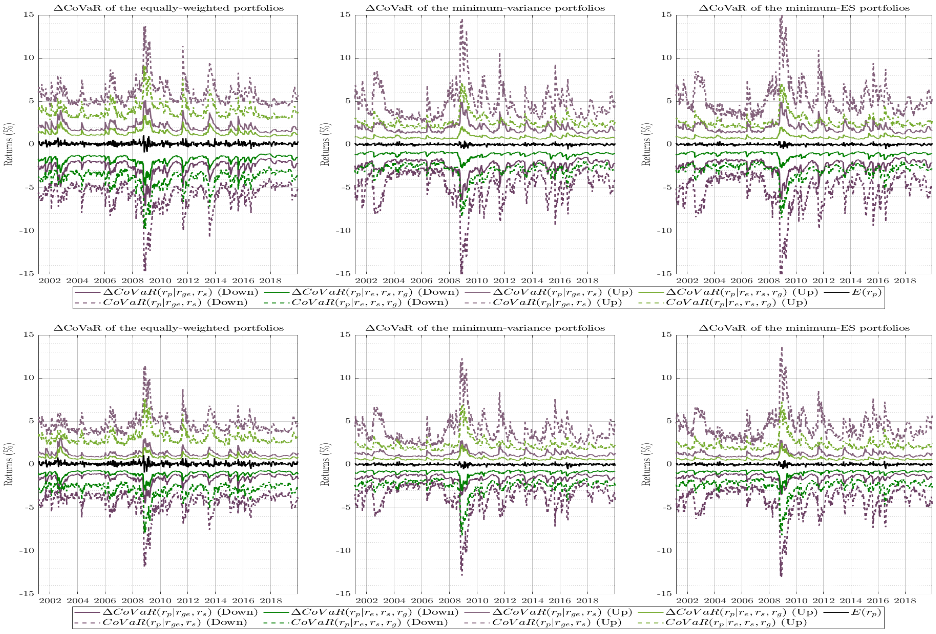
Notes: the figure shows the downside and upside ΔCoVaR of portfolios when gold denominated in euros (violet lines) or gold and exchange rate (green lines) are under or above its VaR, respectively. The dashed lines represents the downside and upside CoVaR of the same portfolios. Panels in upper row refers to portfolios built under Gaussian copula approach, and Panels in lower row refers to portfolios built under Student t copula assumption.

Figure 93: ΔCoVaR of Brazilian portfolios conditional on the situation of exchange rate and gold.



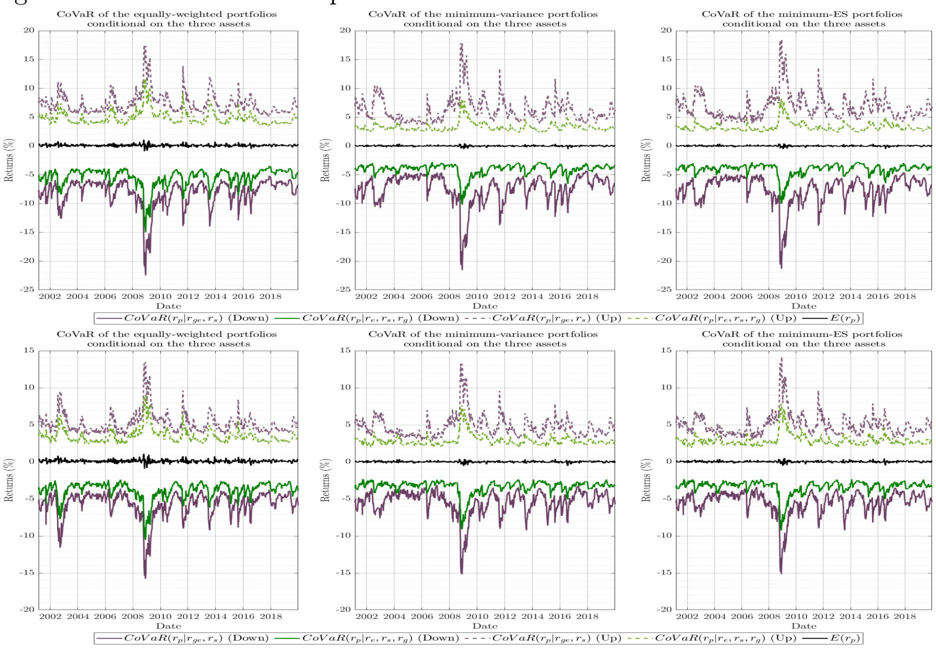
Notes: the figure shows the downside and upside ΔCoVaR of portfolios when gold denominated in euros (violet lines) or gold and exchange rate (green lines) are under or above its VaR, respectively. The dashed lines represents the downside and upside CoVaR of the same portfolios. Panels in upper row refers to portfolios built under Gaussian copula approach, and Panels in lower row refers to portfolios built under Student t copula assumption.

Figure 94: ΔCoVaR of British portfolios conditional on the situation of exchange rate and gold.



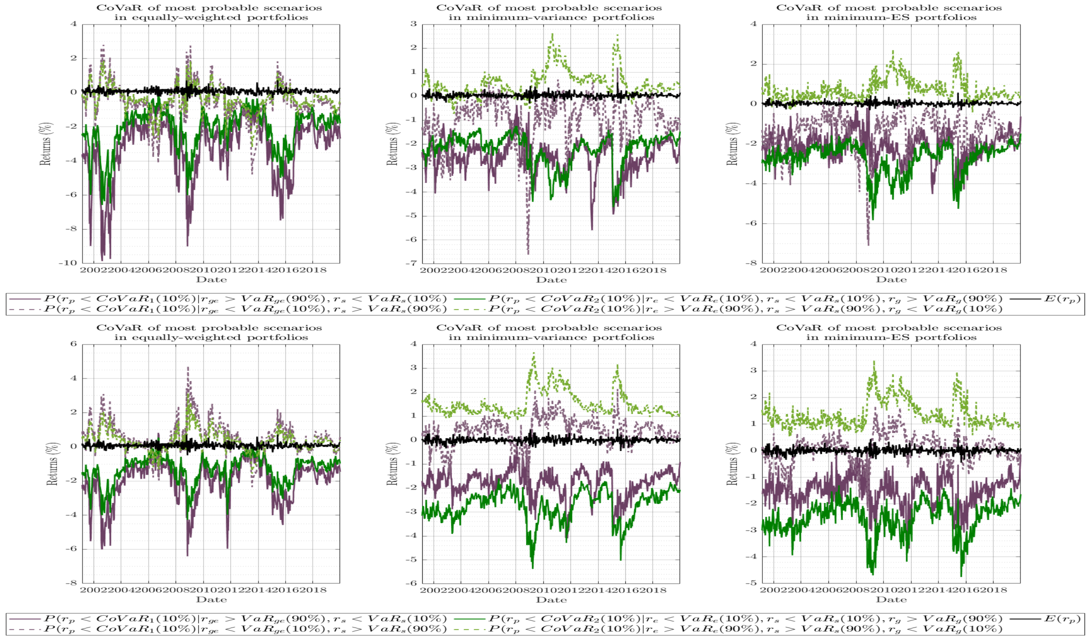
Notes: the figure shows the downside and upside ΔCoVaR of portfolios when gold denominated in euros (violet lines) or gold and exchange rate (green lines) are under or above its VaR, respectively. The dashed lines represents the downside and upside CoVaR of the same portfolios. Panels in upper row refers to portfolios built under Gaussian copula approach, and Panels in lower row refers to portfolios built under Student t copula assumption.

Figure 95: CoVaR of British portfolios conditional on all the assets are at their VaR.



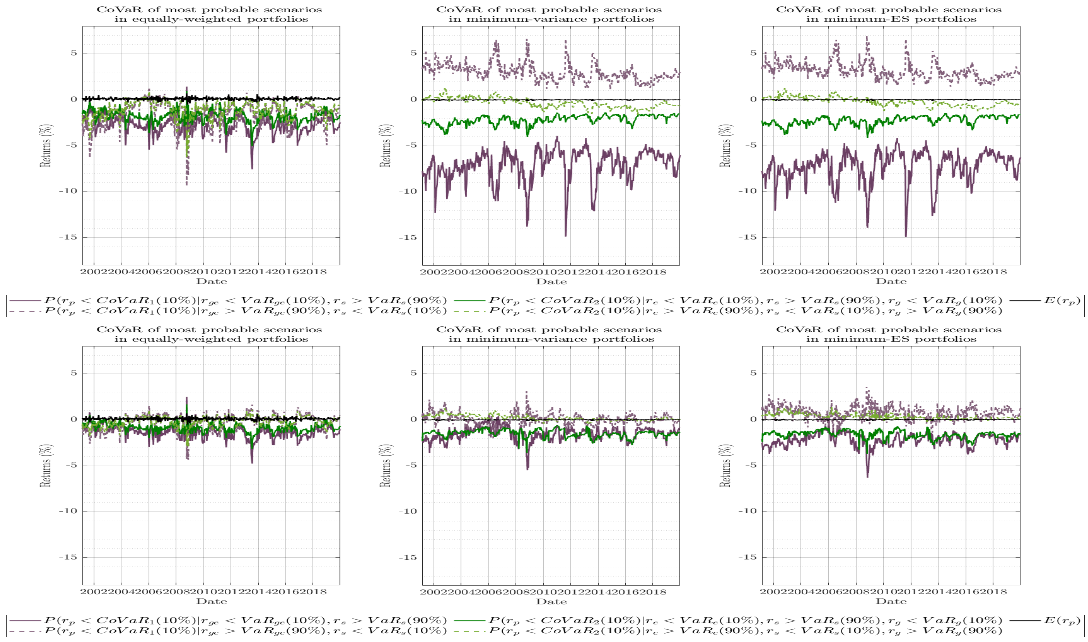
Notes: the figure shows the downside and upside CoVaR of portfolios when all of the assets are under or above its VaR, respectively. Violet lines refers to portolios without extra-investment in exchange rate, while green lines represents those which do have it. Panels in upper row refers to portfolios built under Gaussian copula approach, and Panels in lower row refers to portfolios built under Student t copula assumption.

Figure 96: CoVaR of European portfolios in the most probable scenarios.



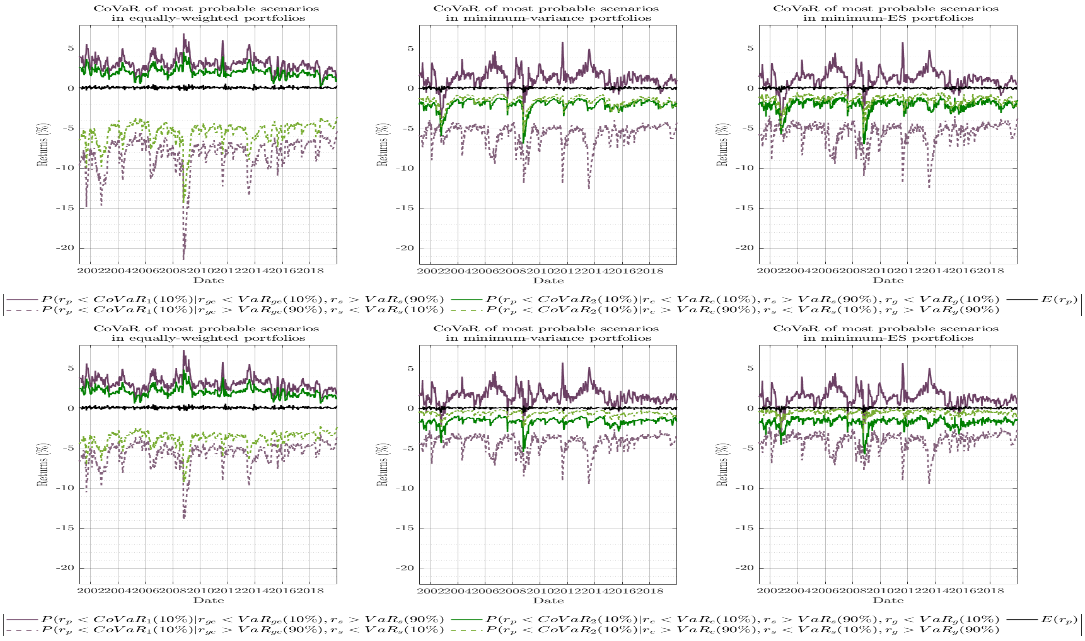
Notes: the figure shows the CoVaR of portfolios in the most probable situations, i.e. $P(R_p < CoVaR(\beta)|R_e < VaR(\alpha), R_s < VaR(\alpha), R_g > VaR(1-\alpha))$ and $P(R_p < CoVaR(\beta)|R_e > VaR(1-\alpha), R_s > VaR(1-\alpha), R_g < VaR(\alpha))$. Violet lines refers to portolios without extra-investment in exchange rate, while green lines represents those which do have it. Panels in upper row refers to portfolios built under Gaussian copula approach, and Panels in lower row refers to portfolios built under Student t copula assumption.

Figure 97: CoVaR of Japanese portfolios in the most probable scenarios.



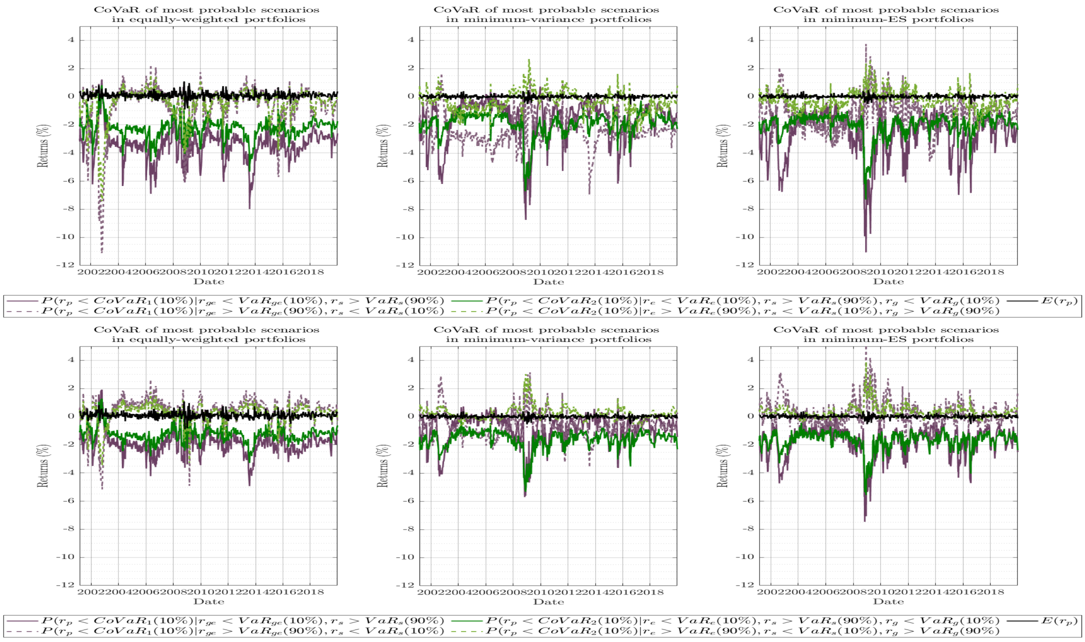
Notes: the figure shows the CoVaR of portfolios in the most probable situations, i.e. $P(R_p < CoVaR(\beta)|R_e < VaR(\alpha), R_s > VaR(1-\alpha), R_g < VaR(\alpha))$ and $P(R_p < CoVaR(\beta)|R_e > VaR(1-\alpha), R_s < VaR(\alpha), R_g > VaR(1-\alpha))$. Violet lines refers to portolios without extra-investment in exchange rate, while green lines represents those which do have it. Panels in upper row refers to portfolios built under Gaussian copula approach, and Panels in lower row refers to portfolios built under Student t copula assumption.

Figure 98: CoVaR of Brazilian portfolios in the most probable scenarios.



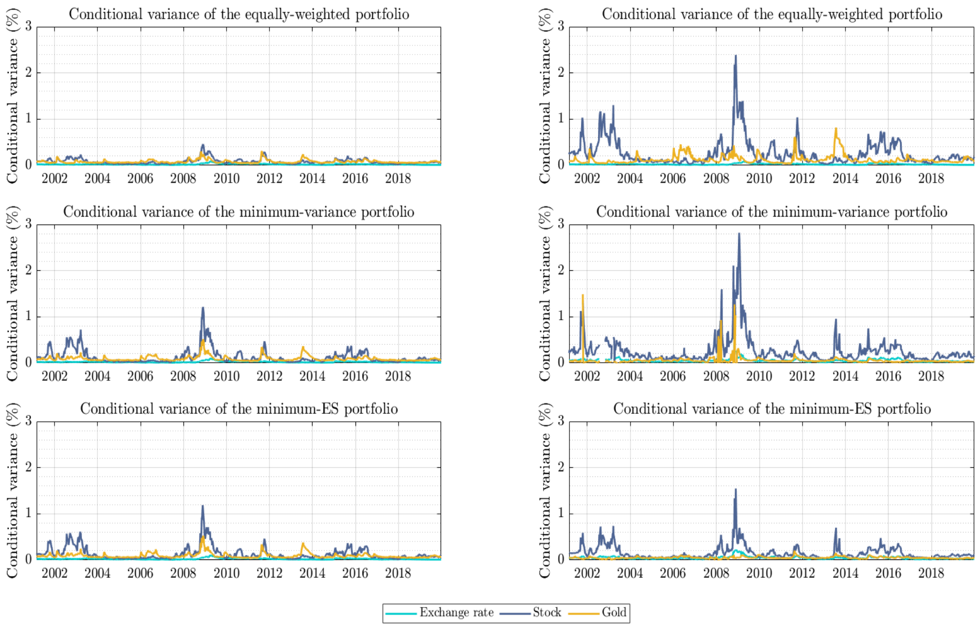
Notes: the figure shows the CoVaR of portfolios in the most probable situations, i.e. $P(R_p < CoVaR(\beta)|R_e < VaR(\alpha), R_s > VaR(1-\alpha), R_g > VaR(1-\alpha))$ and $P(R_p < CoVaR(\beta)|R_e > VaR(1-\alpha), R_s < VaR(\alpha), R_g < VaR(\alpha))$. Violet lines refers to portolios without extra-investment in exchange rate, while green lines represents those which do have it. Panels in upper row refers to portfolios built under Gaussian copula approach, and Panels in lower row refers to portfolios built under Student t copula assumption.

Figure 99: CoVaR of British portfolios in the most probable scenarios.



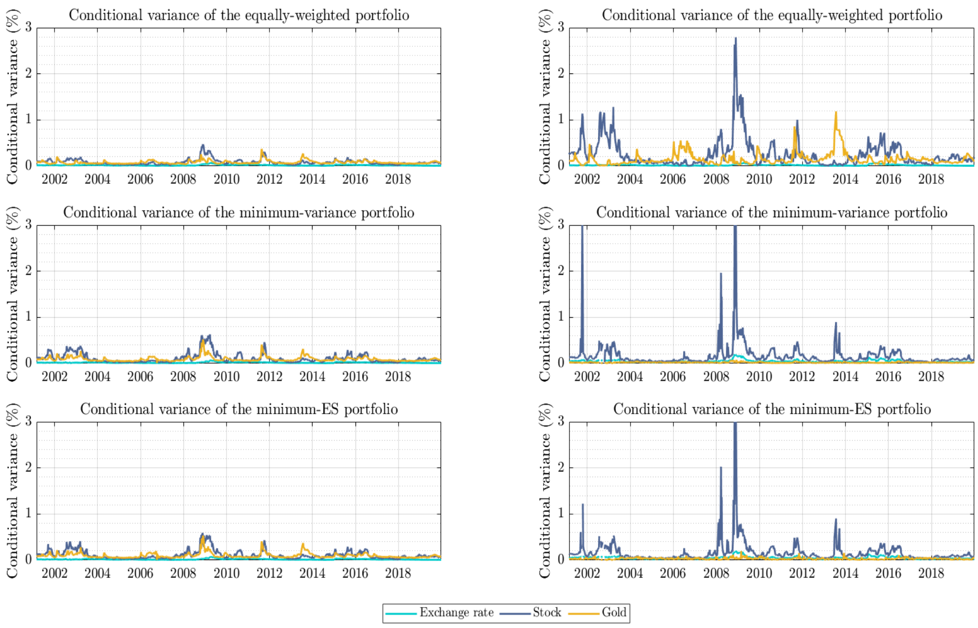
Notes: the figure shows the CoVaR of portfolios in the most probable situations, i.e. $P(R_p < CoVaR(\beta)|R_e < VaR(\alpha), R_s > VaR(1-\alpha), R_g < VaR(\alpha))$ and $P(R_p < CoVaR(\beta)|R_e > VaR(1-\alpha), R_s < VaR(\alpha), R_g > VaR(1-\alpha))$. Violet lines refers to portolios without extra-investment in exchange rate, while green lines represents those which do have it. Panels in upper row refers to portfolios built under Gaussian copula approach, and Panels in lower row refers to portfolios built under Student t copula assumption.

Figure 100: Conditional variance of different portfolios in Europe (assuming Gaussian copula model).



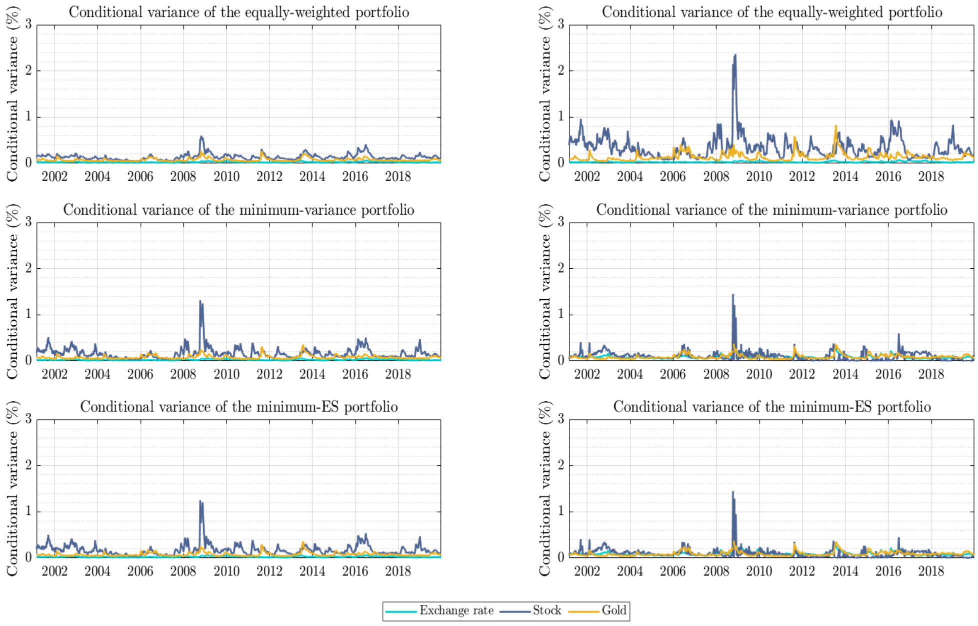
Note: left Panels figures show the conditional variance of each asset in portfolios without extra-investment in exchange rate, while right Panels present the conditional variance in portfolios with this extra-investment.

Figure 101: Conditional variance of different portfolios in Europe (assuming Student t copula model).



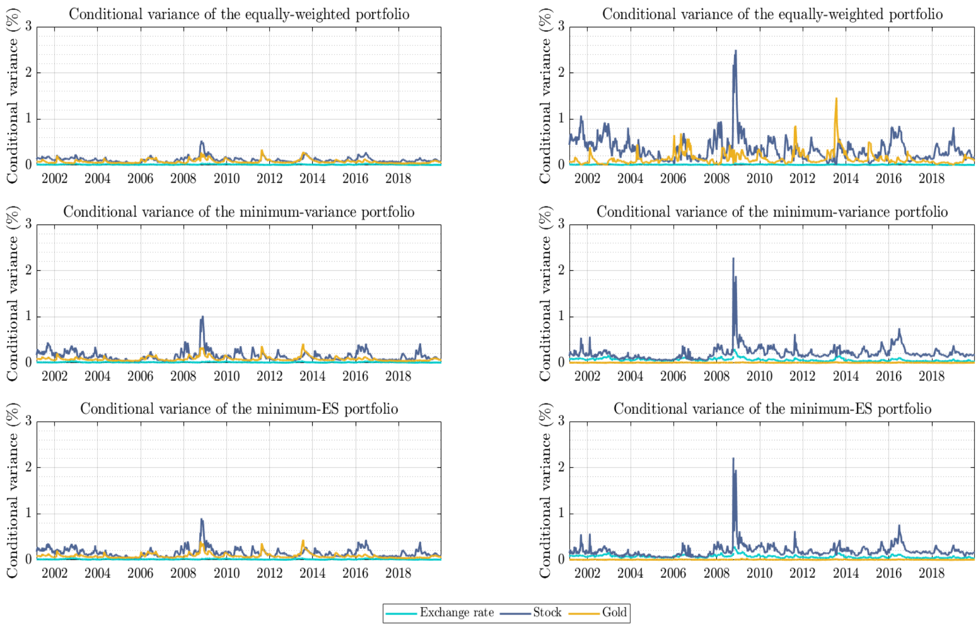
Note: left Panels figures show the conditional variance of each asset in portfolios without extra-investment in exchange rate, while right Panels present the conditional variance in portfolios with this extra-investment.

Figure 102: Conditional variance of different portfolios in Japan (assuming Gaussian copula model).



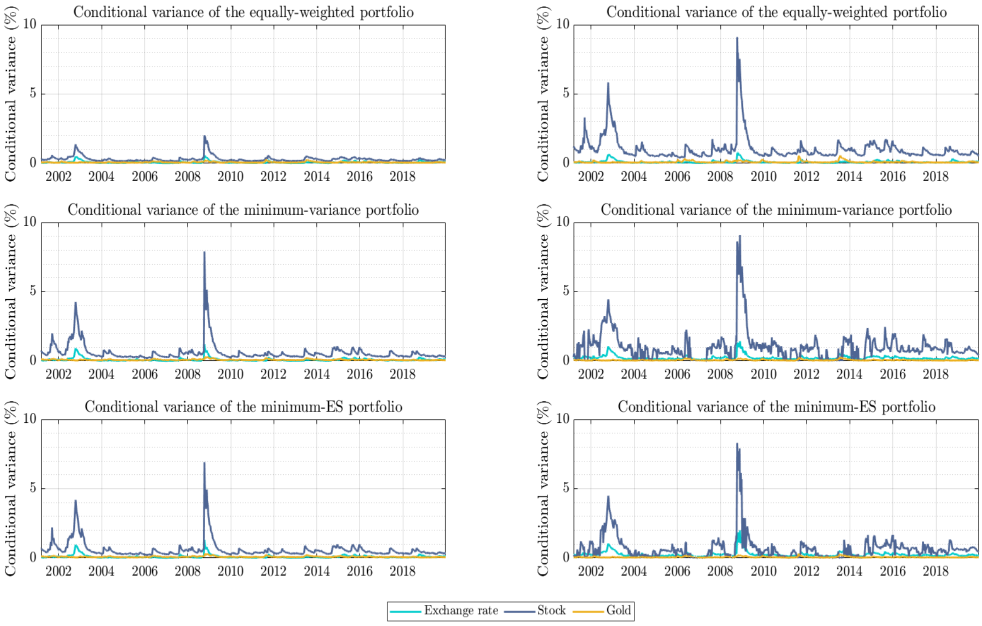
Note: left Panels figures show the conditional variance of each asset in portfolios without extra-investment in exchange rate, while right Panels present the conditional variance in portfolios with this extra-investment.

Figure 103: Conditional variance of different portfolios in Japan (assuming Student t copula model).



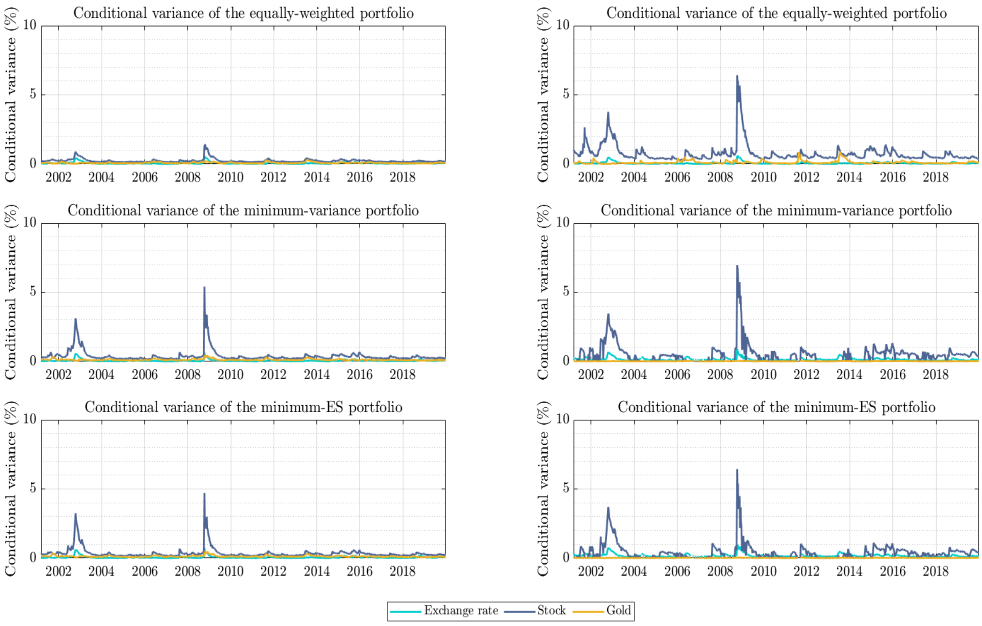
Note: left Panels figures show the conditional variance of each asset in portfolios without extra-investment in exchange rate, while right Panels present the conditional variance in portfolios with this extra-investment.

Figure 104: Conditional variance of different portfolios in Brazil (assuming Gaussian copula model).



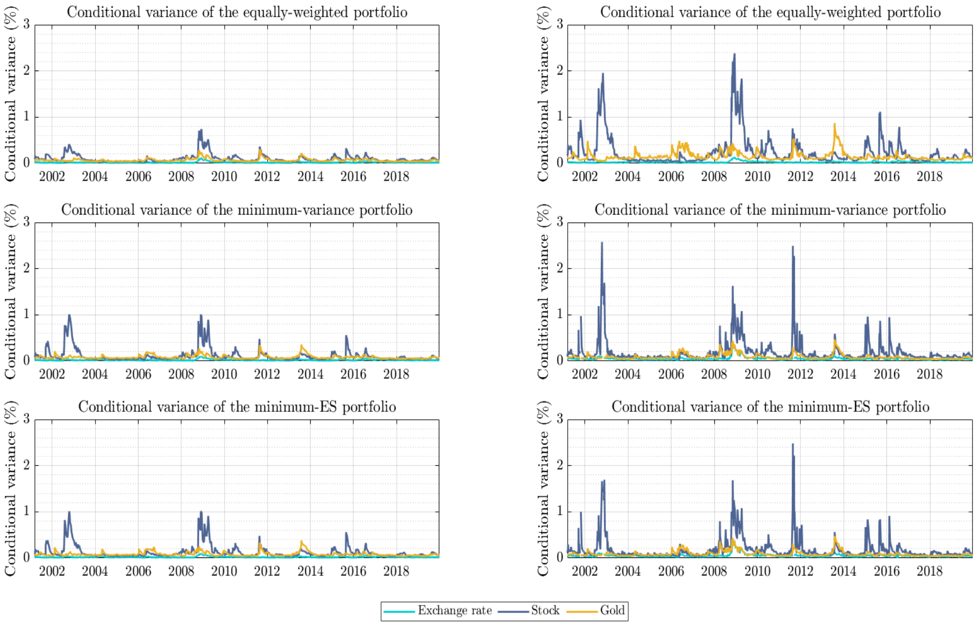
Note: left Panels figures show the conditional variance of each asset in portfolios without extra-investment in exchange rate, while right Panels present the conditional variance in portfolios with this extra-investment.

Figure 105: Conditional variance of different portfolios in Brazil (assuming Student t copula model).



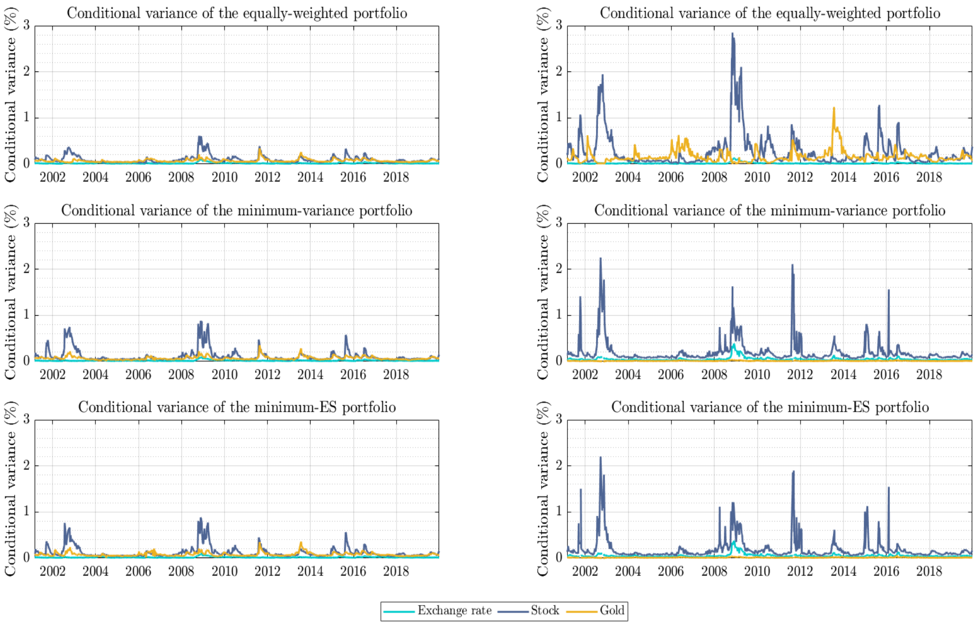
Note: left Panels figures show the conditional variance of each asset in portfolios without extra-investment in exchange rate, while right Panels present the conditional variance in portfolios with this extra-investment.

Figure 106: Conditional variance of different portfolios in UK (assuming Gaussian copula model).



Note: left Panels figures show the conditional variance of each asset in portfolios without extra-investment in exchange rate, while right Panels present the conditional variance in portfolios with this extra-investment.

Figure 107: Conditional variance of different portfolios in UK (assuming Student t copula model).



Note: left Panels figures show the conditional variance of each asset in portfolios without extra-investment in exchange rate, while right Panels present the conditional variance in portfolios with this extra-investment.

Figure 108: Conditional Diversification Benefits of different portfolios in Europe (assuming Gaussian copula model).

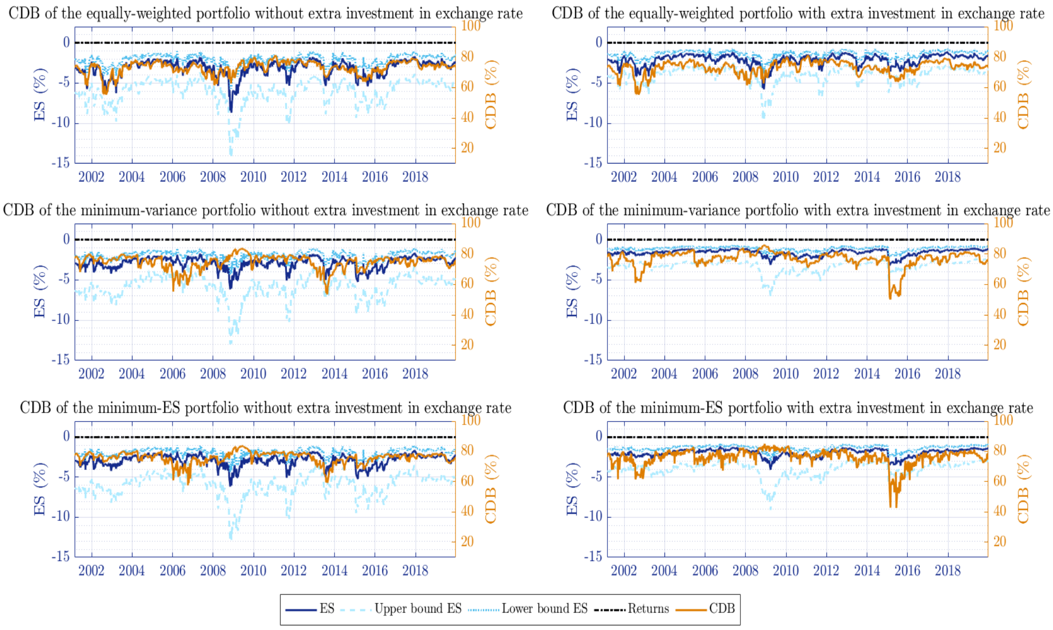


Figure 109: Conditional Diversification Benefits of different portfolios in Japan (assuming Gaussian copula model).

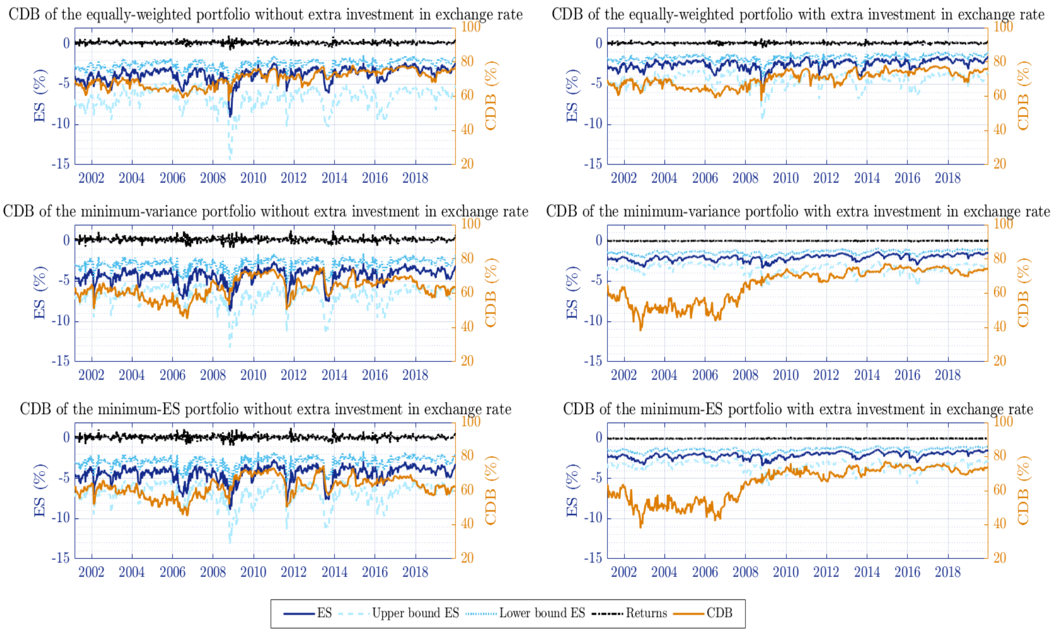


Figure 110: Conditional Diversification Benefits of different portfolios in Brazil (assuming Gaussian copula model).

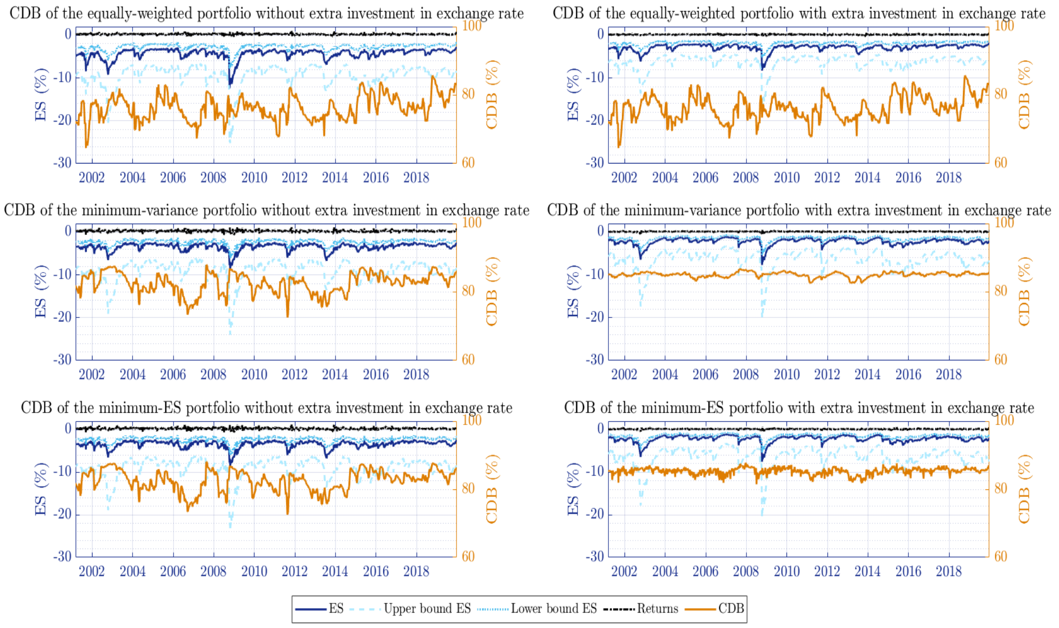


Figure 111: Conditional Diversification Benefits of different portfolios in UK (assuming Gaussian copula model).

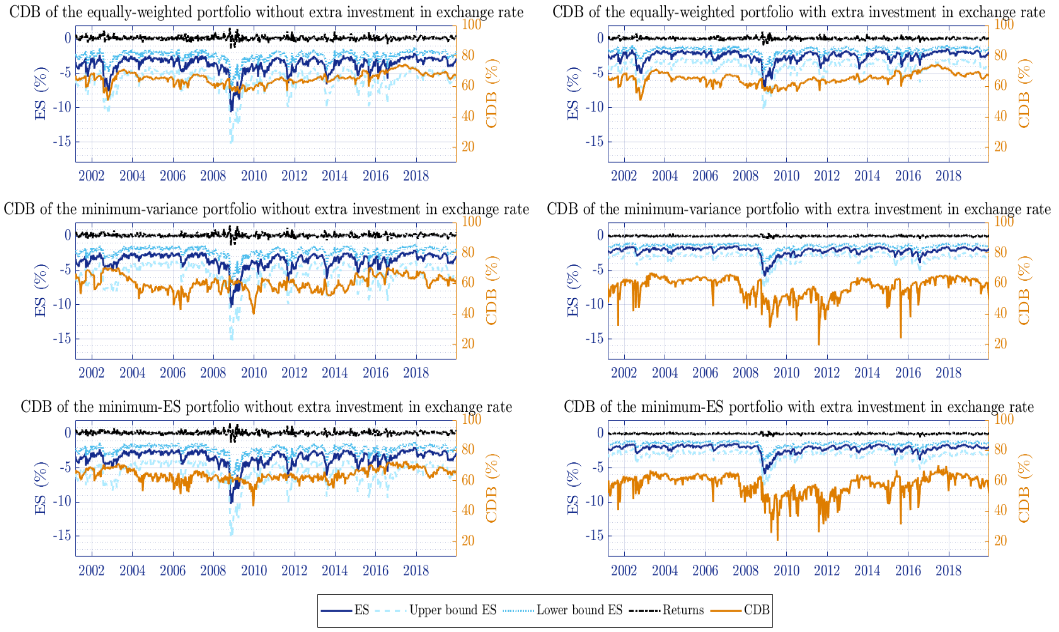
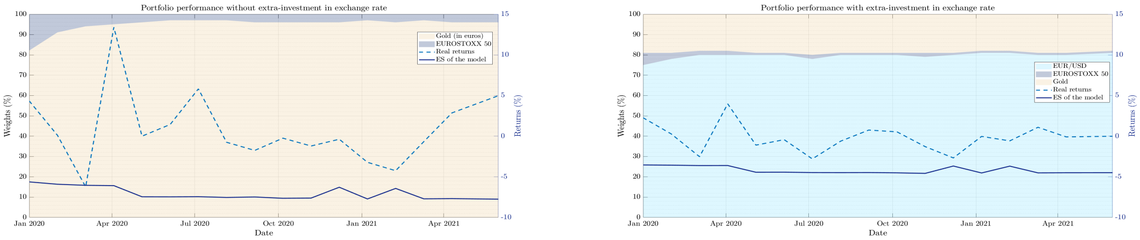
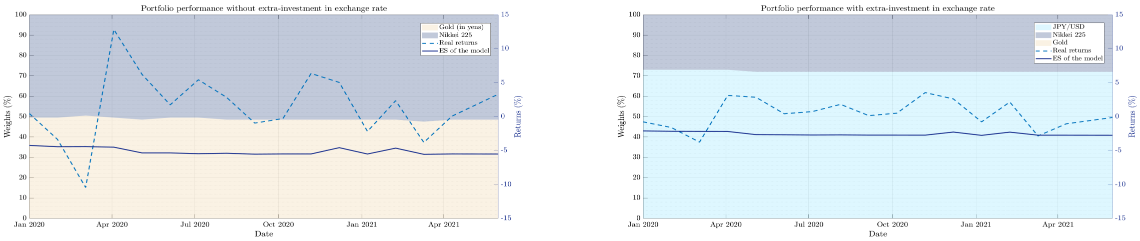


Figure 112: Out-of-sample performance of minimum-ES portfolio in Europe (assuming Gaussian copula model).



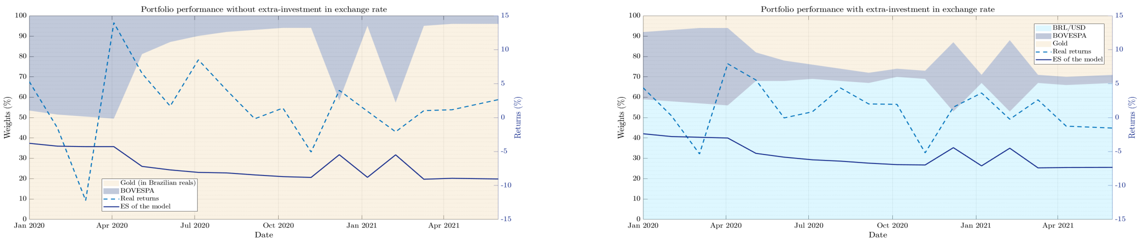
Note: left (right) Panel shows the portfolios without (with) extra investment in the exchange rate.

Figure 113: Out-of-sample performance of minimum-ES portfolio in Japan (assuming Gaussian copula model).



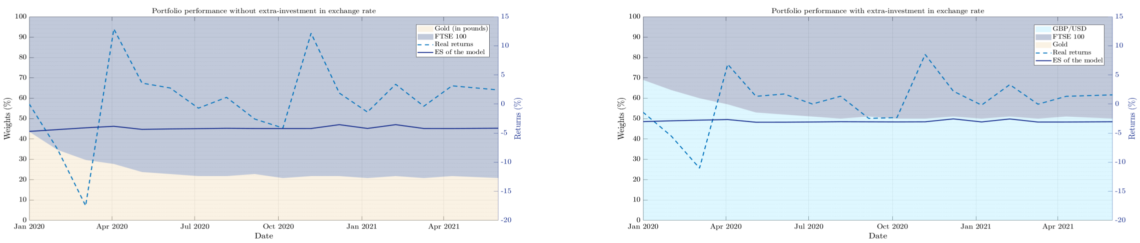
Note: left (right) Panel shows the portfolios without (with) extra investment in the exchange rate.

Figure 114: Out-of-sample performance of minimum-ES portfolio in Brazil (assuming Gaussian copula model).



Note: left (right) Panel shows the portfolios without (with) extra investment in the exchange rate.

Figure 115: Out-of-sample performance of minimum-ES portfolio in UK (assuming Gaussian copula model).



Note: left (right) Panel shows the portfolios without (with) extra investment in the exchange rate.