LIQUIDITY RISK: STUDYING THE RESILENCE OF SPANISH INVESTMENT FUNDS TO REDEMPTION SHOCKS

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Abstract

This work aims to analyse the resilience of Spanish mutual funds to liquidity shocks, which we measure as redemption requests by investors. Just as the banking sector is capable of producing a contagion effect in the event of an economic shock, the investment fund sector can also transmit or aggravate economic shocks depending on the decisions taken by its participants. To measure resilience of Spanish mutual funds to liquidity shocks we first analyse the ability to absorb redemptions, measuring the liquidity of the underlying assets of investment funds. Then we study the relationships between redemptions and fund categories and use dependencies to simulate severe but plausible shocks. Finally, we contrast these simulated shocks with the redemption absorption capacity of funds in terms of the Redemption Coverage Ratio. The main conclusion we draw is that Spanish mutual funds are sufficiently resilient to cope with potential redemption shocks.

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Contents

1.Introduction

The Financial Crisis of 2007-08 which sparked the Great Recession in most of the national economies of the world between 2007 and 2009, revealed the severe consequences of economic crisis in a world as globalized as the one we live in today. So much so that the transmission and the contagious effect in the financial system can lead to consequences for the global economy.

To explain the effects of this type of crisis, it is first important to point out the difference between what, accordingly to ESRB, are systemic and non-systemic companies. So therefore, the systematic companies are the ones that due to their interrelationship between different segments of the economy can spread a financial issue to another company. Many credit companies show systemic character because of their function by channelling savings into investment. Thus, in the same way that credit companies fulfil this function, mutual funds have their own objective of channelling savings into investments by offering to their participants sustainable returns over the long term.

Because of this potential systemic character in the investments through collective investment vehicles, it is key to understand the interdependence between investors savings and corporate financing and how this can also lead to a potential global crisis as well.

Specifically, according to ESMA in 2019 [18] the investment under the management of investment funds domiciled in Europe added 14.2 trillion €, this shows the importance of an appropriate risk management in this type of investments to face that in an eventual shock the consequences spread across the investors and markets. Throughout this work the focus will be in the potential liquidity shocks.

The liquidity risk can be defined as the possibility that an entity may not be able to meet its payments commitments or that, on the other hand, in order to meet the payments, it may have resort to obtaining funds on very poor terms.

Let's imagine a situation where a redemption shock occurs in a specific collective investment institution, consequence of uncertainty scenario in the global markets. In this case, the fund managers will be forced to undo some of their positions in order to attend their obligations towards their participants. This situation could result in a decline of the company's stock prices in the markets, and before this uncertainty increasement, more participants could request to redeem their investment, thus creating a domino effect. It can be imagined that the effect can be contagious to other collective investment institutions, spreading further the shock and affecting the channelling of savings into corporate financing that would have to look for financing in other financing methods, or in the other hand, stop the company growth, affecting then the global economy. Inside this paradigm, it is also known the "first mover advantage", which could aggravate more the situation, generating incentives to anticipate to other participants and reimburse their investment in better conditions that the next one will find.

For all this mentioned before, it is very important to research the effects of the liquidity shocks in collective investment institutions, making researches about extreme but plausible events and how these events can affect the participants and contagious to the financial system and by consequence to the national economies.

It is also important to understand the relationship between different types of crises. The Financial Crisis of 2007-08 started as a financial crisis of bank character derivate of real-estate bubble and the exposition to low quality debt, but resulted in global crisis that affected the national economies triggering a sovereign debt crisis in Europe. Recently we face the COVID Crisis which started as a health situation, a sudden non-financier shock that where most of the governments of the world decreed a lockdown of the society to fight against the COVID crisis but causing an offer shock than are now after two years ago still affecting the supply chain.

2.Literature Review

While there is not an agreed in the description about systemic risk, there have been multiple working papers in relation with systemic risk and there have been defined multiple indicators that pretend to measure it or the marginal effect of certain companies which are considered as systemic. Many of these indicators are conditional metrics that try to analyse the system Value at Risk (VaR), conditionate on certain companies facing difficulties, or the lack of capital expected for certain company conditional on systemic distress. This is how Brownlees and Engle (2012) [12] define the SRISK.

Therefore, it is important to first define systemic risk and systemic institution. According to ESRB (2010), systemic risk is "a risk of disruption of the financial system, which may have serious negative repercussions on the domestic market and the real economy. All types of financial intermediaries, markets and infrastructures can be systemically important to some degree"; according to FSB (2011) [21], a systemic institution or Systemically Important Financial Institutions can be defined as "financial institutions whose distress or disorderly failure, because of their size, complexity and systemic interconnectedness, would cause significant disruption to the wider financial system and economic activity".

According to Acharya (2009) [2], a financial crisis is systemic if many entities fail together or the failure of one entity spreads to others. Therefore, we can distinguish between shocks that affect the financial system as a whole or unexpected shocks that happen to a certain company of a systemic nature and spread to the rest of the entities.

One of the most important contributions in the systemic risk literature is that of Adrian, Tobias and Bruennermeier (2011) [4], in which they propose the Conditional Value at Risk (CoVaR), a risk measure that can be defined as the VaR of the financial system conditional on institutions being in distress. As previously mentioned, this is a conditional VaR measure, where VaR is defined as the maximum percentage loss in monetary units (MU) at a given time horizon with a given probability:

$$VaR_{i,t}^{\alpha} = F_{i,t}^{-1}(\alpha) \quad (1)$$

where $F_{i,t}^{-1}$ is the inverse of the distribution function of the price returns of institution i, and α is the confidence level, $\alpha \in (0,1)$.

Equivalently, VaR can be expressed as

$$Pr(r_{i,t} \leq VaR_{i,t}^{\alpha}) = \alpha \quad (2)$$

This is a risk measure that focuses on a specific company in isolation.

The CoVaR is a conditional measure, it reflects comovements or conditional movements; more specifically, it is the VaR of the entire financial system j conditional on the institution i being in distress.

$$Pr\left(r_{j,t} \leq CoVaR_t^{\alpha,\tau} \middle| r_{i,t} \leq VaR_{i,t}^{\alpha}\right) = \tau \tag{3}$$

Where $CoVaR_t^{\alpha,\tau}$ is the CoVaR of the system j, defined by the τ-quantile, $\tau \in (0,1)$, of the conditional probability.

From the above definition the authors also define the ΔCoVaR as the contribution of institution i to the financial system j, as a difference of the VaR of the financial system j conditional on institution i being in distress and the VaR of the financial system j conditional on the company i being in normal circumstances.

$$\Delta CoVaR_t^{\tau,\alpha} = CoVaR_t^{\tau|r_i=VaR_i^{\alpha}} - CoVaR_t^{\tau|r_i=Median_i} \tag{4}$$

From this last definition they deal with the Forward ΔCoVaR which can be used for financial stability monitoring, and as a basis for (countercyclical) macroprudential policy. Here they incorporate certain characteristics of companies such as leverage, maturity mismatch, market-to-book or size. In this case they use quantile regression to try to predict from these indicators the possible contribution to systemic risk.

Girardi and Ergün (2013) [22] take the CoVaR idea and obtain a time-varying CoVaR using a GARCH model. Then they incorporate by means of a GARCH DCC the calculation of the joint system-institution distribution for each entity. To collect the third and fourth order moments of the distributions they use a t-asymmetric distribution.

Another measure of systemic risk proposed by Brownless and Engle (2012) [12] is the Marginal Expected Shortfall (MES), which, based on Acharya et al. (2010) and using a bivariate GARCH DCC model of the institution and the market, they estimate the MES, which is nothing more than the expected loss of the institution in the tail, conditioned on the market being in distress. It is a tail-based risk measure that has been proposed as a component (or relation) of several methodologies aimed at identification of systemic risk exposures of banking financial institutions

We can define the Expected Shortfall (ES) of the financial system j as the average loss of the financial system when the returns are below the VaR.

$$ES_{j,t}^{\alpha} = E\left(r_{j,t} \middle| r_{j,t} \leq VaR_{j,t}^{\alpha}\right) \tag{5}$$

If we assume that the system is represented by an index composed of institutions where each of them represents a percentage w_i. As defined by the authors Acharya et al. (2010) [1], the MES is the loss of the institution i when the system j is in distress.

$$MES_{i,t}^{\alpha} = \frac{\partial ES_{j,t}^{\alpha}}{\partial w_i} = E\left(r_{i,t} \middle| r_{j,t} \leq VaR_{j,t}^{\alpha}\right) \quad (6)$$

Note that MES can also be interpreted as the change in the ES of the system j if the weight w of the institucion i changes.

Based on this idea, the same authors introduce another measure called SRISK, which is a function of the MES. SRISK measures the expected capital shortage for a certain company conditional on the system being in a systemic crisis.

A remarkable article that collects these measures and compares them for a large sample of US financial institutions is the one proposed by Benoit et al (2013) [11], where they conclude that it depends on the objective of the researcher it is more appropriate to use one measure or another. To identify the most systemic company it would be advisable to use SRISK or ΔCoVaR due to the properties of these measures. On the other hand, if the researcher's objective is to predict the contribution of a particular institution to the overall risk of the financial system, it would be more advisable to use MES or SRISK, since ΔCoVaR is largely determined by its VaR calculated in isolation.

Another proposal that tries to make an analogy of Component VaR (CVaR) to systemic risk is the proposal of Banulescu (2013) [10], where he introduces the Component Expected Shortfall (CES) measure, trying to address the main drawbacks of SRISK and MES. The CES of financial institution measures the firm's 'absolute' contribution to the ES of the financial system. Where CES could be calculated as the product of the MES by the weight of the institution in the financial system:

$$CES_{i,t}^{\alpha} = w_i \frac{\partial ES_{j,t}^{\alpha}}{\partial w_i} = w_i E\left(r_{i,t} \middle| r_{j,t} \leq VaR_{j,t}^{\alpha}\right) \quad (7)$$

Generally previous measures tried to relate companies such as banks or insurers to systemic risk however Dunne and Shaw (2017) [16] use these marginal risk metrics to capture mutual fund exposures to generalized industry-wide tail events. In this study they found evidences that the level of leverage, the use of derivatives, the level of redemptions or the openness of funds are important factors to measure the exposure of the mutual fund industry to systemic tail risk.

For its part, the International Monetary Fund (IMF) in chapter 3 of its April 2015 report [25], highlights the systemic importance of collective investment vehicles and that large managers do not have to contribute more to systemic risk. One of the points that they emphasize is to address liquidity risk since surprise flows (especially outflows) and the relationship of these with market volatility (measured through the VIX), have an impact on companies returns and generate a contagion effect, causing a cycle in which investors start to request redemptions, and this in turn, generates a worse performance of the fund therefore investors who had remained previously have now incentives to request redemptions. In addition to the relationship between fund performance and redemptions, the report also shows a positive relationship between redemptions in funds which invests in more illiquid markets, as well as a relationship between redemptions and investor type or a negative relationship between redemptions and fees.

This is not new, as Sirri and Tufano (1998) [35] have already shown evidence of the relationship between net flows and past performance (they find an asymmetry), market volatility, fees or fund size.

For this reason, ESMA also focuses in the systemic impact of the asset management sector when market stress raises. In Stress simulation for investment funds report (2019) [18], ESMA proposes a wide variety of options to generate redemption stress scenarios (which we will use as a reference in this document), as well as to measure the resilience of investment funds to these stress scenarios in terms of Redemption Coverage Ratio (RCR). In the first place, the regulator proposes three alternatives to treat the flows, that of heterogeneity (treating each fund using only its flows), that of homogeneity (assuming that all funds present similar repayments) or adding by fund style. Secondly, ESMA proposes two methods to estimate the distribution, the first approach would be to use the empirical distribution directly, the second would be to estimate a theoretical distribution. Finally, the regulator details several options to calibrate the redemption shock, among which are the VaR approach, the ES approach and its conditional versions and tail dependencies using Extreme Value Theory (EVT). All this set of options would come from the historical approach, however the possibility of using the event study approach (using periods of stress for the analysis) as well as the expert judgment (which would consist of proposing ex-ante shocks based on at the investigator's discretion). Several options are also given to measure the liquidity of funds. We will focus on the idea of High Quality Liquid Assets HQLA (under the Basel III approach [8]). Since finally, to measure the resilience of investment funds to liquidity shocks, the RCR is proposed, which would be the relationship between the liquid assets in the portfolio and the expected outflows, that is, the relationship between the HQLA and the redemption shock estimations; this measure is very useful since it is intuitive and easy to interpret.

Following some of these ideas, Javier Ojea published a study in the CNMV bulletin for the second quarter of 2020 [15] in which he generates severe but plausible redemption scenarios for Spanish investment funds. This study uses a background category approach, first fitting the series with an AR(4)-GARCH(1,1) model and then estimating dependency using copulas. Once the model is defined, he simulates series of net flows, and from these series, obtains the shock of redemptions. This study will be also a reference this document.

3.Data

3.1. Sample and data analysis

For this study we have obtained data from 101 Spanish funds of different categories between 01/08/2019 and 31/03/2021. Due to the difficulty of obtaining the positions of the investment portfolios throughout the period, we have assumed that the portfolios maintain stable investments over time. The positions and data of the assets have been obtained from the quarterly or annual reports, from the public documentation published by the managers on their websites, form Morningstar and through Reuters Refinitiv.

Firstly the investment funds for which we have not been able to obtain complete information or that presented an investment in other funds greater than 10% have been discarded. Next, we have obtained the positions of the funds and we have obtained for each asset its type of asset, the market spread bid ask, the amount issued, its rating, its severity, the quote currency, the market capitalization, and the volume traded. Once this analysis has been carried out, we have filtered by the criteria established above, we have discarded the funds that have some restrictive redemption policy (these could bring on bias to the sample) and we have filtered the funds that we could not get complete information about their assets. One these filters have applied; we have obtained a final amount of 73 investment funds that has been the final sample to carry out the study.

We have categorized the funds in management styles according to their categories in their Key Investor Information (KIID), grouping them into 4 categories: European Equity (EE), International Equity (EI), Mixed Funds (MX) and Fixed Income Funds (FI). The EE funds includes funds that have at least 75% exposure in European stocks; the EI includes funds that have at least 75% exposure in non-European stocks; the MX funds includes fund that according to their investment policy they can invest in Fixed Income and Stocks; the FI funds include funds that must maintain at least 80% of the investments on fixed income either corporate or government. The distribution of these funds will be as follows: 17 EE funds, 16 EI funds, 16 MX funds and 24 FI funds.

Figure 1 Funds Distribution by Category

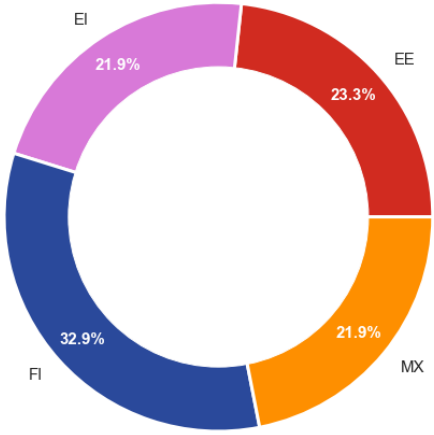

Figure 1 Represents the distribution of the filtered final sample. The weight of each category is shown in the pie chart. International Equity funds (EI) represented with pink weighs 21.9% of the sample, European Equity funds (EE) represented with red weighs 23.3% of the sample, Mixed Funds (MX) represented with orange weighs 21.9% and Fixed Income funds represented with blue weighs 32.9% of the sample.

We have also grouped the redemptions by fund category, in this case each fund presents a series of 435 redemptions between 01/08/2019 and 31/03/2021. Grouping by fund category we have 7,395 redemption data for management style EE, 6,960 for EI, 6,960 for MX and 10,440 for FI, which means that the total sample of redemptions presents 31,755 data. **Table 1** shows the fund categories that have fixed income (MX and FI), presents a greater mean that funds that have not. All fund categories have positive skewness and the EE funds is the category that have the greater kurtosis, it means that the redemptions are mostly grouped in low redemptions as **Figure 2 (a2)** shows. **Table 1** also shows how the fund categories with fixed income presents the greater interquartile range (IQR), this is due to the fact that these categories are the ones with the greatest dispersion of data.

Table 1 Redemption Statistics

	EE	EI	MX	FI
μ	0,0007	0,0006	0,0014	0,0019
σ	0,0010	0,0005	0,0011	0,0017
s	7,1861	2,3311	2,4906	2,8904
κ	74,2368	7,5427	8,7956	11,3138
$q_{0.99}$	0,58%	0,27%	0,63%	0,97%
$q_{0.95}$	0,19%	0,17%	0,34%	0,58%
IQR	0,05%	0,05%	0,10%	0,14%
n	17	16	16	24
AuM (m)	2055	5546	1056	1500

Table 1 shows the statics of redemptions (measured as percentage of the AuM) by fund category. The fund categories are European Equity (EE), International Equity (EI), Mixed funds (MX) and Fixed Income funds (FI). μ is the mean, σ is the standard deviation, s is the skewness, κ is the kurtosis, q refers to several quantiles of the distribution, IQR is the interquartile range, n is the number of funds included in each category and AuM is the total of Assets under Management by fund category in millions.

Redemption data are calculated as a ratio of assets under management (AuM). In this case, the EE funds show an AuM of EUR 2.055 million, the EI funds of EUR 5.546 million, the MX funds an AuM of EUR 1.056 million and the FI funds of EUR 1.500 million. Which means that we are looking at a total of EUR 10.158 million. The series of redemptions in monetary units for each day have been added, as well as the series of AuM, obtaining a single series of redemptions on AuM for each fund category.

$$Redemption_{C,t} = \frac{\sum_{i \in C} Redemtion_{i,t}}{\sum_{i \in C} AuM_{i,t-1}} \tag{8}$$

Figure 2 Redemption Series

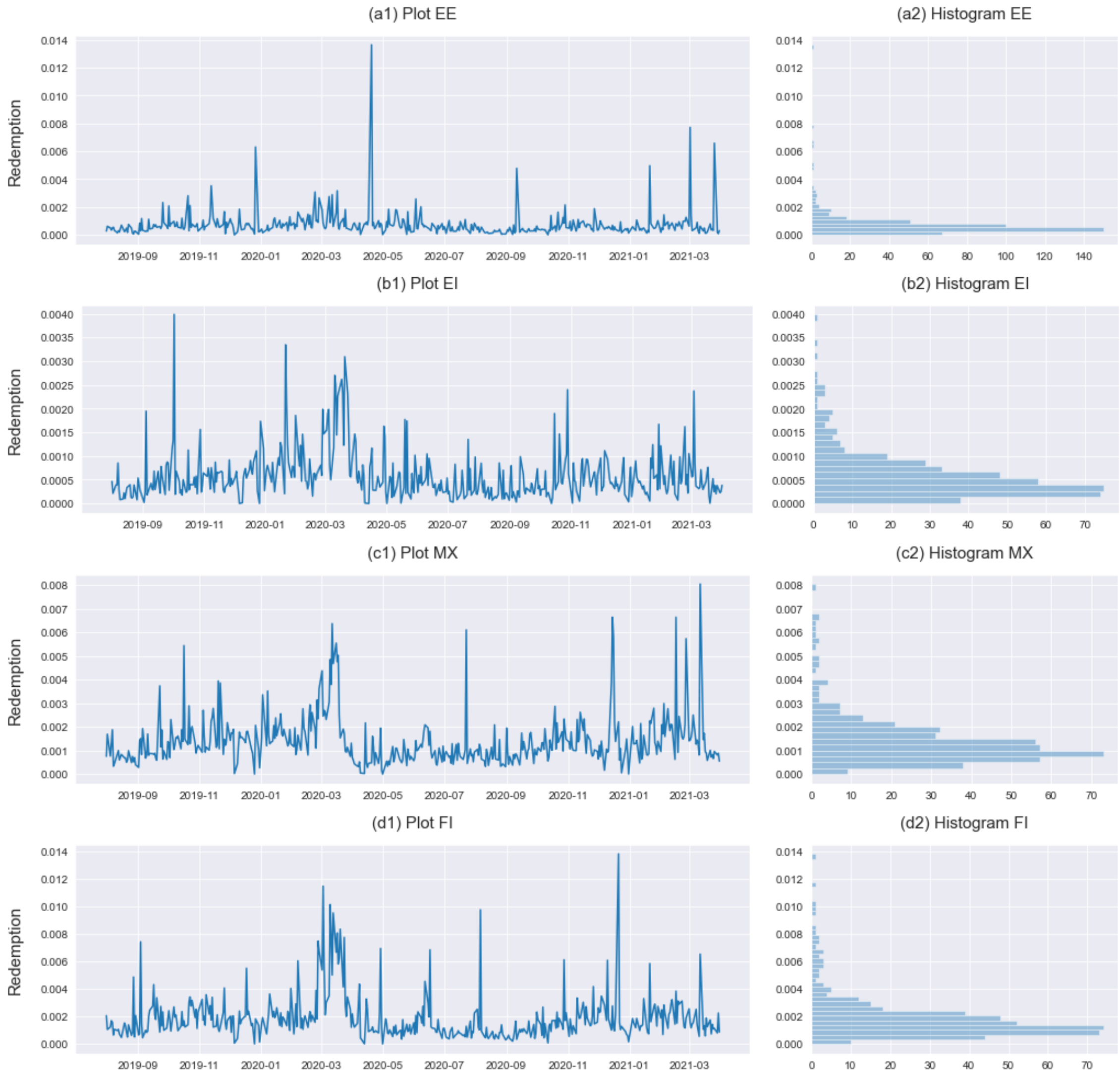

Figure 2 shows the daily redemption series aggregated by fund category as **(8)**. (a1) are the European Equity (EE) redemption series; (a2) is the histogram of (a1); (b1) are the international Equity (EI) redemption series; (b2) is the histogram of (b1); (a3) are the Mixed funds (MX) redemption series; (c1) is the histogram of (c2); (d1) are the Fixed Income (FI) redemption series; (d2) is the histogram of (d1).

3.2. Tail data analysis

As the objective of the study is going to be to create redemption shocks, we are going to focus the analysis on the tails of the data distribution to analyse the possible systemic effect of liquidity risk on investment funds. For this objective we are going to create an aggregate category (AL) that we will define as the system, therefore, this category will be like a proxy of the universe of investment funds.

The most direct approach to study the redemptions would be to calculate the VaR **(1)** of the redemption series for each fund category. This would be an unconditional risk measure since it analyses the series by themselves, without taking into account their relationship with the other fund categories. For this reason, it is also interesting to move from unconditional measures to conditional measures, so we can measure risk from a global perspective, taking into account the relationships and dependencies between different market variables. This approach is more accurate since when we talk about systemic risk it is very common to be contagion between different sectors. Bearing in mind that we study investment funds by fund category, we can assume that the different asset classes in which the funds invest, will be affected jointly in the event of a global shock. This would cause effects on the funds returns, which in turn could affect the redemption of the different investment funds, as indicated in their studies Sirri and Tufano (1998) [35] or the IMF in its April 2015 publication [25].

The first conditional measure that we are going to use will be the Conditional Value at Risk (CoVaR). As it is a conditional variable, one variable must be taken into account as a function of another. In this case, we want to measure the redemptions by fund category in a distress redemption scenario. For this reason, we have created a new category, Aggregated Funds (AL) which is the grouping of all funds in a single category as **(8)** with $C = 1$. To calculate the CoVaR, a threshold must be set for the conditional variable, the threshold on which to define the stress scenario can be the VaR[1] **(1)** of the system (AL) at a given α. Within this conditional scenario, the conditional quantile τ of the fund category redemption is selected, which will be the CoVaR[2] **(3)**. The Conditional Expected Shortfall (CoES) is the mean of the redemptions over the CoVaR.

$$CoES_{i,t}^{\tau,\alpha} = \frac{1}{\tau}\int_{\tau}^{1} CoVaR_t^{\tau,\alpha} \qquad (9)$$

Figure 3 shows a scatter plot between Aggregated Funds and each fund category, x axis represent AL redemptions and y axis represent the fund category redemptions. On left axis unconditional and conditional distribution from fund category are shown. The conditional distribution is located to the right (upper) of the unconditional distribution this is because the conditional distribution is composed by the fund category redemptions conditionate that AL redemptions are higher than α (e.g. $\alpha = 0.95$), we focus on the points in the upper right quadrant of the scatterplot. The τ quantile of these data is the CoVaR for each fund category, and the average of redemptions over the τ quantile is the CoES for each fund category.

[1] Note that VaR is defined in **(1)** as left tail risk since the general definition is about negative returns, in this study we are interested in the positive tail of redemptions so the equation must be reversed.
[2] Same as [1].

Figure 3 Conditional risk measures

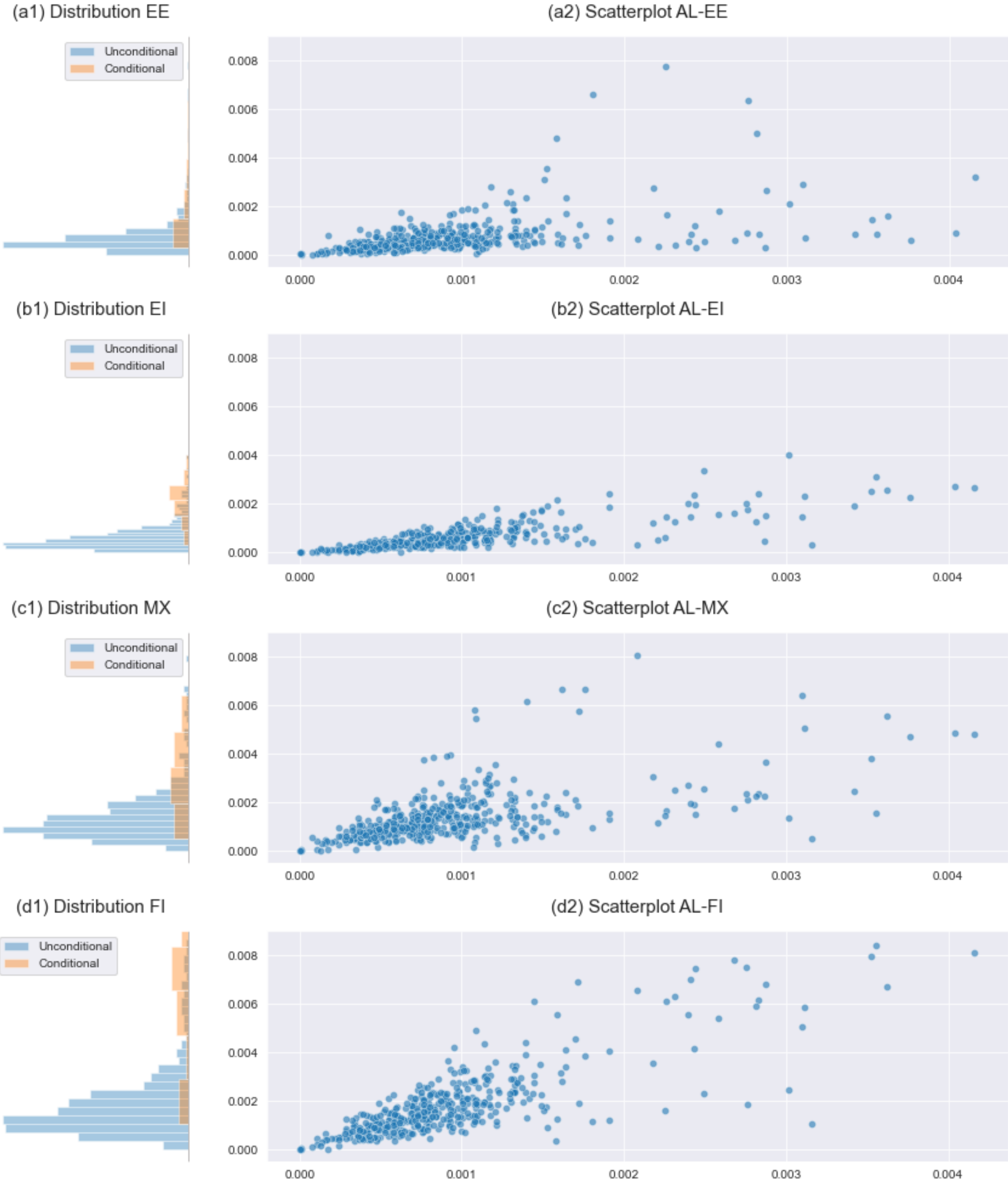

Figure 3 shows a scatterplot between Aggregated Funds (AL) redemptions and each fund category. (a1) represent the unconditional distribution of European Equity (EE) in blue bars, and the conditional distribution of EE when AL is over 0.95 quantile in orange bars. (a2) the blue points represent the scatterplot between AL redemptions represented in the x axis and EE redemptions in the y axis. The same representation for (b1) and (b2) for International Equity (EI) redemptions, for (c1) and (c2) for Mixed funds (MX) redemptions and for (d1) and (d2) for Fixed Income (FI) redemptions.

There are different ways to estimate the CoVaR, we used the quantile regression due to its simplicity and the efficient use of the data, the way in which we use this method is detailed in **Appendix A**. **Table 2** shows the VaR, CoVaR and CoES for each fund category, in general, the redemption shocks are very low, the FI funds are the ones with the highest CoES, with expected conditional redemptions of 1.38%, with values in these ranges the fund managers should not present problems in liquidating 1.38% of their portfolio, so we can guess that there are no liquidity problems in the selected sample. We can also see how VaR for EE redemptions is low 0.19% in relation to the rest of funds, however this category has a relatively a high CoVaR 1.25% and CoES 1.36%, it is due to the fact that the sample presents heavy tails, and it is indicated by the relatively high values of skewness and kurtosis of the EE redemptions in relation to the rest of the categories, as we previously analysed in **Table 1**.

Table 2 Redemptions Risk Measures ($\boldsymbol{\alpha = 0.95; \tau = 0.95}$)

	VaR	CoVaR	CoES
EE	0,19%	1,25%	1,36%
EI	0,17%	0,39%	0,40%
MX	0,34%	0,62%	0,64%
FI	0,58%	1,35%	1,38%

Table 2 shows the risk measures calculated for each fund category redemptions. The categories are European Equity funds (EE), International Equity funds (EI), Mixed funds (MX) an Fixed Income funds (FI). VaR is calculated with a $\alpha = 0.95$. CoVaR and CoES are calculated using quantile regression approach at $\tau = 0.95$.

4.Measurement of the liquidity

In order to measure resilience to redemption shocks for investment funds, we will need a measure of their liquidity. As we have anticipated above, we used the measure of High Quality Liquid Assets (HQLA) as proposed by ESMA In Stress simulation for investment funds report (2019) [18]. This measure is based on the measure with the same name used by banks under Basel III [8] for liquidity regulatory requirements. However, the way of calculating the HQLA proposed by ESMA is very simple since it is based exclusively on the category of the asset (cash, corporate bond, equity, government bond, securitised...) and its credit ratings or credit quality step (CQS1 to CQS3). This measure, therefore, is not at all representative of the liquidity of the assets since it assumes the same liquidity for all equity and for the fixed income it is based only on the credit quality step and asset class. For this reason, we propose a model that includes other factors that we consider to be more representative of the liquidity of the assets in which mutual funds invest.

The proposed model is based on several ideas presented by the CNMV in its technical guide published in January 2022 [14], since we consider that this approach is more exhaustive and takes into account liquidity measures that are widely used in the financial sector, such as trading volumes or bid ask-spreads. For this, it is interesting to analyse the composition of investment funds by style in **Figure 4**.

Figure 4 Funds Composition by Asset Class

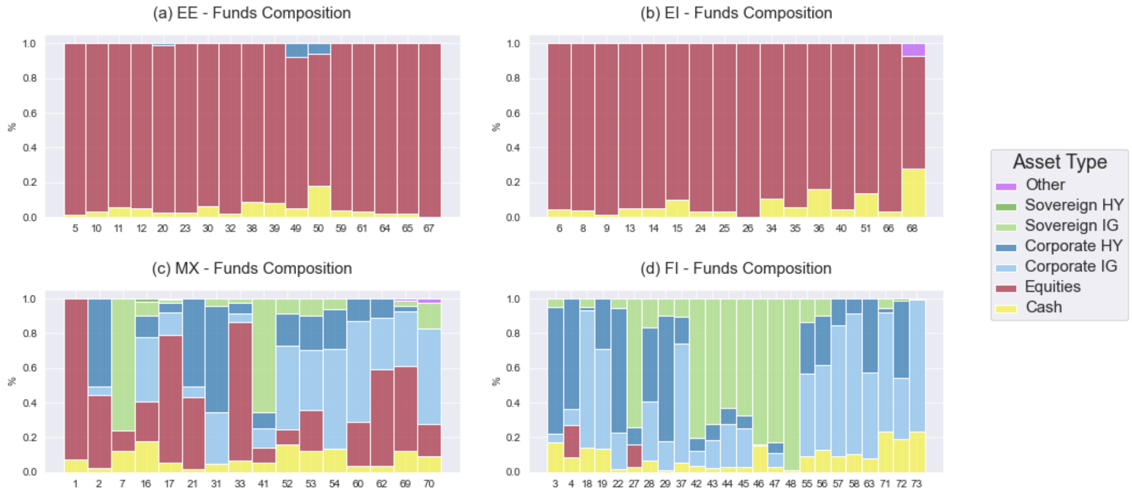

Figure 4 shows the composition of the funds categories by asset class. Each asset class is represented by bar colours established in the legend. (a) represent the composition of the funds categorised as European Equity (EE), x axis represents the fund label that is a number 1 to 73, y label represents the percentage of the asset class held by each fund. The same representation for (b) for International Equity funds (EI), for (c) for Mixed funds (MX) and for (d) for Fixed Income funds (FI).

For fixed income, CNMV suggests that it is necessary to take into account the bid-ask spread of prices, the quality of the bid-ask spread quotes, the asset type, amount over the total issue and its percentage in the investment portfolio and credit quality of the issuer among other things.

Our model for the fixed income consists of applying four haircuts (H_h) to each asset held by each fund. These haircuts are weighted according to the measures that we consider more relevant **Table 3**. The credit quality is the first haircut (H1), this haircut is progressively higher as credit quality worsens, a threshold has been established based on the ratings considered High yield to which the highest haircut will be applied in line with CQS of Basel III. The percentage of investment over the total amount issued is the second haircut (H2), this haircut is greater the more position you have over the total issued, the idea of this haircut is that the higher the percentage of issuance you keep in the portfolio, a greater liquidity loss you may have if in a certain moment the company goes through distress and you want to liquidate the position, being able to lose the entire investment for not being able to liquidate it. The bid-ask is the third haircut (H3), the haircut will be higher the wider the bid-ask spread on the asset, this haircut measures the sensitivity to the loss that the fund manager will assume in the event of not being able to sell an asset at a given moment in time, if in period one the fund manager cannot sell an asset because there is no volume to buy the position, it is possible that in period two he will sell that asset but at a different price than the one he would have sold in the previous period, so the higher is the range, more sensitivity to liquidity loss the fund manager will have. The asset class is the last haircut (H4), the idea of this haircut is that the more complex the bond, the more difficult it will be to sell on the market; in this case we have distinguish between the payment structure and the severity of the bond. See **Appendix B**.

Table 3 Haircut Weights

	Fixed Income	Equity
H1	50,0%	20,0%
H2	12,5%	60,0%
H3	12,5%	20,0%
H4	25,0%	

Table 3 shows the haircut weights assigned to each category. Fixed income assets have four haircuts and equity have three haircuts. More details about haircuts in the **Appendix B**.

For equity, the CNMV suggests measures such as volume, bid-ask spread, issuer size and market capitalization, outstanding capital and free float, and the percentage held in the investment fund over outstanding capital.

Our model for equities consists of applying three haircuts (H_h) to each asset held by each fund. These haircuts are weighted according to the measures that we consider more relevant **Table 3**. The first haircut (H1), measures the amount held in the portfolio in percentage over the volume traded, this haircut is based on that you can only place in the market the volume traded in that asset (assuming that a single trader can occupy the whole volume of an asset), if you try to place more, you may not be able to sell the position, assuming potential losses due to illiquidity. The second haircut (H2), measures the percentage of investment over market capitalization (fixed by free float), this haircut is based on that if the company goes through distress and you keep a high percentage of the company in your portfolio, you may not be able to liquidate the position. The last haircut (H3), measures the bid-ask spread (percentage on mid-price) of assets, the haircut will be higher the wider the bid-ask spread on the asset. More details in **Appendix B**.

The haircut applied on cash (equivalents) has been 0%, for other funds investments 50% haircut has been assumed and for other types of assets haircut of 100% has been applied. Once the model to measure the liquidity of investment funds has been defined, the HQLA has been calculated for each investment fund, HQLA is defined as a ratio over AuM.

$$HQLA_i = w_k \left(1 - \sum H_h\right); \qquad w_k = \frac{x_k}{AuM_i} \tag{10}$$

Where w_k is the weight of asset k in the portfolio i, H_h is the discount derived of haircuts on the asset k and x_k is the position of the asset k in the portfolio i

We have not found a direct relationship between HQLA and AuM as shown in **Figure 5**, however, it does seem that, following the proposed methodology, the funds that present a fixed income component generally have lower HQLA, this is due to the fact that many investment funds hold investments in bonds with a credit quality HY **Figure 4**, and this represents a large haircut to the liquidity of these assets.

Figure 5 Investment Funds HQLA

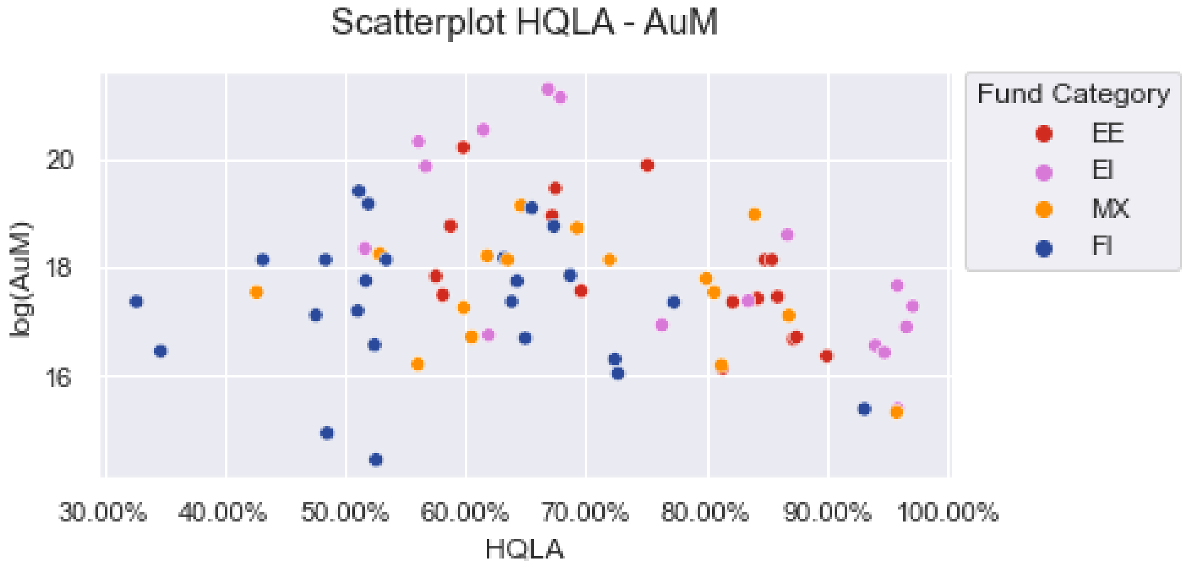

Figure 5 shows a scatterplot between HQLA represented in the x axis and AuM of each fund represented in the y axis. HQLA has been calculated as methodology explained in section **4**. AuM is represented in logarithmic scale. The funds category is represented in the colors indicated by the legend.

5.Estimation of redemption shocks

In this section we will focus on creating severe but plausible shocks by fund category. The most intuitive thing would be to perform a historical analysis by fund style and set a threshold as an α quantile. This would be an unconditional VaR **(2)** measure as proposed by ESMA, we could also calculate the ES of the redemptions above that α quantile **(5)**. The problem with these measures is that they do not take into account certain relationships and dependencies that the market presents when it is in distress.

5.1. TailCoR

The financial and sovereign debt crises have highlighted the importance of queues or rare events. These types of events spread their effect over the system creating tail correlations that can be linear and non-linear. TailCoR is a measure introduced by Sladana Babić, David Veredas et al. in the working paper TailCoR (2020) [7], it is a function for dependence that can be computed under tails that are fatter, equal or thinner than Gaussian. Therefore, it is a measure of correlation (linear or not linear) in the tails.

As Sladana Babić, David Veredas et al. define in their working paper [7], TailCoR is based on the fact that if we draw a scatterplot between two random variables X_j and X_k (properly standardized) and positively related (either linear and/or nonlinearly), most of times the pairs of observations have the same sign , concentrating the points in the north-east and south-west of the scatterplot. By painting a ϕ-degree line that crosses these quadrants, and project all the pairs on this line we produce a new random variable $Z(j,k)$. If the relationship between both variables is strong, these projections will extend along the entire ϕ-degree line, in case of presenting a weak relationship, the projections will be concentrated around the origin.

The TailCoR is equal to the difference between the upper and lower quantiles of $Z(j,k)$. One of the advantages of this measure is that it includes both linear and non-linear relationships, so TailCoR can present high values in cases where the variables X_j and X_k have a strong linear relationship panel (a) of figure **Figure 6** presented by the authors [7], if X_j and X_k present a non-linear relationship, for example they only present a relationship in the tails of the distribution panel (b) of **Figure 6** or if both situations occur simultaneously as shows panel (c) of **Figure 6**.

Figure 6 Diagrammatic Representation of TailCoR

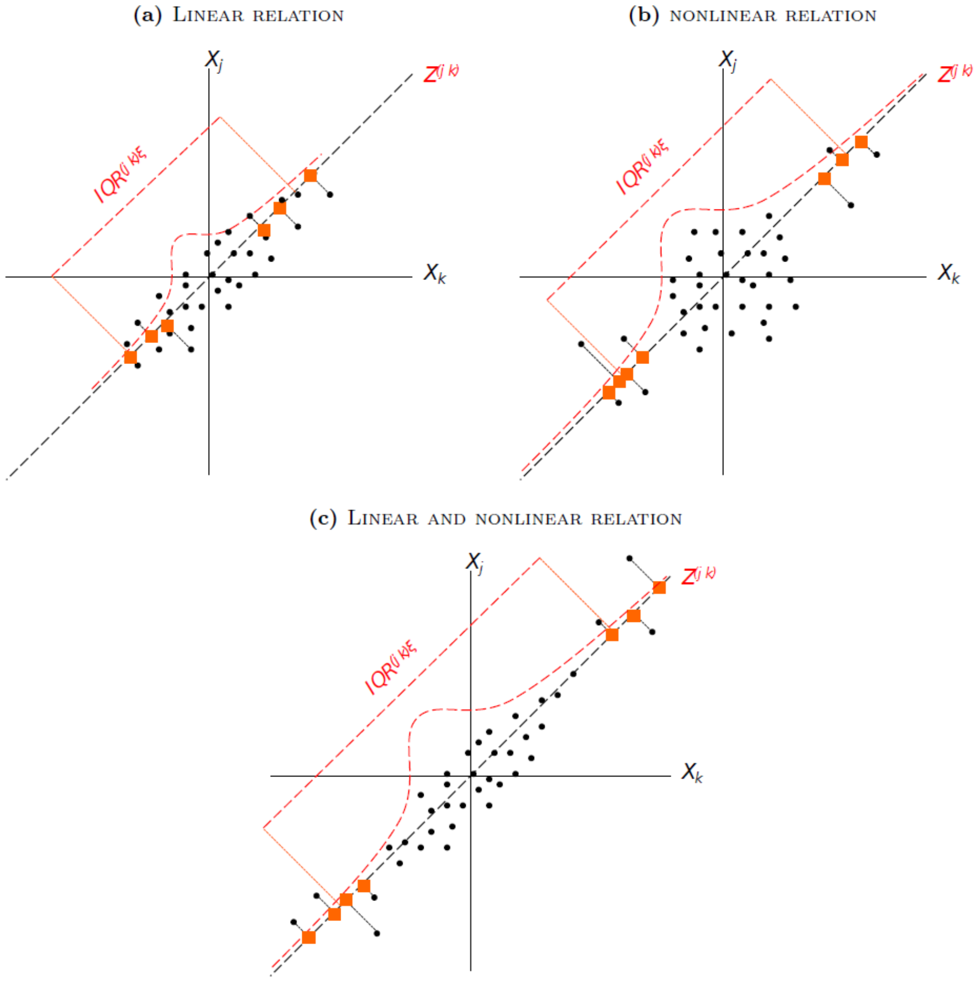

Figure 6 is an extract from the working paper TailCoR (2020) [7] the authors have authorized this figure to be included in this document. (a) Shows the case of linear relationship, (b) displays the case of nonlinear relationship and (c) shows the scenario of both linear and non-linear relations.

The first step will be to model the series of redemptions through an autoregressive process. In this case we have selected an AR(1)-GARCH(1-1), the details of the estimation are included in **Appendix C**. Funds with fixed income assets have the highest redemption median as well the highest IQR, all fund categories have positive skewness and EE redemptions still having the highest kurtosis **Table 4**.

Table 4 Filtered Returns Statistics

	EE	EI	MX	FI
Me	0,0005	0,0004	0,0011	0,0014
$IQR_{0.75}$	0,0005	0,0006	0,0011	0,0015
s	0,0018	0,0014	0,0027	0,0046
κ	2,2339	1,2141	1,4485	2,1520

Table 4 shows the filtered redemption statistics for each fund category. The categories are European Equity funds (EE), International Equity funds (EI), Mixed funds (MX) and Fixed Income funds (FI). All the metrics are quantile-based. Me is the median, $IQR_{0.75}$ is the interquartile range, s is the skewness computed as $(Q^{0.975} – Q^{0.5})$ - $(Q^{0.5}$-$Q^{0.025})$, κ is the excess of kurtosis and computed as $IQR^{0.975}/IQR^{0.75}$ -2.91

Once the series have been filtered by the econometric model, the data has been standardized for the variables X_{jt} and X_{kt}, which in this case will be the pairwise redemption series that we previously analysed **Figure 2**. Standardized variables are defined as:

$$Y_{jt} = \frac{X_{jt} - Q_j^{0.50}}{IQR_j^{\tau}} \tag{11}$$

Where: $X_{jt}\ is\!: the\ jth\ element\ of\ the\ vector\ X_t\ (redemption\ series), Q_j^{\tau}\!: the\ \tau th\ quantile$ $and\ IQR_j^{\tau}\!: the\ \tau th\ interquantile\ range\ \left(Q_j^{\tau} - Q_j^{1-\tau}\right)$

The same process is done for Y_{kt}. So we define the projection of (Y_{jt}, Y_{kt}) onto the 45-degree line if correlation is positive and the projection of (Y_{jt}, Y_{kt}) or onto 315-degree line if negative.

$$Z_t^{(jk)} = \frac{1}{\sqrt{2}}\left(Y_{jt} + Y_{kt}\right)\ if\ correation\ is\ positive \tag{12}$$

$$Z_t^{(jk)} = \frac{1}{\sqrt{2}}\left(Y_{jt} - Y_{kt}\right)\ if\ correation\ is\ negative \tag{13}$$

Then we calculate the interquartile range.

$$IQR^{(jk)\xi} = Q^{(jk)\xi} - Q^{(jk)1-\xi} \tag{14}$$

Where: $\xi\ : the\ tail\ parameter\ that\ tipically\ is\ close\ to\ 1\ and\ 0 < \tau < \xi < 1$

Once we have obtained the interquartile range we can compute the TailCoR.

$$TailCoR^{(jk)\xi} = s_0(\xi,\tau)\, IQR^{(jk)\xi} \tag{15}$$

$s_0(\xi,\tau)$ *is a normalization constant such under Gaussianity and linerar uncorrelation* $TailCoR^{(jk)\xi} = 1$. **Appendix D** shows a tabulation for s_0 for some values of ξ, τ.

Note that TailCoR is not bounded between -1 and 1, so it cannot be interpreted as a traditional correlation, in the working paper by Sladana Babić, David Veredas et al [7], the authors propose a way to rescale the measure which they call $\boldsymbol{TailCoR_{alt}^{(jk)\xi}}$, however this is not within the scope of our study. The authors also point out that TailCoR is not a measure of asymptotic dependence in the tail, so it can be computed for any ξ and for distributions whose tails are Pareto, exponential or even with finite points. As we have discussed before our series of redemptions have similar to an inverse exponential distribution **Figure 2**, so this property of TailCoR can be useful for measuring dependency relationships between redemptions from different fund categories.

The authors also propose a robust measure to estimate the linear and non-linear correlation.

$$\hat{\rho}_{jk,T} = \sin\left(\frac{\pi}{2}\hat{\kappa}_{jk,T}\right) \Rightarrow Lin = \sqrt{1 + |\hat{\rho}_{jk,T}|} \tag{16}$$

Where: $\hat{\kappa}_{jk,T}$*: the Kendall'scorrelation betwwen* Y_{jt} *and* Y_{kt}

$$NLin = \frac{\widehat{IQR}_T^{(jk)\xi}}{\sqrt{1 + |\hat{\rho}_{jk,T}|}} \tag{17}$$

Where: $\widehat{IQR}_T^{(jk)\xi}$*: the interquantile range estimation of the projection*

TailCoR has been calculated for our filtered redemption series as well as the linear and non-linear components. Our sample is made up of daily redemptions between 01/08/2019 and 31/03/2021, having a total of 435 redemptions. As we are interested to study the tails of the data distribution, TailCoR and its decomposition for the full sample has been calculated for a $\tau = 0.75$ as proposed by the authors and $\xi = 0.95$.

All estimates TailCoR for funds categories have values between 2.3 and 1.8 **Figure 7**, as we can expect the diagonal elements are the greatest, this is because the diagonal represents the TailCoR of each fund category with itself, as these elements can be interpreted as a measure of tail risk, we can conclude that Fixed Income funds (FI) have the higher risk of redemption shocks. Expectedly, the category with highest TailCoR is the aggregated funds category (AL), this is because of how this series has been created. More interestingly, are the estimations for FI category, we can appreciate how this category has the mostly dependence with the other categories. On the other hand, the funds categories with less dependency are International Equity funds (EI) and Mixed funds (MX). Note that we cannot differentiate by blocks between fund categories with fixed income assets (FI and MX) and fund categories that do not have (EE, IE).

The nonlinear component estimation has higher values, moving in a range from 3.1 to 4.1, the strongest nonlinear relationships are in general between different categories. The results are very similar to TailCoR, the FI category presents the strongest non-linear relationships and the EI and MX categories are the ones with the weakest relationships, being also between them the weakest relationship of all combinations. Since this element captures the relationships between the tails of the distribution, we can also conclude here that the FI category is the one with the highest risk in the event of a redemption shock.

The estimation of the linear component is the one with the lowest values moving between 1.17 and 1.22 (removing the diagonals and the category AL) and have also the lowest rank. Due to AL category has been constructed that is the one with the strongest linear relationships. In this case we detect how the EE category presents comparatively very low relationships with the rest of the categories. Among the other three categories there are no remarkable linear dependences.

Figure 7 TailCoR and Components

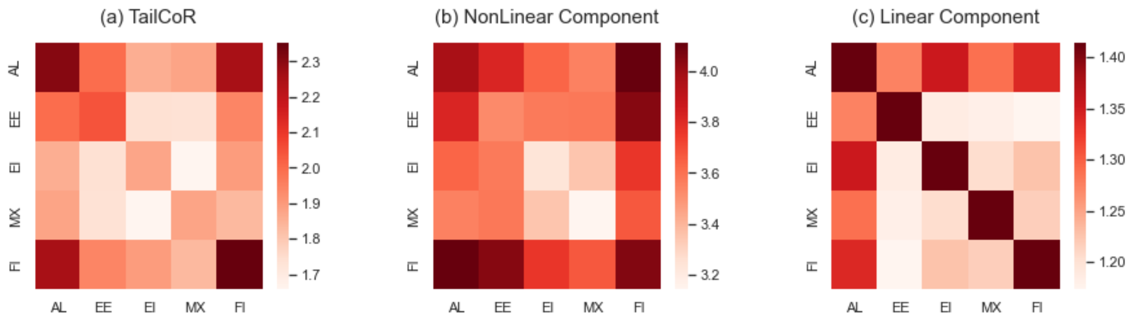

Figure 7 shows TailCoR (a), Nonlinear component (b) and Linear component (c) represented with heat maps for each fund category. The elements on the diagonal (a) and (b) represents the TailCoR and Nonlinear component of each fund category with itself, the elements off the diagonal (a) and (b) represents the TailCoR an Nonlinear component of each fund category with each other. The elements on the diagonal (c) are by definition 1.41, the elements off the diagonal (c) represents the linear component of each fund category with each other.

To evaluate the evolution of TailCoR throughout the sample, we have divided the sample into 87 weeks and computed the TailCoR and its components. We have divided the sample into weeks and not into months since we have a short sample and it is also easier to detect the shocks with shorter samples, preventing the shocks from being compensated with data from remaining distribution.

Figure 8 TailCoR and Components (Weekly Data)

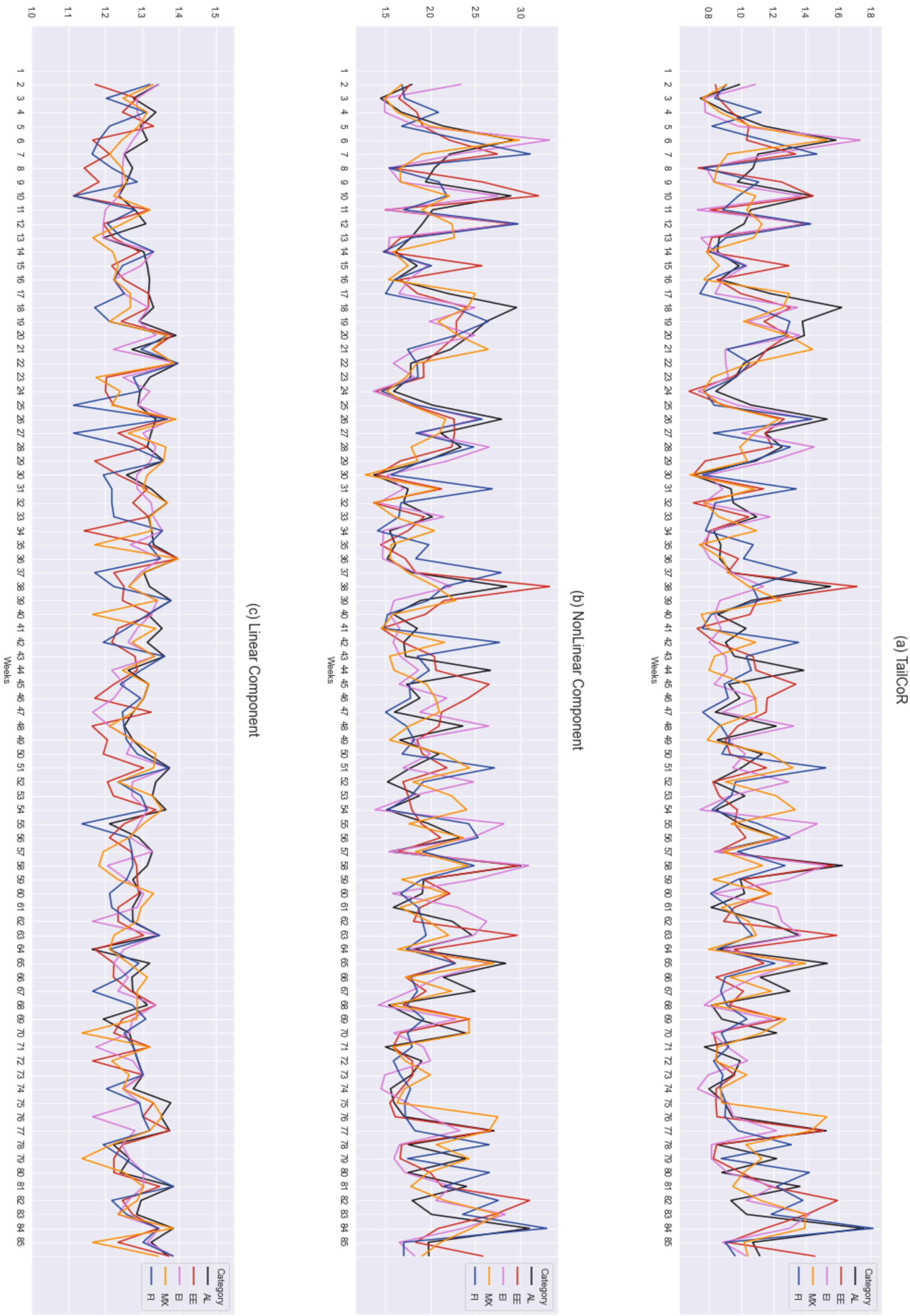

Figure 8 shows the evolution of TailCoR (a), Nonlinear Component (b) and Linear Component (c) for each fund category. Lines represents the cross-categories average TailCoR, Nonlinear Component and Linear Component respectively. The x axis represents the weeks, the y axis shows the estimation of TailCoR, Nonlinear Component and Linear Component respectively. is represented in the colours indicated by the legend.

Due to the data, it is difficult to give a clear interpretation to the results obtained, we can see in **Figure 8** how the TailCoR seems to move in a stable manner between the values 0.7 and 1.8. We must bear in mind that we are working with a relatively short data series and that it also belongs to a period of crisis (COVID-19) from the beginning to the end, which means that we do not have a previous stable period of reference to be able to compare the data between a period of calm and another of crisis. We can see how the linear component is much more stable moving between values of 1.1 and 1.4 for all the series, however the variability of the non-linear component does present much more variability moving in the range of 1.3 to 3.3.

The structure of the TailCoR and the non-linear component are very similar, so we can infer that the most important relationships between the redemptions of the different funds categories are mostly non-linear.

5.2. Simulating redemption shocks

Once analysed the redemptions dependences by fund category, we need to find some way to simulate redemption shocks that pick up those dependencies. The ESMA report (2019) [18] suggests the use of copulas to model the structure dependence to take in account non-linear effects, in this way the joint distribution of net flows could be estimated by choosing a specific copula.

In the previous section we estimated the TailCoR of our sample and shown how it evolved over time. Although it is not possible to interpret the results in terms of comparing the data with the macroeconomic events that occurred during the period, we can analyse the evolution between different weeks. As the authors of TailCoR explain, TailCoR tends to increase in periods of crisis or when certain shocks occur. Therefore, we have decided to choose the 30 weeks in which the TailCoR increased the most to obtain a sample that would serve to approximate the distribution of severe redemption shocks.

Note that as we detect that the non-linear component is the one that provides more information to TailCoR, the series of selected to estimate the distribution should also include this non-linear relationship component. We propose to use Akaike (AIC) and Bayesian Information Criterion (BIC) as model selection method. With these criteria we have studied several distributions with support $x \in (0, \infty)$ that could fit the subsample. Note that shown distributions in **Table 5** are related to each other since the Rayleigh and exponential are special cases form Weibull (see below). The distributions that fits better the sample in terms of information criteria are the beta and Weibull distributions, there are no significant differences between the two, so we have proposed the Weibull due to its statistical properties.

Table 5 Information Criteria

AIC		EE	EI	MIX	RF
	Beta	-2601	-2855	-2456	-2290
	Weibull	-2595	-2833	-2430	-2272
	Rayleigh	-2165	-2778	-2366	-2181
	Exponential	-2597	-2778	-2384	-2240
BIC		EE	EI	MIX	RF
	Beta	-2594	-2849	-2449	-2283
	Weibull	-2594	-2775	-2381	-2237
	Rayleigh	-2588	-2826	-2423	-2265
	Exponential	-2162	-2775	-2362	-2178

Table 5 Shows the values of Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). $AIC = -2logL(\hat{\theta}) + 2k$ where $L(\hat{\theta})$ is the likelihood function of the data when evaluated at the maximum likelihood estimate of θ and k is the number of parameters. $BIC = -2logL(\hat{\theta}) + klog(n)$ where n is the number of observations.

In this case we will use the two-parameter Weibull. The Weibull distribution was discovered by Fréchet in 1927 and is used in many branches such as engineering, medicine, climatology, extreme value theory (EVT) or finance. This function is described according to the shape α and scale β parameters and the support is $x \in (0, \infty)$.

The Weibull distribution with shape parameter α and scale β has density given by equation **(18)** and the cumulative distribution function given by **(19)**.

$$f(x) = \begin{cases} \left(\frac{\alpha}{\beta}\right)\left(\frac{x}{\beta}\right)^{\alpha-1} e^{(-x/\beta)^{\alpha}} & if \;\; x \geq 0 \\ 0 & if \;\; x < 0 \end{cases} \tag{18}$$

$$F(x) = 1 - e^{-(x/\beta)^{\alpha}} \tag{19}$$

The properties of the Weibull are the following.

Expectation	$\beta^{-1}\Gamma(1+\alpha^{-1})$	
Variance	$\beta^{-2}(\Gamma(1+2\alpha^{-1}) - \Gamma(1+\alpha^{-1})^2)$	(20)
Moment $\mathbb{E}\, X^k$	$\beta^{-k}\Gamma(1+k\alpha^{-1})$	
Tail probability $\mathbb{P}(X > x)$	$e^{-(\beta x)^{\alpha}}$, $x \geq 0$	

Where $\Gamma(\cdot)$ is the Gamma function.

If $\alpha \leq 1$, Weibull cumulative distribution function is monotonically decreasing function, if $\alpha = 1$ it is an exponential distribution with mean β and if $\alpha > 1$ Weibull cumulative distribution function is monotonically increasing function to the mode and then monotonically decreasing. A special case of is a Weibull(2, $1/(\sigma\sqrt{2})$) distribution, known as Rayleigh distribution and denoted by Rayleigh(σ). The Weibull parameters have been estimated by maximum likelihood (MLE) as shown in **Appendix E**, the shape parameter estimated for the funds category moves between 1.04 and 1.49 as shown in **Table 6**, so the probability density function (pdf) for EE, EI, and FI will look more like a exponential asymmetric (with some positive skewness) and the MX will present some positive increasing before the decay of the function. These results are interesting because we can see how they are aligned with the results of the **Table 5**, it makes sense that the EE category that presents a shape value very close to 1 has obtained the best relative fit results with the exponential distribution. The parameter scale just scales the fitted values, as the redemptions present very low values, the estimated parameters also present low values.

Table 6 Estimates Weibull Parameters

	EE	EI	MX	FI
Shape ($\hat{\alpha}$)	1,04	1,16	1,49	1,23
Scale ($\hat{\beta}$)	$8.87\text{x}10^{-4}$	$7{,}11\text{x}10^{-4}$	$1{,}59\text{x}10^{-3}$	$2.34\text{x}10^{-3}$

Table 6 shows the shape (α) and scale (β) parameters simulated for each fund category redemptions. Categories are European Equity funds (EE), International Equity funds (EI), Mixed funds (MX) and Fixed Income funds (FI). The parameters have been estimated by maximum likelihood as shown **Appendix E**

Once we have selected the distribution, we have estimated the shape and scale parameters for each fund category, and since the Weibull distribution has an analytical inverse distribution **(21)**, we can simulate it using the inverse Weibull distribution.

$$F_X^{-1}(x) = \frac{\alpha \left(\frac{\beta}{x}\right)^{\alpha} e^{-(\beta/x)^{\alpha}}}{x} \tag{21}$$

We have generated 10.000 paths from 1.000 data. To simulate we have used the inverse transform sampling method **Appendix F**, briefly, this method consists of generating for each path a vector of N random numbers (u) that will be distributed as a uniform. $\{u_n\}^M$, x~U[0,1]. Then we compute for each random u_n number a random variable that will have a Weibull distribution **(22)** with shape $\hat{\alpha}$ and scale $\hat{\beta}$.

$$X = F_X^{-1}(u, \hat{\alpha}, \hat{\beta}) \tag{22}$$

As an example, we present the results of the simulations in **Figure 9**, this shows the scale of the redemption simulated for each fund category; it can also be seen how some of the 10.000 paths always present some value in the highest redemptions for the 1.000 data (there are no gaps at the top of the figures, except at the most extreme points).

Figure 9 Simulated Redemptions

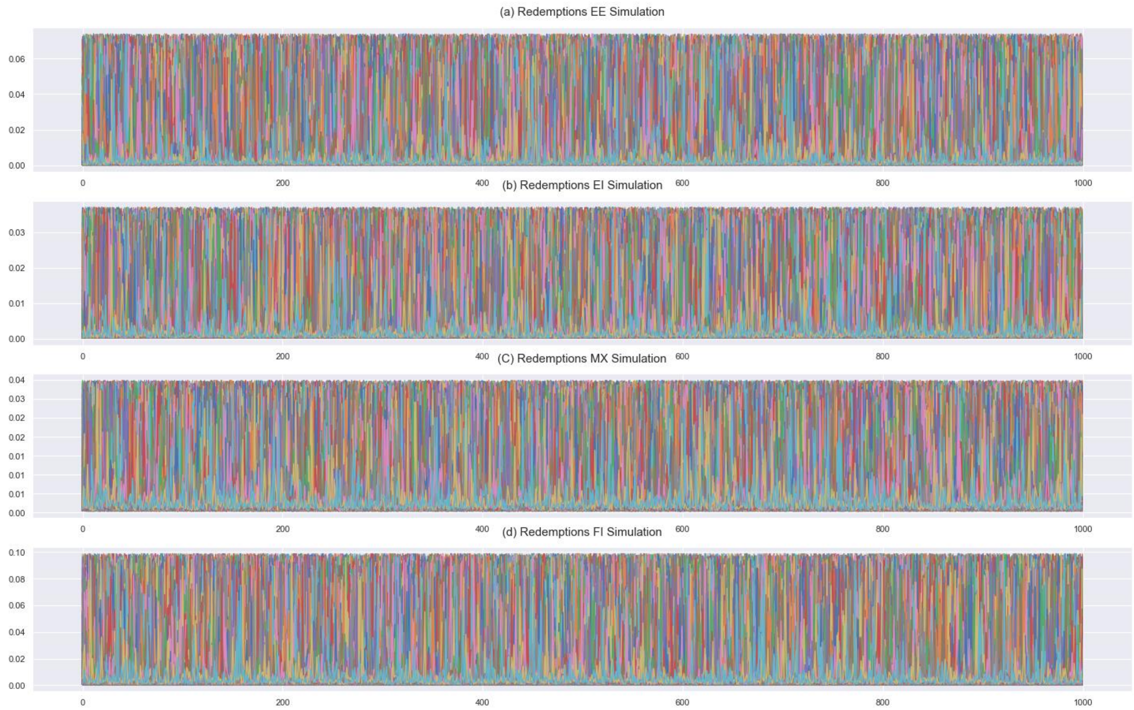

Figure 9 shows the simulated redemptions for each fund category. Categories are European Equity funds (EE) represented in (a), International Equity funds (EI) represented in (b), Mixed funds (MX) represented in (c) and Fixed Income funds (FI) represented in (d). The plot shows 10.000 simulation of 1.000 data. The x axis represents the observation i; $i \in (1{,}1.000)$ of the simulation j; $j \in (1{,}10.000)$, the y axis represents the redemptions as a ratio of AuM.

6.Results

In this section we analyse the simulated redemptions and how, based on these data, we propose redemption shock scenarios. Once we have simulated the redemptions, we have 10.000 samples of 1.000 observation. A sample of simulated redemptions for each fund style is shown in **Figure 10**. We see how the simulated redemptions have slightly higher values than the original series in **Table 7**, this is because we simulated data when the dependences between the series increase. The objective of having followed this method through TailCoR is to obtain redemption shocks, so we were interested in using a subsample as input where the series presented a possible liquidity shock. The FI category still presents the highest redemptions 0.0063, however EE and MX funds now have in mean similar redemptions, this is because the EE redemptions have more dependence with the other series, so when a redemption shock occurs, the redemptions of EE increases more than the redemptions of MX, which has less dependence with the other category funds as you can see in **Figure 7**. The most dependence categories FI and EE have the highest standard deviation, therefore, these categories will present the greatest risk of suffering an unexpected shock of redemptions. It is also interestingly to see how the categories with equity components are the ones that have more kurtosis, however the redemptions of EE are much more volatile than those of EI.

Figure 10 Simulated Redemption Series Sample

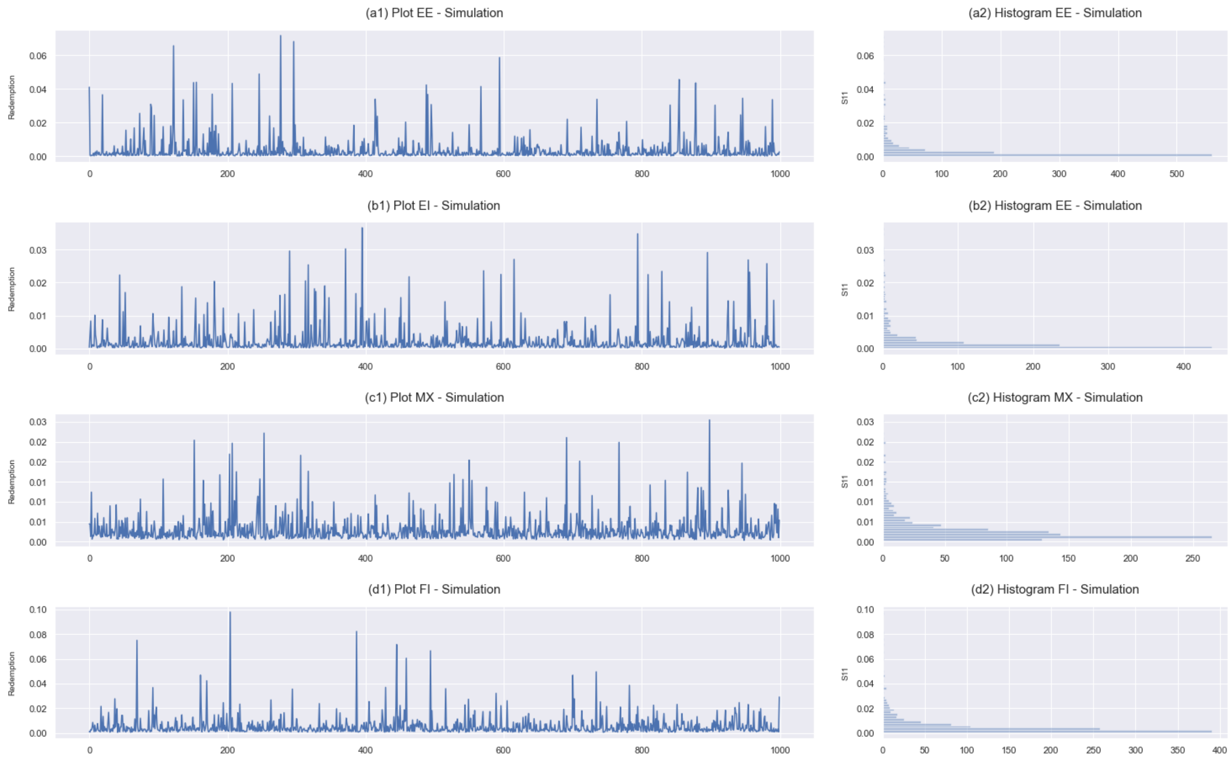

Figure 10 shows a random sample of the simulated redemption series by fund category. Categories are European Equity (EE) redemption series; (a2) is the histogram of (a1); (b1) are the international Equity (IE) redemption series; (b2) is the histogram of (b1); (a3) are the Mixed funds (MX) redemption series; (c1) is the histogram of (c2); (d1) are the Fixed Income (FI) redemption series; (d2) is the histogram of (d1).

Table 7 Simulated Redemption Statistics

	EE	EI	MX	FI
μ	0,0033	0,0021	0,0033	0,0063
σ	0,0068	0,0036	0,0039	0,0100
s	5,2436	4,6780	3,6477	4,4283
κ	34,5713	27,7651	17,1655	24,9853
IQR	0,22%	0,15%	0,23%	0,45%
$VaR_{0.975}$	2,18%	1,25%	1,49%	3,52%
$ES_{0.975}$	3,76%	2,02%	2,14%	5,54%

Table 7 shows the statics of simulated redemptions (measured as percentage of the AuM) by fund category. μ is the mean, σ is the standard deviation, s is the skewness, κ is the kurtosis, q refers to several quantiles of the distribution, IQR is the interquartile range, VaR is the Value at Risk and ES is the Expected Shortfall.

Expectedly, the simulated redemptions show a similar pattern to the redemptions in the original sample but they are shifted a little to the right. As our objective is to study the tail of the distribution we used as input data in periods of stress, in this case when dependencies between the redemptions increase, especially the nonlinear. The histogram shows how the simulated redemptions distribution matches the estimated parameters **Table 6**, being the MX simulated redemption the sample that increase more slowly before reaching the mode.

Figure 11 Histogram Sample-Simulated

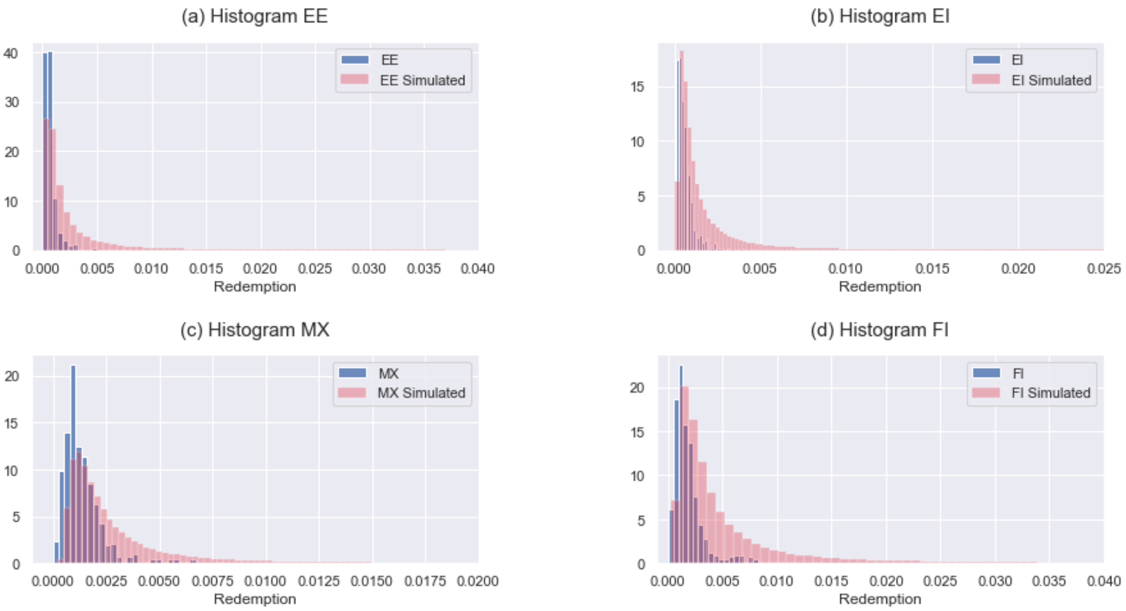

Figure 11 shows the histograms of the sample redemption series in blue, measured as a percentage over AuM, and the redemption simulated in **5.2** in red. Categories are European Equity funds (EE) represented in (a), International Equity funds (EI) represented in (b), Mixed funds (MX) represented in (c) and Fixed Income funds (FI) represented in (d).

We have computed the uncertainty of the redemption shock as the ES of the simulated redemption at a confidence interval of 97.5%. The interquartile range of these shock scenarios is shown in **Figure 12**. As we can see, the interquartile range of the ES of the European Equity redemptions fluctuates between 3.47% and 4.03%, obtaining a point estimate of 3.76%, the maximum ES of the simulations is 5.60.% and the minimum is 2.37%. The ES of international equity redemptions fluctuates between 1.88% and 2.15%, obtaining a point estimate of 2.02%, the maximum ES of the simulations in this case is 2.74% and the minimum is 1.29%. In the case of mixed funds, the interquartile range of the ES of the redemptions moves between 2.03% and 2.26%, the point estimate is 2.14%, the maximum ES of the simulations of redemptions is 2.82% and the minimum of 1.51%. Finally, the fixed income funds show the greatest range in the simulated redemptions, with the ES moving between 5.18% and 5.88%, the point estimate of the ES is 5.54%, however the minimum and maximum are 3.73% and 7.30%, respectively.

If we compare with the results of ESMA and CNMV[18][15], the ES of the redemptions for the equity categories present an interquartile range of between 1.50% and 3.50% approximately, as the mixed funds move between 1.50% and 3.00 % and how Fixed income funds have an interquartile range of between 2.00% and 14.00% (7.00% in the case of the CNMV) for a confidence interval of the ES of 97%, we can see how the results obtained through our methodology are similar to those obtained by the regulators. Note that we have grouped the different categories of these studies to compare them to ours, considering that retail stocks fund (RS) and wholesale stocks funds (WS) are equity funds, that mixed bond funds (MX) are mixed funds and considering that investment-grade corporate bond funds (IG), high yield corporate bond fund (HY), retail sovereign bond funds (RB) and wholesale sovereign bond funds (WB) are Fixed income funds.

Figure 12 Confidence Interval ES of Simulated Redemptions

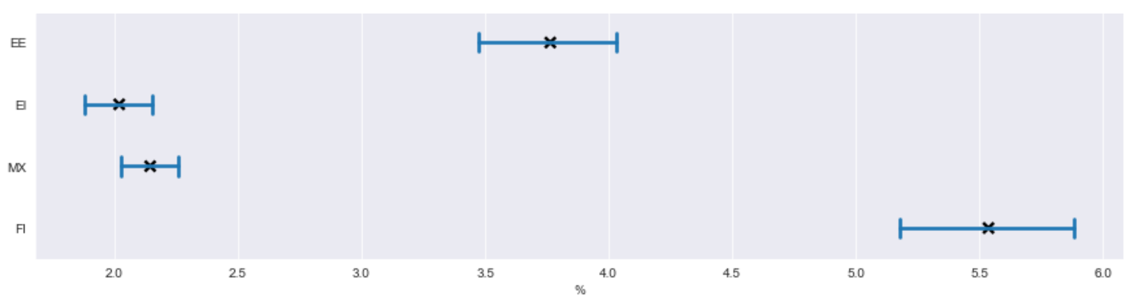

Figure 12 shows the confidence interval for the simulated redemptions Expected Shortfall for each fund category. The calculated point estimated is represented with a black cross. The blue bar indicates the 97.5% confidence interval. The redemption values are represented in the x axis, the funds categories are represented in the y axis.

To measure the resilience of Spanish investment funds by fund style, we use the Redemptions Coverage Ratio (RCR) measure.

$$RCR_i = \frac{HQLA_i}{Redemption\ shock_j} \tag{23}$$

Where subindex i indicates that RCR and HQLA is calculated for each fund $i \in (1{,}73)$, and subindex j indicates that Redemption shock is calculated for each fund category $j \in (1{,}4)$.

This measure is the ratio between the liquid assets of a portfolio measured as a percentage of the AuM and the redemption shock in the denominator, also measured as a percentage of the AuM. One of the advantages of this measure is that it is easy to interpret, since as it is a ratio between the assets and liabilities of an investment fund, we can interpret it as if the investment fund will present liquidity problems when the value of the RCR is less than one. A value less than one means that the investment fund does not have enough liquid assets to cover the redemptions requested by investors.

As HQLA we use the calculated data through the methodology proposed in **4**, while for the redemption shock we use the point estimates of the ES at a confidence interval of 97.5%. While each fund in the sample presents its own HQLA, the redemption shock will be the same for each fund that belongs to each category, so we will assume that all European Equity funds present a redemption shock of 3.76% over AuM.

Once the RCRs for all the investment funds have been calculated, we detect that no investment fund presents RCRs less than one, this is due to the fact that we have a small sample of investment funds and that, in general, all of them have investments in liquid assets. Therefore, we have proposed a stress scenario where we assume that for half a month (10 consecutive days) there are redemptions equal to the ES of your fund style for each investment fund. This kind of stress test scenario has been proposed for Malta Financial Services Authority (MFSA) published in Liquidity Stress Testing for Maltese Retail Investment Funds (2020) [32], where proposed weekly redemption shocks. Given this scenario, we found that some fixed income mutual funds would have liquidity problems, as shown in the **Figure 13**.

Figure 13 Fixed Income Funds RCR

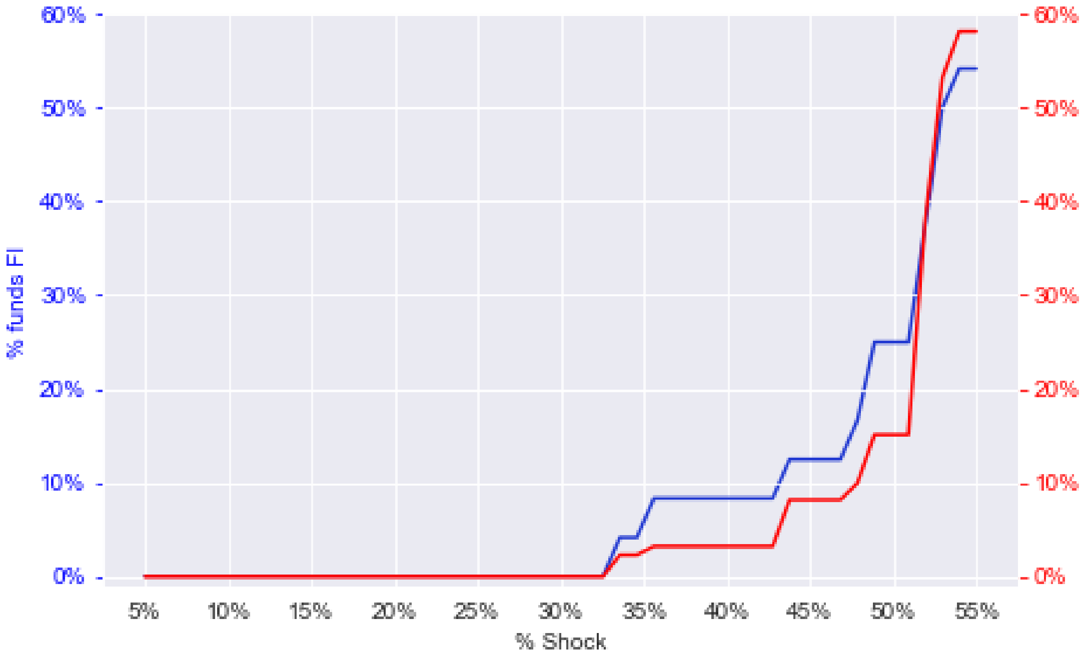

Figure 13 shows the percentage of funds in the category Fixed Income (FI) that have Redemption Coverage Ratio RCR<1 **(23)** in the event a redemption of redemption shocks. The x axis represents the denominator of RCR which is the redemption. The y blue axis represents the percentage of funds that have RCR<1 out of total funds and the y red axis represents the percentage of AuM that have RCR<1.

Figure 13 shows that from redemption shocks of 33%, some fixed income funds would begin to have problems liquidating their assets since they would not have enough liquid assets to cover the net outflows that their investors might request. In general, only 10% of Fixed income funds would have liquidity problems up to a 43% redemption shock, however, from a 44% to 54% shock, we observed that up to 60% of fixed income funds would have problems redeeming their assets to cover the redemptions requested by their investors.

7. Conclusions

In this study we have proposed a method to measure the resilience of investment funds to liquidity shocks. For this purpose, we have used the measure RCR proposed by ESMA [18], that measures the relationship between the assets and liabilities of investment funds.

For the calculation of High Quality Liquid assets (HQLA), improvements have been incorporated with respect to the ESMA and CNMV [15] studies. These studies simplified the liquidity of the assets according to the type of asset and financial rating, we consider that there are other relevant market factors to measure the liquidity of the assets such as volumes or bid-ask spreads, for which we have proposed a scoring model that combines these factors among others, see **4**. In general, we can conclude that the HQLA for the funds analysed are quite high, since almost all the funds present HQLA above 50% of their Assets under management (AuM) **Figure 5**. In general, funds with less liquidity are Fixed Income funds, this is due to the idiosyncrasy of the underlying assets in these funds, which are generally less liquid than equity.

Regarding the part of the liabilities, the objective is to simulate redemption shocks by grouping the funds by fund style. We start with 435 redemptions by fund between August 2019 and March 2021 and the funds have been grouped into four categories European equity (EE), International equity (EI), Mixed funds (MX) and Fixed Income funds (FI). To simulate redemption shocks, conditional statistics must be taken into account, such as the CoVaR or the CoES, since in times of crisis there is a contagion effect between different sectors and precisely these conditional measures capture this type of effects. As we have a limited sample, we are not able to show the granularity between more categories of funds, differentiating between fixed income corporate or sovereign funds or funds with more investment grade or high yield components, as well as between retail and wholesale funds. We do not have excessively long series of redemptions, which served us to analyse between periods of stress and periods of calm. Our series of redemptions picks up the start of the COVID-19 pandemic. For these reasons, starting from the original series, we do not obtain conditional tail statistics that are too severe **Table 2**, either because we do not have a very large sample of investment funds or because there are no significant redemption shocks in the analysed period.

Considering that, to simulate shocks it is necessary to collect this effect of dependence between the series, we have proposed incorporate in the study the TailCoR measure developed by Sladana Babić, David Veredas et al [7] in order to be able to analyse in our sample the effect of the correlation between the different series, or redemptions by fund category. TailCoR is a dependency measure that also disentangles easily between the linear and nonlinear components. Considering that the TailCoR increases in periods of crisis, we have decided to take a subsample for the 30 weeks of the sample in which the

TailCoR increases the most. Therefore, this subsample includes the subperiods where the series increase their relationship due to possible redemption shocks.

We have studied possible distributions that could fit this subsample and due to the characteristics of the distribution we have selected the Weibull distribution **5.2**, which is widely known in the field of extreme value theory. We have simulated 1.000 redemptions and 10.000 paths for each fund category, and from this simulation stress scenarios such as ES at 97.5% confidence level have been calculated. An interesting result is that the EE and FI funds present the higher estimated stress shock (3,76% and 5,54 respectively), and precisely these categories are the ones that present the greatest dependencies (especially non-linear) when we studied the TailCoR **5.1**.

The results in terms of RCR show that Spanish investment funds (within the sample) have a strong resilience to liquidity shocks. None of the fund categories have presented funds in distress, except if we move to a very severe stress scenario, where up to 54% of the funds could present liquidity problems.

As future works, we propose to carry out the study with a large sample and studying other types of distributions to avoid the possible sampling bias, also increasing the granularity of the fund styles since it is very likely that there are significant differences between funds that invest in investment grade or high yield bonds, or between types of investors that have different redemption patterns, such as retail and wholesale. Another interesting analysis would be to study the effects of the second round, and it is how the initial shocks can have a price impact on the market and how the remaining investors can be harmed due to the first-mover advantage effect. Related to this, it must be taken into account that throughout the study homogeneous partial liquidations have been assumed among all the assets of the portfolios, however it is usual for managers to sell the most liquid assets first, so it is possible that the remaining investors are also harmed by this effect, so it would be interesting to include in the study effects on the HQLA of the funds once the redemption shocks have begun.

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Appendix A Quantile regression

The Quantile Regression method is used to model the relationship between variables. Unlike ordinary least squares (OLS) that measures the relationship between the dependent variable with the independent variables on the average, the quantile regression measures the relationship between the dependent variable with the independent variable for different quantiles of the dependent variable. So, the difference between quantile regression and OLS is that with standard regression coefficient estimates the regression line passes through the average, whereas a quantile regression line will pass through a quantile of the points.

With quantile regression we can study the conditional distribution of the dependent variable $F(Y|X)$ throughout the sample, this is interesting because the relationships between variables are not constant across quantiles, meaning that we can expect different values of the dependent variable when the independent variable has values lower or higher than its mean.

More formally quantile regression solves the problem:

$$\min_{(\alpha,\beta)}\left[\sum_{t=1}^{T}\left(q-1_{y_t\leq\alpha+\beta x_t}\right)\left(y_t-(\alpha+\beta x_t)\right)\right]$$

$$\min_{(\alpha,\beta)}\left[q\sum_{y_t\geq\alpha+\beta x_t}\left(y_t-(\alpha+\beta x_t)\right)-(1-q)\sum_{y_t\leq\alpha+\beta x_t}\left(y_t-(\alpha+\beta x_t)\right)\right]$$

Note that the first sum collects positive residuals, while the second sum collects negative residuals, so that both sums enter positively into the objective function. This function generalizes the problem known as Mean Absolute Regression:

$$\min_{(\alpha,\beta)}\sum_{t=1}^{T}|y_t-(\alpha+\beta x_t)|$$

We have computed for the conditional quantiles q 0.05 to 0.95 of each fund category as **Figure A1**, however as we are studying shocks in redemptions we are interested in the regressions for high quantiles, in this case for the quantile q = 0.95. As can be seen, the distribution of the fund category behaves differently depending on the selected quantile, the slope of the regressions between quantiles changes depending on the selected quantile, as we select higher quantiles the slope of the regression increases.

Figure A1 Quantile Regression

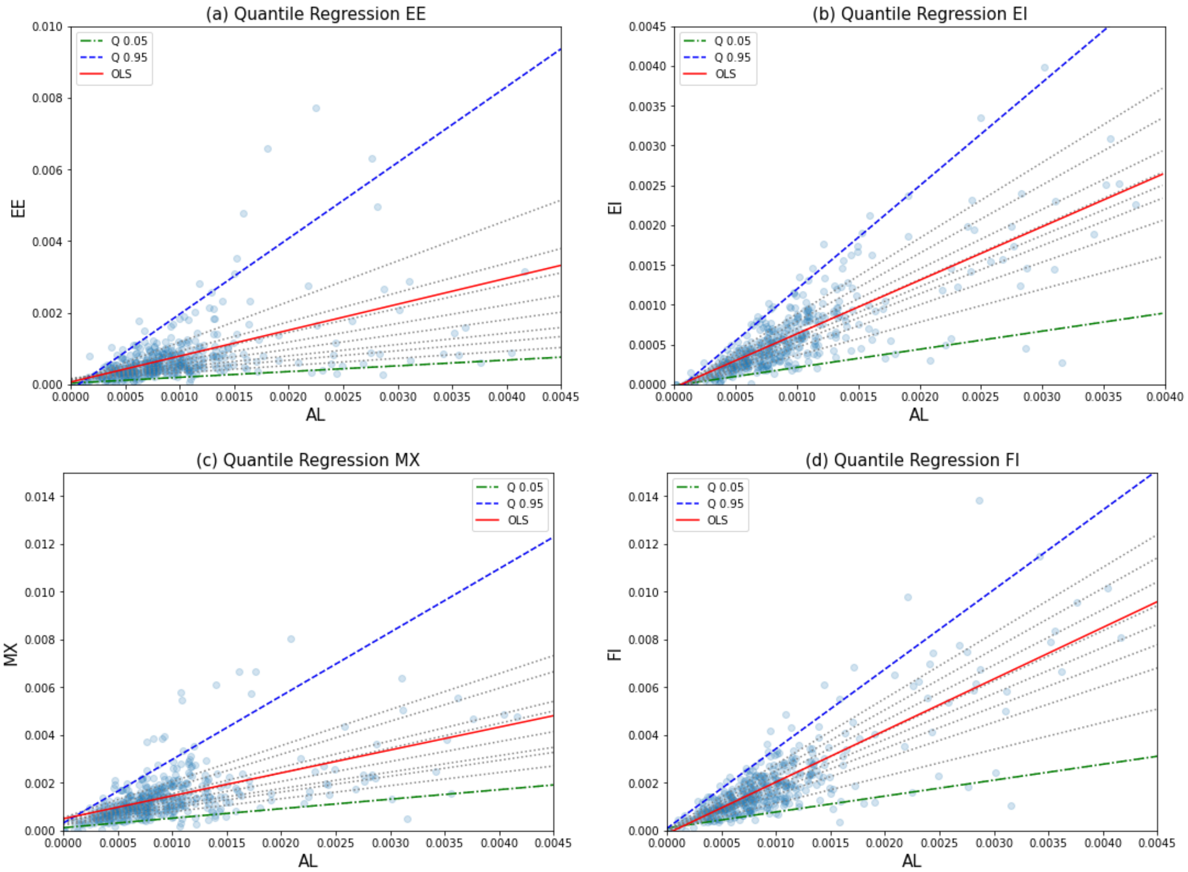

Figure A1 shows the results of the quantile regression estimation for each category fund, for values of q = 0.05 (green dotted line) to values of q = 0.95 (blue dotted line). Red line is the OLS regression. Categories are European Equity funds (EE) represented in (a), International Equity funds (EI) represented in (b), Mixed funds (MX) represented in (c) and Fixed Income funds (FI) represented in (d); Aggregated Funds (AL) redemptions are represented in x axis.

For the quantile regression at q=0.95 the estimated parameters for the different fund categories range from 1.29 to 3.33, all the $\hat{\beta}$ parameters are significantly different from zero **Table A1**. The FI category has the steepest slope and the EI the lowest one, note that **Figure A1** has different scales for each fund category.

Table A1 Quantile Regression Estimates q=0.95

	EE	EI	MX	FI
$\hat{\alpha}$	0,00	0,00	0,00	0,00
$p_{\hat{\alpha}}$	0,00	0,01	0,00	0,50
$\hat{\beta}$	2,12	1,29	2,66	3,33
$p_{\hat{\beta}}$	0,00	0,00	0,00	0,00

Table A1 shows the estimated parameters for quantile regression at q=0.95 for each fund category. $\hat{\alpha}$ is estimation of the intercept parameter, $\hat{\beta}$ is estimation of the slope parameter and $p_{(\alpha,\beta)}$ is the p value of the estimation.

Appendix B Haircuts

In this section is shown the proposed methodology to apply haircuts for equities and fixed income.

For fixed income assets we have proposed to apply four haircuts. H1 refers to the credit quality of the issue, H2 is based on the percentage held over the total issuance, H3 measures the bid-ask spread and H4 sets haircuts thresholds according to the complexity and severity of the issuance. The tables below show the rules in order to apply these haircuts.

Table B1 Fixed Income H1

Rating	Haircut
AAA	0
AA+	5
AA	8
AA-	12,5
A+	15
A	20
A-	25
BBB+	28
BBB	32
BBB-	37,5
BB+	50
BB	50
BB-	50
B+	50
B	50
B-	50
CCC+	50
CCC	50
CCC-	50
NR	50

Table B1 shows the applied haircut to fixed income assets according to their rating. The rating is expressed in S&P scale. If the rating has been obtained on another agency's scale, it has been converted to S&P scale. Note that HY investments will have the worst haircut (high).

Table B2 Fixed Income H2

Rule	Haircut
If x<0,05%	0
If x<25%	3,125
If x<50%	6,25
If x<75%	9,375
If x≥75%	12,5

Table B2 shows the applied haircut to fixed income assets according to the percentage held over the total issuance. x represents the ratio between investment in an asset in MU and total issuance in MU.

Table B3 Fixed Income H3

Rule	Haircut
If x<0,05%	0
If x<0,1%	3,125
If x<0,5%	6,25
If x<1%	9,375
If x≥1%	12,5

Table B3 shows the applied haircut to fixed income assets according to the bid-ask spread of the asset. x represents the difference between bid and ask price, note that since fixed income assets are priced as a percentage, the difference between bid and ask is a percentage.

Table B4 Fixed Income H4

Rule#1	Rule#2	Haircut
Sovereign		0
Simple Corporate	Senior	6,25
Corporate Complex	Senior	12,5
Simple Corporate	Subordinated	18,75
Corporate Complex	Subordinated	25

Table B4 shows the applied haircuts to fixes income assets according to their complexity and severity. Rule#1 difference between corporate and sovereign bonds and in corporate difference between simple and complex. Complex refers to structured bonds and bonds with optionality, simple bonds are regular and floating notes. Rule#2 refers to the severity of the assets, there are two categories senior notes and subordinated notes.

For equities we have proposed to apply three haircuts. H1 measures the amount held in the portfolio in percentage over the volume traded, H2 is based on the percentage of investment over market capitalization and H3 measures the bid-ask spread. The tables below show the rules in order to apply these haircuts.

Table B5 Equities H1

Rule	Haircut
If x<0,5%	0
If x<25%	5
If x<50%	10
If x<75%	15
If x≥75%	20

Table B5 shows the applied haircut to equities according to the amount held in the portfolio in percentage over the volume traded. x represents the ratio the ratio between investment in an asset in MU and volume traded in MU. Volume traded is the average volume of one month.

Table B6 Equities H2

Rule	Haircut
If x<0,05%	0
If x<25%	15
If x<50%	30
If x<75%	45
If x≥75%	60

Table B6 shows the applied haircut to equities according to the percentage held in the portfolio of investment over market capitalization. x represents the ratio the ratio between investment in an asset in MU and its market capitalization in MU. Market capitalization is corrected by free float.

Table B7 Equities H3

Rule	Haircut
If x<0,05%	0
If x<0,10%	5
If x<0,50%	10
If x<1,00%	15
If x≥1,0%	20

Table B7 shows the applied haircut to equities according to the bid-ask spread of the asset. x represents ratio between bid ask spread and mid-price, note the spread has been divided by the mid-price to measure the spread as a ratio.

Appendix C AR(1)-GARCH(1,1)

In this section we show the specified model computed for redemptions in **5.1** Since redemption series present autocorrelation, see **Figure C1**, we assume the redemptions follow an AR(p) model.

$$Redemption_{i,t} = \phi_{i,0} + \sum_{j=1}^{p} \phi_{i,j} Redemption_{i,t-j} + \varepsilon_{i,t}$$

Where p are non-negative integers and $\phi_{i,j}$ are the autoregressive parameters, i refers to a fund category and $\varepsilon_{i,t} = \sigma_{i,t}$. The variance of $\varepsilon_{i,t}$ follows a GARCH(1,1).

$$\sigma_{i,t}^2 = \omega_i + \beta_i \sigma_{i,t-1}^2 + \alpha_i \varepsilon_{i,t-1}^2$$

Where α_i, β_i and ω_i are the GARCH parameters.

The general objective of this volatility model is to reflect the dependence on the series of redemptions, the heavy tails and the volatility clusters. To calculate the parameters, we propose a simultaneous estimation method for the parameters of the autoregressive model and the parameters of the volatility model. We estimated them with the method maximum likelihood. Among several specifications we have selected an AR(1)-GARCH(1,1). This model collects the dynamic structure of the redemptions, as **Table C1** presents the first lag of the redemptions is statically significant for all funds category except for European Equities (EE), these was the expected results as **Figure C1** shows. The parameters of GARCH are not statically significant for any category except for fixed income funds (FI).

Table C1 Parameters AR(1)-GARCH(1,1)

	EE	EI	MX	FI
ϕ_0	0.68	0.43	0.79	1.32
	[0.04]***	[0.04]***	[0.104]***	[0.143]***
ϕ_1	0.0538	0.1688	0.4414	0.2660
	[0.03]	[0.107]***	[0.06]***	[0.07]***
α	0.0000	0.3430	0.3135	0.1029
	[0.00]	[0.248]	[0.210]	[0.04]***
β	0.9890	0.5155	0.1006	0.7371
	[0.00]*	[0.315]	[0.118]	[0.08]***

Table C1 shows the AR(p)-GARCH(p,q) estimated parameters. Data in [ ] is the standard deviation form estimation and * reflects level of significance of the parameters (* = 90% ** = 95% and *** =99%)

Figure C1 Redemption Correlograms

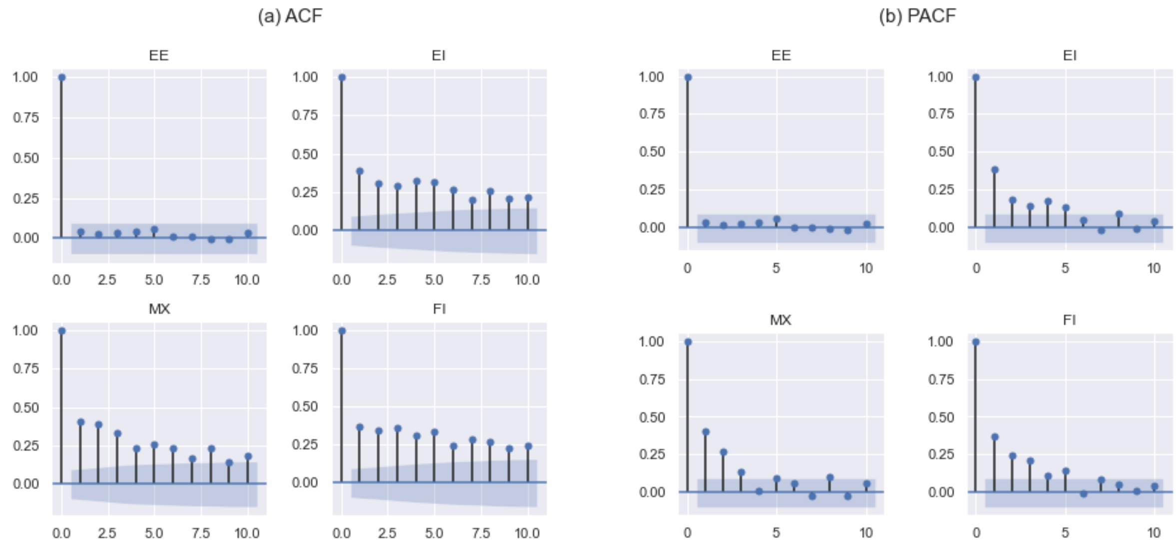

Figure C1 shows the autocorrelation (ACF) of the redemptions for each category funds (a) and the partial correlation (PACF) of the redemptions for each category funds (b).

Appendix D S_0 Tabulation

Table D1 Table D1 S_0 Tabulation

	ξ													
τ	0.700	0.725	0.750	0.775	0.800	0.825	0.850	0.875	0.900	0.925	0.950	0.975	0.990	0.995
0.600	0.483	0.424	0.375	0.335	0.301	0.271	0.244	0.220	0.198	0.176	0.154	0.129	0.109	0.098
0.625	0.607	0.533	0.472	0.422	0.379	0.341	0.307	0.277	0.249	0.221	0.194	0.163	0.137	0.124
0.650	0.735	0.644	0.571	0.510	0.458	0.412	0.372	0.335	0.301	0.268	0.234	0.196	0.166	0.150
0.675	0.865	0.759	0.673	0.601	0.539	0.486	0.438	0.394	0.354	0.315	0.276	0.231	0.195	0.176
0.700	–	0.877	0.778	0.694	0.623	0.561	0.506	0.456	0.409	0.364	0.319	0.267	0.226	0.204
0.725	–	–	0.886	0.791	0.711	0.640	0.577	0.520	0.466	0.415	0.363	0.305	0.257	0.232
0.750	–	–	–	0.893	0.801	0.722	0.651	0.586	0.526	0.468	0.410	0.344	0.290	0.262
0.775	–	–	–	–	0.898	0.808	0.729	0.657	0.589	0.525	0.459	0.385	0.325	0.293
0.800	–	–	–	–	–	0.900	0.812	0.731	0.657	0.585	0.512	0.429	0.362	0.327
0.825	–	–	–	–	–	–	0.902	0.812	0.729	0.649	0.568	0.477	0.402	0.363
0.850	–	–	–	–	–	–	–	0.901	0.809	0.720	0.630	0.529	0.445	0.402
0.875	–	–	–	–	–	–	–	–	0.898	0.799	0.699	0.587	0.494	0.447
0.900	–	–	–	–	–	–	–	–	–	0.890	0.779	0.654	0.551	0.497

Table D1 is an extract from the working paper TailCoR (2020) [7] the authors have authorized this figure to be included in this document. Table D1 shows vales for $S_0(\tau, \xi)$ for a grid of reasonable values for τ and ξ.

Appendix E Maximum likelihood estimation

The maximum likelihood is a method to estimate the parameters of a model distribution maximizing the likelihood function.

$$L = \prod_{i=1}^{n} f(x_i, \hat{\theta})$$

Where $f(x_i, \hat{\theta})$ is a the x_i probability density function with parameters $\hat{\theta}$.

Applying (L) to the Weibull probability density (18), we get the likelihood function to maximize.

$$L(\alpha, \beta) = \prod_{i=1}^{n} \left(\frac{\alpha}{\beta}\right)\left(\frac{x_i}{\beta}\right)^{\alpha-1} e^{[-(x_i/\beta)]^{\alpha}}$$

Taking logarithms of the likelihood function and then differentiating respect both parameters.

$$lnL(\alpha, \beta) = nln\alpha - n\alpha ln\beta - \frac{1}{\beta^{\alpha}}\sum_{i=1}^{n} x_i^{\alpha} + (\alpha - 1)\sum_{i=1}^{n} lnx_i$$

$$\frac{\partial lnL(\alpha, \beta)}{\partial \alpha} = \frac{n}{\alpha} - nln\beta - \frac{\sum_{i=1}^{n} x_i^{\alpha} - ln\beta \sum_{i=1}^{n} x_i^{\alpha}}{\beta^{\alpha}} + \sum_{i=1}^{n} lnx_i = 0$$

$$\frac{\partial lnL(\alpha, \beta)}{\partial \beta} = \frac{n\alpha}{\beta} + \frac{\alpha}{\beta^{\alpha+1}}\sum_{i=1}^{n} x_i^{\alpha} = 0$$

Simplifying the above equations, we get the equations that solving simultaneously is how to get the estimators of shape and scale parameters from a Weibull distribution.

$$\hat{\alpha} = \frac{n}{(1/\hat{\beta})\sum_{i=1}^{n} x_i^{\hat{\alpha}} \, logx_i - \sum_{i=1}^{n} log\, x_i}$$

$$\hat{\beta} = \left[\left(\frac{1}{n}\right)\sum_{i=1}^{n} x_i^{\hat{\alpha}}\right]^{1/\hat{\alpha}}$$

Appendix F Inverse transform sampling

The problem to be solved is how to obtain from a random number generator with a probability density $p_U(u)$, which is generally a uniform one, a random number generator with density $p_X(x)$.

$$P(u) = \begin{cases} 0, & u \ni (0,1) \\ 1, & u \in (0,1) \end{cases}$$

Note that by the probability conservation principle:

$$|p_U(u)du| = |p_X(x)dx| \Rightarrow p_X(x) = p_U(u)\left|\frac{du}{dx}\right|$$

Given the shape of the objective distribution $p_X(x)$, and if we start from a probability distribution U[0,1] for u, we have to solve the following differential equation.

$$p_X(x) = \frac{du}{dx} \Rightarrow du = p_Z(z)dz$$

In terms of the cumulative probability distribution

$$P_X(x) = \int_{-\infty}^{x} dx' p_X(x')$$

solving this:

$$\int_0^u du = \int_{-\infty}^{x} dx'^{p_X}(x') \Rightarrow u = P_X(x) \Rightarrow x = P_X^{-1}(u)$$

$$x = P_X^{-1}(u)$$

Thus, to generate a random variable x with a determinate cumulative distribution function (cfd), we can draw U[0,1] and set $x = P_X^{-1}(u)$. This leads to the following general method illustrated in **Figure F1** and developed analytically above.

Figure F1 Inverse Transform sampling representation