PRICING AND HEDGING IN INCOMPLETE MARKETS DRIVEN BY SHOT NOISE JUMP DIFFUSION MODELS

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Abstract

In this master's thesis we consider a shot-noise jump-diffusion process to model the risky asset's price. This framework gives rise to an infinite set of equivalent martingale measures, which implies market incompleteness and non-uniqueness of arbitrage-free prices. We characterize this set and study several pricing rules based on constant jump risk premia. In particular, we derive the Radon–Nikodym density defining Merton's measure and find numerically that it coincides with the minimal entropy martingale measure. We also establish non-trivial bounds for European-style claims in terms of the Black–Scholes price, and compute indifference prices under exponential utility preferences for fixed hedging strategies. Finally, we illustrate these results through numerical experiments.

Keywords: *contingent claim, incomplete markets, equivalent martingale measure, minimum martingale measure, minimum entropy measure, exponential utility.*

Contents

1 Introduction

Brownian motion has played a central role in financial mathematics since its early development, first proposed by Bachelier in [2] to model the price dynamics of Parisian stocks. Its multiplicative version leads to the well-known Black–Scholes model [5], also referred to as geometric Brownian motion, in which the logarithm of the asset price follows a Brownian path. This process exhibits two features that are particularly relevant to this work: first, it has continuous sample paths, and second, it is scale-invariant—meaning that its statistical properties remain unchanged under time rescaling. While real financial price series may appear continuous over long time horizons, high-frequency intraday data often reveals the presence of abrupt jumps. This discrepancy highlights one of the limitations of the Black–Scholes model: although it may reproduce the correct volatility of log-returns at a given time scale, it fails to adapt appropriately when the scale changes.

Several extensions of the Black–Scholes model have been proposed to address these limitations. Local volatility models, for example, introduce a non-constant diffusion coefficient that depends explicitly on time and the asset price (see [15, 13]), while stochastic volatility models assume that the instantaneous volatility follows its own Itô process (see [26]). These approaches improve upon the statistical realism of the model, allowing for heavier tails in return distributions (see, e.g., [36]), but they rely on non-stationary diffusion coefficients—as in the local volatility case—or on fine tuning of the volatility-of-volatility parameter in stochastic volatility models. Moreover, none of these models can capture the jump-like discontinuities that are empirically observed in asset prices. Such properties, however, arise naturally in jump processes.

Jump-diffusion models were first applied to option pricing by Merton [31], who introduced a framework in which the asset price incorporates a compound Poisson process with i.i.d. Gaussian-distributed jump sizes. This seminal work marked the beginning of a large body of literature on jump processes in financial modeling (see, e.g., [6, 17, 30]). However, Merton's assumption that the effect of each jump is permanent is not entirely realistic. For example, announcements may cause abrupt changes in asset prices, but their effect typically diminishes over time. To account for this feature, jump processes with transient effects—originally introduced in the physical sciences by Campbell [7] and Schottky [34]—have been adapted to financial modeling. The first such application in this context was proposed by Samorodnitsky [33].

In this work, we focus on the shot-noise model introduced by Altmann et al. [1], where the risky asset's price consists of a geometric Brownian motion component and a jump component whose impact decays over time. Specifically, the size of each jump is modulated by a time-decaying factor, so that the effect of past jumps gradually vanishes.

In particular, we show that under this model there is no unique equivalent martingale measure (EMM), but rather infinitely many. We provide an explicit characterization of the set of EMMs and show that this infinite cardinality implies market incompleteness. That is, options are not redundant assets that can be perfectly replicated through self-financing strategies involving only the risk-free asset and the underlying, and moreover, there exist infinitely many arbitrage-free prices, each associated with a different EMM.

In incomplete markets, several pricing methods have been proposed to select a specific price among the infinitely many arbitrage-free candidates. Common approaches include: ignoring the jump risk premium, as proposed by Merton [31]; determining the cheapest portfolio whose terminal value is greater than the payoff almost surely [18, 29]; selecting the minimal martingale measure associated with a strategy that minimizes local quadratic risk [20, 1]; finding the EMM that minimizes the relative entropy with respect to the natural probability measure [22, 24]; or computing utility indifference prices based on exponential preferences [27, 12, 3].

These methods have been developed in a variety of settings, including purely discontinuous processes [16] and continuous models with stochastic volatility [21]. However, aside from the minimal martingale approach, they have not yet been applied in the context of shot-noise processes.

In addition to the findings mentioned above, throughout this work we extend these methods to the shot-noise setting, with the dual objective of pricing and hedging. Specifically, we derive the closed-form expression of the Radon–Nikodym density in Merton's approach and compute the corresponding terminal hedging error; stablish non-trivial bounds for the prices of European-style contingent claims in terms of the Black–Scholes price; characterize the minimal entropy martingale measure; and compute utility indifference prices under different hedging strategies. An additional contribution is the observation that if the jump intensity under a given EMM is constant, then the shot-noise component loses its transient character—effectively eliminating the shot-noise effect from the price dynamics under that EMM.

The rest of this thesis is organized as follows. Section 2 introduces the market model driven by a shot-noise process and discusses some fundamental properties of the process itself, including its representation as a product of Doléans–Dade exponentials, the non-stationarity of its increments, and its non-Markovian nature. Section 3 motivates the construction of arbitrage-free pricing rules as expectations under an equivalent martingale measure, and characterizes the full set of such measures induced by the shot-noise process. Section 4 studies the application of the incomplete-market pricing and hedging methods previously discussed to the shot-noise model, and presents the corresponding results. Finally, Section 5 illustrates the theoretical findings with numerical examples.

2 The Model

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space equipped with the usual filtration assumptions: the filtration $\mathbb{F}$ is right-continuous (i.e., $\mathcal{F}_t = \mathcal{F}_{t+}$), and complete, meaning that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. For simplicity, we assume that the initial σ-algebra $\mathcal{F}_0$ is trivial, in the sense that it contains only events of probability zero or one.

The market consists of two assets: a risk-free asset, with a deterministic return rate r, and a risky asset, both traded up to a finite time horizon $T > 0$.

The risk-free asset evolves deterministically according to

$$dB(t) = r\, B(t)\, dt, \qquad B(0) = 1.$$

We define the discount factor by setting $R := B^{-1}$. Given any process X, we denote its discounted version by $\hat{X} := RX$.

The dynamics of the risky asset's price S are given by the following stochastic differential equation

$$\frac{dS_t}{S_{t^-}} = \mu\, dt + \sigma\, dW_t + \int_{\mathbb{R}} y\, M(dt\, dy) + \frac{\partial}{\partial t} \int_0^{t^-} \int_{\mathbb{R}} ln(1 + y\, \mathsf{d}(t-s))\, M(ds\, dy) dt. \tag{2.1}$$

In this equation, $W = (W_t)_{t\in[0,T]}$ denotes a standard $\mathbb{P}$-Brownian motion. The term M is a Poisson random measure on $[0, T] \times \mathbb{R}$ with $\mathbb{P}$-intensity $\lambda\, F_U(dy)\, dt$, where U is an independent random variable representing the size of the jumps in M, and F_U denotes its law.

The function $\mathsf{d}\colon \mathbb{R}_+ \to [0, 1]$ is deterministic, nonnegative, and nonincreasing. It satisfies $\mathsf{d}(0) = 1$ and models the decay of jump effects over time. A typical choice is $\mathsf{d}(x) = \exp(-cx)$, for $x \geq 0$ and $c \geq 0$.

If we define the compensated Poisson random measure by $\tilde{M}(dt\, dy) := M(dt\, dy) - \lambda F_U(dy)\, dt$, then the stochastic differential equation (2.1) can be rewritten as

$$\frac{dS_t}{S_{t^-}} = \big(\mu + \theta(t^-) + \lambda \mathbb{E}[U_1]\big)\, dt + \sigma\, dW_t + \int_{\mathbb{R}} y\, \tilde{M}(dt, dy), \tag{2.2}$$

where the process θ is defined as

$$\theta(t^-) := \frac{\partial}{\partial t} \sum_{i=1}^{N_{t^-}} \ln(1 + U_i\, \mathsf{d}(t-\tau_i)) = \frac{\partial}{\partial t} \int_0^{t^-} \int_{\mathbb{R}} \ln(1 + y\, \mathsf{d}(t-s))\, M(ds\, dy). \tag{2.3}$$

Here, $N = (N_t)_{t\in[0,T]}$ denotes a Poisson process with constant $\mathbb{P}$-intensity $\lambda > 0$, and $(\tau_i)_{i\in\mathbb{N}}$ its jump times. The random variables $(U_i)_{i\in\mathbb{N}}$ represent the proportional jump sizes of the asset price process S, occurring at the corresponding jump times τ_i. They are assumed to be independent and identically distributed, with common law F_U, independent of the Brownian motion W, and to have finite second moment.

The equality in equation (2.3) follows directly from the definition of the stochastic integral with respect to the Poisson random measure M, which corresponds to the sum of the integrand evaluated at the jump sizes and their respective jump times.

An explicit solution to the stochastic differential equation (2.1), or equivalently to its rewritten form (2.2), is given in the following proposition.

Proposition 2.1. *The unique càdlàg*[1] *solution $S = (S_t)_{t\in[0,T]}$ to the stochastic differential equation (2.1) is given by*

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}\prod_{i=1}^{N_t}\left(1 + U_i\,\mathsf{d}(t-\tau_i)\right). \tag{2.4}$$

Proof. It is enough to verify that expression (2.4) satisfies the SDE given in (2.1). To this end, let us define $S_t^{BS}J_t := S_t^{BS}e^{L_t} := S_t$, where

$$S_t^{BS} := S_0\exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right) \quad\Longrightarrow\quad dS_t^{BS} = S_t^{BS}(\mu\,dt + \sigma\,dW_t),$$

$$L_t := \sum_{i=1}^{N_t}\ln(1 + U_i\,\mathsf{d}(t-\tau_i)) \implies dL_t = \frac{\partial}{\partial t}\sum_{i=1}^{N_{t-}}\ln(1 + U_i\,\mathsf{d}(t-\tau_i))\,dt + \Delta L_t,$$

where $\Delta L_t = \ln(1+U_{N_t})dN_t$, since $\mathsf{d}(0) = 1$. Applying Itô's formula for semimartingales (see, e.g., prop 8.19 in [9]), we compute:

$$dJ_t = e^{L_{t^-}}\left(\theta(t^-)\,dt + (e^{\Delta L_t} - 1)\right).$$

Using the product differentiation rule for semimartingales (see, e.g., prop 8.11 in [9]), we obtain:

$$dS_t = S_{t^-}\left[(\mu + \theta(t^-))\,dt + \sigma\,dW_t + (e^{\Delta L_t} - 1)\right].$$

Since $\Delta L_t = \ln(1 + U_{N_t})dN_t$, we may write $e^{\Delta L_t} - 1$ as $U_{N_t}\,dN_t$. Finally, note that $U_{N_t}\,dN_t$ can be expressed as an integral with respect to the Poisson random measure M, that is,

$$U_{N_t}\,dN_t = \int_{\mathbb{R}} y\,M(dt\,dy).$$

□

The model described by equation (2.4) is designed to capture three specific features: (i) standard geometric Brownian motion dynamics in the absence of jumps, (ii) abrupt, proportional jumps in the asset price occurring at random times, and (iii) a "fade-away" effect that causes the impact of each jump to decay over time.

If no jump has occurred up to time t (i.e., $t < \tau_1$), the asset price evolves according to the geometric Brownian motion (GBM) used in the Black–Scholes model. At the first jump time $t = \tau_1$, the price can be expressed as $S_t = S_{t^-}(1 + U_1)$, where S_{t^-} is just the GBM evaluated just before the jump. In this case, the jump size U_1 reflects the relative percentage change caused by the jump.

[1] A function $f : [0,T] \to \mathbb{R}$ is said to be càdlàg if it is right-continuous with left limits.

To incorporate a more realistic post-jump behavior, the model introduces a decay function d, which modulates the jump size over time. Then, if no further jumps occur (i.e., $\tau_1 < t < \tau_2$), we have

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right) (1 + U_1\, \mathsf{d}(t - \tau_1)).$$

Proceeding in this way, the final expression for the price process—accounting for the successive jumps of the Poisson process—takes the form shown in equation (2.4).

Remark 2.1. By representing the jump term in equation (2.4) as an exponential of a stochastic integral with respect to the Poisson random measure M, the price process can be rewritten in terms of a product of Doléans-Dade exponentials, each of which is a martingale. In particular,

$$S(t) = S(0) \exp\left(\int_0^t \left(\mu + \int_{\mathbb{R}} \mathsf{d}(t-s)\, y\, \lambda\, F_U(dy)\right) ds\right) \mathcal{E}(\sigma W)(t)\, \mathcal{E}(\mathsf{d}\, y\, \tilde{M})(t), \quad (2.5)$$

where

$$\mathcal{E}(\sigma\, W)(t) := \exp\left\{\int_0^t \sigma\, dW_s - \frac{1}{2}\int_0^t \sigma^2\, ds\right\},$$

$$\mathcal{E}(\mathsf{d}\, y\, \tilde{M})(t) := \exp\left\{\int_0^t \int_{\mathbb{R}} \ln\left(1 + \mathsf{d}(t-s)\, y\right)\, M(ds\, dy) - \int_0^t \int_{\mathbb{R}} \mathsf{d}(t-s)\, y\, \lambda\, F_U(dy)\, ds\right\}.$$

Under this representation, it is easy to check that all moments of S are finite if the moment generating function of U exists.

Remark 2.2. Although the "continuous" random part of S evolves with constant volatility σ, the total "volatility" of the process is not constant. The intuition behind this lies in the fact that the quadratic variation of jump processes is generally random. For instance, in the case of a Poisson process N, one has $[N, N]_t = N_t$ (see e.g., Example 8.4 in [9]), which highlights that the variability accumulated through jumps is path-dependent.

One of the defining properties of a Lévy process is having stationary increments, meaning that the unconditional law of the increment over $[t, t+h]$ depends only on the length h, not on the initial time t. This does not generally hold in our setting due to the decay function d. To illustrate this, consider a compound Poisson process $X_t := \sum_{i=1}^{N_t} U_i$, where the U_i are i.i.d. and independent of the Poisson process N. The increment over $[t, t+h]$ is $X_{t+h} - X_t = \sum_{j=1}^{N_{t+h}-N_t} U_{N_t+j}$. Since $N_{t+h} - N_t \stackrel{d}{=} N_h$ and the U_i are i.i.d.—i.e., $(U_{N_t+j})_{j=1,\dots,N_{t+h}-N_t} \stackrel{d}{=} (U_j)_{j=1,\dots,N_h}$—it follows that $X_{t+h} - X_t \stackrel{d}{=} X_h$.

In contrast, if jump sizes are weighted by a time-dependent decay function—that is, $X_t := \sum_{i=1}^{N_t} \mathsf{d}(t - \tau_i)\, U_i$—, the increment over $[t, t+h]$ is no longer distributed as X_h, since the weights depend on the interval. The process thus lacks stationary increments and is not a Lévy process.

Moreover, S is not Markovian in general (see Lemma 2.2 in [1]). Intuitively, as seen in Equation (2.1), the dynamics depend not only on the value S_{t^-}, but also on the process $\theta(t^-)$, which incorporates the entire past trajectory through the history of jumps. As a result, the evolution of S cannot be fully characterized by its current state alone, and the Markov property fails in general, i.e., $\mathbb{E}[f(S_T) \mid \mathcal{F}_t] \neq \mathbb{E}[f(S_T) \mid S_t]$.

3 About Market Incompleteness

This section introduces the concepts of pricing rules, arbitrage, and market completeness, and explores their connection to equivalent martingale measures, drawing primarily from [9] and, for technical details, from [25, 10, 11]. In Subsection 3.3, we characterize the set of equivalent martingale measures for the price process given by (2.1), and show that, since this set is not a singleton[2], it gives rise to infinitely many arbitrage-free prices. This motivates the need to explore alternative pricing methods.

3.1 Arbitrage-Free Pricing Rules

Let $\mathcal{H}$ be the generic space of European contingent claims with maturity at time $T > 0$. Formally, we require that any $H \in \mathcal{H}$ be non-negative and belong to $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, that is, H must be $\mathcal{F}_T$-measurable and $\mathbb{P}$-square integrable. Further specifications of the set $\mathcal{H}$ will be provided in later sections if needed. Typical examples of these claims include the terminal value of the asset $H = S_T$, indicator functions $H = \mathbf{1}_A$ for $A \in \mathcal{F}_T$, standard European options, as well as more general terminal payoffs depending on the entire path, e.g., $H = f((S_t)_{t \le T})$.

The central problem in this context is how to assign a price to contingent claims $H \in \mathcal{H}$. A pricing rule is defined as an operator that assigns a price $V_t(H)$ at time t to each claim $H \in \mathcal{H}$ for every $t \in [0, T]$. Typically, the minimal requirements for a pricing rule are the following: that $V_t(H)$ can be computed using the information available at time t (i.e., the process $(V_t(H))_{t \in [0,T]}$ is adapted to the filtration $\mathbb{F}$); that it satisfies positivity, i.e.,

$$\forall\, \omega \in \Omega,\ H(\omega) \ge 0 \ \Rightarrow\ V_t(H) \ge 0, \quad \forall\, t \le T;$$

and that it is linear in the claims, i.e., $V_t(\Sigma_i H_i) = \Sigma_i V_t(H_i)$, although this assumption will be relaxed in Subsection 4.5 on Rational Pricing and Hedging.

Relying on linearity, we motivate now that our pricing rule can be expressed as an expectation under some probability measure. To this end, we define a disjoint family of sets $(A_i)_{i \ge 1}$ as the states of nature (i.e., $\cup_{i \ge 1} A_i = \Omega$). In this way, any claim $H \in \mathcal{H}$ can be expressed as the sum of the amounts α_i it pays in each state A_i, that is, $H = \Sigma_i \alpha_i \mathbf{1}_{A_i}$. Since this is nothing but the sum of digital options, we can apply linearity and write $V_0(H) = \Sigma_{i \ge 1} \alpha_i V_0(\mathbf{1}_{A_i})$.

Note that the price at $t = 0$ of a claim $H = 1$ is simply the price of the risk-free asset at time 0, $R(T)$. We now define a probability measure [3] as follows:

$$\mathbb{Q}(A) = \frac{V_0(\mathbf{1}_A)}{V_0(1)} = V_0(\mathbf{1}_A)\, R(T)^{-1}, \quad \forall A \in \mathcal{F}_T.$$

[2] A set is a singleton if and only if its cardinality is 1, i.e., it is a set with exactly one element.

[3] Note that $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$ since $0 \le \mathbf{1}_A \le 1$ implies $0 \le \mathbb{Q}(A) \le 1$ for all $A \in \mathcal{F}_T = \mathcal{F}$, and setting $A = \Omega$ yields $\mathbb{Q}(\Omega) = 1$. Moreover, if we assume that impossible claims worth nothing, then $\mathbb{Q}$ is an absolutely continuous measure with respect to the natural probability $\mathbb{P}$, i.e., $\forall A \in \mathcal{F} : \mathbb{P}(A) = 0 \implies \mathbb{Q}(A) = 0$. Conversely, it makes sense as well to assume that $\mathbb{Q}(A)$=0 only for impossible claims. Therefore the artificial measure $\mathbb{Q}$ is in fact absolutely continuous with respect to $\mathbb{P}$, meaning that they agree in what are sure events and impossible events.

Using the linearity of the pricing rule and the identity $V_0(\mathbf{1}_A) = R(T)\,\mathbb{Q}(A)$, it follows that the price of the claim H can be computed as an expectation under $\mathbb{Q}$:

$$V_0(H) = \sum_{i\geq 1} \alpha_i\, R(T)\,\mathbb{Q}(A_i) = \sum_{i\geq 1} \alpha_i\, R(T)\,\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{A_i}] = \mathbb{E}^{\mathbb{Q}}[R(T)H].$$

If we additionally require time consistency—that is, the value at time $t = 0$ of a claim with payoff H coincides with the value at time $t = 0$ of a claim whose payoff is $V_t(H)$ at time t—then any pricing rule satisfying these properties can be expressed at time t as the discounted expected value of the claim under a risk-neutral measure $\mathbb{Q}$, conditinonal on $\mathcal{F}_t$. Namely:

$$V_t(H) = R(t)^{-1}\mathbb{E}^{\mathbb{Q}}[R(T)\,H \mid \mathcal{F}_t], \tag{3.1}$$

where $\mathbb{Q}$ is the so-called *risk-neutral* or *pricing measure* (this is a well know result, se e.g. [9]). A more formal treatment of this result will be presented later in Proposition 3.1.

Another fundamental requirement for any pricing rule is the absence of arbitrage opportunities. Arbitrage is inherently linked to self-financing strategies involving trading the underlying asset with price process S. Therefore, in order to properly understand this concept— and later apply it to hedging arguments— we must temporarily step away from the topic of linear pricing rules and focus on investment strategies involving direct trading in S. Indeed, when modeling a risky asset whose price follows a stochastic process S, it is not only derivative products written on S that are of interest, but also strategies involving dynamic trading of the asset itself.

In our setting, we consider dynamic portfolios resulting from buying or selling units of the underlying asset whose price is given by S. Specifically, suppose an investor trades at discrete times $\tau_0 = 0 < \tau_1 < \cdots < \tau_n = T$, selecting a position ϕ_i in the underlying during the interval $(\tau_i, \tau_{i+1}]$. Then, the terminal gain from the investment strategy is given by:

$$\sum_{i=1}^{n} \phi_{i-1}\left(S_{\tau_i} - S_{\tau_{i-1}}\right). \tag{3.2}$$

This simple example allows us to introduce an investment strategy defined by a simple predictable process, and the associated stochastic integral $\int_0^t \phi\, dS$. In practice, the trading times τ are not deterministic, since the investor adjusts the composition of the portfolio depending on the evolution of S. Moreover, the investor does not know in advance the future composition of the portfolio; instead, they must wait until time τ_i to decide their position for the interval $(\tau_i, \tau_{i+1}]$. That is, ϕ_i is a random variable measurable with respect to $\mathcal{F}_{\tau_i -}$. In addition, when the portfolio composition is updated at time $t = \tau_i$, the position remains ϕ_{i-1} at that instant, and it is only immediately after the transaction (i.e., for $t > \tau_i$) that the new position ϕ_i becomes effective. Therefore, the process ϕ defining the investment strategy is càglàd[4], and hence predictable.

We now present a formal definition of a simple predictable process:

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[4] An stochastic process $(X_t)_{t\in[0,T]}$ is said to be càglàd if its sample paths are left-continuous with right limits.

Definition 3.1. *A stochastic process* $(\phi_t)_{t\in[0,T]}$ *is called a* simple predictable process *if it can be written in the form*

$$\phi_t = \phi_0 \mathbf{1}_{\{t=0\}} + \sum_{i=1}^{n} \phi_{i-1}\, \mathbf{1}_{(\tau_{i-1},\tau_i]}(t). \tag{3.3}$$

Here, $0 = \tau_0 < \tau_1 < \cdots < \tau_n = T$ *are stopping times, and each* ϕ_i *is a bounded* $\mathcal{F}_{\tau_i^-}$*-measurable random variable.*

For now, we denote by Φ the generic space of admissible investment strategies. Formally, we require that any $\phi \in \Phi$ belongs to $L(\hat{S})$, that is, ϕ must be an $\mathbb{F}$-predictable and $\hat{S}$-integrable process. In this way, the associated discounted gains process $\left(\int_0^t \phi\, d\hat{S}\right)_{t\in[0,T]}$ is well-defined. Moreover, we require that every process $\phi \in \Phi$ can be approximated (uniformly in time) by a sequence $(\phi^n)_{n\in\mathbb{N}}$ of simple predictable processes of the form given in equation (3.3), so that the gains associated with ϕ can also be approximated by those of simple strategies. This condition is imposed in order to take advantage of the fact that the processes S and $\hat{S}$ are semimartingales.

Definition 3.2. *A càdlàg, non-anticipative process S is a* semimartingale *if the stochastic integral of simple predictable processes (as defined in equation* (3.2)*) satisfies the following continuity property: for every* $\phi^n, \phi \in \Phi$, *if*

$$\sup_{(t,\omega)\in[0,T]\times\Omega} |\phi_t^n(\omega) - \phi_t(\omega)| \xrightarrow[n\to\infty]{} 0 \quad \textit{then} \quad \int_0^T \phi^n dS \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^T \phi\, dS.$$

Furtheremore, by the semimartingale property, the associated gains process will also be a semimartingale.

We now introduce the notion of a self-financing strategy and its associated gains and cost processes. Consider a portfolio that allocates ϕ^B units in the risk-free asset and ϕ^S units in the risky asset with price process S. The associated investment strategy is denoted by $\phi := (\phi_t^B, \phi_t^S)$, and we define the price process of the traded assets by the $\mathbb{R}^2$-valued semimartingale $X := (B, S) = (R^{-1}, S)$.

The cumulative *gains process* associated with the risky strategy ϕ^S is given by: $G_t(\phi) := \int_0^t \phi \cdot dX$. The *value* or *worth* of the portfolio at time t is defined as: $\Pi_t(\phi) := \phi_t \cdot X_t = \phi_t^B R_t^{-1} + \phi_t^S S_t$. We then define the *cost process* $C = (C_t)_{t\in[0,T]}$ as the difference between the portfolio value and the gains accumulated up to time t: $C_t := \Pi_t(\phi) - G_t(\phi)$.

A strategy ϕ is said to be *self-financing* if the cost process is null $\mathbb{P}$-a.s., that is, $C_t = 0$ for all $t \in [0, T]$ almost surely. In this case, the portfolio value can be expressed as:

$$\Pi_t(\phi) = \int_0^t \phi \cdot dX = \phi_0 \cdot X_0 + \int_{0+}^t \phi \cdot dX = \Pi_0(\phi) + \int_{0+}^t \phi^B dR^{-1} + \int_{0+}^t \phi^S dS. \tag{3.4}$$

Recall that the discounted value of any process Y with respect to the risk-free asset was defined as $\hat{Y}_t := R_t Y_t$. Applying Itô's lemma to $\widehat{\Pi}(\phi)$ and noting that $\phi_t^B = R(t)\,(\Pi_t(\phi) - \phi_t^S S_t)$, we can express the *discounted portfolio value* as:

$$\hat{\Pi}_t(\phi) = \hat{\Pi}_0(\phi) + \int_0^t \phi^S d\hat{S}. \tag{3.5}$$

Now that the notion of a self-financing strategy has been introduced, we proceed to formally define the concept of an arbitrage opportunity in this setting.

Definition 3.3. *An* arbitrage opportunity *is a self-financing strategy ϕ that can generate a strictly positive final gain without any possibility of immediate loss. Therefore, ϕ satisfies: (1) $\mathbb{P}(\forall t \in [0,T],\ \Pi_t(\phi) \geq 0) = 1$, (2) $\mathbb{P}(\Pi_T(\phi) > \Pi_0(\phi)) \neq 0$.*

Recall that linearity implied that the pricing rule could be written as an expectation under some measure $\mathbb{Q}$ (see equation (3.1)). We now examine how the absence of arbitrage restricts the choice of $\mathbb{Q}$. Specifically, $\mathbb{Q}$ must be a probability measure equivalent to $\mathbb{P}$, denoted $\mathbb{Q} \sim \mathbb{P}$, under which the discounted price process is a martingale.

Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]})$ are said to be *equivalent* if they are mutually absolutely continuous, that is, for every $t \in [0,T]$ and every $A \in \mathcal{F}_t$,

$$\mathbb{Q}(A) = 0 \iff \mathbb{P}(A) = 0.$$

This condition ensures that both measures agree on which events are null at every time t. Suppose, without loss of generality (w.l.o.g.), that $r = 0$, and that $\mathbb{Q} \not\sim \mathbb{P}$. Then, there exists a set $A \in \mathcal{F}_T$ such that $\mathbb{P}(A) > 0$ but $\mathbb{Q}(A) = 0$. Consider the contingent claim $H = \mathbf{1}_A$. Under $\mathbb{Q}$, its price would be

$$V_0(H) = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_A] = \mathbb{Q}(A) = 0,$$

while the payoff is non-negative with strictly positive $\mathbb{P}$-probability. This would constitute an arbitrage opportunity. A symmetric argument shows that absolute continuity must also hold in the other direction. Therefore, the pricing measure $\mathbb{Q}$ must be equivalent to $\mathbb{P}$.

To motivate the condition that the discounted price process must be a $\mathbb{Q}$-martingale under arbitrage-free pricing rules, consider two alternative strategies: the first consists in buying the risky asset at time t at price S_t, and simply holding it until maturity, thereby receiving S_T at time T; the second strategy invests the same amount S_t in the risk-free asset, yielding a terminal payoff of $(R(t)/R(T))\, S_t$ at T. Supose r constant w.l.o.g. Since both strategies require the same initial cost at time t, the absence of arbitrage—together with the pricing rule given in equation (3.1)—implies that the expected discounted payoff of both positions must coincide under $\mathbb{Q}$:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} S_T \mid \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \cdot e^{r(T-t)} S_t \mid \mathcal{F}_t\right].$$

Where the RHS is just S_t. By multiplying both sides by $R(t) \equiv e^{-rt}$, we obtain the $\mathbb{Q}-$martingale property for $\hat{S}$.

This intuition is formalized in the following proposition.

Proposition 3.1. *In a market described by a probability measure $\mathbb{P}$ on scenarios, any arbitrage-free linear pricing rule V can be represented as*

$$V_t(H) = R(t)^{-1}\mathbb{E}^{\mathbb{Q}}[R(T)\, H \mid \mathcal{F}_t],$$

where $\mathbb{Q}$ is an equivalent martingale measure: a probability measure on the market scenarios such that $\mathbb{P} \sim \mathbb{Q}$ and $\mathbb{E}^{\mathbb{Q}}[\hat{S}_T \mid \mathcal{F}_t] = \hat{S}_t$.

So far, we have developed the intuition that an arbitrage-free pricing rule can be constructed *if* an equivalent martingale measure exists. Proving the converse implication is more subtle (see, e.g., [25]). The result in both directions is known as the *Fundamental Theorem of Asset Pricing.*

Proposition 3.2. Fundamental Theorem of Asset Pricing. *The financial market described by the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with a risky asset whose price process is $(S_t)_{t\in[0,T]}$, is arbitrage-free if and only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted price process $(\widehat{S}_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-martingale.*

Remark 3.1. To be more precise, when dealing with price processes modeled as bounded semimartingales, one should replace the notion of arbitrage opportunity in the fundamental theorem with the concept of *Free Lunch with Vanishing Risk* (see Theorem 1.1 in [10]). Furthermore, in the general setting of unbounded semimartingales, the notion of equivalent martingale measure must also be replaced by that of an *equivalent sigma-martingale measure*[5] (see Main Theorem in [11]).

3.2 Completeness of the Market

The assumption of market completeness is frequently made, often implicitly, in many classical models of financial mathematics. It originates from the Black–Scholes framework, where every contingent claim can be perfectly replicated by a dynamic self-financing trading strategy. However, this property is far from universal and depends heavily on the chosen model for the asset price. In what follows, we formalize the notion of completeness and examine its implications within the general theory of no-arbitrage pricing.

Definition 3.4. *Let $X := (S, R^{-1})$. The financial market described by the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with a risky asset whose price process is $(S_t)_{t\in[0,T]}$, is said to be complete if, for every contingent claim $H \in \mathcal{H}$, there exists a self-financing strategy $\phi := (\phi^B, \phi^S)$, with $\phi^S \in \Phi$, that perfectly hedges H. That is,*

$$H = \phi_0 \cdot X_0 + \int_0^T \phi_t \cdot dX_t \quad \mathbb{P}\text{-}a.s. \tag{3.6}$$

Recall that if $\mathbb{Q}$ is an equivalent martingale measure (EMM), then, since $\mathbb{Q} \sim \mathbb{P}$, any $\mathbb{P}$-null set where perfect replication fails is also $\mathbb{Q}$-null. Moreover, using the representation of the discounted portfolio in equation (3.5), the condition (3.6) in the definition of market completeness is equivalent to:

$$\hat{H} = \Pi_0 + \int_0^T \phi_t^S \, d\hat{S}_t \quad \mathbb{Q}\text{-a.s.} \tag{3.7}$$

[5] A process $X = (X_t)_{t\geq 0}$ taking values in $\mathbb{R}^d$ is called a *sigma-martingale* if there exists a $\mathbb{R}^d$-valued martingale M and a non-negative, predictable, M-integrable process φ, such that $X = \varphi \cdot M$.

Now, taking expectations under $\mathbb{Q}$, and assuming ϕ^S is such that $\int_0^T \phi_t^S \, d\hat{S}_t \in L^2(\mathbb{Q})$, we obtain that, since $\hat{S}$ is a $\mathbb{Q}$-martingale, it holds:

$$\mathbb{E}^{\mathbb{Q}}[\hat{H}] = \Pi_0.$$

In the absence of arbitrage, any strategy that perfectly hedges the claim H—that is, satisfies either (3.6) or (3.7)—must have the same initial cost Π_0. Therefore, the time-zero value of the contingent claim H, computed under any EMM using the market completeness condition, must coincide and equal Π_0. This requires that all equivalent martingale measures are equal; that is, there exists a unique equivalent martingale measure. These ideas are formalized in the following theorem.

Proposition 3.3. Second Fundamental Theorem of Asset Pricing. *A market defined by the asset $(S_t)_{t\in[0,T]}$, described as a stochastic processes on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, is complete if and only if there exists a unique martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.*

Remark 3.2. As in the previous fundamental theorem, a fully rigorous formulation in a general setting turns out to be significantly more difficult (see Theorem 1.17 in [8]). In order to keep the problem analytically more tractable, we adopt the standard formulation of the fundamental theorems of asset pricing, and rely on the classical notions of arbitrage and equivalent martingale measure.

3.3 Equivalent Martingale Measures Set

The previous theorem shows that a structural property of the market—namely, the ability to hedge any contingent claim using only cash and the underlying risky asset—depends on a statistical property of the model used to describe the risky asset's price, specifically the uniqueness of the equivalent martingale measure. In general, models involving jump-diffusion dynamics do not admit a unique EMM, and hence the corresponding markets are incomplete.

In the following proposition, we characterize the set of equivalent martingale measures for the shot-noise process.

Proposition 3.4. *The set of equivalent martingale measures with $\mathbb{P}$-square integrable density for the shot-noise process, $\mathcal{Q}$, is the set of measures $\mathbb{Q}^\gamma$ whose Radon–Nikodym derivative, $L^\gamma(t)$, is given by*

$$L^\gamma(t) = \mathcal{E}(\gamma_c W)(t)\, \mathcal{E}(\gamma_J \tilde{M})(t)\,.$$

where

$$\mathcal{E}(\gamma_c W)(t) = \exp\Big\{ \int_0^t \gamma_c(s)\, dW_s - \frac{1}{2}\int_0^t \gamma_c(s)^2\, ds \Big\}, \tag{3.8}$$

$$\begin{aligned} \mathcal{E}(\gamma_J \tilde{M})(t) = \exp\Big\{ &\int_0^t\int_{\mathbb{R}} \ln\big(1+\gamma_J(s,y)\big)\, \tilde{M}(ds,dy) \\ &- \int_0^t\int_{\mathbb{R}} \big[\gamma_J(s,y) - \ln\big(1+\gamma_J(s,y)\big)\big]\, \lambda F_U(dy)\, ds \Big\}. \end{aligned} \tag{3.9}$$

Here the processes γ_c and γ_J are predictable and satisfy

$$\mu + \theta(t^-) + \sigma\,\gamma_c(t) + \lambda \int_{\mathbb{R}} y\,(1+\gamma_J(t,y))\,F_U(dy) = r \quad \mathbb{P}\otimes dt\text{-}a.s., \tag{3.10}$$

where $\theta(t^-)$ is given by equation (2.3). We further assume that $1+\gamma_J(t,y)>0$, $\mathbb{P}\otimes dt$-a.s., so that L^γ is strictly positive.

Proof. Note that the process L^γ is the Radon–Nikodym derivative $\frac{d\mathbb{Q}^\gamma}{d\mathbb{P}}$, and is therefore, by construction, a strictly positive $\mathbb{P}$-martingale. Thus, we can apply the predictable representation theorem, which ensures the existence of a unique pair of predictable processes (γ_c, γ_J) such that L^γ admits the following representation (see, e.g., [4]):

$$dL_t^\gamma = L_{t-}^\gamma \left(\gamma_c(t)\,dW_t + \int_{\mathbb{R}} \gamma_J(t,y)\,\tilde{M}(dt,dy) \right). \tag{3.11}$$

Define $Y_t := \ln L_t^\gamma$. By applying Itô's formula for semimartingales to Y, we obtain that the solution to the equation (3.11) is the product of Doléans–Dade exponentials expressed in equations (3.8) and (3.8).

By Girsanov's theorem for semimartingales (see, e.g., Theorem 3.24 in [28]), under the probability measure $\mathbb{Q}^\gamma$ given by the density process L^γ, the following hold:

(i) The process W^γ defined as $W_t^\gamma := W_t - \int_0^t \gamma_c(s)\,ds$ is a $\mathbb{Q}^\gamma$-Brownian motion.

(ii) The $\mathbb{Q}^\gamma$-compensator (or intensity measure) of the Poisson random measure M is given by

$$\lambda(1+\gamma_J(t,y))\,F_U(dy)\,dt.$$

Therefore, the $\mathbb{Q}^\gamma$-compensated random measure takes the form

$$\tilde{M}^\gamma(dt,dy) = \tilde{M}(dt,dy) - \lambda\,\gamma_J(t,y)\,F_U(dy)\,dt.$$

Applying these results to equation (2.2), the dynamics of S under $\mathbb{Q}^\gamma$ can be expressed as:

$$\begin{aligned} \frac{dS_t}{S_{t^-}} = &\left(\mu + \theta(t^-) + \lambda\,\mathbb{E}[U_1] + \sigma\,\gamma_c(t) + \lambda \int_{\mathbb{R}} y\;\gamma_J(t,y)\,F_U(dy) \right) dt \\ &+ \sigma\,dW_t^\gamma + \int_{\mathbb{R}} y\,\tilde{M}^\gamma(dt,dy). \end{aligned}$$

Where the drift term can be rewritten as:

$$\mu + \theta(t^-) + \sigma\,\gamma_c(t) + \lambda \int_{\mathbb{R}} y\,(1+\gamma_J(t,y))F_U(dy). \tag{3.12}$$

Finally, in order for the discounted asset price to be a $\mathbb{Q}^\gamma$-martingale, the processes γ_c and γ_J must be such that the $\mathbb{Q}^\gamma$-drift term of the price process, shown in equation (3.12), equals r, $\mathbb{P}\otimes dt$-a.s. □

Remark 3.3. An interesting implication of this proposition is that, under any martingale measure $\mathbb{Q}^\gamma \in \mathcal{Q}$, the dynamics of the price process no longer depend directly[6] on the decay function d, as it is eliminated by the density L^γ. Therefore, the shot-noise price process, given in equation (2.4), can be represented under $\mathbb{Q}^\gamma$ as a product of Doléans-Dade exponentials, which are themselves martingales. In particular,

$$S(t) = S(0)\, R(t)^{-1}\mathcal{E}(\sigma\, W^\gamma)(t)\; \mathcal{E}(y\, \tilde{M}^\gamma)(t),$$

where

$$\mathcal{E}(\sigma\, W^\gamma)(t) = \exp\left\{\int_0^t \sigma\, dW_s^\gamma - \frac{1}{2}\int_0^t \sigma^2\, ds\right\},$$

$$\mathcal{E}(y\, \tilde{M}^\gamma)(t) = \exp\left\{\int_0^t \int_{\mathbb{R}} \ln(1+y)\, M^\gamma(ds, dy) - \int_0^t \int_{\mathbb{R}} y\, \lambda\, (1+\gamma_J(s,y)) F_U(dy)\, ds\right\}.$$

When γ_J is a deterministic function, the $\mathbb{Q}^\gamma$-intensity of the Poisson process N is given by $\lambda\,(1+\gamma_J)$ (see [4]). Since the intensity corresponds to the expected number of jumps per unit of time, the function γ_J can be interpreted as the relative change in the jump intensity under the risk-neutral measure. In this sense, the processes γ_c and γ_J represent the risk premium associated with the continuous diffusion and the jump components, respectively.

In the sequel, we consider a family $\mathcal{G}$ of predictable processes γ_J such that the associated density process L^γ defines a measure in the set $\mathcal{Q}$ of equivalent martingale measures. We then focus on a subset $\Gamma \subset \mathcal{Q}$, consisting of those measures whose density process L^γ is associated with a constant value of γ_J. This restriction is imposed for consistency: in Section 2, we assumed that the $\mathbb{P}$-intensity of the Poisson process N is constant; hence, to preserve constant intensity under $\mathbb{Q}$, the corresponding γ_J must also be constant.

It is clear that the shot-noise process does not admit a unique equivalent martingale measure, since infinitely many predictable pairs (γ_c, γ_J) satisfy equation (3.10). Therefore, the market generated by the asset S is incomplete: contingent claims are not redundant assets that can be perfectly replicated using only cash and the underlying, and under the linear pricing rule given in Proposition 3.1, there are infinitely many arbitrage-free prices for a given claim $H \in \mathcal{H}$. Each of these corresponds to a viable price process V^γ, defined by

$$R(t)V^\gamma(t) = \mathbb{E}^\gamma\left[R(T)H(S_T) \mid \mathcal{F}_t\right],$$

[7] where different choices of $\gamma := (\gamma_c, \gamma_J)$ generate different pricing rules. In this setting, we now explore possible pricing and hedging approaches.

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6 The processes γ_c and γ_J must satisfy equation (3.10), which allows for infinitely many measures under which the entire drift component induced by the decay function d, captured by θ, is absorbed into γ_c. As a result, γ_J does not depend on d, and neither does the price process under $\mathbb{Q}^\gamma$. There also exist many other measures in which γ_J does depend indirectly on d—by absorbing part of θ—, but this will not be the focus of this work.

7 Actually, the notation should be $\mathbb{E}^{\mathbb{Q}^\gamma}$, but we use $\mathbb{E}^\gamma$ since it's too heavy.

4 Derivative Pricing and Hedging Methods

This section reviews several pricing methods in incomplete markets, and apply them to the shot noise model of Section 2. We examine valuation rules arising from specific choices of the parameter γ_J in Proposition 3.4, including the Merton and minimal entropy cases. We also derive non trivial arbitrage-free bounds based on the Black–Scholes function and consider pricing via hedging strategies, such as the minimal martingale measure. Lastly, we study exponential utility pricing and its dependence on the agent's risk aversion, as well as its connections with the other pricing methods discussed.

Throughout this section, we work with contingent claims $H \in \mathcal{H}$, recalling that each $H \in \mathcal{H}$ is assumed to be non-negative and to belong to $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Additional conditions on H will be introduced depending on the chosen pricing method.

4.1 Merton's Aproach

The first application of jump-diffusion processes to option pricing was introduced by Merton in [31]. Although his model considers a relatively simple jump structure for the underlying asset—specifically, a compound Poisson process with Gaussian jumps—we adopt the same method for constructing the equivalent martingale measure. This measure is constructed analogously to the Black-Scholes model (and, more generally, to any one-dimensional continuous diffusion model): by adjusting the drift of the Brownian motion while keeping the jump component unchanged in distribution. In doing so, the risk-neutral measure preserves the original jump intensity and distribution.

Theorem 4.1. *Let S follow the dynamics given in equation* (2.1)*, corresponding to a shot-noise process. The* Merton martingale measure $\mathbb{Q}^0$ *is the unique element of $\mathcal{Q}$ whose Radon–Nikodym density $L^0(t)$ with respect to $\mathbb{P}$ is given by $L^0(t) = \mathcal{E}(\psi^0 W)(t)$, where*

$$\mathcal{E}(\psi^0 W)(t) = \exp\left\{ \int_0^t \psi^0(s)\, dW_s - \frac{1}{2}\int_0^t (\psi^0(s))^2\, ds \right\},$$

and the process ψ^0 is defined by

$$\psi^0(t) = \frac{r - \mu - \theta(t^-) - \lambda \mathbb{E}[U_1]}{\sigma}, \quad \textit{with} \quad \theta(t^-) := \frac{\partial}{\partial t} \sum_{i=1}^{N_{t-}} ln\left(1 + U_i\, \mathsf{d}(t - \tau_i)\right).$$

Proof. The method follows the same structure as in Proposition 3.4. In this case, we apply Girsanov's theorem for Brownian motion, (see e.g., Theorem 5.2.3 in [35]). Under this result, we can represent the Brownian motion W under $\mathbb{P}$ as $dW_t = dW_t^{\mathbb{Q}^0} + \psi^0(t)\, dt$, and express the dynamics of the discounted asset price process $\hat{S}_t := R(t) S_t$ under $\mathbb{Q}^0$ as

$$\frac{d\hat{S}_t}{\hat{S}_{t-}} = \left(\mu + \theta(t^-) + \lambda \mathbb{E}[U_1] - r + \sigma \psi^0(t)\right) dt + \sigma\, dW_t^0 + \int_{\mathbb{R}} y\, \tilde{M}(dt, dy).$$

Therefore, to ensure that $\hat{S}$ is a $\mathbb{Q}^0$-martingale, ψ^0 is choosen such that the drift vanishes i.e.,

$$\mu + \theta(t^-) + \lambda \mathbb{E}[U_1] - r + \sigma \psi^0(t) = 0 \quad \mathbb{P} \otimes dt - a.s.,$$

Solving for $\psi^0(t)$ yields the expression given in the theorem. □

While jumps are present in the dynamics of the underlying asset and thus influence the option price, they do not affect the pricing rule through a change in the jump distribution under the risk-neutral measure, as this measure preserves their original intensity and distribution. In this sense, the pricing rule effectively disregards jump risk as a source of systematic risk, even though jumps remain part of the asset's behaviour. The corresponding Radon–Nikodym density is precisely the one in Proposition 3.4 with $\gamma_J \equiv 0$, where γ_J was interpreted as the risk premium introduced by jumps. This implicitly assumes that jump risk is fully diversifiable. Merton justifies this by assuming that the jump-diffusion components of different assets are uncorrelated, which allows jump risk to be interpreted as idiosyncratic. However, this assumption does not hold in practice: highly diversified portfolios—such as broad market indices—often exhibit large price jumps, particularly downward movements, resulting from highly correlated jumps across their constituent assets. This suggests that jump risk in real markets often contains a significant systematic component, which this approach ignores.

Define Merton's price process V^0 as $R(t)V^0(t) = \mathbb{E}^0\left[R(T)H(S_T) \mid \mathcal{F}_t\right]$. The hedging strategy proposed by Merton is the self-financing portfolio $(\phi^{0,B}, \phi^{0,S})$ defined by

$$\phi^{0,S} = \frac{\partial V^0}{\partial S}, \qquad \phi^{0,B} = \int \phi^{0,S}\, d\hat{S} \,-\, \phi^{0,S}\hat{S}.$$

This strategy hedges only the risk associated with the diffusion part of the underlying process—namely, the component driven by Brownian motion, as captured by the stochastic integral with respect to W in equation (4.3) below.

The initial value of this strategy, denoted Π_0^0, equals the expected discounted payoff under $\mathbb{Q}^0$. By the absence of arbitrage and the martingale property of $\hat{V}^0$ under $\mathbb{Q}^0$, we have $\Pi_0^0 = V_0^0$.

While this strategy captures the average effect of jumps—since the delta incorporates all jumps up to time t—it provides no protection against the realization of individual jumps. As a result, the portfolio remains fully exposed to jump risk.

Consider the operators $\mathscr{L}$ and Λ defined on $C^{1,2}$ functions by

$$\mathscr{L}[f](t,x) = \frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x),$$

$$\Lambda[f](t,x,y) = f(t,(1+y)x) - f(t,x) - yx\frac{\partial f}{\partial x}(t,x).$$

We now use these operators to formulate the hedging error associated with Merton's strategy.

Proposition 4.1. *Let $\mathbb{Q}^0 \in \mathcal{Q}$ be the Merton measure given in Theorem 4.1, and let V^0 denote the corresponding viable price process, defined by $R(t)V^0(t) = \mathbb{E}^0\left[R(T)H(S_T) \mid \mathcal{F}_t\right]$ for some $H \in \mathcal{H}$. Define the hedging error at maturity as $H - \Pi_T^0(H)$. If the decay function d is such that the asset price process S, given by equation (2.1), is Markovian, and H is such that the first and second spatial derivatives of V^0 are bounded, then the hedging error associated with Merton's strategy takes the form:*

$$\int_0^T \int_{\mathbb{R}}^{\infty} R(T)^{-1}R(t)\,\Lambda[V^0](t,S_{t-},y)\,\tilde{M}(dt,dy).$$

Proof. Recall that since $\mathbb{Q}^0 \in \mathcal{Q}$, the dynamics of S under $\mathbb{Q}^0$ are given by

$$\frac{dS_t}{S_{t-}} = r\,dt + \sigma\,dW_t^0 + \int_{\mathbb{R}} y\,\tilde{M}(dt, dy),$$

where the compensated Poisson random measure $\tilde{M}$ is defined as $\tilde{M}(dt, dy) := M(dt, dy) - \lambda F_U(dy)dt$. Supose that H is such that V^0 is a smooth, $C^{1,2}$ function, then we can apply Itô's lemma for semimartingales to the discounted price process $RV^\circ =: \hat{V}^0$. (Otherwise, apply Itô's lemma for convex functions, see e.g. [32]).

$$\begin{aligned}\hat{V}_T^0 - \hat{V}_0^0 = \quad & \int_0^T \frac{\partial}{\partial t}\hat{V}^0(t, S_t)\,dt \\ & + \int_0^T \frac{\partial}{\partial x}\hat{V}^0(t, S_{t-})\,S_{t-}\left(r\,dt + \sigma\,dW_t^\gamma + \int_{\mathbb{R}} y\,\tilde{M}(dt, dy)\right) \\ & + \frac{1}{2}\int_0^T \frac{\partial^2}{\partial x^2}\hat{V}^0(t, S_{t-})\,S_{t-}^2\sigma^2\,dt \\ & + \int_0^T \int_{\mathbb{R}} R(t)\,\Lambda[V^0](t, y, S_{t-})\,M(dt, dy).\end{aligned}$$

By collecting the terms with respect to the Lebesgue measure dt, applying the operator $\mathscr{L}$, and rewriting the last integral using the definition of the compensated Poisson random measure, the above expression becomes:

$$\hat{V}_T^0 - \hat{V}_0^0 = \int_0^T \left(\mathscr{L}[\hat{V}^0](t, S_{t-}) + \int_{\mathbb{R}} R(t)\Lambda\left[V^0\right](t, S_{t-}, y)\lambda\,F_U(dy)\right)dt \tag{4.1}$$

$$+ \int_0^T \int_{\mathbb{R}} R(t)\Lambda\left[V^0\right](t, S_{t-}, y)\,\tilde{M}(dt, dy) \tag{4.2}$$

$$+ \int_0^T S_{t-}\frac{\partial}{\partial x}\hat{V}^0(t, S_{t-})\left[\sigma\,dW_t^0 + \int_{\mathbb{R}} y\,\tilde{M}(dt, dy)\right]. \tag{4.3}$$

From the boundedness of Delta and Gamma, and the fact that $S \in L^2(\Omega, \mathbb{F}, \mathbb{Q}^\gamma)$, the stochastic integrals on the right-hand side are $\mathbb{Q}^0$-martingales.

Recall that under any arbitrage-free pricing rule, the price of the claim at maturity must coincide with the payoff, that is, $V_T^0 = H(S_T)$. This condition implies that $\hat{V}^0$ is a $\mathbb{Q}^0$-martingale by construction:

$$\hat{V}^0(t, S_t) = \mathbb{E}^0\left[R(T)H(S_T) \mid \mathcal{F}_t\right] = \mathbb{E}^0\left[\hat{V}^0(T, S_T) \mid \mathcal{F}_t\right].$$

Therefore, $\hat{V}_T^0 - \hat{V}_0^0$ is equal to its martingale component (i.e., the integral given by (4.1) is equal to 0). Note that the integral in equation (4.3) matches the integral of the partial derivative of V^0 with respect to x with respect to the process $\hat{S}$ under $\mathbb{Q}^0$.

Finally, since Merton's hedging portfolio is self-financing, we can express the discounted hedging error $R(T)[H(S_T) - \Pi_T^0(H)] = R(T)[V_T^\gamma(S_T) - \Pi_T^0(H)]$ as follows:

$$\begin{aligned}\hat{H}(S_T) - \hat{\Pi}_T^0(H) &= \hat{H}(S_T) - \left(\hat{V}_0^0 + \int_0^T \frac{\partial}{\partial x} V^0(t, S_{t-})\, d\hat{S}_t\right) \\ &= \hat{V}_T^0 - \hat{V}_0^0 - \int_0^T \frac{\partial}{\partial x} V^0(t, S_{t-})\, d\hat{S}_t \\ &= \int_0^T \int_{\mathbb{R}} R(t)\Lambda\left[V^0\right](t, S_{t-}, y)\, \tilde{M}(dt, dy).\end{aligned}$$

Multiplying both sides by $R(T)^{-1}$ yields the error stated in the proposition.

□

This result is to be expected. In the hedging portfolio proposed by Merton, only the changes in the derivative's price caused by the continuous part of the underlying are hedged—implicitly assuming a complete market. Consequently, the hedging error results from the accumulation of mismatches driven by jumps in the underlying, that is, by the *jump compensation term* that arises when applying Itô's lemma for semimartingales, instead of its continuous version.

Furthermore, the hedging error is a $\mathbb{Q}^0$-martingale, and since Merton's Radon–Nikodym density does not modify the compensator of the jump measure, it remains a martingale under $\mathbb{P}$ as well. Consequently, it is a random variable with zero $\mathbb{P}$-expectation, which implies that in some trajectories the strategy will over-hedge the contingent claim (resulting in a negative hedging error), while in others it will under-hedge it (yielding a positive error), depending on the size and frequency of the jumps along each path of the process S.

Remark 4.1. In the proof of Proposition (4.1), we used the fact that the discounted price process is a $\mathbb{Q}^0$-martingale. Consequently, its representation is given solely by its martingale component. Therefore, the integral in (4.1) vanishes. This provides a way to determine the price of the derivative by solving an integro-differential equation (IDE):

$$\begin{cases} \mathscr{L}[\hat{V}^0](t, S_{t-}) + \displaystyle\int_{\mathbb{R}} R(t)\Lambda\left[V^0\right](t, S_{t-}, y)\lambda\, F_U(dy) = 0, & \text{for } t < T, \\ V^0(T, S_T) = H(S_T). \end{cases}$$

This IDE extends immediately to any measure $\mathbb{Q}^\gamma$ by replacing λ with $\lambda(1 + \gamma_J)$. Moreover, if γ_J is constant, the IDE reduces to a PDE.

4.2 Super-Hedging

As seen in Merton's hedging portfolio, a strategy covering only the continuous diffusion component leads to a systematic hedging error for a given contingent claim $H \in \mathcal{H}$.

In fact, any strategy that ignores the jump risk will typically fail to replicate it. In contrast, the approach developed in this section adopts a different perspective: we focus on self-financing strategies $\phi := (\phi^B, \phi^S)$ that super-replicate the claim H, meaning that their terminal value, given by $\Pi_T(\phi)$ as in (3.4), is greater than or equal to H $\mathbb{P}$-a.s. Strategies satisfying this inequality are referred to as superhedging strategies for the contingent claim H.

We define the super-hedging cost at time $t = 0$ as the minimal initial capital among all superhedging strategies for H. That is,

$$V_0^{\mathrm{SUP}}(H) := \inf_{\phi\in\Phi} \left\{\Pi_0(\phi) \,:\, \mathbb{P}\left(\Pi_T(\phi) \geq H\right) = 1\right\}. \tag{4.4}$$

V^{SUP} can then be interpreted as the cost of fully hedging the claim. It represents the selling price in the case where the seller does not wish to bear any risk. If some risk is accepted, the price would lie below V^{SUP}, so this quantity serves as an upper bound.

We now take the perspective of a buyer of the claim to derive a lower bound using the superhedging cost introduced in (4.4). A buyer will receive the payoff H at maturity T, and, under this aproach, will choose a hedging portfolio with terminal value $\Pi_T(\phi)$ such that the total payoff at maturity is almost surely non-negative:

$$H + \Pi_T(\phi) \geq 0 \quad \mathbb{P}\text{-a.s.} \quad \Longrightarrow \quad \Pi_T(\phi) \geq -H \quad \mathbb{P}\text{-a.s.}$$

The minimal cost at time $t = 0$ of such a hedging strategy is given by $V_0^{\mathrm{SUP}}(-H)$. Let p denote the purchase price of the claim. The total cost of the position (claim + hedge) is then $p + V_0^{\mathrm{SUP}}(-H)$, which must be non-negative in order to prevent arbitrage, since the terminal payoff is almost surely non-negative:

$$p + V_0^{\mathrm{SUP}}(-H) \geq 0 \quad \Longrightarrow \quad p \geq -V_0^{\mathrm{SUP}}(-H).$$

Therefore, in the absence of arbitrage, the price of the claim must lie within the no-arbitrage bounds: $p \in \left[-V_0^{\mathrm{SUP}}(-H),\ V_0^{\mathrm{SUP}}(H)\right]$.

Recall that we denote $X := (R^{-1}, S)$. For $t > 0$, we express the superhedging cost as follows:

$$V_t^{\mathrm{SUP}}(H) := \inf_{\phi\in\Phi} \left\{\hat{\Pi}_t(\phi) \,:\, \mathbb{P}\left(\Pi_T(\phi) = \Pi_t(\phi) + \int_t^T \phi \cdot dX \geq H\right) = 1\right\}. \tag{4.5}$$

The no-arbitrage bounds derived at time $t = 0$ still apply in this setting, replacing $V^{\mathrm{SUP}}(H)$ by $V_t^{\mathrm{SUP}}(H)$, and analogously for $-H$.

Solving the optimization problem given in (4.5) is non-trivial, and in addition, it would require providing an explicit characterization of the set of predictable admissible strategies Φ. To overcome this difficulty, a well-known duality approach is often applied, originally proposed by [18] and later generalized by Kramkov (see Theorem 3.2 in [29]). This result allows the superhedging cost to be expressed as the essential supremum, over the set $\mathcal{Q}_a^L$ of absolutely continuous local martingale measures, of the expected discounted payoff. Mathematically, expression (4.4) coincides with:

$$\operatorname*{ess\,sup}_{\mathbb{Q}^\gamma\in\mathcal{Q}_a^L} \mathbb{E}^\gamma[\hat{H}]. \tag{4.6}$$

However, this representation is not entirely suitable for our framework either, since we do not have an explicit description of the full set $\mathcal{Q}_a^L$. Since every equivalent measure is also absolutely continuous, and every martingale is also a local martingale, it follows that the set of equivalent martingale measures $\mathcal{Q}$, provided in Proposition 3.4, is contained in the set of absolutely continuous local martingale measures, that is, $\mathcal{Q} \subseteq \mathcal{Q}_a^L$. Therefore, using the subset $\mathcal{Q}$ would only yield a lower bound for the true superhedging cost. Moreover, the measure $\mathbb{Q}^\gamma$ that solves the dual problem depends on the claim H, meaning it would have to be recomputed for each specific payoff.

In order to obtain analytical expressions for the bounds arising from the superhedging approach, we adopt the idea introduced by Bellamy and Jeanblanc in [4], which consists in bounding the prices of a claim H using the convexity of the Black–Scholes function with respect to the stock price (i.e., the spatial variable).

Definition 4.1. *Let $H \in \mathcal{H}$ be a convex function. The* Black–Scholes function $V^{BS}(t,x)$ *associated with H is defined as*

$$R(t)\, V^{BS}(t,x) := \mathbb{E}\left[R(T)\, H(X_T) \mid X_t = x\right],$$

with terminal condition

$$V^{BS}(T,x) = H(x),$$

where X denotes a geometric Brownian motion with constant drift r, i.e.,

$$dX_t = X_t\,(r\,dt + \sigma\,dW_t).$$

Given that the volatility σ is constant in our model, if the payoff function H is convex and has bounded one-sided derivatives, then the associated Black–Scholes function $V^{BS}(t,x)$ belongs to the class $C^{1,2}$, is convex in the spatial variable x, and has bounded first and second spatial derivatives (Delta and Gamma); see [19].

We define $\mathcal{H}^C \subset \mathcal{H}$ as the subset of claims H that satisfy the following properties: H is convex, having bounded one-sided derivatives, fulfills $H(x) \leq x\ ,\forall x \geq 0$, satisfies $H(0) = 0$, and the function $g(x) := x - H(x)$ is bounded. For instance, the payoff of a European call option, given by $H(x) = (x-K)^+$ for some $K \geq 0$, belongs to $\mathcal{H}^C$.

Similarly, we define $\mathcal{H}^P \subset \mathcal{H}$ as the subset of convex and bounded claims H having bounded one-sided derivatives and satisfy $H(x) \leq H(0)\ ,\forall x \geq 0$. The payoff of a European put option, given by $H(x) = (K-x)^+$ for some $K \geq 0$, belongs to $\mathcal{H}^P$.

In the following theorem, we present explicit bounds for the prices of convex European-style contingent claims, expressed in terms of the Black–Scholes function.

Theorem 4.2. *Let $\mathbb{Q}^\gamma \in \mathcal{Q}$ be the equivalent martingale measure given by L^γ, and let V^γ be the associated viable price process such that $R(t)\, V^\gamma(t) = \mathbb{E}^\gamma\left[R(T)\, H(S_T) \mid \mathcal{F}_t\right]$, for some $H \in \mathcal{H}^C \cup \mathcal{H}^P$. If the asset price process S follows the dynamics given in* (2.1)*, then:*

1. *The error with respect to the Black–Scholes price is given by* $\Lambda[V^{BS}]$. *Specifically,*

$$R(t)\,V^\gamma(t) = R(t)\,V^{BS}(t,S_t) + \epsilon^\gamma(t)$$

where

$$\epsilon^\gamma(t) := \mathbb{E}^\gamma\left[\int_t^T \int_{\mathbb{R}} R(s)\,\Lambda[V^{BS}](s,y,S_{s-})\,\lambda\,(1+\gamma_J(s,y)\,)\,F_U(dy)\,ds\,\middle|\,\mathcal{F}_t\right].$$

2. *The lower bound of any viable price is given by the Black–Scholes function evaluated at* S_t*:*

$$V^{BS}(t,S_t) \le V^\gamma(t), \;\; \forall\gamma_J \in \mathcal{G}$$

3. *If* $H \in \mathcal{H}^C$*, then the upper bound is* S_t*, i.e.,* $V^\gamma(t) \in [V^{BS}(t,S_t), S_t]$, $\;\forall\gamma_J \in \mathcal{G}$

4. *If* $H \in \mathcal{H}^P$*, then the upper bound is* $H(0)$*, i.e.,* $V^\gamma(t) \in \left[V^{BS}(t,S_t), H(0)\right], \forall\gamma_J \in \mathcal{G}$

Proof. Recall that, under any $\mathbb{Q}^\gamma \in \mathcal{Q}$, the dynamics of S take the form

$$\frac{dS_t}{S_{t^-}} = r\,dt + \sigma\,dW_t^\gamma + \int_{\mathbb{R}} y\,\tilde{M}^\gamma(dt\,dy).$$

Since $H \in \mathcal{H}^C \cup \mathcal{H}^P$, V^{BS} is smooth and $C^{1,2}$. We may then apply Itô's formula for semimartingales to the process $R\,V^{BS}$. Substituting the dynamics dS_t and the quadratic variation $[S^c, S^c]_t$ under the measure $\mathbb{Q}^\gamma$, we obtain:

$$\begin{aligned}
R(T)V^{BS}(T,S_T) = \;& R(t)V^{BS}(t,S_t) \\
&+ \int_t^T \frac{\partial}{\partial s} RV^{BS}(s,S_s)\,ds \\
&+ \int_t^T \frac{\partial}{\partial x} RV^{BS}(s,S_{s^-})\,S_{s^-}\left(r\,ds + \sigma\,dW_s^\gamma + \int_{\mathbb{R}} y\,\tilde{M}^\gamma(ds,dy)\right) \\
&+ \frac{1}{2}\int_t^T \frac{\partial^2}{\partial x^2} RV^{BS}(s,S_{s^-})\,S_{s^-}^2\sigma^2\,ds \\
&+ \int_t^T \int_{\mathbb{R}} R(s)\,\Lambda[V^{BS}](s,y,S_{s^-})\,M^\gamma(ds,dy).
\end{aligned}$$

By grouping the terms with respect to the Lebesgue measure ds, applying the operator $\mathscr{L}$ to the integrand, and using the decomposition of the Poisson random measure $M^\gamma(ds\,dy) = \hat{M}^\gamma(ds\,dy) + \lambda(1+\gamma_J)\,F_U(dy)\,ds$, we can rewrite the expression above as follows:

$$
\begin{aligned}
R(T)V^{BS}(T,S_T) = \; & R(t)V^{BS}(t,S_t) \\
& + \int_t^T \mathscr{L}\left[R\,V^{BS}\right](s,S_{s^-})\,ds \\
& + \int_t^T \int_{\mathbb{R}} R(s)\,\Lambda[V^{BS}](s,y,S_{s^-})\,\lambda\,(1+\gamma_J(s,y)\,)\,F_U(dy)\,ds \\
& + \int_t^T \frac{\partial}{\partial x} RV^{BS}(s,S_{s^-})S_{s^-}\left(\sigma\,dW_s^\gamma + \int_{\mathbb{R}} y\,\tilde{M}^\gamma(ds,dy)\right) \\
& + \int_t^T \int_{\mathbb{R}} R(s)\,\Lambda[V^{BS}](s,y,S_{s^-})\,\tilde{M}^\gamma(ds,dy).
\end{aligned}
$$

Note that the integrand of the first integral on the right-hand side, $\mathscr{L}[R\,V^{BS}]$, corresponds to the well-known Black–Scholes PDE (that holds for the geometric Brownian motion in complete markets), and hence vanishes.

Thanks to the boundedness of the Delta and Gamma of the Black–Scholes function, and to the fact that S has finite second moment, each of the stochastic integrals appearing on the right-hand side, when taken from time 0 up to any $t \leq T$, belongs to $L^2(\Omega, \mathcal{F}_t, \mathbb{Q}^\gamma)$ and therefore defines a $\mathbb{Q}^\gamma$-martingale. It follows that their increments over the interval $[t,T]$—that is, the integrals from t to T that appear on the right-hand side of the equation above—have zero $\mathbb{Q}^\gamma$-conditional expectation with respect to $\mathcal{F}_t$.

Taking conditional expectations under $\mathbb{Q}^\gamma$, we deduce that:

$$
\begin{aligned}
\mathbb{E}^\gamma\left[R(T)\,V^{BS}(T,S_T)\mid\mathcal{F}_t\right] &= R(t)\,V^{BS}(t,S_t) \\
&\quad + \mathbb{E}^\gamma\left[\int_t^T \int_{\mathbb{R}} R(s)\,\Lambda[V^{BS}](s,y,S_{s^-})\,\lambda\,(1+\gamma(s,y))\,F_U(dy)ds \mid \mathcal{F}_t\right] \\
&\equiv R(t)\,V^{BS}(t,S_t) + \epsilon^\gamma(t). \qquad (4.7)
\end{aligned}
$$

Finally, note that the left-hand side of equation (4.7), after applying the terminal condition $V^{BS}(T,S_T) = H(S_T)$, coincides with the definition of $R(t)\,V^\gamma(t)$ given in the statement of the theorem.

To prove point (2), it suffices to verify the non-negativity of $\epsilon^\gamma(t)$ for all $t \in [0,T]$. By construction, $R(t)$ is strictly positive, and $\lambda\,(1+\gamma_J(t,y))$ is non-negative $\mathbb{P}\otimes dt$-a.s., as shown in Proposition 3.4. It is therefore enough to show that $\Lambda[V^{BS}]$ is non-negative. To do so, we expand V^{BS} via a second-order Taylor expansion in the spatial variable. Specifically, we obtain:

$$
\Lambda[V^{BS}](s,y,S_{s^-}) = \frac{1}{2}\,y^2 S_{s^-}^2\,\frac{\partial^2}{\partial x^2}V^{BS}(s,S_{s^-}) + o\left((yS_{s^-})^2\right).
$$

Since V^{BS} is convex in x, its gamma is non-negative, and therefore so is $\Lambda[V^{BS}]$.

To prove (3), we use that $H(x) \leq x$. Specifically, we have:

$$
R(t)\,V^\gamma(t) \equiv \mathbb{E}^\gamma\left[R(T)\,H(S_T)\mid\mathcal{F}_t\right] \leq \mathbb{E}^\gamma\left[R(T)\,S_T\mid\mathcal{F}_t\right] = R(t)\,S_t,
$$

where the last equality follows from the fact that the discounted price process $R\,S$ is a $\mathbb{Q}^\gamma$-martingale. Dividing both sides by $R(t)$, we obtain the upper bound.

For (4), apply the same argument, now using that $H(x) \leq H(0)$.

□

The upper bounds derived in points (3) and (4) are natural and intuitive. For instance, in the case of a European call option, the cheapest way to superhedge the claim is to hold one unit of the underlying asset, leading to a cost equal to S_t. In contrast, the lower bound given by the Black–Scholes function is less obvious and thus more interesting. Nonetheless, this bound is still reasonable: in the absence of jumps, our model reduces to the classical Black–Scholes framework, under which the price process V^γ equals V^{BS}. The presence of jumps introduces additional risk, which must be compensated by increasing the hedging cost, thereby justifying the positive adjustment to the Black–Scholes price.

Lemma 4.1. *$\forall C > 0$, $\mathbb{Q}^\gamma\left(\mathcal{E}(y\,\tilde{M}^\gamma)_T \geq C\right) \xrightarrow[\gamma_J \to \infty]{} 0$.*

Proof. By Markov's inequality, we obtain that $\mathbb{Q}^\gamma\left(\mathcal{E}(y\,\tilde{M}^\gamma)_T \geq C\right) \leq \frac{1}{C^a}\mathbb{E}^\gamma\left[\mathcal{E}(y\,\tilde{M}^\gamma)^a_T\right]$ for $a > 0$. Thus, it is enough to prove that $\mathbb{E}^\gamma[\mathcal{E}(y\,\tilde{M}^\gamma)^a_T] \longrightarrow 0$ for some a>0. To this end, define $y_a := (1+y)^a - 1$, and compute its stochastic exponential $\mathcal{E}(y_a\,\tilde{M}^\gamma)$:

$$
\begin{aligned}
\mathcal{E}(y_a\,\tilde{M}^\gamma)_T = \exp\Bigg\{&\int_0^T \int_{\mathbb{R}} a\ln(1+y)\, M^\gamma(ds\,dy) \\
&- \int_0^T \int_{\mathbb{R}} [(1+y)^a - 1 \pm ay]\ \lambda(1+\gamma_J)F_U(dy)\,ds\Bigg\} \\
= \mathcal{E}(y\,\tilde{M}^\gamma)^a_T &\exp\left\{-\int_0^T \int_{\mathbb{R}} Z(y)\,\lambda(1+\gamma_J)F_U(dy)\,ds\right\}, \qquad (4.8)
\end{aligned}
$$

where $Z(y) := (1+y)^a - 1 - a\,y \leq 0$ for $1 - a > 0$, with equality only at $y = 0$.

Since the jump size distribution is not concentrated on $\{0\}$, and since $(1+\gamma_J) > 0 \quad \mathbb{P}\otimes dt$-a.s., the integral in the exponential on the right-hand side of equation (4.8)—which equals $\lambda\,(1+\gamma_J)\mathbb{E}[Z(U_1)]\,T$—is strictly negative. Multiplying both sides of equation equation (4.8) by $\exp\{\lambda\,(1+\gamma_J)\mathbb{E}[Z(U_1)]\,T\}$ gives:

$$
\mathcal{E}(y\,\tilde{M}^\gamma)^a_T = \mathcal{E}(y_a\,\tilde{M}^\gamma)_T \cdot \exp\left(\lambda(1+\gamma_J)\,\mathbb{E}[Z(U_1)]\,T\right).
$$

Moreover, taking unconditional expectation on both sides yields: $\mathbb{E}^\gamma[\mathcal{E}(y\,\hat{M}^\gamma)^a_T] = 1 \cdot \exp\{\lambda(1+\gamma_J)\,\mathbb{E}[Z(U_1)]\,T\}$, which tends to 0 when γ_J tends to infinity.

□

The next result confirms that the no-arbitrage prices generated by the family of equivalent martingale measures associated with our shot-noise model are well spread within the interval between the Black–Scholes price and the superhedging bound. In particular, we establish that the lower end of this interval is attained as $\gamma_J \searrow -1$, and that the upper bound is attained as $\gamma_J \to \infty$.

Theorem 4.3. *Let $\mathbb{Q}^\gamma \in \Gamma \subset \mathcal{Q}$, and let $H \in \mathcal{H}^C \cup \mathcal{H}^P$. Then:*

(1) The lower bound for the viable price process V^γ is given by the Black–Scholes price, in the sense that

$$V^\gamma(t) \searrow V^{BS}(t, S_t) \quad \text{as } \gamma_J \searrow -1.$$

(2) Moreover, if $H \in \mathcal{H}^C$, then the upper bound S_t is attained in the limit as $\gamma_J \to \infty$, that is,

$$V^\gamma(t) \nearrow S_t \quad \text{as } \gamma_J \to \infty.$$

Proof. Let V_x^{BS} denote the partial derivative of V^{BS} with respect to x. By point (1) in Theorem 4.2, it is enough to prove that $\epsilon^\gamma(t) \searrow 0$ $\mathbb{P}$-a.s. as $\gamma_J \searrow -1$. To this end, we exploit the convexity of V^{BS} with respect to x, which is guaranteed by the assumption $H \in \mathcal{H}^C \cup \mathcal{H}^P$. Fix $s > 0$. By convexity, we have

$$y\, x\, V_x^{BS}(s, x) \le V^{BS}(s, (1+y)x) - V^{BS}(s, x) \le y\, x\, V_x^{BS}(s, (1+y)x),$$

for all $x > 0$ and $y > -1$. Subtracting $y\, x\, V_x^{BS}(s, x)$ from all sides yields

$$0 \le \left|\Lambda[V^{BS}](s, y, x)\right| \le x|y| \left|V_x^{BS}(s, (1+y)x) - V_x^{BS}(s, x)\right|.$$

Since V_x^{BS} is uniformly bounded by some constant $C > 0$, we get that $\left|\Lambda[V^{BS}](s, y, x)\right| \le 2\, C\, x\, |y|$. Applying this bound for $\Lambda[V^{BS}]$, we obtain:

$$0 \le \epsilon^\gamma(t) \le 2\, C\, \lambda\, (1 + \gamma_J)\, \mathbb{E}^\gamma \left[\int_t^T \int_{\mathbb{R}} R(s)\, S_{s-}\, |y|\, F_U(dy)\, ds \,\middle|\, \mathcal{F}_t \right].$$

Using Fubini's theorem, and the $\mathbb{Q}^\gamma$-martingale property of RS, it follows that

$$\begin{aligned} \epsilon^\gamma(t) &\le 2C\lambda(1 + \gamma_J)\, \mathbb{E}[|U_1|] \int_t^T \mathbb{E}^\gamma \left[R(s)\, S_{s-} \,\middle|\, \mathcal{F}_t \right] ds \\ &= 2C\lambda(1 + \gamma_J)\, \mathbb{E}[|U_1|]\, R(t)\, S_{t-}\, (T - t), \end{aligned}$$

which clearly tends to 0 a.s. as $\gamma_J \searrow -1$.

Let us now prove point (2). Assume $t = 0$ without loss of generality. By the $\mathbb{Q}^\gamma$-martingale property of the discounted price process, its representation as a product of Doléans-Dade exponentials—that is, $S(T) = R(T)^{-1} \mathcal{E}(\sigma W^\gamma)_T\, \mathcal{E}(y \tilde{M}^\gamma)_T$—and noting that $H \in \mathcal{H}^{\mathcal{C}}$ implies $H(x) = x - g(x)$, it follows that

$$\begin{aligned} V_0^\gamma \equiv \mathbb{E}^\gamma \left[R(T) H(S_T)\right] &= \mathbb{E}^\gamma \left[R(T) S_T\right] - \mathbb{E}^\gamma \left[R(T) g(S_T)\right] \\ &= R(0) S_0 - \mathbb{E}^\gamma \left[R(T) g\left(R(T)^{-1} \mathcal{E}(\sigma W^\gamma) \mathcal{E}(y\, \tilde{M}^\gamma) \right) \right] \\ &= S_0 - R(T)\, \mathbb{E}^\gamma \left[G\left(\mathcal{E}(y\, \tilde{M}^\gamma) \right) \right], \end{aligned} \tag{4.8}$$

where $G(x) = \mathbb{E}^\gamma \left[g\left(x \cdot R(T)^{-1}\mathcal{E}(\sigma W^\gamma)\right)\right]$ is a continuous and bounded function. By the dominated convergence theorem, the continuity of G, and Lemma 4.1, we obtain that $\mathbb{E}^\gamma[G(\mathcal{E}(y\,\tilde{M}^\gamma))] \xrightarrow[\gamma_J \to \infty]{} \mathbb{E}^\gamma\left[G(0)\right] = 0$. Therefore, the right-hand side of equation (4.8) converges to S_0 as $\gamma_J \to \infty$.

□

The intuition behind this result is closely linked to the interpretation of $\lambda(1+\gamma_J)$ as the jump intensity under the pricing measure $\mathbb{Q}^\gamma$—that is, the expected number of jumps per unit of time. As $\gamma_J \searrow -1$, this intensity vanishes, and under $\mathbb{Q}^\gamma$ it is as if there were no jumps. As a result, the price corresponds to the purely continuous part of the model, and the price V^γ converges to the Black–Scholes function V^{BS}.

As the jump intensity tends to infinity, the continuous diffusion component becomes negligible. Since the compensated Poisson random measure is a martingale, it jumps "very fast" around its initial value. As a result, the process S remains essentially flat, appearing not to deviate from its initial level. The upper bound limit result can also be established for $H \in \mathcal{H}^P$.

4.3 Locally Risk-Minimizing Strategies

In contrast to the previous pricing methods, the approach considered in this section has already been developed for the shot-noise model by Altman et al. [1], based on the framework of Föllmer and Schweizer [20], whose exposition we follow in presenting the method.

Working in discrete time, the aim is to construct a strategy that minimizes the local (period-by-period) squared hedging error within the class of L^2-admissible strategies. This leads to a recursive scheme based on conditional regressions. The corresponding value process defines a linear pricing rule under the minimal martingale measure.

In the section on Arbitrage-Free Pricing Rules, we introduced simple predictable processes and their stochastic integrals in equations (3.3) and (3.2), following the convention in [9], which is arguably more intuitive from a financial perspective. However, from now on, we follow the convention in [20], according to which a (simple) predictable process $(\phi_t^S)_{t=0,\dots,T}$ is defined so that ϕ_t^S is $\mathcal{F}_{t-1}$-measurable and denotes the number of risky assets held during the interval $(t-1, t]$.

Accordingly, we define a general trading strategy as a pair $\phi := (\phi^{\mathrm{B}}, \phi^S)$, where ϕ^{B} is adapted and ϕ^S is predictable. The discounted gain process is then given by the stochastic integral

$$\hat{G}(\phi)_t := \int_0^t \phi_u^S \, d\hat{S}_u = \sum_{k=1}^t \phi_k^S \, (\hat{S}_k - \hat{S}_{k-1}).$$

The (discounted) value of the portfolio, $\widehat{\Pi}_t(\phi)$, still follows the same definition as before, namely $\widehat{\Pi}_t(\phi) := \phi_t^{\mathrm{B}} + \phi_t^S \hat{S}_t$. Up to this point, we have considered self-financing strategies, for which the cost process—defined as the difference between the portfolio value and the gains process—is almost surely zero, which implied $\widehat{\Pi}_t(\phi) = \widehat{G}_t(\phi)$ $\mathbb{P}$-a.s. However, in the present approach, we do not impose this self-financing condition, so the portfolio value satisfies $\widehat{\Pi}_t(\phi) = \widehat{G}_t(\phi) + \widehat{C}_t(\phi)$, where $C_t(\phi)$ denotes the cumulative cost process.

The framework of Föllmer and Schied requires that $H \in L^2(\mathbb{P})$ and $\hat{S}_t \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ for all t, conditions which are satisfied in our setting, as discussed in previous sections. Since no longer restricted to self-financing strategies, it is defined the set of admissible generalized strategies over which we will search for one that locally minimizes the squared hedging error. This set is referred to as the class of L^2-admissible strategies.

Definition 4.2. *A generalized strategy* (ϕ^B, ϕ^S) *is said to be* L^2*-admissible if, given* $H \in \mathcal{H}$*, it satisfies* $\hat{\Pi}_T(\phi) = \hat{H}$ $\mathbb{P}$*-a.s., with* $\hat{\Pi}_t(\phi), \hat{G}_t(\phi) \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ *for all* t*.*

That is, we consider strategies that replicate the claim H at maturity and whose discounted value and gains processes have finite conditional variance.

We define the *local risk* of a generalized strategy $(\phi^{\mathrm{B}}, \phi^S)$ at time t as the conditional mean squared increment of the discounted cost process:

$$\begin{aligned} R_t^{\mathrm{loc}}(\phi^{\mathrm{B}}, \phi^S) &:= \mathbb{E}^{\mathbb{P}}\left[\left(\hat{C}_{t+1}(\phi) - \hat{C}_t(\phi)\right)^2 \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\left(\hat{\Pi}_{t+1}(\phi) - \hat{\Pi}_t(\phi) - \phi_{t+1}^S\left(\hat{S}_{t+1} - \hat{S}_t\right)\right)^2 \mid \mathcal{F}_t\right]. \end{aligned} \tag{4.9}$$

The increment of the cost process represents the amount of capital that needs to be added to the portfolio in order to compensate for the part of the change in its value that is not covered by the gains from trading during that period. When the discounted cost process is a $\mathbb{P}$-martingale, the local risk corresponds to the conditional variance of these capital injections.

The aim is to identify the unique strategy within the class of L^2-admissible strategies that minimizes the local risk at each time step. This leads to the following definition.

Definition 4.3. *An* L^2*-admissible strategy* (ϕ^{B*}, ϕ^{S*}) *is called a* locally risk-minimizing strategy *if for all* $t = 0, 1, \ldots, T-1$ *and for every* L^2*-admissible strategy* (ϕ^B, ϕ^S) *such that*

$$\hat{\Pi}_{t+1}(\phi) = \hat{\Pi}_{t+1}(\phi^*) \equiv \phi_{t+1}^{B*} + \phi_{t+1}^{S*}\hat{S}_{t+1},$$

it holds that

$$R_t^{loc}(\phi^{B*}, \phi^{S*}) \leq R_t^{loc}(\phi^B, \phi^S), \quad \mathbb{P}\text{-}a.s.$$

The condition $\hat{\Pi}_{t+1}(\phi) = \hat{\Pi}_{t+1}(\phi^*)$ might initially seem artificial, but we will now see that it is essential for identifying the locally risk-minimizing strategy via backward induction.

For t+1=T, it requires that the strategies considered finally hedge the claim H, although at some cost. And given backward induction, it requires that given the optimal strategy to hedge H has been set from t+1 onwards and the value of the porfolio is determined, the portofolios considered also hedge this value. Therefore, it is just an application of Bellman's optimality principle.

Specifically, at time $t = T-1$, the local risk takes the following form:

$$R^{\text{loc}}_{T-1}(\phi^{\text{B}}, \phi^{S}) = \mathbb{E}^{\mathbb{P}}\left[\left(\hat{\Pi}_T(\phi) - \hat{\Pi}_{T-1}(\phi) - \phi^S_T\left(\hat{S}_T - \hat{S}_{T-1}\right)\right)^2 \,\middle|\, \mathcal{F}_{T-1}\right].$$

Due to the L^2-admissibility condition, we must minimize this quantity among strategies (ϕ^B, ϕ^S) satisfying $\hat{\Pi}_T(\phi) = \hat{H} \equiv \phi^{\text{B}}_T + \phi^S_T \hat{S}_T$. It is easy to see that minimizing the local risk is equivalent to minimizing the mean squared error in a simple linear regression of $\hat{\Pi}_T(\phi)$ on the regressor $\hat{S}_T - \hat{S}_{T-1}$, with $\hat{\Pi}_{T-1}(\phi)$ as intercept and ϕ^S_T as slope. The solution to this problem is well-known: the ordinary least squares (OLS) estimators provide the values of ϕ^{S*}_T and $\hat{\Pi}_{T-1}(\phi^*)$. The component $\phi^{\text{B}*}_T$ is then determined from the L^2-admissibility condition $\hat{H} = \hat{\Pi}_T(\phi^*)$.

In the next step, since $\hat{\Pi}_{T-1}(\phi^*)$ is now known, we minimize R^{loc}_{T-2} among all L^2-admissible strategies satisfying $\hat{\Pi}_{T-1}(\phi^*) = \phi^{\text{B}}_{T-1} + \phi^S_{T-1}\hat{S}_{T-1}$, proceeding analogously.

Following this approach, we can obtain the strategy that minimizes the local risk at each time step. Denote $\Delta X_k := X_k - X_{k-1}$. The locally risk-minimizing strategy is thus constructed recursively, and the expressions for the optimal estimators (given by the ordinary least squares solution) take the explicit form:

$$\hat{\Pi}_T(\phi^*) := \hat{H},$$

$$\phi^{S*}_{t+1} := \frac{\operatorname{cov}\left(\Delta\hat{\Pi}_{t+1}(\phi^*),\, \Delta\hat{S}_{t+1} \,\middle|\, \mathcal{F}_t\right)}{V(\Delta\hat{S}_{t+1} \,|\, \mathcal{F}_t)}, \quad \phi^{B*}_{t+1} := \hat{\Pi}_{t+1}(\phi^*) - \phi^{S*}_{t+1}\hat{S}_{t+1}, \tag{4.10}$$

$$\hat{\Pi}_t(\phi^*) := \mathbb{E}^{\mathbb{P}}\left[\hat{\Pi}_{t+1}(\phi^*) \,\middle|\, \mathcal{F}_t\right] - \phi^{S*}_{t+1} \cdot \mathbb{E}^{\mathbb{P}}\left[\Delta\hat{S}_{t+1} \,\middle|\, \mathcal{F}_t\right].$$

Remark 4.2. Note that in equation (4.10), subtracting $\hat{\Pi}_t(\phi^*)$, which is an $\mathcal{F}_t$-measurable random variable, does not affect the conditional covariance.

Remark 4.3. The strategy defined in equation (4.10) is not always guaranteed to be an L^2-admissible strategy and, therefore, a locally risk-minimizing strategy. For this to hold, combining Proposition 10.10 and Remark 10.12 in [20], it is sufficient that the increments $\Delta\hat{S}_{t+1}$ are not $\mathcal{F}_t$-measurable, which is indeed the case in our model for S.

By Theorem 10.9 in [20], the strategy described in (4.10) is *mean self-financing*, meaning that its cost process is a $\mathbb{P}$-martingale, i.e., it satisfies $\mathbb{E}^{\mathbb{P}}[\hat{C}_{t+1}(\phi^*) - \hat{C}_t(\phi^*) \mid \mathcal{F}_t] = 0$, $\mathbb{P}$-a.s. for all t. This implies that the strategy indeed minimizes the conditional variance of capital contributions. Moreover, the theorem also ensures that the cost process and the discounted price process are *strongly orthogonal*, that is, the conditional covariance of their increments is null $\mathbb{P}$-a.s.

4.3.1 Minimun Martingale Measure

An equivalent martingale measure $\mathbb{Q}^{\mathrm{MM}} \in \mathcal{Q}$ is called a *minimal martingale measure* if its Radon–Nikodym derivative L^{MM} is square-integrable with respect to $\mathbb{P}$, and for every square-integrable $\mathbb{P}$-martingale M that is strongly orthogonal to $\hat{S}$, the process M is also a $\mathbb{Q}^{\mathrm{MM}}$-martingale.

Given a minimal martingale measure $\mathbb{Q}^{\mathrm{MM}}$, there exists a unique locally risk-minimizing strategy $\phi^* := (\phi^{\mathrm{B}*}, \phi^{S*})$ such that its discounted value process satisfies

$$\hat{\Pi}_t(\phi^*) = \mathbb{E}^{\mathrm{MM}}[\hat{H} \mid \mathcal{F}_t], \quad \text{for all } t = 0, \ldots, T. \tag{4.11}$$

Therefore, the value of this locally risk-minimizing strategy can be interpreted as the linear pricing rule associated with the minimal martingale measure.

Consider the Doob decomposition of the arithmetic return of the price process, which we denote by r. That is, we write

$$r_t = \Delta M_t + \mu_t,$$

where (M_t) is a $\mathbb{P}$-martingale and $\mu_t := \mathbb{E}[r_t \mid \mathcal{F}_{t-1}]$ is the conditional expectation of the return.

As shown by Dothan in [14], the density of the minimal martingale measure (in discrete time) can be computed as:

$$L_T = \prod_{t=1}^{T} \left[1 - \frac{\mu_t \, \Delta M_t}{\mathbb{E}_{\mathbb{P}}(\Delta^2 M_t \mid \mathcal{F}_{t-1})} \right].$$

The explicit values of the decomposition, and hence of the density, for the shot noise process can be found in [1].

Taking into account the Doob decomposition and the representation result in (4.11), the locally risk-minimizing strategy given in (4.10) can be rewritten as:

$$\phi_t = \frac{\mathbb{E}^{\mathbb{P}} \left[\Delta \hat{S}_t \cdot \Delta \mathbb{E}^{\mathrm{MM}}[\hat{H} \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_{t-1} \right]}{\mathbb{E}^{\mathbb{P}}[(\Delta \hat{S}_t)^2 \mid \mathcal{F}_{t-1}]}.$$

4.4 Minimum Entropy Measure

In the case of the minimal martingale measure, we obtained a linear pricing rule derived from a hedging strategy. In contrast, here we introduce a measure that does not originate from any explicit hedging strategy, but rather from minimizing a notion of distance—specifically, the *relative entropy*—with respect to the historical probability measure $\mathbb{P}$.

The main references for this section are [22], [9], and [24].

Definition 4.4. *Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F}, \mathbb{F})$. The* relative entropy *of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined as*

$$\mathbf{H}(\mathbb{Q} \mid \mathbb{P}) = \begin{cases} \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], & \text{if } \mathbb{Q} \ll \mathbb{P}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Relative entropy, or Kullback–Leibler divergence, is often used as a measure of proximity between two equivalent probability measures. In our setting, given a specific stochastic model for the asset price process $(S_t)_{t\in[0,T]}$, we define the *Minimal Entropy Measure* (MEM) as the element $\mathbb{Q}^{ME} \in \mathcal{Q}$ that minimizes the relative entropy with respect to $\mathbb{P}$, that is:

$$\mathbf{H}(\mathbb{Q}^{ME} \mid \mathbb{P}) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbf{H}(\mathbb{Q} \mid \mathbb{P}) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]. \tag{4.12}$$

The minimal entropy measure has a natural interpretation from the perspective of information theory: it is the martingale measure that introduces the least amount of information relative to the historical model, while still ensuring the no-arbitrage condition is satisfied. Therefore, $\mathbb{Q}^{ME}$ is the martingale measure closest to the natural probability measure $\mathbb{P}$.

More generally, there exists a broad literature on the selection of equivalent martingale measures via the minimization of functionals of the form

$$\mathbb{E}^{\mathbb{P}}[f\left(\tfrac{d\mathbb{Q}}{d\mathbb{P}}\right)],$$

where $f : (0, \infty) \to \mathbb{R}$ is a strictly convex function[8]. The MEM corresponds to the case $f(x) = x \ln x$. Other choices, such as $f(x) = x^2$, lead to different criteria, like minimizing the L^2-norm of the density.

Although these optimal measures provide arbitrage-free linear pricing rules, they are not generally associated with any hedging strategy and thus lack a clear financial interpretation. This contrasts with the minimal martingale measure, which does arise from a well-defined hedging approach.

However, we focus on the minimal entropy measure due to its close connection with exponential utility. As shown in the next section, the price derived from the MEM coincides with the limit case of utility indifference pricing arising when risk aversion is vanishing and preferences are exponential. In this context, the density of the MEM is proportional to the marginal utility of terminal wealth [see [22]].

The structure of the MEM density, the conditions for its existence, and the distribution of the underlying asset have been explicitly derived in the case of Lévy processes (see, e.g., [23]). Unfortunately, extending these results to a general semimartingale framework proves to be considerably more challenging from both analytical and numerical perspectives.

[8] Strict convexity ensures the existence and uniqueness of the optimal measure under suitable integrability conditions.

To overcome this difficulty, we restrict ourselves to the subset $\Gamma \subset \mathcal{Q}$—previously introduced as the class of equivalent martingale measures with constant jump risk premium γ_J. Within this family, we numerically solve the entropy minimization problem stated in equation (4.12) by optimizing over the parameter γ_J, and construct the corresponding MEM density accordingly.

More precisely, in this setting, the minimal entropy measure $\mathbb{Q}^{\text{ME}}$ is the element in Γ whose Radon–Nikodym density is given by Proposition 3.4, corresponding to the choice $\gamma_J \equiv \gamma_J^{\text{ME}}$, defined as

$$\gamma_J^{\text{ME}} = \underset{\gamma_J \in (-1,\infty)}{\arg\min} \left\{ \frac{1}{M} \sum_{m=1}^{M} L_m^{\gamma} \ln\left(L_m^{\gamma}\right) \right\},$$

where L_m^{γ} denotes the Radon–Nikodym density at time T, evaluated along the mth simulated path under $\mathbb{P}$, and M is the total number of simulated trajectories.

4.5 Rational Pricing with Exponential Utility-Based Preferences

This section is divided into two parts. In the first, we introduce the concept of indifference pricing, as originally proposed by [27]. In the second, we present a robust dual representation, following [12] and [3], which transforms the pricing problem into a selection of an equivalent martingale problem.

Unlike the superhedging approach, which treats all scenarios in Ω with positive probability equally, here we assign different weights to each outcome depending on the losses incurred, and aim to minimize the weighted average loss. This idea is formalized through the notion of expected utility. In our case, we use a utility function of the CARA type[9], specifically $U(x) = -\exp(-\alpha x)$, where $\alpha > 0$. The parameter α represents the risk aversion coefficient and increases with the agent's degree of risk aversion.

4.5.1 Indifference Pricing

The indifference price, introduced by [27], is based on the economic notion of the certainty equivalent. For an agent with initial wealth x, that is the fixed amount of cash whose utility equals the expected utility of receiving a random payoff H.

However, in our context, the agent can invest dynamically in self-financing strategies involving the underlying asset. As a result, a fixed initial amount does not yield a deterministic utility, since the agent will not simply hold it, but will invest it in a self-financing portfolio $(\phi^{\text{B}}, \phi^{S}), \phi^{S} \in \Phi$, with initial value x, receiving at maturity a

[9] CARA stands for Constant Absolute Risk Aversion

discounted terminal wealth given by $\hat{\Pi}_T(\phi) = x + \int_0^T \phi^S \, d\hat{S}$. The agent will then choose the strategy that maximizes the expected utility of their final wealth.

$$u(x\,;\alpha) := \sup_{\phi^S \in \Phi} \mathbb{E}^{\mathbb{P}} \left[-\exp\left(-\alpha \left(x + \int_0^T \phi^S \, d\hat{S} \right) \right) \right] \tag{4.13}$$

If the agent now purchases a derivative with payoff H at price π, their resulting utility is

$$u(x - \pi + \hat{H}; \alpha) := \sup_{\phi^S \in \Phi} \mathbb{E}^{\mathbb{P}} \left[-\exp\left(-\alpha \left(x - \pi + \int_0^T \phi^S \, d\hat{S} + \hat{H} \right) \right) \right] \tag{4.14}$$

Whenever there exists a unique solution $\pi^b(H) \equiv \pi^b(H; \alpha)$ to the equation

$$u(x; \alpha) = u(x - \pi^b(H) + \hat{H}; \alpha), \tag{4.15}$$

we refer to this value $\pi^b(H)$ as the *utility indifference price* (more precisely, the buying price) for the contingent claim H. This quantity is precisely what makes the agent indifferent between holding the contingent claim with a random payoff and not holding it. If, instead, the agent sells the claim, the same reasoning leads to the definition of the *indifference selling price* $\pi(H)$, characterized as the solution to the equation

$$u(x; \alpha) = u(x + \pi(H) - \hat{H}; \alpha).$$

In particular, we adopt the exponential utility function because, since x is $\mathcal{F}_0$-measurable, this term cancels out from both sides of equation (4.15), and thus the resulting price does not depend on it.

However, computing this price requires an explicit characterization of the set Φ, which is which is by no means a trivial task. For this reason, we will restrict our analysis to the indifference price associated with specific strategies seen in the previous sections. Moreover, this difficulty motivates the robust dual representation with respect to Φ proposed in [12], which we present below.

4.5.2 Entropic Penalty Pricing

First, note that our price process is locally bounded—because the geometric Brownian motion has continuous paths, and in the jump component J_t the number of jumps N_t is almost surely finite for each $t > 0$, while the jump sizes themselves are bounded almost surely.

Next, let $\mathcal{Q}_e^L$ denote the family of equivalent measures under which the discounted price process is a local martingale, and let $\mathcal{Q}_f$ denote the family of absolutely continuous measures with finite relative entropy under which the discounted price is a local martingale as well. In previous sections we showed that there are measures on $\mathcal{Q}$ with finite entropy; hence $\mathcal{Q}_e^L \cap \mathcal{Q}_f \neq \varnothing$.

If we impose the additional condition on the claim H:

$$\mathbb{E}\left[e^{(\alpha+\varepsilon)H}\right] < \infty \quad \text{and} \quad \mathbb{E}\left[e^{-\varepsilon H}\right] < \infty \quad \text{for some } \varepsilon > 0,$$

then all the hypotheses needed to apply the robust duality result of [12] are satisfied, and the result takes the form

$$\sup_{\phi^S \in \Phi} \mathbb{E}\Big[-\exp\big(-\alpha\big(\int_0^T \phi_t^S \, d\hat{S}_t - \hat{H}\big)\big)\Big] \;=\; -\exp\Big(\sup_{\mathbb{Q}\in\mathcal{Q}_f} \left\{\mathbb{E}_{\mathbb{Q}}[\alpha\hat{H}] - \mathbf{H}(\mathbb{Q} \mid \mathbb{P})\right\}\Big).$$

Applying the dual representation to the definition of the indifference price, we obtain (see (4.6) in [12]):

$$\pi(H;\alpha) = \sup_{\mathbb{Q}\in\mathcal{Q}_f} \left\{\mathbb{E}^{\mathbb{Q}}[\hat{H}] - \frac{1}{\alpha}\left(\mathbf{H}(\mathbb{Q} \mid \mathbb{P}) - \mathbf{H}(\mathbb{Q}^{ME} \mid \mathbb{P})\right)\right\}. \tag{4.16}$$

This price admits a natural interpretation: it corresponds to selecting the measure that maximizes the expected value of the claim, while penalizing deviations from the measure $\mathbb{P}$ via an entropy-based correction term. This formulation also leads to a well-known result connecting the indifference price to the superhedging price and the minimal entropy price (see [3], [12], [9]).

Specifically, as the risk aversion parameter α tends to infinity, the correction term vanishes, and the optimization problem in equation (4.16) reduces to the superhedging problem described in equation (4.6). Consequently, the indifference price converges to the superhedging price.

On the other hand, as $\alpha \to 0$, the influence of the expected payoff in the objective function becomes negligible, and the optimization favors the choice of the measure closest to $\mathbb{P}$ in the sense of relative entropy. That is, the optimizer converges to the minimal entropy martingale measure $\mathbb{Q}^{\mathrm{ME}}$. In this sense, the minimal entropy price can be viewed as the price resulting from a risk-neutral agent with exponential utility preferences.

While the theoretical results above are insightful due to their connection with alternative pricing approaches, they are of limited practical use in full generality. In practice, we only have explicit access to the set $\mathcal{Q}$, and not to the entire class of local martingale measures with finite entropy. As a result, we can only compute a lower bound for the indifference price.

5 Numerical Experiments and Simulations

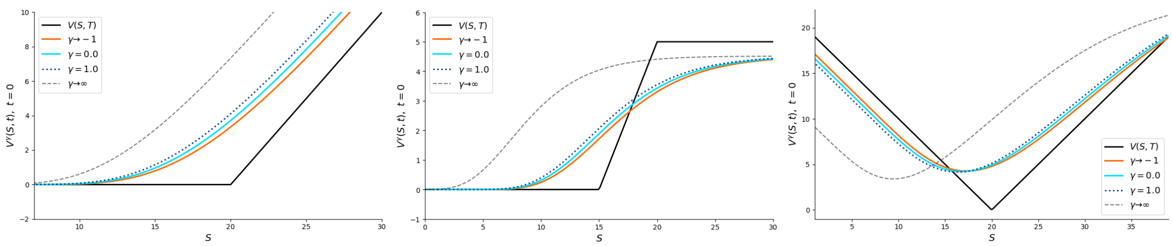
This section is divided into two parts. In the first part, we focus on the case where the risky asset price is Markovian. Relying on the result established in Remark 4.1, we compute the price of a European call option with strike $K = 20$, as well as its spatial derivatives—delta and gamma—for different values of the jump risk premium γ_J, by numerically solving a PDE. In the second part, the analysis turns to the non-Markovian setting. A single sample path of the asset price is considered, and the option price for this path is computed for various values of γ_J using Monte Carlo methods, with comparisons to the closed-form Black–Scholes solution. Indifference prices based on exponential utility preferences are also computed under different hedging strategies.

Throughout this section, the parameter configuration proposed by Altmann et al. [1] is adopted, with a slight modification to the jump-size distribution. Specifically, the jump sizes are initially assumed to be i.i.d. and uniformly distributed on the interval $[-0.25, -0.05]$, although this distribution will be modified later in the section.

Numerically, we find that the minimal entropy martingale measure is the EMM given by the Radon–Nikodym density obtained when the jump risk premium is zero, i.e., $\gamma_J \equiv 0$. Therefore, the Merton measure and the minimal entropy measure are the same.

In the Markovian setting, Figure 1 displays the numerical solution of the pricing PDE for three option structures: a European call option with strike price $K = 20$, a bull spread with lower and upper strikes $K_1 = 15$ and $K_2 = 20$, and a straddle centered at $K = 20$.

As predicted by the theoretical results, we observe that the option price increases with the jump risk premium γ_J, and that the Black–Scholes price (obtained for $\gamma_J \to -1$) serves as a lower bound. Some numerical instabilities, however, arise in the straddle case to the left of the strike.



European Call Option | Bull Spread | Straddle

Figure 1: Prices for a European call option (left), bull spread (center), and straddle (right), computed at time $t = 0$ for different values of the jump risk premium γ_J.

Moreover, we compute the delta and gamma of the call option using the PDE approach. As shown in Figure 2, the delta increases with the jump risk premium γ_J. To understand this behavior, consider the limiting case $\gamma_J \to \infty$: in this limit, as established in Theorem 4.3, the option price converges to S_t, whose spatial derivative is identically one for all $S > 0$.[10] As for the gamma, we observe that the entire curve shifts to the left as γ_J increases. This is consistent with the fact that a higher jump risk premium causes the delta to reach the value one more rapidly, and thus the gamma—being the spatial derivative of the delta—drops to zero sooner.

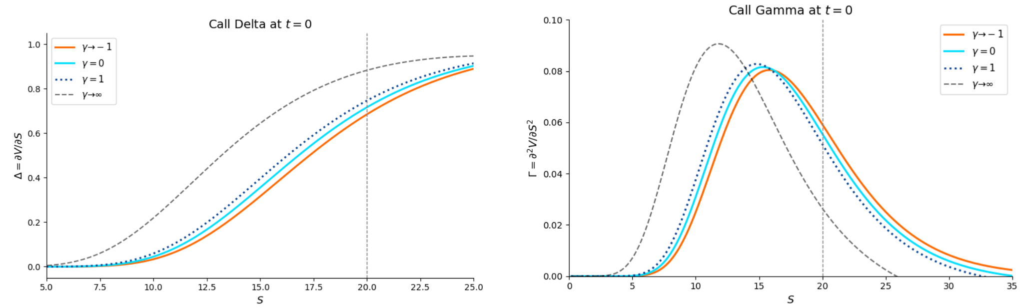


Figure 2: Call-option delta (left) and gamma (right) at $t = 0$ for different values of the jump risk premium γ_J.

We now turn to the non-Markovian setting. The shot-noise model is capable of generating arbitrarily heavy tails and pronounced skewness in the distribution of the log-price under $\mathbb{Q}^\gamma$. In particular, Figure 3 displays the log-return density under $\mathbb{Q}^\gamma$ for two representative values of the jump risk premium: $\gamma_J \to -1$, and $\gamma_J \to \infty$, using the same jump-size distribution as in the previous experiments. As γ_J increases, the left tail of the distribution becomes significantly heavier, illustrating how the jump risk premium affects both the kurtosis and the skewness of returns under the pricing measure.

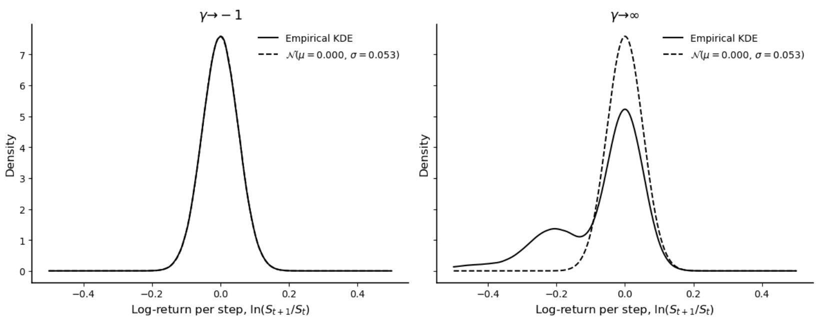


Figure 3: Density of the shot-noise model log-returns under $\mathbb{Q}^\gamma$ for $\gamma_J \to -1$ (left) and $\gamma_J \to \infty$ (right).

Since the theoretical framework developed in this work allows for jump sizes that are unbounded almost surely, we now consider a distribution consistent with this feature. Specifically, we assume that the jump sizes are i.i.d. and follow a lognormal distribution shifted one unit to the left, that is,

$$1 + U_i \sim \log\mathcal{N}\left(\mu_U, \sigma_U^2\right),$$

where we set $\mu_U \equiv -\frac{\sigma_U^2}{2}$ to ensure that $\mathbb{E}[U_1] = 0$. With this specification, the jump sizes lie in the interval $(-1, \infty)$, $\mathbb{P}$-almost surely.

[10] In the plots, the case $\gamma_J \to \infty$ is approximated by a large but finite value of γ_J that avoids numerical instabilities. As a result, some deviations from the exact theoretical limit are observed.

Figure 4 illustrates one simulated trajectory of the shot-noise process (left), together with the corresponding jump component (right). It is worth noting that, unlike a Poisson process, the effect of each jump gradually fades over time due to the decay function d, instead of remaining flat.

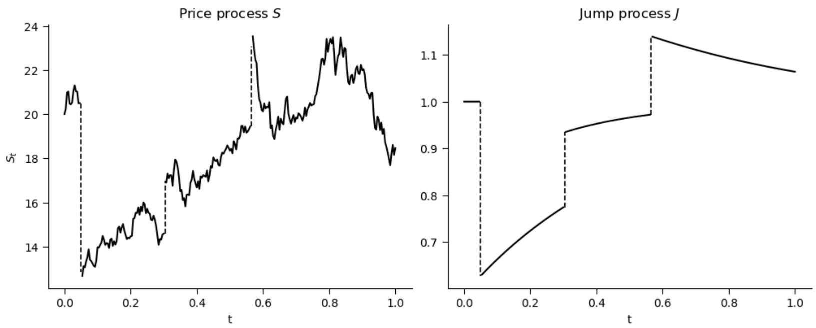


Figure 4: Sample path of the shot-noise price process under the natural probability measure (left), and its corresponding jump component (right).

Figure 5 shows the price of a European call option with strike $K = 20$ for the trajectory displayed above, computed via Monte Carlo simulation under $\mathbb{Q}^{\gamma}$ for various values of the jump risk premium γ_J. Following the first jump, the underlying price recovers rapidly—partly due to the effect of the decay function d—whereas the option price remains nearly flat for a longer period. This contrast clearly illustrates the result stated in Remark 3.3: for fixed γ_J, the underlying price process under $\mathbb{Q}^{\gamma}$ does not depend on the decay function d, and hence, the option price—computed through an expectation under $\mathbb{Q}^{\gamma}$—does not depend on it either.

The right-hand panel of Figure 5 shows the difference between the option price computed under $\mathbb{Q}^{\gamma}$ for various values of the jump risk premium γ_J, and the closed-form Black–Scholes price. Consistent with the theoretical results, the Black–Scholes value is obtained in the limit as $\gamma_J \to -1$, and as γ_J increases, the option price rises, making the difference positive and increasingly larger. All price curves eventually coincide at maturity, as expected from the fact that arbitrage-free pricing rules must match the payoff of the contingent claim at the terminal date.

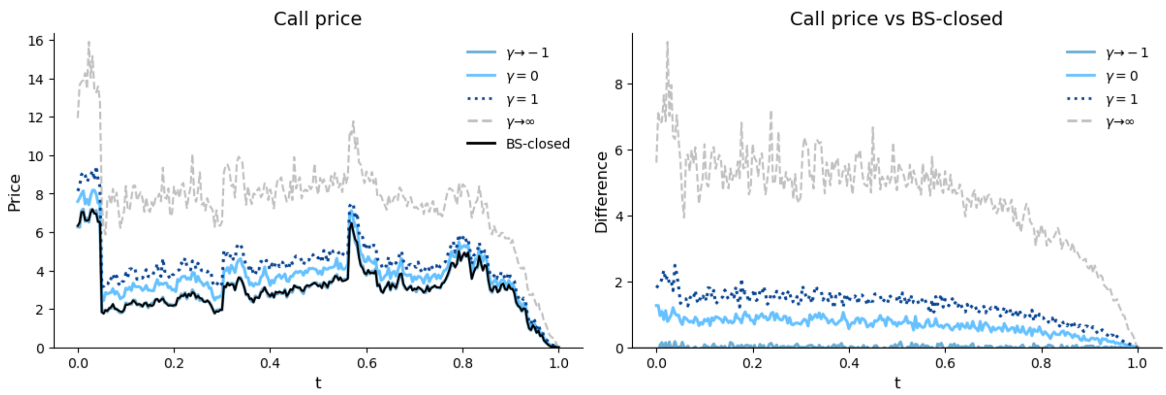


Figure 5: Left: evolution of the European call price along the sample path in Figure 4, computed for different values of the jump risk premium γ_J. Right: difference between these prices and the Black–Scholes closed-form solution.

Finally, Table 1 reports the buyer and seller indifference prices at time $t = 0$ for the same call option, computed under various hedging strategies, since explicitly characterizing the full admissible set Φ and taking the supremum was not practically attainable, as previously discussed. The risk aversion parameter α is fixed at 6.5% throughout the computations.

Specifically, we consider two types of strategies. The first consists of self-financing portfolios that hedge only the variation in the option price caused by the continuous part of the underlying's diffusion, using sensitivities computed under different values of the jump risk premium. That is, we evaluate the indifference price associated with the strategy

$$\phi^{S,\gamma} = \frac{\partial V^\gamma}{\partial x},$$

for $\gamma_J = 0$, $\gamma_J = 1$, and for the limiting cases $\gamma_J \to -1$ and $\gamma_J \to \infty$.

Secondly, we consider the locally risk-minimizing strategy associated with the minimal martingale measure (MMM), as discussed in Section 4.3. In this setting, to ensure that the Radon–Nikodym density of the MMM remains non-negative and does not lead to arbitrage opportunities, the jump sizes must be restricted to have a constant sign (see Proposition 3.5 in [1]). For this reason, we return to the case in which jump sizes follow a uniform distribution, as assumed at the beginning of this section.

Strategy	Buying Price	Selling Price	Mean Hedging Error
$\gamma_J \to -1$	4.05	7.06	1.32
$\gamma_J = 0$	4.02	7.02	0.83
$\gamma_J = 1$	3.99	6.98	0.33
$\gamma_J \to \infty$	3.85	6.77	-3.02
MMM	4.52	6.95	0.00

Table 1: Buying and selling indifference prices and mean hedging error under different hedging strategies.

We observe that, as the jump risk premium γ_J increases, the indifference prices decrease. The intuition behind this result appears to be that a higher value of γ_J implies that the chosen strategy hedges a greater portion of the option's risk. Consequently, the indifference price for a risk-averse agent is lower, as the associated strategy carries less risk. A similar effect seems to occur in the case of the minimal martingale measure, since the corresponding strategy replicates the option's terminal payoff almost surely.

It is also worth highlighting the mean hedging error at maturity, which, as previously defined, is given by $H(S_T) - \Pi_T^\gamma$. In particular, we observe that as the jump risk premium increases, the error decreases in magnitude, yet remains positive—indicating that the strategy is underhedging. In the limiting case $\gamma_J \to \infty$, the error becomes negative, meaning that the strategy is super-hedging the claim. For the minimal martingale measure, the hedging error is identically zero by construction.

6 Conclusion

This master's thesis has studied contingent claim valuation and hedging in incomplete markets driven by shot-noise jump-diffusion processes. These models capture more realistically the dynamics of financial asset prices, where announcements or unexpected events may induce abrupt jumps whose effects gradually vanish over time, instead of having a permanent impact. A key feature of this framework is that it does not yield a unique equivalent martingale measure (EMM). As a result, the market is incomplete: in general, contingent claims cannot be perfectly replicated using self-financing strategies composed solely of cash and the underlying asset, and arbitrage-free prices are no longer unique.

Within this setting, we have characterized the full set of equivalent martingale measures and developed several arbitrage-free pricing rules associated with particular choices within this family—most notably by assuming a constant jump risk premium.

In particular, we have derived an explicit expression for the Radon–Nikodym density defining Merton's measure, corresponding to the case where the jump risk premium vanishes, and we have found numerically that this measure coincides with the minimal entropy martingale measure. Furthermore, we have shown that the indifference price based on exponential utility preferences converges to this value in the limit as the agent's risk aversion tends to zero.

Additionally, we have established non-trivial bounds for European-style contingent claims, with the Black–Scholes price emerging as the lower bound. We have also computed indifference prices under exponential utility preferences for fixed hedging strategies. Furthermore, we have shown that when the jump risk premium is constant, the resulting price under the corresponding EMM loses the vanishing effect, as it no longer depends on the decay function.

A possible extension of this work would be to explicitly characterize the sets of absolutely continuous measures and equivalent measures under which the price process is a local martingale. Such a characterization would allow for the application of the robust duality approach in [12], enabling the computation of true utility indifference prices—rather than lower bounds or approximations based on fixed strategies, as done in this thesis.

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