Dynamics of a map with a power-law tail: Description of an order-to-chaos transition[1]

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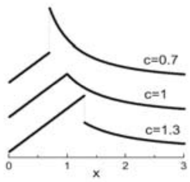
Introduction

We analyze a one-dimensional piecewise continuous discrete model proposed originally in studies on population ecology [2]. The map is composed of a linear part and a power-law decreasing piece, and has three parameters. The system presents both regular and chaotic behavior. Particularly interesting is the description of the abrupt transition order-to-chaos mediated by an attractor made of an infinite number of limit cycles with only a finite number of different periods. It is shown that the power-law piece in the map is at the origin of this type of bifurcation.

The VGH map

The VGH (Varley-Gradwell-Hassell) map has the following form

$$x_{n+1} = \begin{cases} r x_n, & x_n \le c \\ r x_n^{1-b}, & x_n > c \end{cases}$$



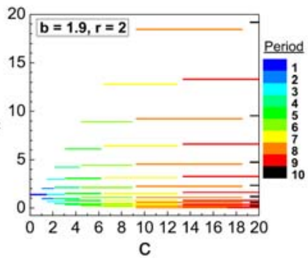
This map accepts an exact linearization, which is useful for some purposes, in terms of the following parameters

$$z \equiv 2\log(x)/\log(r) \quad , \quad \xi \equiv 2\log(c)/\log(r)$$

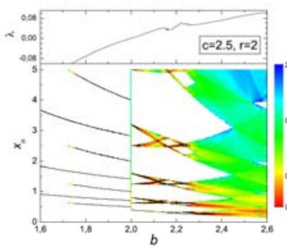
$$z_{n+1} = \begin{cases} z_n + 2, & z_n \le \xi \\ (1-b) z_n + 2, & z_n > \xi \end{cases}$$

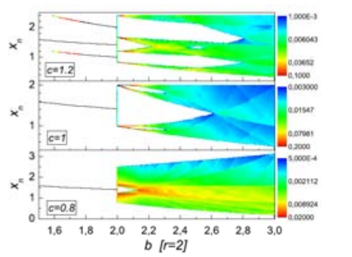
Dynamics of the map

For fixed b<2, the attractor consists of a finite number of coexisting limit cycles.



The chaotic regime corresponds to b>2, independently of the values of c and r, without stable regularity windows embedded.





For **b=2** a **sudden order-to-chaos transition** occurs. All initial conditions are fixed, periodic or eventually periodic. Thus, the attractor in b=2 consists of an **infinity of coexisting limit cycles**. Their periods are allowed to take only a restricted set of values, which depend on x_0, c and r. For a detailed study see [1].

Lyapunov exponent

The lyapunov exponent of the VGH map in its two forms can be written as follows:

$$\lambda_x = \ln r + \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=0}^{n-1}\left[\ln|1-b| - b\ln|x_k|\right]\theta(x_k - c)\right\}$$

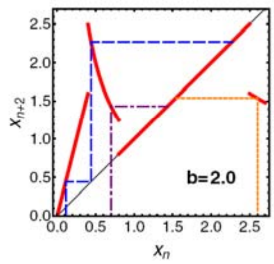
$$\lambda_z = \ln|1-b| \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=0}^{n-1}\theta(z_k - \xi)\right\}$$

The mathematical equivalence of these two expressions, $\lambda=\lambda_x=\lambda_z$, leads to the following result for the statistics of points $x_k>c$ in the attractor.

$$\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=0}^{n-1}\ln(x_k)\,\theta(x_k - c)\right\} = \frac{\ln r}{b}$$

It is clear to see from these expressions that $\lambda=0$ if b=2, independently of the rest of the parameters.

Order-to-chaos transition: an explanation



In the cobweb plot of $f^{[2]}$ we can see a piece which is **colinear with the bisectrix**. This piece conveys the existence of a continuous interval of fixed points, x ∈ [0.8 , 2.5] approximately.

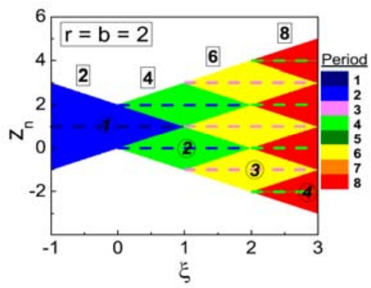
Whenever an infinite set of trajectories of period n>2 is observed, with b=2, it is because $f^{[n]}$ has one or more segments co-linear with the bisectrix.

This phenomenon may take place whenever the piecewise continuous map f has a piece defined by a power-law function. The general expression of a term of $f^{[n]}$ can be written as

$$r^{p(b)} x^{(1-b)^m}, \quad m \le n$$

Since b>1, the conditions for this term to be co-linear with the bisectrix are: even m, b=2 and p(2)=0.

In principle, several iterates can have co-linear terms simultaneously for the same value of c. This gives rise to coexisting cycles of different periods, a fact that is illustrated in the following diagram.



We can conclude that any one-dimensional map defined as a piecewise function, with a power-law piece in its definition, is a candidate to exhibit a transition of the present type.

References

[1] V. Botella-Soler, J.A. Oteo, J. Ros : *Dynamics of a map with a power law tail.* J.Phys. A: Math. Theor. **42** (2009) 385101

[2] G.C. Varley, G.R. Gradwell, M.P. Hassell : *Insect Population Ecology: An Analytical Approach*. Oxford: Blackwell. (1973)