Finite groups with only one non-linear irreducible representation

<table>
<thead>
<tr>
<th>Journal:</th>
<th>Communications in Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript ID:</td>
<td>LAGB-2011-2469.R1</td>
</tr>
<tr>
<td>Manuscript Type:</td>
<td>Original Papers</td>
</tr>
<tr>
<td>Date Submitted by the Author:</td>
<td>n/a</td>
</tr>
<tr>
<td>Complete List of Authors:</td>
<td>Dolfi, Silvio; Universita' di Firenze, Dipartimento di Matematica U.Dini Navarro, Gabriel; Universitat de Valencia, Departament d'Algebra</td>
</tr>
<tr>
<td>Keywords:</td>
<td>Representations, Finite Groups</td>
</tr>
</tbody>
</table>
FINITE GROUPS WITH ONLY ONE NON-LINEAR
IRREDUCIBLE REPRESENTATION

by

Silvio Dolfi
Dipartimento di Matematica
Università di Firenze
50134 Firenze, Italy
E-mail: dolfi@math.unifi.it

and

Gabriel Navarro
Departament d’Àlgebra
Facultat de Matemàtiques
Universitat de València
46100 Burjassot. València SPAIN
E-mail: gabriel.navarro@uv.es

ABSTRACT. Let $\mathbb{K}$ be an algebraically closed field. We classify the finite groups having exactly one irreducible $\mathbb{K}$-representation of degree bigger than one. The case where the characteristic of $\mathbb{K}$ is zero, was done by G. Seitz in 1968.

The first author is partially supported by MURST project “Teoria dei Gruppi e Applicazioni”. The second author is partially supported by the Ministerio de Educación y Ciencia proyecto MTM2010-15296, Programa de Movilidad, and Prometeo/Generalitat Valenciana.

Key words: Representations, Finite Groups.
1. Introduction. Let $K$ be an algebraically closed field. In 1968, G. Seitz classified the finite groups having exactly one irreducible $K$-representation of degree bigger than one for fields $K$ of characteristic zero. Apparently, the case where the field $K$ has characteristic $p$ has not been achieved until now.

Since $O_p(G)$, the largest normal $p$-subgroup of $G$, is contained in the kernel of every irreducible $K$-representation, in order to obtain our classification, it is necessary to assume that $O_p(G) = 1$. The following is the main result of this note.

THEOREM A. Suppose that $K$ is an algebraically closed field of characteristic a prime $p$. Let $G$ be a finite group with $O_p(G) = 1$. Then $G$ has exactly one irreducible $K$-representation of degree bigger than one up to equivalence if and only if $G$ is solvable and one of the following situations occur:

(I) $G$ is a doubly transitive Frobenius group, with Frobenius complement $H$ such that $H'$ is a $p$-group;

(II) $p \neq 2$ and $G$ is an extraspecial 2-group;

(III) $p = 2$ and $G = A\Gamma(q^2)$ is the affine semi-linear group, with $q$ a Mersenne prime;

(IV) $p = 2$ and $G$ is isomorphic to one of the two groups $G_1$ or $G_2$ described below.

These groups $G_1$ and $G_2$ in case (IV) of Theorem A, have orders 1200 and 12960, respectively, and were discovered by B. Huppert in his classification [H1] of the solvable doubly-transitive groups. In the GAP library of primitive groups, these groups can be accessed via $G_1 = \text{PrimitiveGroup}(5^2, 18)$ and $G_2 = \text{PrimitiveGroup}(3^4, 71)$. (See [GAP].)

Recall that a Frobenius group with kernel $K$ and complement $H$ is called doubly transitive if $H$ acts transitively on $K \setminus \{1\}$. The doubly transitive Frobenius groups were classified by H. Zassenhaus. (See Chapter XII.9 of [HB2].)

We finally mention that our proof of Theorem A does not use the Classification of Finite Simple Groups.

2. Proofs. Instead of working with representations, we find more convenient to work with Brauer characters. So we choose a maximal ideal $M$ of the ring $R$ of the algebraic integers, with $M$ containing $pR$, and we set $F = R/M$. We know that $F$ is the algebraic closure of $\mathbb{Z}_p$, and we have consistently defined a set of irreducible Brauer characters $\text{IBr}(G)$ for every finite group $G$. In general, we use the notation of [I] and [N]. In particular, $\text{IBr}(G/G')$ is the set of linear Brauer characters of $G$.

If $K$ is an algebraically closed field of characteristic $p$, then $K$ contains $F$, and since the $F$-representations of $G$ are absolutely irreducible, in order to prove Theorem A, one needs to study $F$-representations only. The following, therefore, includes Theorem A of the introduction.

Recall that we denote by $A\Gamma(q^n)$ the group of all affine semi-linear transformations of the field $GF(q^n)$, i.e. all the mappings $x \mapsto ax^\sigma + b$, where $a, b \in GF(q^n)$, $a \neq 0$ and $\sigma \in \text{Gal}(GF(q^n)/GF(q))$. 

(2.1) **THEOREM.** Let $G$ be a finite group, let $p$ be a prime, and suppose that $O_p(G) = 1$. Then $\text{IBr}(G) = \text{IBr}(G/G') \cup \{\delta\}$, where $\delta \in \text{IBr}(G)$ has degree $\delta(1) > 1$ if and only if $G$ is solvable, and one of the situations (I), (II), (III) or (IV) in Theorem A occur.

**Proof.** Suppose first that $\text{IBr}(G) = \text{IBr}(G/G') \cup \{\delta\}$, where $\delta \in \text{IBr}(G)$ has degree $\delta(1) > 1$. Let $K = G'$ and let $1 \neq \mu \in \text{IBr}(K)$ (note that such a character $\mu$ does exist, as otherwise $K$ would be a $p$-group and, as $O_p(G) = 1$, $G$ would be abelian, against the assumption about $\text{IBr}(G)$). Then $\mu$ lies under some irreducible Brauer character of $G$ which should be $\delta$. By Clifford Theorem [N, (8.7)], we conclude that $\text{IBr}(K) = 1 \cup \{\mu^x \mid x \in G\}$.

Let $T = I_G(\mu)$. Since $G/K$ is abelian, we have that $T = I_G(\mu^x)$ for all $x \in G$. Let $A = G/K$ and $B = T/K$. We have that $A$ acts on $\Gamma = \text{IBr}(K)$ and $\Delta = \text{cl}(K^0)$, the set of $p$-regular classes of $K$. By Brauer’s Lemma on permutation actions, we have that if $a \in A$, then $a$ fixes the same number of elements in $\Gamma$ as in $\Delta$. In fact, if $a \in A - B$, then $a$ fixes exactly one element of $\Gamma$, and if $a \in B$, then $a$ fixes every element of $\Gamma$. Suppose that $C \leq A$. Then we have that either $C$ fixes one element of $\Gamma$ or all the elements of $\Gamma$, according on whether $C$ is not contained in $B$ or is contained in $B$, respectively. Suppose that $C$ is not contained in $B$. Then there is $c \in C$ which is not in $B$. Then $c$ fixes only one Brauer character, so $c$ fixes only one $p$-regular class (the class of the identity). Hence the number of $C$-fixed points in $\Delta$ is only one. If $C$ is contained in $B$, then every Brauer character is fixed by $c$, and therefore every $p$-regular class is fixed by $C$ (because the number of fixed points is the same in both cases). Therefore $C$ fixes all $p$-regular classes.

We then deduce that for every subgroup $C$ of $A$, $C$ fixes the same number of points in $\Gamma$ as in $\Delta$. So, Theorem 13.23 of [I] implies that the two actions are permutation isomorphic. In particular, we have that all the $p$-regular classes of $K$ different from $\{1\}$, are $A$-conjugate. Hence, it follows that only one more prime $q \neq p$ can divide $|K|$. Thus, $|K|$ is only divisible by at most two primes, and therefore $K$ is solvable. So, we conclude that $G$ is solvable.

Write now $N = O_p(K)$ and let $N/L$ be a chief factor of $G$ (clearly, $N \neq 1$ as $K \neq 1$ and $O_p(G) = 1$). Hence, $N/L$ is not a $p$-group. If $1 \neq \nu \in \text{IBr}(N/L)$, then we have that $\nu$ necessarily lies under some $G$-conjugate of $\mu$. Hence, $L$ is in the kernel of all Brauer characters of $K$, and is contained in $O_p(K) = 1$. We deduce that $N$ is an elementary abelian $q$-group, for a prime $q \neq p$. Since $O_p(G) = 1$, it also follows that $C_K(N) = N$.

Now, if $M$ is a normal $p'$-group of $G$ and $N \cap M = 1$, then $M = 1$. This is because all non-trivial irreducible characters of $N \times M$ lie under $\delta$, so they are $G$-conjugate. Therefore, $N$ is the only minimal normal subgroup of $G$. So, $O_p(G) = 1$, if $r \neq q$.

Let $R$ be a Sylow $r$-subgroup of $C_G(N)$, with $r \neq q$. Then since $C_G(N)/N$ is abelian, it follows that $R \times N \triangleleft C_G(N)$. Thus $R$ is normal in $C_G(N)$ and hence $R \subseteq O_p(G) = 1$. We conclude that $Q = C_G(N)$ is a normal $q$-subgroup of $G$. As $K/N$ acts coprimely on $Q$ and trivially on $Q/N$, it follows that $Q = \text{NC}_Q(K/N) = \text{NC}_Q(K)$. Since $C_Q(K)$ is normal in $G$, we deduce that either $K$ centralizes $Q$, or $C_Q(K) = 1$ and hence $Q = N$.

Assume first that $Q \neq N$. Thus $K = C_K(Q) \leq C_K(N) = N$, so $K = N$. Now, as $G/N$ is abelian, $G$ has a normal Sylow $q$-subgroup $Q_0$. But $Z(Q_0)$ is a nontrivial normal subgroup of $G$ and it intersects $N$ nontrivially. By the minimality of $N$, we get that $N$ is central in $Q_0$ and hence $Q_0 = Q$. Observe now that $G/Q$ acts faithfully and coprimely on $Q$, while $G/Q$ acts trivially on $Q/N$. By Maschke’s Theorem we get that $\Phi(Q) \neq 1$, so
because \( N \) is the unique minimal normal subgroup of \( G \) and \( N < Q \). Since \( \Phi(Q) \) is normal in \( G \), it follows that \( N \leq \Phi(Q) \) and hence that \( G/Q \) acts trivially on \( Q/\Phi(Q) \). We conclude that \( G/Q \) acts trivially on \( Q \) as well, and hence \( G = Q \). Now, \( G \) acts transitively on \( \Delta = \text{cl}(N^0) \) and \( N \) is central in \( G \), so we deduce that \( |N| = 2 \) and that \( G \) is a 2-group.

By assumption, we know that \( k = |\text{Irr}(G)| = 1 + |G/N| = 1 + |G|/2 \). As every conjugacy class of \( G \) has order at most \( |G| \), we also have that \( k = |Z| + (|G| - |Z|)/2 \), where \( Z = Z(G) \). It follows that \( |Z| = 2 \) and hence \( N = G' = \Phi(G) = Z(G) \). Thus, we have case (II): \( G \) is an extraspecial 2-group.

So, we can assume that \( C_G(N) = N \). Since \( N \) is the unique minimal normal subgroup of \( G \), then \( F(G) = O_2(G) \) and \( N \) is contained in the center of \( F(G) \). We conclude that \( N = F(G) \) and that \( \Phi(G) = 1 \) (as otherwise \( \Phi(G) = F(G) \), which by III.4.5(d) of [H2] gives \( G = 1 \), a contradiction). Therefore, by III.4.4 of [H2], there exists a complement \( H \) of \( N \) in \( G \).

Now \( G \) acts transitively (by right multiplication) on the set \( \Omega \) of the (right) cosets of \( H \) in \( G \) and, by II.2.2 of [H2], the action of the stabilizer \( H \) on \( \Omega - \{H\} \) is isomorphic to the conjugation action of \( H \) on \( \Omega - \{1\} \). Since \( C_H(N) = 1 \), the action of \( G \) on \( \Omega \) is faithful. As a consequence of Brauer’s Permutation Lemma (see [I, (6.33)]) the number of orbits of the action of \( H \) on \( N \) and on \( \text{Irr}(N) \) is the same. By Clifford Theorem, every nonprincipal character of \( N \) lies under \( \delta \) and hence \( H \) acts transitively on \( \text{Irr}(N) - \{1\} \).

We conclude that \( H \) acts transitively on \( \Omega - \{1\} \) and hence on \( \Omega - \{H\} \). Therefore, \( G \) is a doubly transitive permutation group on \( \Omega \).

We denote by \( q^n \) the order of the regular normal subgroup \( N \). Huppert’s classification [H1] of the solvable doubly transitive permutation groups yields that (up to permutation isomorphism) either \( H \) is a subgroup of the semilinear group \( \Gamma(q^n) \) (the subgroup of semilinear transformations \( x \mapsto ax^\sigma \) of \( \text{A}_q(q^n) \)), or \( G \) is one of the following groups (for convenience, we use GAP notation): \( G = \text{PrimitiveGroup}(q^n, i) \), with

\[
(q^n, i) \in \{(3^2, 6), (3^2, 7), (5^2, 15), (5^2, 18), (5^2, 19), (7^2, 25), (7^2, 29), (11^2, 39), (11^2, 42), (23^2, 59), (3^4, 71), (3^4, 90), (3^4, 99)\}.
\]

In the following, we denote by \( C \) the centralizer \( C_H(x) \) of a nontrivial \( x \in N \). Assume that \( C \) is cyclic. Then by Brauer Permutation Lemma, \( C \leq I_H(\varphi) \) for some nonprincipal character \( \varphi \in \text{Irr}(N) \). As \( q^n - 1 = |H:C| = |H:I_H(\varphi)| \), we deduce that \( I_H(\varphi) = C \). Write \( I = I_G(\varphi) \). So \( I/N \cong I_H(\varphi) \) is cyclic and then \( \varphi \) extends to a character \( \tilde{\varphi} \in \text{Irr}(I) \) of \( N \). Hence, by [N, (8.7), (8.20)] the map \( \beta \mapsto \beta \tilde{\varphi} \) is a bijection of \( \text{Irr}(I/N) \) onto \( \text{Irr}(I|\varphi) \). So, by Clifford Correspondence [N, (8.9)] and the assumption on \( \text{Irr}(G) \), we conclude that \( |\text{Irr}(I/N)| = 1 \) and hence that \( C \cong I/N \) is a p-group. Therefore, if \( C \) is cyclic, then \( C \) is a p-group.

Consider first the case \( H \leq \Gamma(q^n) \). Then \( H \) has a cyclic normal subgroup \( H_0 \) acting Frobenius on \( V \) and such that \( H/H_0 \) is a cyclic group of order dividing \( n \). So, we see that \( C \) is a cyclic and that \( |C| = p^n \) is a divisor of \( n \). If \( C \) is normal in \( H \), then \( C = 1 \) because \( H \) acts faithfully on \( N \) and transitively on \( \Omega - \{1\} \). In this case, \( G \) is a doubly transitive Frobenius group: we have case (I). Thus we can assume \( |C| \neq 1 \neq |H'| \).

Assume that there exists a Zsigmondy prime divisor \( s \) of \( q^n - 1 \) (see [HB1, (IX.8.3)]). Then \( s \neq p \), because \( s > n \geq p^a \geq p \). Moreover, if \( s \) is a Sylow s-subgroup of \( H \), then \( S \)
is normal in $H$ because $S \leq H_0$. We also observe that $C_H(S) = H_0$; this follows because $s$ does not divide the order of the multiplicative group of any proper subfield of $GF(q^n)$.

Let $P$ be a Sylow $p$-subgroup of $H$ such that $C \leq P$. Since $H' \simeq K/N$ is a $p$-group, then $P$ is normal in $H$ and hence $P \leq C_H(S) = H_0$. It follows that $C = C \cap H_0 = 1$, against the assumption $C \neq 1$.

We can hence assume that there exists no Zsigmondy prime divisor of $q^n - 1$. Thus, either $(q, n) = (2, 6)$ or $q$ is a Mersenne prime and $n = 2$.

If $q^n = 2^6$, then $p$ divides 6 and $p \neq 2$. Thus $p = 3$, and $H$ has both a normal Sylow 3-subgroup $P$ and a normal Sylow 7-subgroup $T$; so $P \leq C_H(T)$. Considering the order of the fixed fields of the Galois automorphisms of $GF(2^6)$, we see that $[H : C_{H_0}(T)]$ divides 2, so $P \leq H_0$. As $C \leq P$, it follows that $C = 1$. So $q^n \neq 2^6$.

If $q$ is a Mersenne prime and $n = 2$, then $p = 2 = |C|$ and $|C|(q^n - 1)$ divides $|H|$, so we conclude that $H = \Gamma(q^2)$; this is case (III).

Finally, we are left with the list (*) of the ‘exceptional’ doubly transitive solvable groups. One can check that the only groups in this list having a point stabilizer $H$ with $H'$ of prime power order, are the following:

$\text{PrimitiveGroup}(q^n, i)$, with $(q^n, i) \in \{(3^2, 6), (5^2, 15), (5^2, 18), (11^2, 39), (3^4, 71)\}$.

However, $G = \text{PrimitiveGroup}(3^2, 6)$ does not satisfy the assumption on Brauer characters, for any prime $p$. In fact, writing $G = NH$ where $N$ is the regular normal subgroup of $G$ and $H$ is a point stabilizer, we see that $H'$ has order 2, while $C = C_H(x)$ has order 3, for an $x \in N$, $x \neq 1$. Since $C$ is cyclic, it should be a 2-group, as shown above.

If $G$ is either $\text{PrimitiveGroup}(5^2, 15)$ or $\text{PrimitiveGroup}(11^2, 39)$, then $G$ is a doubly transitive Frobenius group, and we are back to case (I).

Finally, the groups $G_1 = \text{PrimitiveGroup}(5^2, 18)$ and $G_2 = \text{PrimitiveGroup}(3^4, 71)$ are precisely the groups mentioned in case (IV).

Conversely, we show that all the groups in cases (I) to (IV) have exactly one nonlinear irreducible Brauer character in characteristic $p$, for the corresponding primes $p$.

First, we consider case (II). Here, $G$ is an extraspecial 2-group and, for every odd prime $p$, we have $O_p(G) = 1$ and $\text{IBr}(G) = \text{IBr}(G/G') \cup \{\delta\}$, with $\delta(1) = 2^n$, where $|G| = 2^{n+1}$, $n \geq 1$.

In all the remaining cases (I), (III) and (IV), $G$ is a doubly transitive permutation group on some set of $q^n$ elements, $q$ a prime, $n \geq 1$. Precisely, $G = NH$ where $N$ is the regular normal subgroup of $G$, $|N| = q^n$, $H$ is a complement of $N$ in $G$ and $H$ acts transitively on $N - \{1\}$ by conjugation. Moreover, we know that $H'$ is a $p$-group for a suitable prime $p$ such that $O_p(G) = 1$. (Note that any prime $p \neq q$ will do when $H$ is abelian, while in cases (III) and (IV), we have $p = 2$.) Observe also that, for any nontrivial element $x \in N$, the centralizer $C = C_H(x)$ has either order 1 (case (I)) or order 2 (cases (III) and (IV)). Let $\varphi \in \text{IBr}(N)$ be a nonprincipal character of $N$. By $[I, (6.33)]$ the action of $H$ on $\text{IBr}(N) - \{1\}$ is transitive, so $|I_G(\varphi)/N| = |I_H(\varphi)| = |H|/(q^n - 1) = |C| \leq 2$. Hence, there exists an unique character $\hat{\varphi} \in \text{IBr}(I|\varphi)$ and $\hat{\varphi}$ extends $\varphi$: this follow by Green’s Theorem $[N, (8.11)]$ in the cases (III) and (IV), when $|I/N| = 2 = p$. Hence, $\text{IBr}(G|\varphi) = \{\hat{\varphi}^G\}$, by Clifford Correspondence. Note now that every nonlinear $\delta \in \text{IBr}(G)$ lies over some nonprincipal $\varphi \in \text{IBr}(N)$, because $(G/N)' \simeq H'$ is a $p$-group. As all the nontrivial characters $\varphi \in \text{IBr}(N)$ are $G$-conjugate, by Clifford Theorem we conclude that
IBr(G) contains exactly one nonlinear character. The proof is complete.

We conclude with a couple of remarks concerning the nonzero characteristic \( p \) that might appear in Theorem A.

For the groups of types (III) and (IV), necessarily \( p = 2 \). On the other hand, for groups \( G \) of type (II) there is a unique nonlinear character in IBr(G) for any odd characteristic \( p \) (and also in characteristic 0).

Finally, let \( G \) be a group of type (I) and let \( q^n \) be the order of its Frobenius kernel. If the Frobenius complement \( H \) of \( G \) is abelian, then IBr(G) contains a unique nonlinear character for every \( p \neq q \) (and also in characteristic 0). By the Zassenhaus’ classification (see XII.9.8 in [HB2]), the commutator subgroup \( H' \) is a nontrivial \( p \)-group if and only if:

(i) \( p = 2 \) and \( H \) is as in cases (a) or (c) of [HB2, XII.9.8]; or

(ii) \( H \leq \Gamma(q^n) \) and there exists a divisor \( k \) of \( n \) such that

\[
|H'| = \frac{q^n - 1}{k(q^n/k - 1)} = p^a
\]

for some \( a \geq 1 \), and the following two conditions hold: (1) every prime divisor of \( k \) divides \( q^n/k - 1 \); and (2) if 4 divides \( k \), then 4 divides \( q^n/k - 1 \). (See [HB2, (XII.9.7)].)

It is natural to ask what primes \( p \) can show up in case (ii), or if there are strong restrictions to those. Our machine computations seem to point out that many primes do satisfy these conditions.

We finish by noticing that the groups appearing in Seitz’s classification [S] are exactly the groups of type (I) such that \( H' = 1 \) and the groups of type (II) of Theorem A.

REFERENCES


