Characters of finite groups

Carolina Vallejo Rodríguez

Universidad Carlos III de Madrid - Instituto de Ciencias Matemáticas

3rd BYMAT conference

1. Symmetries and Groups

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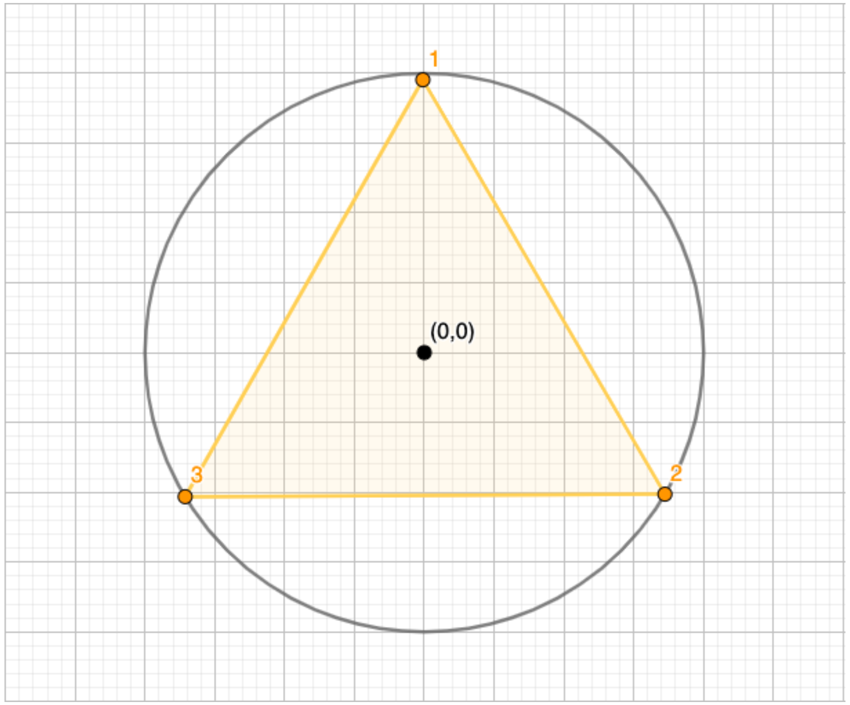
Symmetries



Group of Symmetries

The symmetries $\mathrm{Sym}(\mathrm{T})$ of the triangle $\mathrm{T} \subseteq \mathbb{R}^2$ can be described as permutations of its set of vertices $\{1, 2, 3\}$

1. Symmetries and Groups

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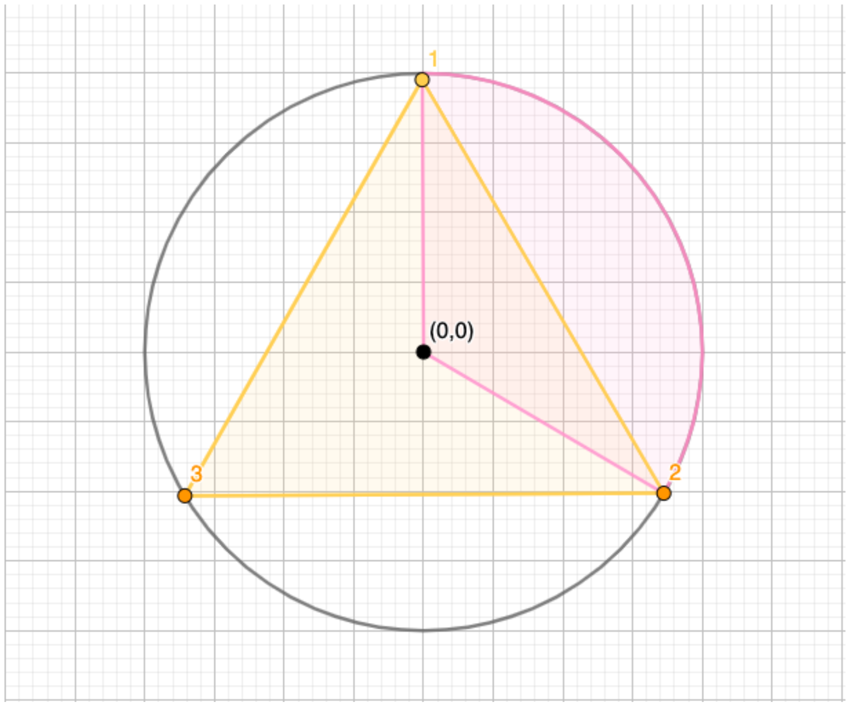

Group of Symmetries

Rotations...

$$\mathrm{Sym}(\mathrm{T}) = \{1, (1\ 2\ 3), (1\ 3\ 2), \ldots\}\,.$$

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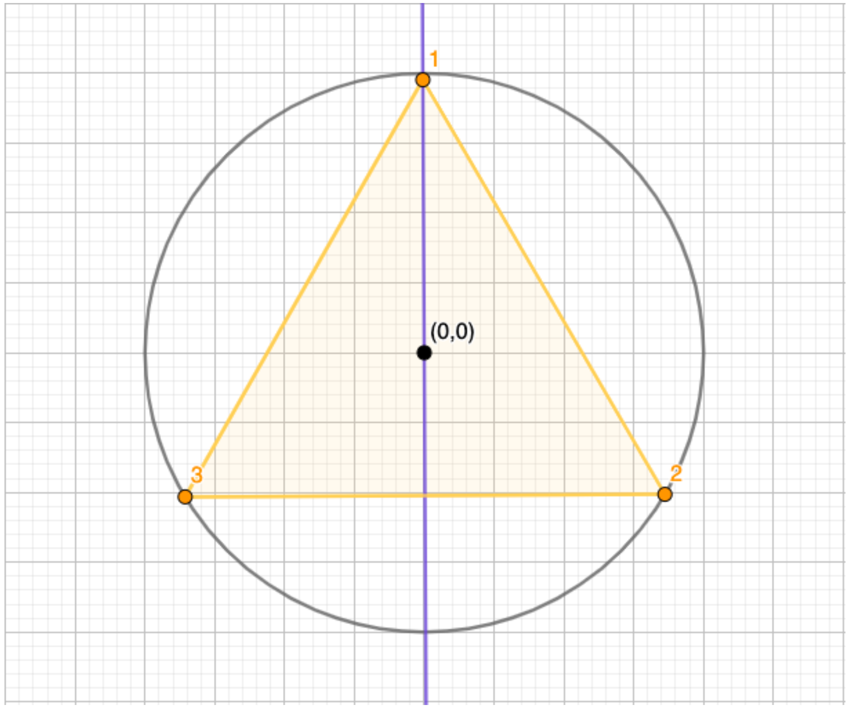

Group of Symmetries

... and reflections.

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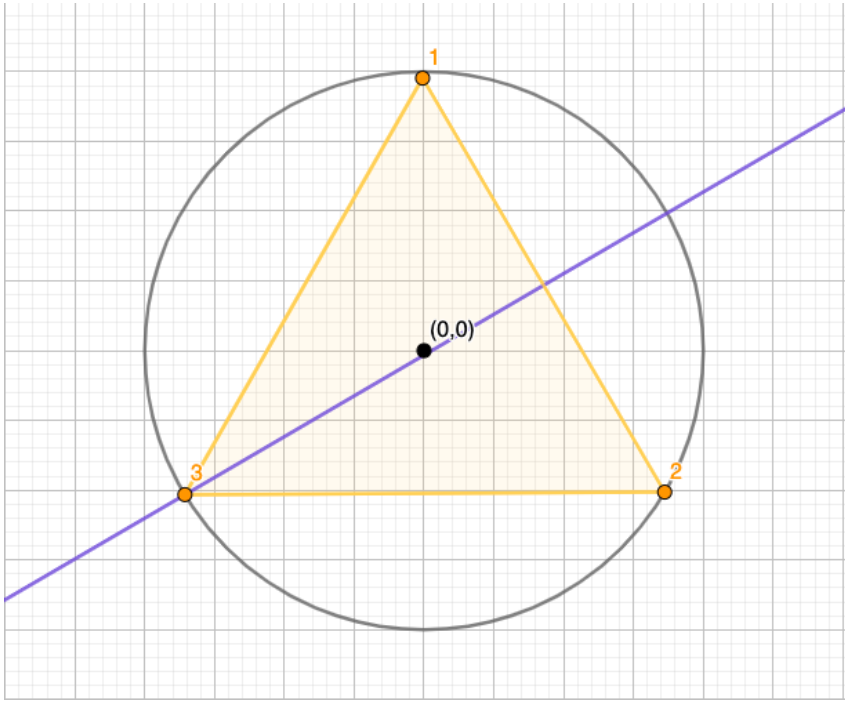

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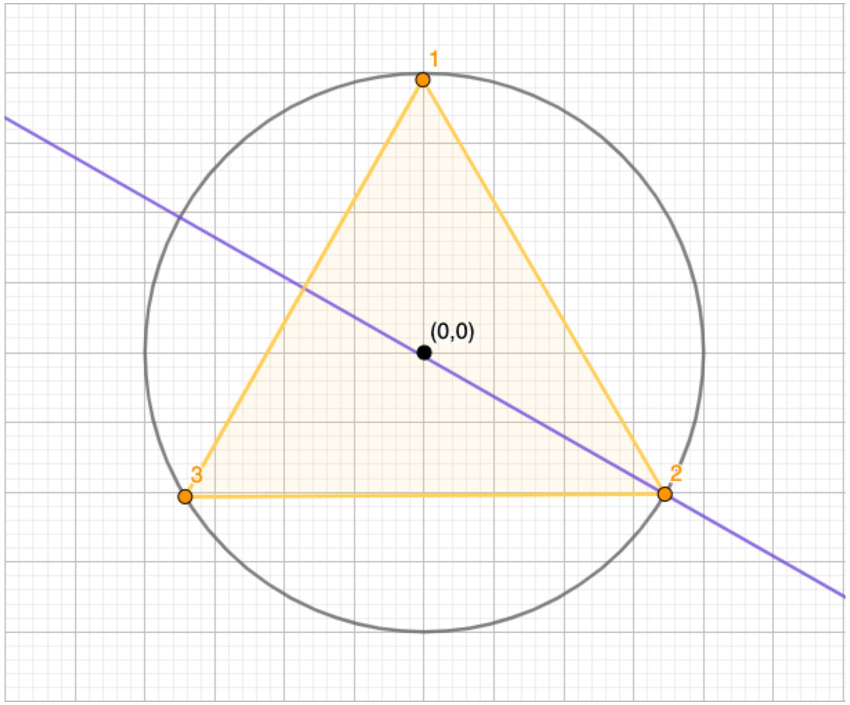

Group of Symmetries

Rotations and reflections.

$$\mathrm{Sym}(\mathrm{T}) = S_3 = \{1, (1\ 2\ 3), (1\ 3\ 2), (2\ 3), (1\ 2), (1\ 3)\}.$$

- Klein's Erlangen Program (1872).

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  "Geometry is **its** group of symmetries."

- Take M your favorite mathematical object.

  $\mathrm{Aut}(M) = \{\phi\colon M \to M \mid \phi \text{ is a bijective morphism of } M\}$ is a group.

  $\mathrm{Aut}(M)$ acts (as automorphisms) on M.

E. T. Bell (1938).

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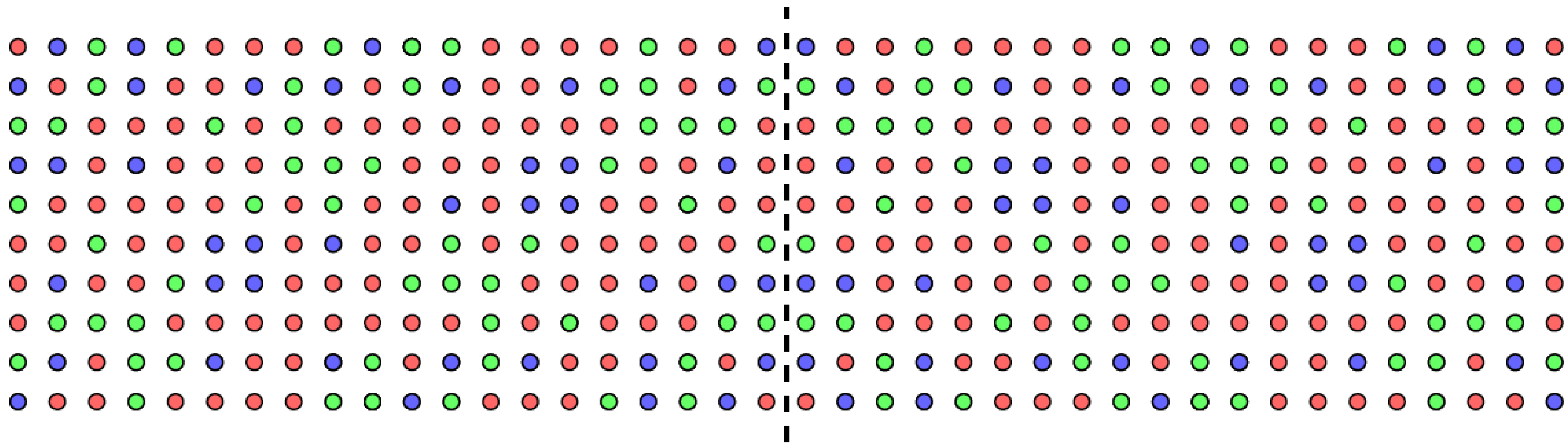

Finite groups are

- natural,
- ubiquitous in Mathematics, and
- extremely useful.

Their systematic study didn't start until the end of the XIX century.

Why did it take so long to realize the importance of the notion of group?

A possible answer: There was not an abstract definition!

E. Galois (around 1830) studied the complex roots of rational polynomials.

- Importance of the **interaction** between the symmetries (in his case permutations of the roots respecting their algebraic properties.)
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Definitions

- A subgroup N of G is normal if for every $g \in G$ then $g^{-1}Ng = N^g = N$.
- A group G is simple if it does not have proper normal subgroups.
- Jordan-Hölder theorem asserts that simple groups are the atomic constituents of finite groups.
- Solvable groups are groups in which every simple atomic group constituent is cyclic.

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Once we have an axiomatic definition, how to study abstract finite groups (very much simplified)?

- Via their actions on sets: Permutation Groups.

$$\alpha \colon G \to \mathrm{S}_\Omega \text{ homomorphism.}$$

- Via their realizations as groups of matrices (linear actions on complex vector spaces): Representation Theory.

$$\rho \colon G \to \mathrm{GL}_n(\mathbb{C}) \text{ homomorphism.}$$

2. Character Theory of Finite Groups

$$\rho\colon G \to \mathrm{GL}_n(\mathbb{C}) \text{ representation } \longleftrightarrow V = \mathbb{C}^n \text{ vector space with a linear } G\text{-action.}$$

$$\rho \text{ irreducible } \longleftrightarrow V \text{ has no proper } G\text{-invariant subspace.}$$

Irreducible representations are the building blocks of the representations.

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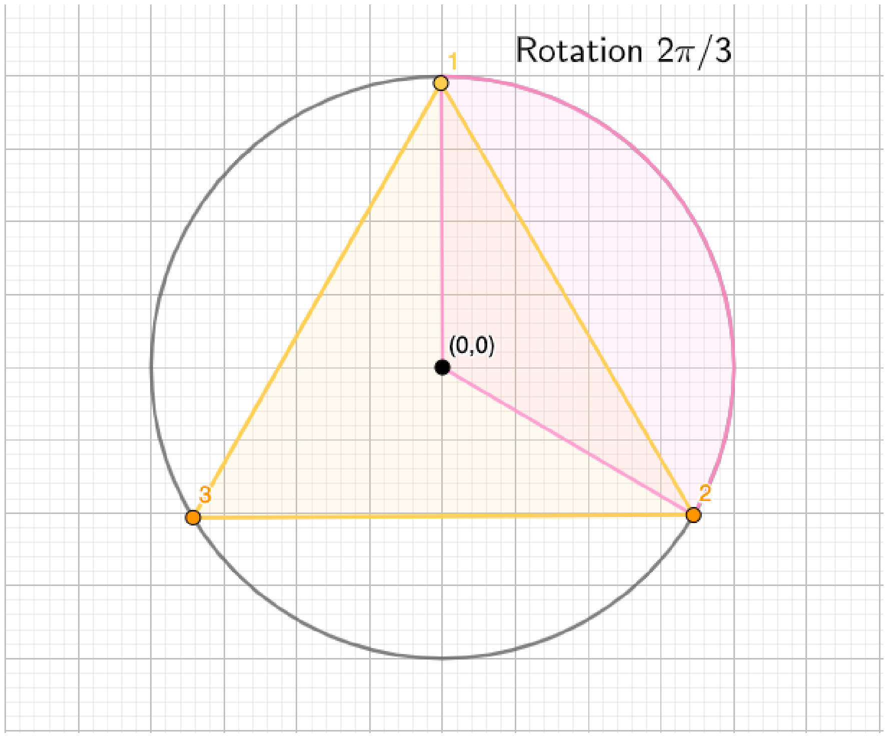
Examples

- The trivial representation of G is $1_G\colon G \to \mathbb{C}^\times$ with $1_G(g) = 1$ for every $g \in G$.
- In general, $\mathrm{Hom}(G, \mathbb{C}^\times)$ are irreducible representations.

The degree of the representation ρ is n (the dimension of the underlying space V).

A more specific example

S_3 as the group of symmetries of a regular triangle.



$$\rho\colon S_3 \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

$$(1\ 2\ 3) \longmapsto \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

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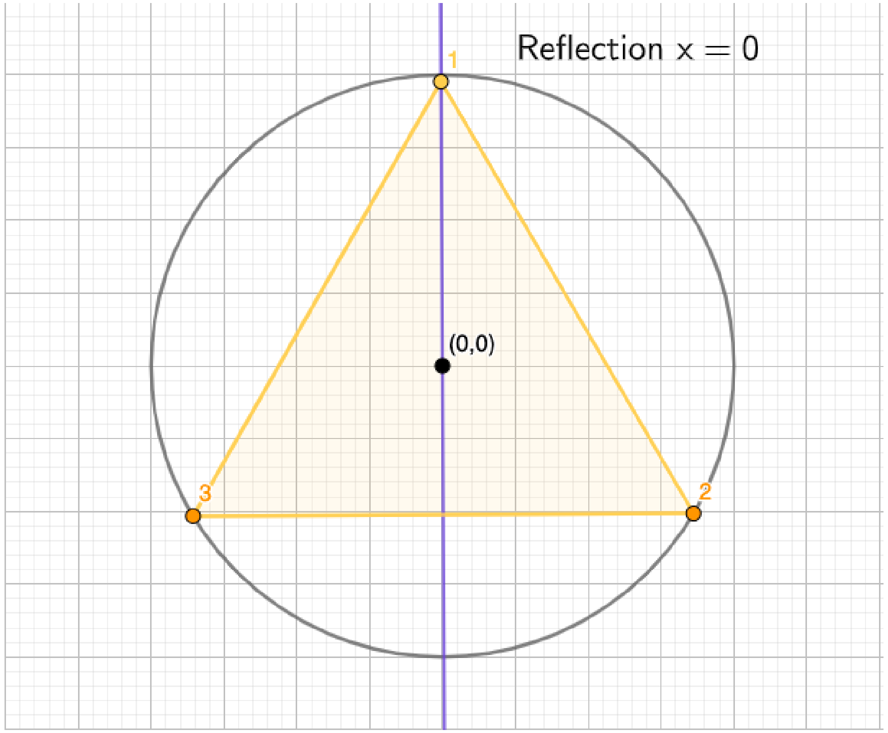

$$\rho\colon S_3 \longrightarrow \mathrm{GL}_2(\mathbb{C})$$
$$(1\ 2\ 3) \longmapsto \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$
$$(2\ 3) \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

ρ is an irreducible representation of S_3 of degree 2.

Representations can get cumbersome!

The (Fischer-Griess) Monster group $\mathbb{M}$

- $\mathbb{M}$ is a simple group.
- $\mathbb{M}$ has $2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$ elements.
- The smallest degree of a nontrivial irreducible representation is $d = 196883 = 47 \cdot 59 \cdot 71$.

We actually study the trace of representations...

Given a representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ of degree n, its trace

$$\begin{aligned} \chi\colon G &\to \mathbb{C} \\ g &\mapsto \mathrm{Tr}(\rho(g)), \end{aligned}$$

is a character of G.

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is a character of G.

- $\chi(1) = n$ is the degree of χ (of ρ).
- $\chi(g^x) = \chi(g)$ for every $g, x \in G$.
- χ is irreducible if ρ is irreducible $\Leftrightarrow \chi \neq \chi_1 + \chi_2$.

Every character can be written as a sum of the elements of $\mathrm{Irr}(G)$, the set of irreducible characters of G.

Examples

- The trivial character of G is 1_G, the trivial representation of G.
- The characters $\mathrm{Hom}(G, \mathbb{C}^\times) \subseteq \mathrm{Irr}(G)$ are all the irreducible characters of degree 1 of G. These are the only characters that are group homomorphisms.
- The irreducible character associated to the representation ρ of S_3 we built before $\chi\colon \mathsf{S}_3 \to \mathbb{C}$ is determined by $\chi(1) = 2$, $\chi((1\ 2\ 3)) = -1$ and $\chi((2\ 3)) = 0$.

Properties of (irreducible) characters

- Any representation ρ of G is determined up to *isomorphism* by its character $\chi = \mathrm{Tr} \circ \rho$.
- Characters are complex functions on the conjugacy classes of G.
- Every character can be written uniquely as a sum of $\mathrm{Irr}(G)$.
- $|\mathrm{Irr}(G)| = k$, the number of conjugacy classes of G.

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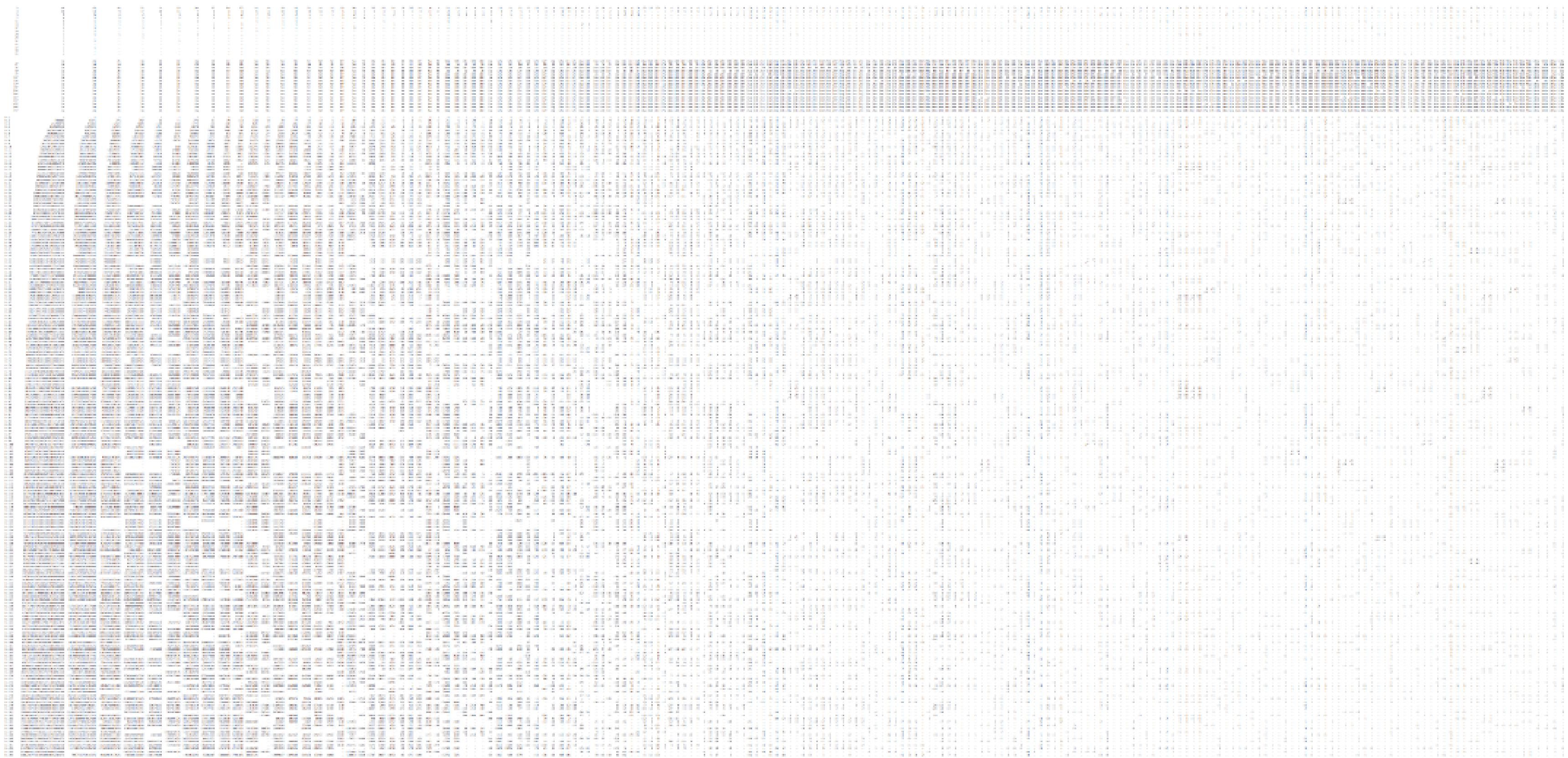
In particular, we can display all the information on the values of $\mathrm{Irr}(G)$ in a $(k \times k)$ matrix $\mathrm{X}(G)$ known as the character table of G.

The character table of S_3

Classes S_3:	1	$(1\ 2\ 3)$	$(2\ 3)$
1_{S_3}	1	1	1
sign	1	1	-1
χ	2	-1	0

$$X(S_3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}.$$

The character table of the Monster group $\mathbb{M}$



The character table of the Monster group $\mathbb{M}$

A bit closer...

	ind	1A	2A	2B	3A	3B	3C	4A	4B	4C	4D	5A	5B	6A	6B	6C	6D	6E	6F	7A	7B	8A	8B	8C	8D	8E	8F	
		@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	
		808017424794512875886459904961710757005754368000000000	8309562962452852382355161088000000	139511839126336328171520000	3765617127571985163878400	1429615077540249600	272237831663616000	8317584273309696000	26489012826931200	48704929136640	8244323942400	1365154560000000	94500000000	774741019852800	2690072985600	481579499520	130606940160	1612431360	278691840	28212710400	84707280	792723456	778567680	143769600	23592960	12582912	3096576	
		p power	A	A	A	A	A	B	A	B	B	A	A	AA	BB	AB	BA	BB	CB	A	A	C	A	B	C	C	D	
		p' part	A	A	A	A	A	A	A	A	A	A	A	AA	BB	AB	BA	BB	CB	A	A	A	A	A	A	A	A	
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	χ_1
χ_2	+	196883	4371	275	782	53	-1	275	51	19	-13	133	8	78	77	14	-3	5	-1	50	1	35	11	-1	-5	3	-1	χ_2
χ_3	+	21296876	91884	-2324	7889	-130	248	1772	-52	-20	12	626	1	273	286	-47	30	-2	-8	153	6	92	-12	0	4	-4	0	χ_3
χ_4	+	842609326	1139374	12974	55912	-221	-248	6878	782	-82	78	2451	-49	1000	923	104	-53	-13	5	426	-15	222	54	26	6	-2	-6	χ_4
χ_5	+	18538750076	8507516	123004	249458	1598	248	28796	2652	380	156	6326	76	2354	1990	370	86	46	-8	798	14	354	132	0	20	12	0	χ_5
χ_6	+	19360062527	9362495	-58305	297482	1508	-247	35903	-833	63	-65	8152	27	2378	2862	-246	-52	-12	9	1072	-6	511	-65	-25	-1	-1	7	χ_6
χ_7	+	293553734298	53981850	98970	1055310	-3927	3876	94874	1274	-102	-454	17423	-77	5070	4785	78	129	-39	35	1700	-15	714	34	-26	-14	-22	6	χ_7
χ_8	+	3879214937598	337044990	-690690	4751823	-4173	-3876	345598	-3874	-258	286	54473	98	12495	13299	-561	-141	51	-36	4249	35	1646	-154	26	-10	14	-6	χ_8
χ_9	+	36173193327999	1354188159	2864511	12616074	18954	0	701823	20383	-897	351	91124	-126	20682	18954	1674	-54	-54	0	5355	-35	1607	375	27	-25	-17	27	χ_9
χ_{10}	+	125510727015275	3215883115	1219435	24688454	-25375	248	1223531	19499	-661	-1385	145275	-350	33670	28105	70	217	-47	-8	7497	0	2379	219	-53	-5	11	-21	χ_{10}
χ_{11}	+	190292345709543	2814161895	10249191	17144568	-8073	27000	734183	22151	-281	-1209	78793	168	12024	14103	2808	279	-9	120	3451	70	1075	351	-53	15	23	-21	χ_{11}
χ_{12}	+	222879856734249	3864184921	-7194631	26057022	11310	34503	1187881	-39831	1577	1001	132374	-126	18174	24726	-1986	678	30	-57	6154	-20	1993	-583	25	25	9	-7	χ_{12}
χ_{13}	+	1044868466775133	9223504989	-15756195	47292301	84448	31123	1762307	-46019	-2723	637	170258	133	11661	28392	-2163	888	-48	147	6272	0	1869	-507	25	21	-19	-7	χ_{13}
χ_{14}	+	1109944460516150	9697078070	26155830	40851749	97733	61256	1379126	87126	4662	-1002	123025	525	43109	18381	3621	989	309	72	4200	35	966	845	26	-66	38	-6	χ_{14}
χ_{15}	+	2374124840062976	22509152496	4100096	110509112	-1456	-61504	4100096	8192	4096	0	392976	476	62776	64064	56	-1376	-16	-64	14553	0	4096	0	0	0	0	0	χ_{15}
χ_{16}	o	5980616927734375	-2720265625	39914375	1603525	200200	-116375	-9625	-20825	8295	1575	0	0	1925	0	-315	1520	-360	105	0	0	7	55	-25	-105	71	7	χ_{16}
χ_{17}	o	5980616927734375	-2720265625	39914375	1603525	200200	-116375	-9625	-20825	8295	1575	0	0	1925	0	-315	1520	-360	105	0	0	7	55	-25	-105	71	7	χ_{17}
χ_{18}	+	15178147608537368	72990279960	-29873896	255751145	33254	-30628	7620888	-44200	-2280	728	620368	-132	99177	88166	-3607	-858	38	92	17850	-84	4408	-440	0	40	-8	0	χ_{18}
χ_{19}	+	36660528852077425	128459630705	47061105	387316348	-227227	30380	9660529	62705	-1215	6097	712425	-700	112892	93093	2940	869	-219	-84	17493	0	4033	617	53	25	-95	21	χ_{19}
χ_{20}	+	50359800576579350	143552415510	71276310	371703759	-95718	120498	9004822	98294	-4074	1078	627475	1225	50895	79002	7119	1242	90	-78	13728	0	3206	638	274	14	38	-14	χ_{20}
χ_{21}	+	251098487132187500	336140827500	339595500	599239034	251810	-147250	11393900	512396	16860	4940	653125	0	146874	67210	12922	-870	-110	-50	10879	50	1660	2212	376	20	-36	56	χ_{21}
χ_{22}	+	290568421805921077	444043625365	45093685	948362611	35400	-147497	20222773	-122059	4917	-7371	1269827	-798	50675	147784	3955	-2056	256	-41	23324	0	5173	-331	-299	-75	53	21	χ_{22}
χ_{23}	+	336041615485526050	536115345090	-288233790	1087588073	-68263	61256	22648514	-153374	-2622	-4446	1397925	1050	211497	155649	-13335	2193	201	72	25704	175	4818	-1254	-26	10	50	6	χ_{23}
χ_{24}	+	2500435234254428856	1864421481144	319494840	2635262973	691236	116622	42900152	249016	-5448	-8008	2180231	-2394	245181	206052	6909	604	-156	-114	28322	0	4536	1144	-376	120	-72	-56	χ_{24}
χ_{25}	+	2985480825407204125	1612726093525	-389717475	1933299590	-57343	779495	28797725	-204867	7709	7293	1358500	1000	121132	115977	-6356	1657	-335	-281	15078	-161	2541	-715	377	-155	77	57	χ_{25}
χ_{26}	o	3503934660075044981	-89143381839	-755265515	0	0	767637	184437	161109	39285	16837	-8019	4356	0	0	0	0	0	405	153	-288	-27	-99	-351	45	-123	-63	χ_{26}
χ_{27}	o	3503934660075044981	-89143381839	-755265515	0	0	767637	184437	161109	39285	16837	-8019	4356	0	0	0	0	0	405	153	-288	-27	-99	-351	45	-123	-63	χ_{27}
χ_{28}	+	3605718753580653125	2239938073125	288185125	2906660652	189540	0	43954725	345253	-20955	4005	2131250	0	241812	189540	5076	-540	-540	0	25227	-155	3429	837	-27	5	-27	-27	χ_{28}
χ_{29}	+	8456839343580310400	3618137095040	753384320	3560377184	-1117324	-30628	46447488	903552	-13440	11648	1954150	-2100	487136	149084	10592	-4372	-28	92	18852	-160	1664	1792	0	0	128	0	χ_{29}
χ_{30}	+	8754193822112578125	3831845698125	-959677675	4307266557	-917665	27000	60123725	-883507	20045	9165	2678125	0	117117	228735	-19971	1935	135	120	26452	90	4109	-2739	53	-115	13	21	χ_{30}
χ_{31}	+	28585990950721540625	7669080680625	-804759375	6880196379	1056405	-813750	82024625	-832559	6045	-11375	3200000	0	-13221	248325	-3045	-4635	-75	330	23324	0	3521	-1495	221	25	-95	-35	χ_{31}
χ_{32}	+	30815545786259524745	3436802874505	1656343689	2715826566	2163434	809627	25226377	437801	-63607	-22295	842745	2870	-348920	66066	18312	3106	474	-357	2058	0	985	1441	-299	-175	57	21	χ_{32}
χ_{33}	+	31556817307122659605	8003575389525	2741767509	6723221931	-821106	-895869	76528981	1062517	-7083	15925	2943480	2730	126699	211926	26283	-2010	318	387	21301	280	2533	2925	25	-35	69	-7	χ_{33}
χ_{34}	+	47377503606643784400	7170718631440	6518654480	3475562246	3624405	-30628	26404368	2365584	59832	45968	702525	3150	695110	26741	38918	373	1685	92	3723	195	-644	3376	624	240	208	48	χ_{34}
χ_{35}	+	49609712911192813665	4216672209945	5047513185	731909772	6751602	116622	3899489	1538001	59017	15041	-39710	4290	437580	858	12584	7146	1506	-114	-442	-246	49	1177	677	-55	273	133	χ_{35}

The character table of $\mathbb{M}$ was computed (by Fischer, Livingston and Thorne) before its existence was proven.

Applications of Character Theory (to Group Theory)

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Burnside's $p^a q^b$-theorem (1904)

Every group of order $p^a q^b$ is solvable.

- A proof without the use of Character Theory was not found until 1972 by Bender.

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Feit-Thompon's odd order theorem (1963)

Every group of odd order is solvable.

- The paper is 255 pages long. A mixture of techniques from group theory (Fitting, Hall, Sylow...) and character theory (Burnside, Frobenius,...).

Thompson was awarded a Fields Medal in 1970 for his work on groups of odd order.

Richard Brauer, at the ICM (1970).
"The central outstanding problem in the theory of finite groups today is that of determining the simple finite groups. One may say this problem goes back to Galois. In any case Camille Jordan must have been aware of it."

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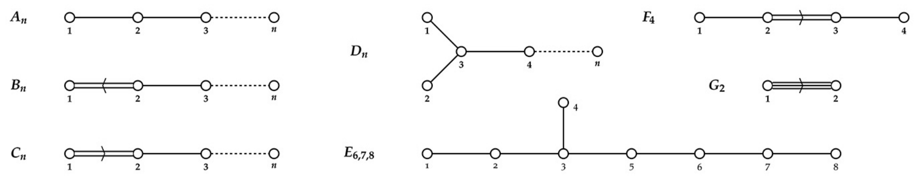
The Feit-Thompson theorem made the community believe that a complete classification of finite simple groups could be possible. And they were right...

The Classification of Finite Simple Groups (CFSG) - 2004

Dynkin Diagrams of Simple Lie Algebras



$0, C_1, \mathbb{Z}_1$ **1** 1																	C_2 2
$A_1(4), A_1(5)$ A_5 60	$A_2(2)$ $A_1(7)$ 168											$^2A_3(4)$ $B_2(3)$ 25 920	$C_3(3)$ 4 585 351 680	$D_4(2)$ 174 182 400	$^2D_4(2^2)$ 197 406 720	$G_2(2)'$ $^2A_2(9)$ 6 048	C_3 3
$A_1(9), B_2(2)'$ A_6 360	$^2G_2(3)'$ $A_1(8)$ 504											$B_2(4)$ 979 200	$C_3(5)$ 228 501 000 000 000	$D_4(3)$ 4 952 179 814 400	$^2D_4(3^2)$ 10 151 968 619 520	$^2A_2(16)$ 62 400	C_5 5
A_7 2 520	$A_1(11)$ 660	$E_6(2)$ 214 841 575 522 005 575 270 400	$E_7(2)$ 7 997 476 042 075 799 759 100 487 262 680 802 918 400	$E_8(2)$ 337 804 753 143 634 806 261 388 190 614 085 595 079 991 692 242 467 651 576 160 959 909 068 800 000	$F_4(2)$ 3 311 126 603 366 400	$G_2(3)$ 4 245 696	$^3D_4(2^3)$ 211 341 312	$^2E_6(2^2)$ 76 532 479 683 774 853 939 200	$^2B_2(2^3)$ 29 120	Tits* $^2F_4(2)'$ 17 971 200	$^2G_2(3^3)$ 10 073 444 472	$B_3(2)$ 1 451 520	$C_4(3)$ 65 784 756 654 489 600	$D_5(2)$ 23 499 295 948 800	$^2D_5(2^2)$ 25 015 379 558 400	$^2A_2(25)$ 126 000	C_7 7
$A_3(2)$ A_8 20 160	$A_1(13)$ 1 092	$E_6(3)$ 7 257 703 347 541 463 210 028 258 395 214 643 200	$E_7(3)$ 1 271 375 236 818 136 742 240 479 751 139 021 644 554 379 203 770 766 254 617 395 200	$E_8(3)$ 18 830 052 912 953 932 311 099 032 439 972 660 332 140 886 784 940 152 038 522 449 391 826 616 580 150 109 878 711 243 949 982 163 694 448 626 420 940 800 000	$F_4(3)$ 5 734 420 792 816 671 844 761 600	$G_2(4)$ 251 596 800	$^3D_4(3^3)$ 20 560 831 566 912	$^2E_6(3^2)$ 14 636 855 916 969 695 633 965 120 680 532 377 600	$^2B_2(2^5)$ 32 537 600	$^2F_4(2^3)$ 264 905 352 699 586 176 614 400	$^2G_2(3^5)$ 49 825 657 439 340 552	$B_2(5)$ 4 680 000	$C_3(7)$ 273 457 218 604 953 600	$D_4(5)$ 8 911 539 000 000 000 000	$^2D_4(4^2)$ 67 536 471 195 648 000	$^2A_3(9)$ 3 265 920	C_{11} 11
A_9 181 440	$A_1(17)$ 2 448	$E_6(4)$ 85 528 710 781 342 640 103 833 619 055 142 765 466 746 880 000	$E_7(4)$ 111 131 458 114 940 385 379 597 233 477 884 941 280 664 199 527 155 056 307 251 745 263 504 588 800 000 000	$E_8(4)$ 191 797 292 142 671 717 754 639 757 897 512 906 421 357 507 604 216 557 533 558 287 598 236 977 154 127 870 984 484 770 345 340 348 298 409 697 395 609 822 849 492 217 656 441 474 908 160 000 000 000	$F_4(4)$ 19 009 825 523 840 945 451 297 669 120 000	$G_2(5)$ 5 859 000 000	$^3D_4(4^3)$ 67 802 350 642 790 400	$^2E_6(4^2)$ 85 696 576 147 617 709 485 896 772 387 584 983 695 360 000 000	$^2B_2(2^7)$ 34 093 383 680	$^2F_4(2^5)$ 1 318 633 155 799 591 447 702 161 609 782 722 560 000	$^2G_2(3^7)$ 239 189 910 264 352 349 332 632	$B_2(7)$ 138 297 600	$C_3(9)$ 54 025 731 402 499 584 000	$D_5(3)$ 1 289 512 799 941 305 139 200	$^2D_4(5^2)$ 17 880 203 250 000 000 000	$^2A_2(64)$ 5 515 776	C_{13} 13
A_n $\frac{n!}{2}$	$PSL_{n+1}(q), L_{n+1}(q)$ $A_n(q)$ $\frac{q^{n(n+1)/2}}{(n+1,q-1)}\prod_{i=1}^{n}(q^{i+1}-1)$	$E_6(q)$ $\frac{q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)}{(3,q-1)}$	$E_7(q)$ $\frac{q^{63}}{(2,q-1)}\prod_{i=1, i\neq 2,8}^{9}(q^{2i}-1)$	$E_8(q)$ $q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	$F_4(q)$ $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	$G_2(q)$ $q^6(q^6-1)(q^2-1)$	$^3D_4(q^3)$ $q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	$^2E_6(q^2)$ $\frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{(3,q+1)}$	$^2B_2(2^{2n+1})$ $q^2(q^2+1)(q-1)$	$^2F_4(2^{2n+1})$ $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	$^2G_2(3^{2n+1})$ $q^3(q^3+1)(q-1)$	$O_{2n+1}(q), \Omega_{2n+1}(q)$ $B_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$PSp_{2n}(q)$ $C_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$O_{2n}^{+}(q)$ $D_n(q)$ $\frac{q^{n(n-1)}(q^n-1)}{(4,q^n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$O_{2n}^{-}(q)$ $^2D_n(q^2)$ $\frac{q^{n(n-1)}(q^n+1)}{(4,q^n+1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$PSU_{n+1}(q)$ $^2A_n(q^2)$ $\frac{q^{n(n+1)/2}}{(n+1,q+1)}\prod_{i=2}^{n+1}(q^i-(-1)^i)$	$\mathbb{Z}_p$ C_p p

M_{11} 7 920	M_{12} 95 040	M_{22} 443 520	M_{23} 10 200 960	M_{24} 244 823 040	$J(1), J(11)$ J_1 175 560	HJ J_2 604 800	HJM J_3 50 232 960	J_4 86 775 571 046 077 562 880	HS 44 352 000	McL 898 128 000	F_7, HHM, HTH He 4 030 387 200	Ru 145 926 144 000
Sz Suz 448 345 497 600	$O'NS, O\text{–}S$ $O'N$ 460 815 505 920	$\cdot 3$ Co_3 495 766 656 000	$\cdot 2$ Co_2 42 305 421 312 000	$\cdot 1$ Co_1 4 157 776 806 543 360 000	F_5, D HN 273 030 912 000 000	LyS Ly 51 765 179 004 000 000	F_3, E Th 90 745 943 887 872 000	$M(22)$ Fi_{22} 64 561 751 654 400	$M(23)$ Fi_{23} 4 089 470 473 293 004 800	$F_{3+}, M(24)'$ Fi_{24}' 1 255 205 709 190 661 721 292 800	F_2 B 4 154 781 481 226 426 191 177 580 544 000 000	F_1, M_1 M 808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000

- Alternating Groups
- Classical Chevalley Groups
- Chevalley Groups
- Classical Steinberg Groups
- Steinberg Groups
- Suzuki Groups
- Ree Groups and Tits Group*
- Sporadic Groups
- Cyclic Groups

Alternates† Symbol Order‡

*The Tits group $^2F_4(2)'$ is not a group of Lie type, but is the (index 2) commutator subgroup of $^2F_4(2)$. It is usually given honorary Lie type status.

The groups starting on the second row are the classical groups. The sporadic suzuki group is unrelated to the families of Suzuki groups.

†For sporadic groups and families, alternate names in the upper left are other names by which they may be known. For specific non-sporadic groups these are used to indicate isomorphims. All such isomorphisms appear on the table except the family $B_n(2^m) \cong C_n(2^m)$.

‡Finite simple groups are determined by their order with the following exceptions:
$B_n(q)$ and $C_n(q)$ for q odd, $n > 2$;
$A_8 \cong A_3(2)$ and $A_2(4)$ of order 20160.

3. Brauer's Problem 12.

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> It has been said by E. T. Bell that "wherever groups disclosed themselves or could be introduced, simplicity crystallized out of comparative chaos." This may often be true, but, strangely enough, it does not apply to group theory itself, not even when we restrict ourselves to groups of finite order. We are reminded of the educators who want to educate the world and cannot handle their own children. A tremendous effort has been made by mathematicians for more than a century to clear up the chaos in group theory. Still, we cannot answer some of the simplest questions.

This is the start of a landmark survey article by Brauer (1963) containing a long list of deep problems on Character Theory.

This list still guides our research today!

Brauer's Problem 12

How much does $\mathrm{X}(G)$ know about the Sylow subgroups of G? (And more generally about local subgroups.)

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- Normal structure of G and character tables of quotient groups (solvability, nilpotency, simplicity,...).

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Brauer's Problem 12 asks about the p-local structure of G (much harder question)

$$P \in \mathrm{Syl_p(G)},\ \mathbf{N}_{\mathrm{G}}(\mathrm{P}) = \{\mathrm{g} \in \mathrm{G} \mid \mathrm{P}^{\mathrm{g}} = \mathrm{P}\} \ \text{ and } \ \mathbf{C}_{\mathrm{G}}(\mathrm{P}) = \{\mathrm{g} \in \mathrm{G} \mid [\mathrm{g}, \mathrm{P}] = 1\}\,.$$

What does $\mathrm{X}(\mathrm{G})$ know about $P \in \mathrm{Syl_p}(\mathrm{G})$?

As Sylow theory is a cornerstone in Group Theory, Global-Local theory is today a cornerstone in Representation and Character Theory.

Property or invariant	$\mathrm{X}(G)$
$\|P\|$	✓ (elementary)
P normal in G	✓ (elementary)
$\mathbf{N}_G(P) = P$	✓ p odd [Navarro-Tiep-Turull, 2007] (using CFSG) ✓ $p = 2$ [Schaeffer Fry, 2019] (using CFSG)
$\mathbf{N}_G(P) = P\mathbf{C}_G(P)$	✓ p odd [Navarro-Tiep-V., 2019] (using CFSG) ✓ $p = 2$ [Schaeffer Fry-Taylor, 2018] (using CFSG)
$\|\mathbf{N}_G(P)\|$	We don't know!

What does $\mathrm{X}(\mathrm{G})$ know about $P \in \mathrm{Syl}_{\mathrm{p}}(\mathrm{G})$?

Property or invariant	$\mathrm{X}(G)$
P cylic (1-generated)	✓ [Kimmerle-Sandling, 1995] (using CFSG) ✓ [Navarro, 2003] (elementary proof) ✓ $p \in \{2, 3\}$ [Rizo-Schaeffer Fry-V., 2020] (using CFSG)
P abelian	✓ [Kimmerle-Sandling, 1995] (CFSG) ✓ [Kessar-Malle, 2013], [Malle-Navarro, 2020] (CFSG)
P 2-generated	✓ $p = 2$ [Navarro-Rizo-Schaeffer Fry-V., 2020] (CFSG) $p = 3$ conjecturally yes [Navarro-Rizo-Schaeffer Fry-V., 2020] $p \geqslant 5$? We don't know!!

What's the key for some much progress in the last decade?

4. Global-Local Conjectures

p prime dividing $|G|$.

Global side: $\mathrm{Irr}_{p'}(G) = \{\chi \in \mathrm{Irr}(G) \mid (\chi(1), p) = 1\}$.

Local side: $\mathrm{Irr}_{p'}(\mathbf{N}_G(P)) = \{\psi \in \mathrm{Irr}(\mathbf{N}_G(P)) \mid (\psi(1), p) = 1\}$.

Philosophy: Certain invariants on the character theory of G (global) can be computed looking at $\mathbf{N}_G(P)$ (local).

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The McKay Conjecture (1971)

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The McKay Conjecture (1971)

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))| \,.$$

If $\mathbf{N}_G(P) = P$, then McKay predicts $|\mathrm{Irr}_{p'}(G)| = k(P/P') = |P : P'|$ is a power of p. But this property does not characterize groups with a self-normalizing Sylow.

Take $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q})$. Then the group $\mathcal{G}$ acts naturally on $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$. These actions are **not** permutation isomorphic.

The following is a revolutionary conjecture in the field. ($\mathbb{Q}_p$ stands for the field of p-adic numbers below.)

Take $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q})$. Then the group $\mathcal{G}$ acts naturally on $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$. These actions are **not** permutation isomorphic.

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The McKay-Navarro Conjecture (2004)

The actions of $\mathcal{H}_p = \mathrm{Gal}(\mathbb{Q}_p(e^{2\pi i/|G|})/\mathbb{Q}_p)$ on $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ are permutation isomorphic.

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The Mckay-Navarro Conjecture is behind most of the results contained in the tables above!

It is a source of inspiration for unveiling new local properties in the character table.

Progress on these conjectures

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Isaacs-Malle-Navarro, 2007: To prove the McKay conjecture in full generality, it is enough to verify the inductive McKay statement on finite simple groups. (New perspective: Use the CFSG.)

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Navarro-Späth-V., 2020: To prove the McKay-Navarro conjecture in full generality, it is enough to verify the inductive McKay-Navarro statement on finite simple groups.

Research groups in France, Germany, USA and Spain are currently working on this statement!

Thanks for your attention!