J. Group Theory **19** (2016), 1129–1136
DOI 10.1515/jgth-2016-0011

Feit numbers and p'-degree characters

Carolina Vallejo Rodríguez

Communicated by Robert M. Guralnick

Abstract. Suppose that χ is an irreducible complex character of a finite group G and let f_χ be the smallest integer n such that the cyclotomic field $\mathbb{Q}_n$ contains the values of χ. Let p be a prime, and assume that $\chi \in \mathrm{Irr}(G)$ has degree not divisible by p. We show that if G is solvable and $\chi(1)$ is odd, then there exists $g \in \mathbf{N}_G(P)/P'$ with $o(g) = f_\chi$, where $P \in \mathrm{Syl}_p(G)$. In particular, f_χ divides $|\mathbf{N}_G(P) : P'|$.

Introduction

Suppose that G is a finite group and let $\chi \in \mathrm{Irr}(G)$ be an irreducible complex character of G. Among the different numbers that one can associate to the character χ (such as the degree $\chi(1)$, or the determinantal order $o(\chi)$ of χ), we are concerned here with the so-called *Feit number* f_χ of χ, which is the smallest possible integer n such that the field of values of χ is contained in the cyclotomic field $\mathbb{Q}_n$ (obtained by adjoining a primitive n-th root of unity to $\mathbb{Q}$). Since $\chi(g)$ is a sum of $o(g)$-roots of unity for $g \in G$, notice that f_χ is always a divisor of $|G|$.

The number f_χ is a classical invariant in character theory that has been studied by Burnside, Blichfeldt and Brauer, among others. But it was W. Feit who, following work of Blichfeldt, made an astonishing conjecture that remains open until today: If G is a finite group and $\chi \in \mathrm{Irr}(G)$, then there is $g \in G$ of order f_χ (see for instance [2]). This conjecture was proven to be true by G. Amit and D. Chillag in [1] for solvable groups.

Our aim in this note is to come back to the Amit–Chillag theorem to prove a global/local (with respect to a prime p) variation (for odd-degree characters with p'-degree).

Theorem A. *Let p be a prime and let G be a finite solvable group. Let $\chi \in \mathrm{Irr}(G)$ be of degree not divisible by p, and let $P \in \mathrm{Syl}_p(G)$. If $\chi(1)$ is odd, then there exists an element $g \in \mathbf{N}_G(P)/P'$ such that $o(g) = f_\chi$. In particular, the Feit number f_χ divides $|\mathbf{N}_G(P) : P'|$.*

This research is partially supported by the Spanish Ministerio de Educación y Ciencia MTM2013-40464-P.

It is unfortunate that we really need to assume that $\chi(1)$ is odd, as $G = \mathrm{GL}(2, 3)$ shows us: if $\chi \in \mathrm{Irr}(G)$ is non-rational of degree 2, then $f_\chi = 8$; but the normalizer of a Sylow 3-subgroup of G has exponent 6. Also, Theorem A is not true outside solvable groups, as shown by $G = \mathrm{A}_5$, $p = 2$, and any $\chi \in \mathrm{Irr}(G)$ of degree 3 (which has $f_\chi = 5$).

We shall conclude this note by recording the fact that Feit's conjecture is also true for Brauer characters, but only for solvable groups. (The group $G = \mathrm{A}_8$ has (2-)Brauer characters $\varphi \in \mathrm{IBr}(G)$ with $f_\varphi = 105$ but no elements of that order.)

1 Proofs

We begin with a well-known elementary lemma.

Lemma 1.1. *Let $N \lhd G$ be finite groups. Let $\chi \in \mathrm{Irr}(G)$ and let $P \in \mathrm{Syl}_p(G)$ for some prime p. If $\chi(1)$ is not divisible by p, then some constituent of χ_N is P-invariant and any two are $\mathbf{N}_G(P)$-conjugate.*

Proof. Let $\theta \in \mathrm{Irr}(N)$ be under χ, and let G_θ be the stabilizer of θ in G. Since $|G : G_\theta|$ is not divisible by p, we have that $P^g \subseteq G_\theta$ for some $g \in G$. Hence $\varphi = \theta^{g^{-1}}$ is a P-invariant constituent of χ_N. Let $\tau \in \mathrm{Irr}(N)$ be P-invariant under χ. Then $\tau = \varphi^x$ for some $x \in G$ by Clifford's theorem. Hence $P, P^x \subseteq G_\tau$ and there exists some $t \in G_\tau$ such that $P = P^{xt}$. Since $\varphi^{xt} = \tau^t = \tau$, the result follows. □

Let $\chi \in \mathrm{Irr}(G)$. We write $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) : g \in G)$, the field of values of χ. We will use the following well-known result about Gajendragadkar special characters. We recall that if G is a p-solvable group, then $\chi \in \mathrm{Irr}(G)$ is p-special if $\chi(1)$ is a power of p and every subnormal constituent of χ has determinantal order a power of p.

Lemma 1.2. *Let G be a finite p-solvable group and let $P \in \mathrm{Syl}_p(G)$. Then restriction yields an injection from the set of p-special characters of G into the set of characters of P. In particular, if χ is p-special, then $\mathbb{Q}(\chi) = \mathbb{Q}(\chi_P) \subseteq \mathbb{Q}_{|G|_p}$ and f_χ is a power of p.*

Proof. This statement is a particular case of [3, Proposition 6.1]. See also [3, Corollary 6.3]. □

The following result will help us to control fields of values under certain circumstances.

Lemma 1.3. *Let G be a finite group, let q be a prime and let ζ be a primitive q-th root of unity. Suppose that G is q-solvable and $\chi \in \mathrm{Irr}(G)$ is q-special. If $\chi \neq 1$, then $\zeta \in \mathbb{Q}(\chi)$.*

Proof. Let Q be a Sylow q-subgroup of G. By Lemma 1.2, we have that $\psi \mapsto \psi_Q$ is an injection from the set of q-special characters of G into the set $\mathrm{Irr}(Q)$. In particular, $\mathbb{Q}(\chi) = \mathbb{Q}(\chi_Q)$. Of course, $\chi_Q \neq 1$. Thus, we may assume that G is a q-group. We also may assume that χ is faithful by modding out by $\ker(\chi)$. Choose $x \in \mathbf{Z}(G)$ of order q. We have that $\chi_{\langle x \rangle} = \chi(1)\lambda$, where $\lambda \in \mathrm{Irr}(\langle x \rangle)$ is faithful. Hence $\lambda(x^i) = \zeta$ for some integer i. In particular, $\zeta \in \mathbb{Q}(\chi)$. □

The following observation will be used later.

Lemma 1.4. *Suppose that λ is a linear character of a finite group G, and let $P \in \mathrm{Syl}_p(G)$. Let $\mathbf{N}_G(P) \subseteq H \leq G$ and let $\nu = \lambda_H$. Then $o(\lambda) = o(\nu)$.*

Proof. If $\lambda = 1_G$, there is nothing to prove. We may assume λ is non-principal and hence $G' < G$. We have that $P \subseteq PG' \lhd G$. By the Frattini argument, we have that $G = G'\mathbf{N}_G(P) = G'H$. Since $G' \subseteq \ker(\lambda)$ and $\ker(\nu) = \ker(\lambda) \cap H$, the result follows. □

The proof of Theorem A requires the use of a *magical* character; the canonical character associated to a *character five* defined by Isaacs in [4]. We summarize the properties of ψ below.

Let $L \subseteq K \lhd G$ with $L \lhd G$ and K/L abelian. Let $\theta \in \mathrm{Irr}(K)$ and $\varphi \in \mathrm{Irr}(\theta_L)$. Suppose that θ is the unique irreducible constituent of φ^K (in this case we say that φ is *fully ramified* with respect to K/L or equivalently that θ is *fully ramified* with respect to K/L) and φ is G-invariant. Then we say that $(G, K, L, \theta, \varphi)$ is a *character five*.

Theorem 1.5. *Let $(G, K, L, \theta, \varphi)$ be a character five, and suppose that K/L is a q-group for some odd prime q. Then there exist a character ψ of G such that $K \subseteq \ker(\psi)$ and a subgroup $U \leq G$ such that*

(a) *$UK = G$ and $U \cap K = L$,*

(b) *$\psi(g) \neq 0$ for every $g \in G$, $\psi(1)^2 = |K : L|$ and the determinantal order of ψ is a power of q,*

(c) *if $K \subseteq W \leq G$, then the equation $\xi_W = \psi_W \xi_0$ for characters $\xi \in \mathrm{Irr}(W|\theta)$ and $\xi_0 \in \mathrm{Irr}(W \cap U|\varphi)$ defines a one-to-one correspondence between these two sets,*

(d) *if* $K \subseteq W \leq G$, *then* $\xi \in \mathrm{Irr}(W|\theta)$ *and* $\xi_0 \in \mathrm{Irr}(W \cap U|\varphi)$ *correspond in the sense of* (c) *if and only if* $\xi_0^G = \overline{\psi}_W \xi$, *where* $\overline{\psi}$ *denotes the complex conjugate of* ψ,

(e) *if* K/L *is elementary abelian, then* $\mathbb{Q}(\psi) \subseteq \mathbb{Q}_q$.

Proof. For parts (a)–(c) see [9, Theorem 3.1]. Part (d) follows from [4, Corollary 9.2] (since the complement U provided by [9] is "good" not only for G/L but also for every W/L where $K \subseteq W \leq G$). For part (e), by [4, Theorem 9.1] and [4, discussion at the end of p. 619], the values of the character ψ are $\mathbb{Q}$-linear combinations of products of values of the bilinear multiplicative symplectic form $\ll\cdot,\cdot\gg_\varphi : K \times K \to \mathbb{C}^\times$ associated to φ (defined in [4, beginning of Section 2]). The values of $\ll\cdot,\cdot\gg_\varphi$ are values of linear characters of cyclic subgroups of K/L. Since K/L is q-elementary abelian, we do obtain that $\mathbb{Q}(\psi) \subseteq F$. □

We can finally prove Theorem A, which we restate here.

Theorem 1.6. *Let* p *be a prime and let* G *be a finite solvable group. Let* $\chi \in \mathrm{Irr}(G)$ *be of degree not divisible by* p, *and let* $P \in \mathrm{Syl}_p(G)$. *If* $\chi(1)$ *is odd, then there exists an element* $g \in \mathbf{N}_G(P)/P'$ *such that* $o(g) = f_\chi$. *In particular, the Feit number* f_χ *divides* $|\mathbf{N}_G(P) : P'|$.

Proof. By the Amit–Chillag theorem [1], we may assume that p divides $|G|$. We proceed by induction on $|G|$.

Let $N \lhd G$. If $\theta \in \mathrm{Irr}(N)$ is P-invariant and lies under χ, then we may assume that θ is G-invariant. Let $\psi \in \mathrm{Irr}(G_\theta|\theta)$ be the Clifford correspondent of χ. By the character formula for induction, $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\psi)$ and $\chi(1) = |G : G_\theta|\psi(1)$. Thus the character ψ satisfies the hypotheses of the theorem in G_θ and f_χ divides f_ψ. If $G_\theta < G$, then by induction there exists some $g \in \mathbf{N}_{G_\theta}(P)/P' \leq \mathbf{N}_G(P)/P'$ (notice that the P-invariance of θ implies $P \subseteq G_\theta$) such that $o(g) = f_\psi$. Hence, some power of g has order f_χ and we may assume $G_\theta = G$.

We claim that we may assume that χ is primitive. Otherwise, suppose that χ is induced from $\psi \in \mathrm{Irr}(H)$ for some $H < G$. In particular, p does not divide $|G : H|$ and so H contains some Sylow p-subgroup of G, which we may assume is P. Again by the character formula for induction, the degree $\psi(1)$ is an odd p'-number and f_χ divides f_ψ. By induction there is $g \in \mathbf{N}_H(P)/P' \leq \mathbf{N}_G(P)/P'$ such that $o(g) = f_\psi$. Thus some power of g has order f_χ, as claimed.

By [5, Theorem 2.6] the primitive character χ factorizes as a product

$$\chi = \prod_q \chi_q,$$

where the χ_q are q-special characters of the group G for distinct primes q.

Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}(\chi))$. Then

$$\prod_q \chi_q^\sigma = \prod_q \chi_q.$$

By using the uniqueness of the product of special characters (see [3, Proposition 7.2]), we conclude that $\chi_q^\sigma = \chi_q$ for every q. Hence f_{χ_q} divides f_χ for every q, and since the integers f_{χ_q} are coprime also $\prod_q f_{\chi_q}$ divides f_χ. Notice that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\chi_q : q) \subseteq \mathbb{Q}_{\prod_q f_{\chi_q}}$ by elementary Galois theory. This implies the equality

$$f_\chi = \prod_q f_{\chi_q}.$$

Now, consider $K = \mathbf{O}^{p',p}(G) < G$. Notice that $PK = \mathbf{O}^p(G) \lhd G$. By the Frattini argument $G = PK\mathbf{N}_G(P) = K\mathbf{N}_G(P)$. If $K = 1$, then $P \lhd G$ and we are done in this case. We may assume that $K > 1$. Let K/L be a chief factor of G. Then K/L is an abelian p'-group. If $H = \mathbf{N}_G(P)L$, then $G = KH$ and $K \cap H = L$, by a standard group theoretical argument. Furthermore, all the complements of K in G are G-conjugate to H. Finally, notice that $\mathbf{C}_{K/L}(P) = 1$ using that $H \cap K = L$.

We claim that for every q, there exists some q-special $\chi_q^* \in \mathrm{Irr}(H)$ such that $f_{\chi_q^*}$ is equal to f_{χ_q} and $\chi_q^*(1)$ is an odd p'-number.

If $q \in \{2, p\}$, then the character $\lambda = \chi_q$ is linear (because χ has odd p'-degree). Let $\lambda^* = \lambda_H$. Then λ^* is q-special (since λ is linear and q-special, this is straightforward from the definition) and $f_{\lambda^*} = f_\lambda$ by Lemma 1.4.

Let $q \neq p$ be an odd prime and write $\eta = \chi_q$. We work to find some character $\eta^* \in \mathrm{Irr}(H)$ of odd p'-degree with $f_{\eta^*} = f_\eta$. By Lemma 1.1, let $\theta \in \mathrm{Irr}(K)$ be some P-invariant constituent of η_K and let $\varphi \in \mathrm{Irr}(L)$ be some P-invariant constituent of η_L. By the second paragraph of the proof, we know that both θ and φ are G-invariant and hence φ lies under θ. By [8, Theorem 6.18] one of the following holds:

(a) $\theta_L = \sum_{i=1}^t \varphi_i$, where the $\varphi_i \in \mathrm{Irr}(L)$ are distinct and $t = |K : L|$,

(b) $\theta_L \in \mathrm{Irr}(L)$, or

(c) $\theta_L = e\varphi$, where $\varphi \in \mathrm{Irr}(L)$ and $e^2 = |K : L|$.

Notice that the situation described in case (a) cannot occur here, because φ is G-invariant.

In the case described in (b), we have $\varphi = \theta_L \in \mathrm{Irr}(L)$. Then restriction defines a bijection between the set of irreducible characters of G lying over θ and the set of irreducible characters of H lying over φ (by [7, Corollary (4.2)]). Write $\xi = \eta_H$. By [7, Theorem A], we know that ξ is q-special. We claim that $\mathbb{Q}(\eta) = \mathbb{Q}(\xi)$.

Clearly, $\mathbb{Q}(\xi) \subseteq \mathbb{Q}(\eta)$. If $\sigma \in \mathrm{Gal}(\mathbb{Q}(\eta)/\mathbb{Q}(\xi))$, then notice that φ is σ-invariant because ξ_L is a multiple of φ. Now, φ is P-invariant, and because $\mathbf{C}_{K/L}(P) = 1$, there is a unique P-invariant character over φ (by [8, Problem 13.10]). By uniqueness, we deduce that $\theta^\sigma = \theta$. Now, η^σ lies over θ and restricts to ξ, so we deduce that $\eta^\sigma = \eta$, by the uniqueness in the restriction. Thus $\mathbb{Q}(\eta) = \mathbb{Q}(\xi)$. We write $\eta^* = \xi$.

Finally, we consider the situation described in (c). Since θ_L is not irreducible, then $|K : L|$ is not a q'-group, by [8, Corollary 11.29]. Hence K/L is q-elementary abelian and e is a power of q. By Theorem 1.5 (and using that all the complements of K/L in G/L are conjugate), there exists a (not necessarily irreducible) character ψ of G such that:

(i) ψ contains K in its kernel, $\psi(g) \neq 0$ for every $g \in G$, $\psi(1) = e$ and the determinantal order of ψ is a power of q,

(ii) if $K \subseteq W \leq G$ and $\xi \in \mathrm{Irr}(W|\theta)$, then $\xi_{W\cap H} = \psi_{W\cap H}\xi_0$ for a unique irreducible character ξ_0 of $W \cap H$,

(iii) the values of ψ lie on $\mathbb{Q}_q$.

In particular, $\eta_H = \psi\eta_0$, so that $\eta_0 \in \mathrm{Irr}(H|\varphi)$ (where we are viewing ψ as a character of H). We claim that η_0 is q-special. First notice that $\eta_0(1) = \eta(1)/e$ is a power of q. Now, we want to show that whenever S is a subnormal subgroup of H, the irreducible constituents of $(\eta_0)_S$ have determinantal order a power of q. Since $(\eta_0)_L$ is a multiple of φ, which is q-special, we only need to control the irreducible constituents of $(\eta_0)_S$ when $L \subseteq S \lhd\lhd H$, by using [3, Proposition 2.3]. We have that $K \subseteq SK \lhd\lhd G$. Write

$$\eta_{SK} = a_1\gamma_1 + \cdots + a_r\gamma_r,$$

where the $\gamma_i \in \mathrm{Irr}(SK)$ are q-special because η is q-special and $a_i \in \mathbb{N}_0$. By using property (ii) of ψ, we have that $\eta_S = \psi_S(\eta_0)_S$ also decomposes as

$$\begin{aligned}\eta_S &= a_1\psi_S(\gamma_1)_0 + \cdots + a_r\psi_S(\gamma_r)_0\\ &= \psi_S(a_1(\gamma_1)_0 + \cdots + a_r(\gamma_r)_0).\end{aligned}$$

Since ψ never vanishes on G, we conclude that $(\eta_0)_S = a_1(\gamma_1)_0 + \cdots + a_r(\gamma_r)_0$. It suffices to see that $o((\gamma_i)_0)$ is a power of q for every γ_i constituent of η_{SK}. Just notice that

$$\det((\gamma_i)_S) = \det(\psi_S(\gamma_i)_0) = \det(\psi_S)^{(\gamma_i)_0(1)}\det((\gamma_i)_0)^e.$$

Since $o(\psi)$, $o(\gamma_i)$, $\gamma_i(1)$ and e are powers of q, we easily conclude that also the determinantal order of $(\gamma_i)_0$ is a power of q. This proves that η_0 is q-special. We claim that $\mathbb{Q}(\eta) = \mathbb{Q}(\eta_0)$ so that the two Feit numbers are the same. Let ζ be a primitive q-th root of unity and write $F = \mathbb{Q}(\zeta)$. Then the values of ψ lie in F.

We next see that η and η_0 are non-principal. This is obvious because θ and φ are fully ramified. Suppose that $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/F)$ stabilizes η. Then

$$\psi\eta_0 = \psi^\sigma\eta_0^\sigma = \psi\eta_0^\sigma.$$

Using that ψ is never zero, we conclude that $\eta_0^\sigma = \eta_0$. Now, by part (d) of Theorem 1.5, we have that ξ and ξ_0 correspond (as in part (c) of Theorem 1.5) if and only if $(\xi_0)^G = \overline{\psi}\xi$. Hence, if $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/F)$ and $\eta_0^\sigma = \eta_0$, then

$$\overline{\psi}\eta = (\eta_0)^G = (\eta_0^\sigma)^G = \overline{\psi}\eta^\sigma$$

(since $\mathbb{Q}(\overline{\psi})$ is also contained in F). This implies that $\eta^\sigma = \eta$. By Galois theory, we have that $F(\eta) = F(\eta_0)$. By Lemma 1.3, this implies $\mathbb{Q}(\eta) = \mathbb{Q}(\eta_0)$. We set $\eta^* = \eta_0$. The claim follows.

Now, we define $\chi^* = \prod_q {\chi_q}^*$ which has odd p'-degree. The character χ^* is irreducible by [3, Proposition 7.2]. Also $f_{\chi^*} = \prod_q \chi_q^*$ as in the fourth paragraph of this proof. Hence

$$f_{\chi^*} = \prod_q f_{{\chi_q}^*} = \prod_q f_{\chi_q} = f_\chi.$$

By the inductive hypothesis, there exists $g \in \mathbf{N}_H(P)/P' \leq \mathbf{N}_G(P)/P'$ such that $o(g) = f_{\chi^*}$ and we are done. □

It is natural to ask if Feit's conjecture admits a version for (p-)Brauer characters. Using the deep theory in [6], this is easy to prove for solvable groups. Let G be a solvable group. We write G^0 to denote the set of p-regular elements of G (elements whose order is not divisible by p). If $\varphi \in \mathrm{IBr}(G)$, by [6, Corollary 10.3] there exists a canonically defined $\chi \in \mathrm{Irr}(G)$ such that $\chi_{G^0} = \varphi$ (the character χ is canonical as an Isaacs' $B_{p'}$-character). By the uniqueness of the lifting, we have that $\mathbb{Q}(\chi) = \mathbb{Q}(\varphi) \subseteq \mathbb{Q}_{|G|_{p'}}$. By the Amit–Chillag theorem there exists an element $g \in G$ such that $o(g) = f_\chi = f_\varphi$ (of course g is p-regular). However, we have noticed in the introduction that Feit's conjecture does not hold for Brauer characters in general.

Acknowledgments. I would like to thank G. Malle and G. Navarro for useful remarks on a previous version of this paper.

Bibliography

[1] G. Amit and D. Chillag, On a question of Feit concerning character values of finite solvable groups, *Pacific J. Math.* **122** (1986), 257–261.

[2] W. Feit, Some consequences of the classification of finite simple groups, in: *Finite Groups* (Santa Cruz 1979), Proc. Sympos. Pure Math. 37, American Mathematical Society, Providence (1980), 175–181.

[3] P. Gajendragadkar, A characteristic class of characters of finite π-separable groups, *J. Algebra* **59** (1979), 237–259.

[4] I. M. Isaacs, Characters of solvable and symplectic groups, *Amer. J. Math.* **95** (1973), 594–635.

[5] I. M. Isaacs, Primitive characters, normal subgroups, and M-groups, *Math. Z.* **177** (1981), 267–284.

[6] I. M. Isaacs, Characters of π-separable groups, *J. Algebra* **86** (1984), 98–128.

[7] I. M. Isaacs, Induction and restriction of π-special characters, *Can. J. Math.* **38** (1986), no. 3, 576–604.

[8] I. M. Isaacs, *Character Theory of Finite Groups*, American Mathematical Society, Providence, 2006.

[9] G. Navarro, Induction of characters and p-subgroups, *J. Algebra* **250** (2002), 217–228.

Received February 2, 2016; revised February 15, 2016.

Author information

Carolina Vallejo Rodríguez, Departament d'Àlgebra, Universitat de València, 46100 Burjassot, València, Spain.
E-mail: carolina.vallejo@uv.es