Rend. Sem. Mat. Univ. Padova, Vol. 131 (2014)
DOI 10.4171/RSMUP/131-15

A criterium for monomiality

Carolina Vallejo Rodríguez

Abstract - Let G be a solvable group. An odd degree rational valued character χ of G is induced from a linear character of some subgroup of G. We extend this result to odd degree characters χ of G that take values in certain cyclotomic extensions of $\mathbb{Q}$.

Mathematics Subject Classification (2010). 20C15.

Keywords. Monomial characters, characters of finite groups.

1. Introduction

It follows from a well-known result of R. Gow [1] that an odd-degree rational-valued irreducible character of a solvable group is monomial. In this note, we slightly generalize Gow's result.

If χ is an irreducible complex character of a finite group G, let

$$n(\chi) = \gcd(n \mid \mathbb{Q}(\chi) \subseteq \mathbb{Q}_n)\,,$$

where $\mathbb{Q}(\chi)$ is the smallest field containing the values of χ, and we denote by $\mathbb{Q}_n$ the n-th cyclotomic field.

Theorem A. *Let G be a solvable group. Let $\chi \in \mathrm{Irr}(G)$. If $\chi(1)$ is odd and $\gcd(\chi(1), n(\chi)) = 1$, then there exist a subgroup $U \subseteq G$ and a linear character λ of U such that $\lambda^G = \chi$. Moreover, if μ is a linear character of some subgroup $W \subseteq G$ and $\mu^G = \chi$, then there exists $g \in G$ such that $W = U^g$ and $\mu = \lambda^g$.*

(*) Indirizzo dell'A.: Departamento de Álgebra, Facultad de Ciencias Matemáticas C/ Dr. Moliner, 50 C.P. 46100 Burjassot, Spain.
E-mail: carolina.vallejo@uv.es
This research is partially supported by Proyecto MTM2010-15296 of the Spanish Government.

Theorem A is not true if $\chi(1)$ is even or if G is not solvable ($SL_2(3)$ has a rational valued non-monomial character of degree 2, and A_6 has a rational valued non-monomial character of degree 5).

I would like to thank M. Isaacs and G. Navarro for simplifications of earlier versions of the proof of this result.

2. Proof of Theorem A

We use the notation of [2]. First, we prove by induction on $|G|$ that χ is monomial.

STEP 1. We may assume χ is faithful and that there is no proper subgroup $H < G$ and $\psi \in \mathrm{Irr}(H)$ such that $\psi^G = \chi$ and $\mathbb{Q}(\psi) \subseteq \mathbb{Q}(\chi)$.

Let $K = \ker\chi$. If $K > 1$, all the hypotheses hold in G/K so by induction we are done. For the second part $\psi(1)$ divides $\chi(1)$, and $n(\psi)$ divides $n(\chi)$, so $\gcd(\psi(1), n(\psi)) = 1$, and the inductive hypothesis applies.

STEP 2. $\mathbf{F}(G) = \prod\limits_{p \nmid \chi(1)} \mathbf{O}_p G$.

Let p be a prime. Suppose that p divides $\chi(1)$. In particular p is odd, and p does not divide $n(\chi)$. Let M be a normal p-subgroup of G. Since $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{n(\chi)}$ we have that $\mathbb{Q}_{|M|} \cap \mathbb{Q}(\chi) = \mathbb{Q}$. Hence χ_M is rational-valued. If $\xi \in \mathrm{Irr}(M)$, then $[\chi_M, \xi] = [\chi_M, \bar{\xi}]$. Since $\chi(1)$ is odd, there exists a real irreducible constituent ξ of χ_M. Since $|M|$ is odd, we have that $\xi = 1_M$, by Burnside's theorem. By Step 1, we know that χ is faithful and we conclude $M = 1$.

STEP 3. $F = \mathbf{F}(G)$ is abelian.

Let M be a normal p-subgroup of G, where p does not divide $\chi(1)$. It then follows that the irreducible constituents of χ_M are linear. Let $\lambda \in \mathrm{Irr}(M)$ be under χ. We have that $M' \subseteq \ker\lambda^g = \ker\lambda^g$ for every $g \in G$. Then $M' \subseteq \mathrm{core}_G(\ker(\lambda)) \subseteq \ker(\chi) = 1$, so that M is abelian. Hence F is abelian by Step 2.

STEP 4. Let $N \triangleleft G$ and let $\theta \in \mathrm{Irr}(N)$ be under χ. Let $g \in G$. Then $\theta^g = \theta^\sigma$ for some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$. Also θ is faithful.

Let $T = I_G(\theta)$ be the stabilizer of θ in G, and write T^* for the semi-inertia subgroup of θ, this is $T^* = \{g \in G | \theta^g = \theta^\sigma \text{ for some } \sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})\}$. By part (b) of Lemma 2.2 of [3], if $\psi \in \mathrm{Irr}(T|\theta)$ is the Clifford correspondent for χ, then $\eta = \psi^{T^*} \in \mathrm{Irr}(T^*)$ induces χ and $\mathbb{Q}(\eta) = \mathbb{Q}(\chi)$. By Step 1, we have that $T^* = G$. So that every G-conjugate of θ is actually a Galois conjugate. Thus $\ker \theta^g = \ker \theta$ for every $g \in G$. It follows that $\ker(\theta) \lhd G$ and $\ker(\theta)$ is contained in $\ker(\chi)$ by Clifford's theorem. So θ is faithful by Step 1.

Step 5. If $\lambda \in \mathrm{Irr}(F)$ is under χ, then $\lambda^G = \chi$.

Let $\lambda \in \mathrm{Irr}(F)$ be under χ. If $y \in G$ is such that $\lambda^y = \lambda$, then we have $[x, y] \in \ker(\lambda)$ for every $x \in F$. By step 4, λ is faithful, so the element y centralizes F. Since F is self-centralizing, necessarily $y \in F$. We have proved $I_G(\lambda) = F$. This implies λ^G is irreducible and thus $\lambda^G = \chi$. This finishes the proof that χ is monomial.

Now, we work by induction on $|G|$ to show that if U and V are subgroups of G and $\lambda \in \mathrm{Irr}(U)$ and $\mu \in \mathrm{Irr}(V)$ are linear such that $\lambda^G = \chi = \mu^G$, then the pairs (U, λ) and (V, μ) are G-conjugate. Since $K = \ker(\chi) \subseteq \mathrm{core}_G(\ker(\lambda)) \cap \mathrm{core}_G(\ker(\mu))$ we may assume that χ is faithful, for if $K > 1$ then we can work in G/K. If p is a prime not dividing $\chi(1)$, then $\mathbf{O}_p(G)$ is contained in both U and V, because $|G : U| = \chi(1) = |G : V|$. By Step 2 (which only required that χ is faithful), we have that

$$\mathbf{F}(G) = \prod_{p \nmid \chi(1)} \mathbf{O}_p(G) \subseteq U \cap V .$$

Now λ_F and μ_F are both under χ. So that $\mu_F = (\lambda_F)^g$ for some $g \in G$ by Clifford's theorem. We may assume that $\mu_F = \lambda_F = \nu$, by replacing the pair (U, λ) by some G-conjugate. Thus U and V are contained in $T = I_G(\nu)$ and also in T^*, the semi-interia subgroup of ν. Since λ^G and μ^G are irreducible, also λ^T and μ^T are irreducible. By uniqueness of the Clifford correspondent, we deduce that $\lambda^T = \mu^T$. In particular $\lambda^{T^*} = \mu^{T^*} = \psi \in \mathrm{Irr}(T^* | \nu)$. We know that $\mathbb{Q}(\psi) = \mathbb{Q}(\chi)$, again using Lemma 2.2 of [3]. If $T^* < G$, then the result follows by induction. Hence, we may assume $T^* = G$. In particular, arguing as in the first part of the proof, we conclude that $\nu^G = \chi$. This implies that $U = F = V$ and the theorem is proven.

Recently we have given a related criterium for monomiality in which the odd degree hypothesis is replaced by certain oddness related to Sylow normalizers (see [4]). This result and our Theorem A seem to be independent. Under the hypothesis of Theorem A, it can also be proved that in fact χ is **supermonomial**, that is, that every character inducing $\boldsymbol{\chi}$ is monomial.

REFERENCES

[1] R. GOW, *Real valued Characters of Solvable Groups*, Bull. London Math. Soc., 7 (1975), p. 132.
[2] I. M. ISAACS, *Character Theory of Finite Groups*, AMS Chelsea Publishing, 2006.
[3] G. NAVARRO - J. TENT, *Rationality and Sylow 2-subgroups*, Proc. Edinburgh Math. Society. 53 (2010), pp. 787–798.
[4] G. NAVARRO - C. VALLEJO, *Certain monomial characters*, Archiv der Mathematik. 99 (2012), pp. 407–411.

Manoscritto pervenuto in redazione il 27 Marzo 2013.