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Certain monomial characters of p'-degree

Gabriel Navarro and Carolina Vallejo

Abstract. An odd degree rational valued character χ of a solvable group G is induced from a linear character of order 2 of some subgroup of G. We extend this result of Gow to certain cyclotomic fields.

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1. Introduction. A lovely result of Gow [1] from 1975 asserts that if G is a solvable group and $\chi \in \mathrm{Irr}(G)$ is an odd degree irreducible complex character with real values, then χ is necessarily induced from a rational linear character λ of some subgroup U of G (in particular, χ is rational valued). Our aim in this note is to provide a *cyclotomic version* of Gow's result. If n is an integer, here we denote by $\mathbb{Q}_n$ the n-th cyclotomic field and if χ is a character, then $\mathbb{Q}(\chi)$ is the smallest field containing the values of χ.

Theorem A. *Let p be a prime, let G be a p-solvable finite group, and let $P \in \mathrm{Syl}_p(G)$. Let $\chi \in \mathrm{Irr}(G)$ be such that p does not divide $\chi(1)$ and such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{p^a}$ for some $a \geq 0$. If $|\mathbf{N}_G(P)/P|$ is odd, then there exists a subgroup $V \subseteq G$ and a linear character $\lambda \in \mathrm{Irr}(V)$ with $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^a}$ such that $\lambda^G = \chi$. Furthermore, if $\chi = \gamma^G$ for some linear $\gamma \in \mathrm{Irr}(W)$, then $W = V^g$ and $\gamma = \lambda^g$ for some $g \in G$.*

The condition that $|\mathbf{N}_G(P)/P|$ is odd is (unfortunately) necessary: if $p = 3$, then $G = SL(2,3)$ has a rational character of degree 2 which is not monomial. In this case, $\mathbf{N}_G(P)/P$ has order 2. Also, we need to assume that G is p-solvable, even in Gow's original situation: the group $G = A_6$ has two non-monomial rational characters of degree 5.

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2. Proofs. In general, we use the notation of [4]. If $U \subseteq G, \gamma \in \mathrm{Irr}(U)$, and $g \in G$, then $\gamma^g \in \mathrm{Irr}(U^g)$ is the character of U^g satisfying $\gamma^g(u^g) = \gamma(u)$ for $u \in U$. Also, $(U, \gamma)^g = (U^g, \gamma^g)$. If $\chi \in \mathrm{Irr}(U)$ has values in some subfield $F \subseteq \mathbb{C}$, where $F/\mathbb{Q}$ is Galois and $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, then we know that χ^σ, defined by

$$\chi^\sigma(u) = \chi(u)^\sigma$$

for $u \in U$, is also an irreducible character of U (for a proof of this elementary fact see Lemma (2.1) of [6], for instance).

Proof of Theorem A. First we argue by induction on $|G|$ that there exists a pair (V, λ), where V is a subgroup of G and λ is a linear character of V, such that $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^a}$ and $\lambda^G = \chi$. For the reader's convenience, we do this in a series of steps. The first four of these are of general type, which are not obtained by induction, and that shall be used in the second part of the proof.

Step 1. If $N \triangleleft G$, then there exists a P-invariant $\theta \in \mathrm{Irr}(N)$ under χ, and any two of them are $\mathbf{N}_G(P)$-conjugate.
This follows from the following standard argument. Let $\theta_1 \in \mathrm{Irr}(N)$ be under χ, let T_1 be the stabilizer of θ_1 in G, and let $\psi_1 \in \mathrm{Irr}(T_1|\theta_1)$ be the Clifford correspondent of χ over θ_1. (See Theorem (6.11) of [4].) Since χ has p'-degree, we have that $|G : T_1|$ is not divisible by p, and then $P^{h^{-1}} \subseteq T_1$ for some $h \in G$. Then $P \subseteq T = I_G(\theta)$, where $\theta = (\theta_1)^h \in \mathrm{Irr}(N)$. Also, if $\eta \in \mathrm{Irr}(N)$ is also P-invariant under χ, then by Clifford's theorem we have that $\eta^g = \theta$ for some $g \in G$. Then $P, P^g \subseteq T$, and thus $P^{gt} = P$ for some $t \in T$ by Sylow theory. Now $\eta^{gt} = \theta^t = \theta$ are $\mathbf{N}_G(P)$-conjugate.

Step 2. If $N \triangleleft G, \theta \in \mathrm{Irr}(N), g \in G$, and $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$, then we have that $(\theta^\sigma)^g = (\theta^g)^\sigma$. In particular, the stabilizer of θ in G is the stabilizer of θ^σ in G.
This immediately follows from the corresponding definitions.

Step 3. Suppose that $N \triangleleft G$, and let $\theta \in \mathrm{Irr}(N)$ be P-invariant under χ. If $\bar{\theta}$ is also an irreducible constituent of χ_N, where $\bar{\theta}$ is the complex-conjugate of θ, then $\theta = \bar{\theta}$.
By Step 2, we have that $\bar{\theta}$ is also P-invariant, and therefore there exists $g \in \mathbf{N}_G(P)$ such that $\bar{\theta} = \theta^g$, by Step 1. Now, g^2 fixes θ (also using Step 2). Now since $\mathbf{N}_G(P)/P$ has odd order by hypothesis, we conclude that $\langle gP\rangle = \langle g^2P\rangle$. Therefore g fixes θ and $\theta = \bar{\theta}$ is real.

Step 4. We have that $N = \mathbf{O}_{p'}(G) \subseteq \ker(\chi)$.
By Step 1, let $\theta \in \mathrm{Irr}(N)$ be P-invariant under χ, let T be the stabilizer of θ in G, and let $\psi \in \mathrm{Irr}(T)$ be the Clifford correspondent of χ over θ. We prove first that θ is real. By Step 3, it suffices to show that $\bar{\theta}$ is under χ. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$ the automorphism induced by complex conjugation. Since $\mathbb{Q}(\theta) \subseteq \mathbb{Q}_{|N|}$ and N is a p'-group, we have that $\mathbb{Q}_{p^a} \cap \mathbb{Q}(\theta) = \mathbb{Q}$. By the Natural Irrationalities theorem on Galois theory, let $\tau \in \mathrm{Gal}(\mathbb{Q}_{p^a}(\theta)/\mathbb{Q}_{p^a})$ be the natural extension of σ. Then $\chi^\tau = \chi$ because χ has values in $\mathbb{Q}_{p^a}$. Therefore, $\theta^\tau = \theta^\sigma = \bar{\theta}$ lies under χ. Hence θ is real.

Next we use the Glauberman correspondence (see Chapter 13 of [4]). This is a natural bijection

$$^* : \mathrm{Irr}_P(N) \to \mathrm{Irr}(\mathbf{C}_N(P)),$$

where $\mathrm{Irr}_P(N)$ is the set of P-invariant irreducible characters of N. In fact, if $\theta \in \mathrm{Irr}_P(N)$, then

$$\theta_{\mathbf{C}_N(P)} = e\theta^* + p\Delta,$$

where $\theta^* \in \mathrm{Irr}(\mathbf{C}_N(P)), p$ does not divide e, and Δ is a character of $\mathbf{C}_N(P)$ or zero. Now, $\overline{\theta^*}$ is an irreducible constituent of $\theta_{\mathbf{C}_N(P)}$ with p'-multiplicity, and by uniqueness we deduce that θ^* is real. But $\mathbf{C}_N(P) = \mathbf{N}_N(P)$ has odd order because $\mathbf{N}_G(P)/P$ has odd order by hypothesis. Thus $\theta^* = 1$ by Burnside's theorem on real characters of groups of odd order, and therefore $\theta = 1$ by the uniqueness of the Glauberman correspondence. Hence $\mathbf{O}_{p'}(G) \subseteq \ker(\chi)$, as claimed.

Step 5. The character χ is faithful. In particular $\mathbf{O}_{p'}(G) = 1$.
Let $L = \ker(\chi)$. Then $\mathbf{N}_{G/L}(PL/L)/(PL/L) \cong \mathbf{N}_G(P)/P\mathbf{N}_L(P)$ has odd order. If $L > 1$, then the existence of (V, λ) is readily obtained by applying the inductive hypothesis in G/L.

Step 6. G is not a p-group.
Otherwise, since p does not divide $\chi(1)$, then we have that χ is linear. In this case, we take $(V, \lambda) = (G, \chi)$.

Step 7. $M = \mathbf{O}_p(G)$ is abelian, and if $Z = \mathbf{Z}(G)$, then $Z < M$.
Let $\nu \in \mathrm{Irr}(M)$ be under χ. Since χ has p'-degree, then we have that ν is linear. Thus M' is contained in the kernel of every G-conjugate of ν. Since χ is faithful, we deduce that M is abelian. Let $Z = \mathbf{Z}(G) \subseteq M$ by Step 5. By Hall-Higman 1.2.3 Lemma, we have that $Z < M$ (because G is not a p-group, by Step 6).

Step 8. Let K/Z be a chief factor of G/Z inside M/Z. Let $\mu \in \mathrm{Irr}(K)$ be P-invariant under χ (which exists by Step 1). Let I be the stabilizer of μ in G, and let $\psi \in \mathrm{Irr}(I|\mu)$ be the Clifford correspondent of χ over μ. Then $I < G$ and $\mathbb{Q}(\mu) \subseteq \mathbb{Q}_{p^a}$.
By Step 7, we have that K is abelian. If $I = G$, then $\chi_K = \chi(1)\mu$ and then, using that μ is linear, faithful and G-invariant, we conclude that $K \subseteq Z$, which is not possible. Thus $I < G$. We also have that $|G : I|$ is not divisible by p. We need to show that $\mathbb{Q}(\mu) \subseteq \mathbb{Q}_{p^a}$.
Write $\chi_Z = \chi(1)\zeta$, where $\zeta \in \mathrm{Irr}(Z)$ is faithful. In particular, if $|Z| = p^b$, then $\mathbb{Q}(\chi)$ contains a p^b-primitive root of unity. Thus $\mathbb{Q}_{p^b} \subseteq \mathbb{Q}_{p^a}$. Since K/Z is elementary abelian, then we have that the exponent of K divides p^{b+1}. Thus $o(\mu)$ divides p^{b+1} and $\mathbb{Q}(\mu) \subseteq \mathbb{Q}_{p^{b+1}}$.
Suppose first that $a = 0$, so that χ is rational. Since χ is real, then μ and $\bar{\mu}$ are P-invariant characters under μ. By Step 3, we have that μ is real. Since μ is linear, then we have that $\mathbb{Q}(\mu) = \mathbb{Q}$, and we are done in this case. So we may assume that $a \geq 1$. Since $\mathbb{Q}_{p^b} \subseteq \mathbb{Q}_{p^a}$ and $a \geq 1$, we now have that $p^b \leq p^a$. In particular, $o(\mu)$ divides p^{a+1}. Let $\langle\sigma\rangle = \mathrm{Gal}(\mathbb{Q}_{p^{a+1}}/\mathbb{Q}_{p^a})$, where σ has order p because $a \geq 1$.

Since $\chi^\sigma = \chi$, by Clifford's theorem it follows that $\mu^\sigma = \mu^g$ for some $g \in G$. Then $I^g = I$ by Step 2. Furthermore, since $o(\sigma) = p$, we have that $g^p \in I$. However, $\mathbf{N}_G(I)/I$ is a p'-group (because $P \subseteq I$), and therefore $g \in I$. Hence $\mu^\sigma = \mu$ and therefore $\mathbb{Q}(\mu) \subseteq \mathbb{Q}_{p^a}$, as desired.

Final Step. Let $\tau \in \mathrm{Gal}(\mathbb{Q}_{p^a}(\psi)/\mathbb{Q}_{p^a})$. Since τ fixes χ and μ, then we have that τ fixes ψ by the uniqueness in the Clifford correspondent. Therefore, we have that $\mathbb{Q}(\psi) \subseteq \mathbb{Q}_{p^a}$. Now, since $P \subseteq I < G$ and $\mathbf{N}_I(P)/P$ has odd order, we can apply induction to ψ and deduce that there exists $V \subseteq I$ and $\lambda \in \mathrm{Irr}(V)$ with $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^a}$ such that $\lambda^I = \psi$. Then $\lambda^G = \chi$, and the first part of the proof is complete.

In the second part of this proof, we prove that if (W, γ) and (V, λ) are any two pairs with $\gamma(1) = 1 = \lambda(1)$ and $\lambda^G = \chi = \gamma^G$, then there is $g \in G$ such that $(W, \gamma) = (V, \lambda)^g$. We do this by arguing by induction on $|G|$. Let $L = \ker(\chi)$. Since $\gamma^G = \chi = \lambda^G$, then $L = \mathrm{core}_G(\ker(\lambda)) = \mathrm{core}_G(\ker(\gamma))$ is contained in $V \cap W$. By induction, we easily may assume that $L = 1$. In particular $\mathbf{O}_{p'}(G) = 1$ by Step 4 in the first part of this proof. Now, let $M = \mathbf{O}_p(G)$. Since $|G : V|$ and $|G : W|$ are not divisible by p (because $\chi(1)$ is not), then we have that $M \subseteq V \cap W$. Now, by Mackey, λ_M and γ_M are two irreducible constituents of χ_M. By Clifford's theorem and replacing (V, λ) by some G-conjugate, there is no loss if we assume that $\lambda_M = \tau = \gamma_M$. Hence V and W are contained in T, the stabilizer of τ in G. By the uniqueness of the Clifford correspondent, we have that $\lambda^T = \gamma^T = \psi$. Since τ and χ have values in some cyclotomic field $\mathbb{Q}_{p^b}$, then we have that ψ has also values in $\mathbb{Q}_{p^b}$ by the uniqueness of the Clifford correspondence. If $T = G$, then $M \subseteq \mathbf{Z}(G)$ (using that τ is linear and χ is faithful) and G is a p-group, by Hall-Higman's Lemma 1.2.3. In this case, $V = W = G$ and $\chi = \gamma = \lambda$. Otherwise, $T < G, \mathbf{N}_T(P)/P$ has odd order, and we apply induction to T and ψ. □

We finish this note with a few remarks. Let G be a finite group, let p be a prime, and let $\mathrm{Irr}_{p'}(G)$ be the set of irreducible characters of G of p'-degree. How many p'-degree irreducible characters does G have with field of values contained in $\mathbb{Q}_{p^a}$? It does not seem easy at all how to answer this question in general. However, if G is p-solvable and $\mathbf{N}_G(P)/P$ has odd order, then it is somewhat remarkable that this number can be computed locally. If we write $\mathcal{X}_{p',p^a}(G) = \{\chi \in \mathrm{Irr}_{p'}(G) \,|\, \mathbb{Q}(\chi) \subseteq \mathbb{Q}_{p^a}\}$, then it can be proved that $|\mathcal{X}_{p',p^a}(G)| = |\mathcal{X}_{p',p^a}(\mathbf{N}_G(P))|$, where $P \in \mathrm{Syl}_p(G)$. This result follows by using the natural correspondences between $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ constructed by Isaacs (in the case where $p = 2$) and by Turull (in the case where $|\mathbf{N}_G(P)|$ is odd) (see [3] and [7] for these two non-trivial theorems). But in fact, it is easier to construct a natural bijection

$$\mathcal{X}_{p',p^a}(G) \to \mathcal{X}_{p',p^a}(\mathbf{N}_G(P))$$

using that $\mathcal{X}_{p',p^a}(G)$ consists of monomial characters (by Theorem A) and the main result of [5].

Also, it can also be proved that the monomial character in our Theorem A is one of the characters that can be obtained via the method proposed in [2].

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Gabriel Navarro and Carolina Vallejo
Departament d'Àlgebra,
Universitat de València,
46100 Burjassot,
Spain
e-mail: gabriel.navarro@uv.es

Carolina Vallejo
e-mail: carolina.vallejo@uv.es

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