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2-Brauer correspondent blocks with one simple module [☆]

Gabriel Navarro [a], Carolina Vallejo [b]

[a] *Department of Mathematics, Universitat de València, Dr. Moliner 50, 46100 Burjassot, Spain*
[b] *ICMAT, Campus Cantoblanco UAM, C/ Nicolás Cabrera, 13-15, 28049 Madrid, Spain*

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One of the main problems in representation theory is to understand the exact relationship between Brauer corresponding blocks of finite groups. The case where the local correspondent has a unique simple module seems key. We study this situation for 2-blocks.

1. Introduction

Let G be a finite group, let p be a prime, and let $\mathbb{F}$ be an algebraically closed field of characteristic p. The (p-)blocks of G are the indecomposable two-sided ideals of the group algebra $\mathbb{F}G$. A block B of G uniquely determines a subset $\mathrm{Irr}(B)$ of the irreducible

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E-mail addresses: gabriel.navarro@uv.es (G. Navarro), carolina.vallejo@icmat.es (C. Vallejo).

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complex characters of G; a p-subgroup D of G, up to G-conjugacy, which is called the defect group of B; and a block b of the local subgroup $\mathbf{N}_G(D)$ with defect group D. To study what properties the algebras B and b share is one the main problems in Group Representation Theory. One of the first cases to study is the case of blocks with one simple module. These are exactly the blocks B for which $B/\mathbf{J}(B)$ is isomorphic to a matrix algebra over $\mathbb{F}$. In general, the semisimple algebra $B/\mathbf{J}(B)$ is a direct sum of $l(B)$ matrix algebras, by Wedderburn's theorem. In spite of the fact that blocks with one simple module have the easiest possible structure, many deep questions remain open about them. For instance, the following consequence of the Alperin Weight Conjecture is still unproven: if B has one simple module, then b has one simple module. However, the converse is not true. How does the hypothesis $l(b) = 1$ affects the character theory of B and conversely? Somewhat surprisingly, the Galois version of the Alperin–McKay conjecture proposed in [7] helps us to partially understand this problem.

In the main result of [11] the authors have proved that $l(b_0) = 1$ if, and only if, the only p-rational character of degree not divisible by p lying in B_0 is the principal one, whenever B_0 is the principal block of G, b_0 its local correspondent and p is odd. (p-Rational characters are those whose values lie in a cyclotomic extension $\mathbb{Q}_n$ of $\mathbb{Q}$ by a root of unity of order n not divisible by p.) For $p = 2$, this is false, and a totally different approach is needed. A single automorphism σ of the group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ holds the key, namely, the automorphism that fixes the 2-power order roots of unity, and squares the odd order roots of unity. (The values of $\chi \in \mathrm{Irr}(G)$ lie in $\mathbb{Q}_{|G|}$ and $\chi^\sigma \in \mathrm{Irr}(G)$, where $\chi^\sigma(g) = \chi(g)^\sigma$ for every $g \in G$.)

Conjecture A. *Let G be a finite group, let $P \in \mathrm{Syl}_2(G)$, let B_0 be the principal 2-block and let b_0 be the principal block of $\mathbf{N}_G(P)$. Then $l(b_0) = 1$ if, and only if, all odd-degree irreducible characters lying in B_0 are fixed by σ.*

Recently, the McKay conjecture has been proved for $p = 2$ by G. Malle and B. Späth in the landmark [5] using the reduction proposed in [4]. It is our hope that the many current investigations on the characters of finite simple groups of Lie type together with our main result below will terminate, sooner than later, in a proof of Conjecture A.

Theorem B. *Let G be a finite group, let $P \in \mathrm{Syl}_2(G)$, let B_0 be the principal 2-block and let b_0 be the principal block of $\mathbf{N}_G(P)$. Assume that Conjecture A holds for every almost simple group H involved in G whose socle S has 2-power index in H. Then Conjecture A is true for G.*

It is important to notice that, unlike the McKay or the Alperin–McKay conjectures, where complicated inductive statements have to be checked for all simple groups, in order to prove Conjecture A it suffices to verify that Conjecture A holds for certain almost simple groups. Conjecture A was introduced as Conjecture 6.3 in [11], and some cases were dealt there.

While proving Theorem B we have to overcome with several difficulties, and these have resulted in some theorems of independent interest. For instance, Theorem C below is a generalization of the Glauberman correspondence that turns out to be quite useful. By a result of R. Brauer, the principal block B_0 of G satisfies $l(B_0) = 1$ if, and only if, G has a normal p-complement. In particular, the principal block b_0 of $\mathbf{N}_G(P)$ satisfies $l(b_0) = 1$ if, and only if, $\mathbf{N}_G(P) = P \times V$.

Theorem C. *Let G be a group and let $P \in \mathrm{Syl}_p(G)$. Suppose that $\mathbf{N}_G(P) = P \times V$. Let $\chi \in \mathrm{Irr}_{p'}(G)$. Then*

$$\chi_V = e\hat{\chi} + p\Delta,$$

where p does not divide e and $\hat{\chi} \in \mathrm{Irr}(V)$. The map defined by $\chi \mapsto \hat{\chi}$ is a surjection from $\mathrm{Irr}_{p'}(G)$ onto $\mathrm{Irr}(V)$.

Conjecture A is the principal block version of Theorem 5.2 of [7], which asserts that G has a self-normalizing Sylow 2-subgroup if, and only if, every odd-degree irreducible character of G is σ-invariant. This conjecture was reduced to almost simple groups in [12] (one of the implications was independently reduced in [10]). There are exciting new results in [13], where the condition given in [12] has been checked for certain families of simple groups. Using Theorem C, it is easy to see that Theorem B implies the main result of [12] (see Theorem 6.7). Our approach in these notes is independent of the one in [12]. The use of blocks makes things more complicated and we shall need, among other things, results on isomorphic blocks due to J.L. Alperin.

2. A Glauberman correspondence

We follow the notation of [2] for ordinary characters and the notation of [6] for modular characters and blocks. In particular, if p is a prime number, and $\mathbf{R}$ is the ring of algebraic integers in $\mathbb{C}$, we choose M a maximal ideal of $\mathbf{R}$ containing $p\mathbf{R}$, with respect to which the Brauer characters of any finite group G are constructed. We also let $^* : \mathbf{R} \to \mathbf{R}/M$ be the canonical ring epimorphism. If $N \triangleleft G$ and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G|\theta)$ is the set of irreducible constituents of the induced character θ^G. Also, G_θ is the stabilizer of θ in G. By a block, we shall mean a p-block. Also, if B is a block of G, we will denote by $\mathrm{Irr}(B)$ the set of irreducible complex characters lying in the block B. Two characters $\alpha, \beta \in \mathrm{Irr}(G)$ lie in the same block if, and only if,

$$\left(\frac{|K|\alpha(x)}{\alpha(1)}\right)^* = \left(\frac{|K|\beta(x)}{\beta(1)}\right)^*$$

for all $x \in G$, where K is the conjugacy class of x. Recall R. Brauer associated to each block B of G a p-subgroup D of G, up to conjugation, which is called a defect group of B, and a block b of the local subgroup $\mathbf{N}_G(D)$, which is called the Brauer first main correspondent of B. In Brauer's notation, $b^G = B$. Sometimes, we will denote by B_0 the principal p-block of G and by b_0 the principal p-block of $\mathbf{N}_G(P)$, where $P \in \mathrm{Syl}_p(G)$, and, in general, we will write $B_0(H)$ to denote the principal block of the group H.

Let $\chi \in \mathrm{Irr}(B)$, where B has defect group D. Then $\chi(1)_p = |G : D|_p p^h$, where the non-negative integer $h = h(\chi)$ is called the height of χ. Height zero characters in B are those whose degree has minimal p-part among the characters in $\mathrm{Irr}(B)$. We write $\mathrm{Irr}_0(B)$ to denote the set of height zero characters of $\mathrm{Irr}(B)$. Note that if B has defect group a Sylow p-subgroup, then $\mathrm{Irr}_0(B) = \mathrm{Irr}_{p'}(G) \cap \mathrm{Irr}(B)$, where in this paper $\mathrm{Irr}_{p'}(G)$ is the set of irreducible characters of G of degree not divisible by p.

We recall that whenever $P\mathbf{C}_G(P) \subseteq H \subseteq \mathbf{N}_G(P)$ and b is a block of H, then the induced block b^G is defined, and every defect group of b is contained in a defect group of b^G (Theorem 4.14 and Lemma 4.13 of [6]).

We start with a version of the Glauberman correspondence, which is essentially what is called the *Alperin argument*. This, in particular, proves Theorem C.

Theorem 2.1. *Let G be a group and let $P \in \mathrm{Syl}_p(G)$. Suppose that $\mathbf{N}_G(P) = P \times V$. Let $\chi \in \mathrm{Irr}_{p'}(G)$.*

(a) We have that

$$\chi_V = e\hat{\chi} + p\Delta,$$

where p does not divide e, for some $\hat{\chi} \in \mathrm{Irr}(V)$, and Δ is a character of V or zero.

(b) We have that $\hat{\chi}$ is the only $\gamma \in \mathrm{Irr}(V)$ such that χ belongs to the induced block $(b_\gamma)^G$, where b_γ is the unique block of $\mathbf{N}_G(P)$ lying over γ.

(c) The map defined by $\chi \mapsto \hat{\chi}$ is a surjection from $\mathrm{Irr}_{p'}(G)$ onto $\mathrm{Irr}(V)$.

(d) We have that $\hat{\chi}$ is the unique character γ of V such that $\chi(1)^*\gamma(v)^* = \gamma(1)^*\chi(v)^*$ for all $v \in V$.

Proof. Write $H = \mathbf{N}_G(P)$ and $C = \mathbf{C}_G(P)$. We have that $H = P \times V$ by hypothesis. If $\gamma \in \mathrm{Irr}(V)$, let b_γ the unique block of H lying over γ, so that $\mathrm{Irr}(b_\gamma) = \mathrm{Irr}(H|\gamma)$. Notice that $\tilde{\gamma} = 1_P \times \gamma \in \mathrm{Irr}(B_\gamma)$. Let $B_\gamma = (b_\gamma)^G$ be the induced block. By the Brauer first main theorem and Theorem 10.20 of [6], we have that $\{B_\gamma \mid \gamma \in \mathrm{Irr}(V)\}$ are all the different blocks with defect group P of G.

Let $\chi \in \mathrm{Irr}_{p'}(G)$. Then χ lies in a unique B_γ. Let $x \in V$ and let $K = \mathrm{Cl}_G(x)$. Since $P \subseteq \mathbf{C}_G(x)$, we have that p does not divide $|K|$. By Lemma 4.16 of [6], we have that $K \cap C = \mathrm{Cl}_C(x)$. Moreover $|K| \equiv |K \cap C| \bmod p$, by using that P acts on K with fixed points $K \cap C$. We have that

$$\left(\frac{|K|\chi(x)}{\chi(1)}\right)^* = \lambda_{B_\gamma}(\hat{K}) = \lambda_{b_\gamma}(\widehat{K \cap C}) = \lambda_{\tilde{\gamma}}(\widehat{K \cap C}) = \left(\frac{|K \cap C|\gamma(x)}{\gamma(1)}\right)^*.$$

Since p does not divide $\chi(1)\gamma(1)$, we deduce that

$$\chi(x)^*\gamma(1)^* = \gamma(x)^*\chi(1)^*,$$

for every $x \in V$. This reasoning, together with Problem 4.5 of [6], also proves that $\chi \in \mathrm{Irr}_{p'}(G)$ lies in B_γ if, and only if, $\chi(x)^*\gamma(1)^* = \gamma(x)^*\chi(1)^*$ for every $x \in V$.

Let $\delta \in \mathrm{Irr}(V)$. Then we have that

$$\begin{aligned}\gamma(1)^*|V|^*[\chi_V, \delta]^* &= \sum_{x\in V} \gamma(1)^*\chi(x)^*\delta(x^{-1})^* \\ &= \chi(1)^* \sum_{x\in V} \gamma(x)^*\delta(x^{-1})^* \\ &= \chi(1)^*|V|^*[\gamma, \delta]^*.\end{aligned}$$

Since $\gamma(1)^*|V|^*$ and $\chi(1)^*|V|^*$ are nonzero, we have that $[\chi_V, \delta] \equiv 0 \mod p$ if $\delta \neq \gamma$, and $[\chi_V, \gamma] \neq 0 \bmod p$. Notice that $\mathrm{Irr}_{p'}(B_\gamma)$ is exactly the set of $\chi \in \mathrm{Irr}_{p'}(G)$ such that $\hat{\chi} = \gamma$. Since $\mathrm{Irr}_{p'}(B_\gamma)$ is not empty by elementary block theory, this also proves that the map is surjective. $\square$

Notice that $\chi \mapsto \hat{\chi}$ is injective if, and only if, $|\mathrm{Irr}_{p'}(B_\gamma)| = 1$ for all $\gamma \in \mathrm{Irr}(V)$. By Problem 3.11 of [6], this happens if, and only if, G is a p'-group.

Later, we shall need the following useful consequence.

Corollary 2.2. *Let G be a group and let $P \in \mathrm{Syl}_p(G)$. Suppose that $\mathbf{N}_G(P) = P \times V$. Then $\beta \in \mathrm{Irr}_{p'}(G)$ belongs to the principal block of G if, and only if, $\beta(x)^* = \beta(1)^*$ for every $x \in V$. In particular, if β lies in the principal block and $N \triangleleft G$ is contained in $\ker\beta$, then β lies in the principal block of G/N.*

Proof. The first part follows from Theorem 2.1(d). The second statement follows from the first since V/N is a normal p-complement for $\mathbf{N}_{G/N}(PN/N)$. $\square$

In general, Corollary 2.2 is false without the hypothesis on Sylow normalizers.

3. Extending characters

Suppose that $N \triangleleft G$ and $\theta \in \mathrm{Irr}_{2'}(N)$ extends to G. It is known that if θ lies in the principal 2-block of G, then there is not necessarily an extension of θ in the principal 2-block of G. However, if θ is σ-invariant, then we can show this, and more than this. Throughout this section $p = 2$.

The next lemma appears as Lemma 3.4 in [12]. We give a proof of it to help the reader to get acquainted with our methods in this section. Recall that σ is the only Galois automorphism that fixes 2-roots of unity and squares odd roots of unity. By elementary character theory, notice that

$$\det(\chi^\tau) = \det(\chi)^\tau$$

for every character χ of G and $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$.

Lemma 3.1. *Let G be a group and let $p = 2$. Suppose that $N \lhd G$ and $\theta \in \mathrm{Irr}_{p'}(N)$ extends to G. If θ is σ-invariant and $|G : N|$ is a power of p, then some extension of θ is σ-invariant. In particular, every extension of θ is σ-invariant.*

Proof. Suppose first that θ is linear. Let $\chi \in \mathrm{Irr}(G)$ be any extension of θ. Write $\chi = \chi_2\chi_{2'}$ and $\theta = \theta_2\theta_{2'}$, where the order of $\chi_{2'}$ is odd and the order of χ_2 is a power of 2. Then $(\chi_2)_N = \theta_2$ and χ_2 is σ-invariant. It suffices to see that $\theta_{2'}$ has a σ-invariant extension. By the uniqueness of the decomposition of θ as its 2-$2'$-parts notice that $\theta_{2'}$ is σ-invariant, so we may assume the order of θ is odd. Hence $\theta = \theta^\sigma = \theta^2$ implies $\theta = 1_N$ and we are done in this case because G/N is a 2-group.

In the general case, let $\lambda = \det\theta$. Then λ is σ-invariant. By the first part of this proof, let $\mu \in \mathrm{Irr}(G)$ be a σ-invariant extension of λ. By Lemma 6.24 of [2], there is a unique extension $\chi \in \mathrm{Irr}(G)$ of θ such that $\det\chi = \mu$. By uniqueness, χ is σ-invariant, as desired.

The last claim follows from Gallagher's theorem Corollary 6.18 of [2] since every linear character of G/N has 2-power order. $\square$

The odd-degree hypothesis in the above lemma is necessary as shown by D_{24}. In the next result, the case where G/N has odd order is studied.

Theorem 3.2. *Suppose that $N \lhd G$, let $p = 2$ and let $\theta \in \mathrm{Irr}_{p'}(N)$ be G-invariant and σ-invariant.*

(a) If $|G : N|$ is odd, then θ extends to G, and it has a unique σ-invariant extension to G.
(b) If θ lies over under $\chi \in \mathrm{Irr}_{p'}(G)$, then θ extends to G.

Proof. These are Theorem 2.1 and Corollary 2.2 of [10]. $\square$

In the next result, we remove the hypothesis that G/N is a 2-group in Lemma 3.1.

Theorem 3.3. *Let G be a group and set $p = 2$. Suppose that $N \lhd G$ and that $\theta \in \mathrm{Irr}_{p'}(N)$ extends to G. If θ is σ-invariant, then some extension of θ to G is σ-invariant.*

Proof. We argue by induction on $|G : N|$. Let χ be any extension of θ to G.

First assume that G/N is solvable. Let K/N be a chief factor. If K/N has odd order, then by Theorem 2.1 of [10], there exists a unique σ-invariant extension $\tau \in \mathrm{Irr}(K)$ of θ. We claim that τ extends to G. By uniqueness, notice that τ is G-invariant. Let $P \in \mathrm{Syl}_2(G)$. By Corollary 4.2 of [3], we have that restriction defines a bijection

$$\mathrm{Irr}(PK|\tau) \to \mathrm{Irr}(PN|\theta)\,.$$

Therefore, there exists $\gamma \in \mathrm{Irr}(PK|\tau)$ such that $\gamma_{PN} = \chi_{PN}$. Since $\chi_N = \theta$, then $\gamma_K = \tau$. Hence, τ extends to PK. Suppose now that Q/K is a Sylow q-subgroup of G/K for some odd prime q. By Theorem 3.2(a) we have that τ has a unique σ-invariant extension $\rho \in \mathrm{Irr}(Q)$. We conclude that τ extends to every Sylow subgroup of G/K. By Corollary 11.31 of [2], τ extends to G. In this case, we are done by induction. If K/N is a 2-group, then χ_K is σ-invariant by Lemma 3.1. Again we are done by induction.

In the general case, let M/N be the solvable residual of G/N, so that M/N is perfect and G/M is solvable. We have that θ has a unique extension to M, using Gallagher's theorem and the fact that M/N has a unique linear character. This extension is $\tau = \chi_M$. By uniqueness, we have that τ is σ-invariant. Since G/M is solvable, then we are done by the first part of the proof. $\square$

Theorem 3.4. *Let G be a group and let $p = 2$. Suppose that $\psi \in \mathrm{Irr}_{p'}(G)$ is σ-invariant and $\psi_N = \theta \in \mathrm{Irr}(N)$. If θ lies in the principal block of N, then ψ lies in the principal block of G.*

Proof. Since ψ lies in a block of maximal defect, by Problem 4.5 of [6], it suffices to show that

$$\left(\frac{|K|\psi(x)}{\psi(1)}\right)^* = |K|^*,$$

for every conjugacy class $K = \mathrm{Cl}_G(x)$ of a p-regular element $x \in G$ with p not dividing the index $|G : \mathbf{C}_G(x)|$. Since p does not divide $\psi(1)$, it is enough to show that

$$\psi(x)^* = \psi(1)^*.$$

Let $H = N\langle x\rangle$ and let $P \in \mathrm{Syl}_p(G)$ be such that $P \subseteq \mathbf{C}_G(x)$. Let $Q = P \cap N \in \mathrm{Syl}_p(N)$. We have that $Q \subseteq \mathbf{C}_N(x) \subseteq \mathbf{C}_H(x)$. Since H/N is a p'-group, it follows that $Q \in \mathrm{Syl}_p(H)$. In particular, if $L = \mathrm{Cl}_H(x)$, then p does not divide $|L|$. Moreover, if we write $Y = \langle x\rangle$, then $H = NY$, where Y is a p'-group centralizing Q.

By Theorem 3.2(a), we have that θ has a unique σ-invariant extension to H, which by hypothesis is ψ_H. On the other hand, by Theorem 3.2 of [9], there is a unique $\hat{\theta} \in \mathrm{Irr}(H)$ in the principal block that lies over θ. In fact, $\hat{\theta}$ extends θ. By uniqueness $\hat{\theta}$ is σ-invariant, and we conclude $\psi_H = \hat{\theta}$ lies in the principal block of H. Hence

$$\left(\frac{|L|\psi(x)}{\psi(1)}\right)^* = |L|^*,$$

which implies $\psi(x)^* = \psi(1)^*$ as wanted. $\square$

Thanks to Theorem 3.4, we can show that the extension of θ given by Theorem 3.3 lies in the principal block provided that θ lies in the principal block.

Corollary 3.5. *Suppose that $N \triangleleft G$. Suppose that $\theta \in \mathrm{Irr}(N)$ has odd degree, extends to G and is σ-invariant. If θ is in the principal 2-block of N, then there exists an extension $\psi \in \mathrm{Irr}(G)$ of θ which is σ-invariant and lies in the principal 2-block of G.*

Proof. By Theorem 3.3, we know that there exists a σ-invariant $\psi \in \mathrm{Irr}(G)$ such that $\psi_N = \theta$. Now apply Theorem 3.4. $\square$

4. Semisimple by 2-groups

In this section, we study groups of the form $G = NP$, where $N \triangleleft G$ is direct product of non-abelian simple groups transitively permuted by G and $P \in \mathrm{Syl}_p(G)$ for $p = 2$.

Lemma 4.1. *Suppose that N is the direct product of distinct subgroups $S_1, \ldots, S_t$ of N. Assume that A acts as automorphisms on N and permutes $\Omega = \{S_1, \ldots, S_t\}$ transitively. Write $S = S_1$. Let $B = \mathbf{N}_A(S_1)$, let $a_i \in A$ such that $S^{a_i} = S_i$, so that $\{a_1, \ldots, a_t\}$ is a complete set of representatives of right cosets of B in A.*

(a) The map $f\colon \mathbf{C}_S(B) \to \mathbf{C}_N(A)$ given by $c \mapsto \prod_i c^{a_i}$ is a group isomorphism.
(b) The map $\mathrm{Irr}_B(S) \to \mathrm{Irr}_A(N)$ given by $\psi \mapsto \psi^{a_1} \times \cdots \times \psi^{a_t}$ is a bijection.

Proof. Let $\Delta = \{a_1, \ldots, a_t\}$ so that A acts on Δ. Write $a_i \cdot a = a_{\sigma(i)}$, where $Ba_ia = Ba_{\sigma(i)}$. If $c \in \mathbf{C}_S(B)$, then $c^{a_ia} = c^{a_{\sigma(i)}}$, and it follows that f is a well-defined group homomorphism. Because it is a direct product the map is injective. Suppose that $d = x_1 \ldots x_t$ is A-invariant, where $x_i \in S$. Then x_1 is fixed by B. It is also clear that $x_i = x_1^{a_i}$. The second part is Lemma (4.1)(ii) of [8]. $\square$

Lemma 4.2. *Suppose that $G = NP$, where $P \in \mathrm{Syl}_p(G)$, and $N \triangleleft G$ is a direct product of subgroups $\Omega = \{S_1, \ldots, S_t\}$ which are permuted transitively by P. Let $H = \mathbf{N}_G(S_1)$, and $P_1 = H \cap P$. Then $\mathbf{N}_G(P)$ has a normal p-complement if, and only if, $\mathbf{N}_{S_1P_1}(P_1)$ has a normal p-complement.*

Proof. Write $S = S_1$, $N \subseteq H \subseteq G$, and notice that $P_1 \in \mathrm{Syl}_p(H)$. Also, we can choose representatives $\{x_1, \ldots, x_t\}$ of right cosets of P_1 in P such that $S^{x_i} = S_i$, and $x_1 = 1$. Now, write $Q = P \cap N = P_1 \cap N$, and $R = Q \cap S = P \cap S = P_1 \cap S \in \mathrm{Syl}_p(S)$. Notice too that $\mathbf{N}_G(P)/P$ is naturally isomorphic to $C/Q = \mathbf{C}_{\mathbf{N}_G(Q)/Q}(P)$ and that $\mathbf{N}_G(P) = CP$. By the same reason, $\mathbf{N}_{SP_1}(P_1)/P_1$ is naturally isomorphic to $B/R = \mathbf{C}_{\mathbf{N}_S(R)/R}(P_1)$ and $\mathbf{N}_{SP_1}(P_1) = BP_1$. Now, $Q = R^{x_1} \times \cdots \times R^{x_t}$, and $\mathbf{N}_N(Q) = \mathbf{N}_S(R)^{x_1} \times \cdots \times \mathbf{N}_S(R)^{x_t}$. Now, P acts naturally on the group $\Gamma = (\mathbf{N}_S(R)/R)^{x_1} \times \cdots \times (\mathbf{N}_S(R)/R)^{x_t}$ which is P-isomorphic to $\mathbf{N}_N(Q)/Q$.

Let $y = z_1^{x_1} \ldots z_t^{x_t} \in \mathbf{N}_N(Q)$, where $z_i \in \mathbf{N}_S(R)$. By Lemma 4.1(a), we have that $y \in \mathbf{N}_G(P)$ if, and only if, there is $zR \in \mathbf{C}_{\mathbf{N}_S(R)/R}(P_1)$ such that $z_iR = zR$ for all i, and where z can be chosen p-regular.

Assume first that $\mathbf{N}_{S_1P_1}(P_1)$ has a normal p-complement, and let $y \in \mathbf{N}_G(P)$ be p-regular, so that $y \in \mathbf{N}_N(P) \subseteq \mathbf{N}_N(Q)$. Write as before $y = z_1^{x_1} \dots z_t^{x_t}$, where $z_i \in \mathbf{N}_S(R)$ is p-regular. We know that $z_i = zr_i$ for some $r_i \in R$ and some $zR \in \mathbf{C}_{\mathbf{N}_S(R)/R}(P_1)$, where z is p-regular, by the previous paragraph. Since $\mathbf{N}_{S_1P_1}(P_1)$ has a normal p-complement, we deduce that $[z, P_1] = 1$. Since $z_i = zr_i$ is p-regular and $[z, R] = 1$, we deduce that $r_i = 1$ for all i. Hence, we easily check that $y = z^{x_1} \dots z^{x_t}$ is fixed by P using that $[z, P_1] = 1$. We deduce that $\mathbf{N}_G(P)$ has a normal p-complement.

Conversely, assume that $\mathbf{N}_G(P)$ has a normal p-complement, and let $y \in \mathbf{N}_{SP_1}(P_1)$ be p-regular. Thus $y \in B$, where $B/R = \mathbf{C}_{\mathbf{N}_S(R)/R}(P_1)$. Let $z = y^{x_1} \dots y^{x_t} \in \mathbf{N}_N(Q)$. Again zQ is fixed by P, and we deduce that z is a p-regular element in $\mathbf{N}_G(P)$. Therefore $[z, P] = 1$. This implies that $[y, P_1] = 1$. □

In the next lemma, we use that if G_1 and G_2 finite groups, then $\mathrm{Irr}(B_0(G_1 \times G_2)) = \mathrm{Irr}(B_0(G_1)) \times \mathrm{Irr}(B_0(G_2))$. (This follows directly from the definition of the principal block, see Definition 3.1 of [6].)

Lemma 4.3. *Suppose that $G = NP$, where $P \in \mathrm{Syl}_2(G)$ and $N \triangleleft G$ is the direct product of non-abelian simple subgroups $S_1, \dots, S_t$ of N transitively permuted by P. Let $H = \mathbf{N}_G(S_1)$ and $P_1 = H \cap P$. Then every odd degree character in $\mathrm{Irr}(B_0(G))$ is σ-invariant if, and only if, every odd degree character in $\mathrm{Irr}(B_0(S_1P_1))$ is σ-invariant.*

Proof. Write $p = 2$. Note that $P_1 \in \mathrm{Syl}_p(H)$. Let $x_i \in P$ be such that $S_1^{x_i} = S_i$ and $\{x_1, \dots, x_t\}$ is a right transversal of P_1 in P. By Lemma 4.1(b), we have that the map $\mathrm{Irr}_{P_1}(S_1) \to \mathrm{Irr}_P(N)$ given by $\theta \mapsto \theta^{x_1} \times \cdots \times \theta^{x_t}$ is a bijection.

Assume first that every odd degree $\tau \in \mathrm{Irr}(B_0(S_1P_1))$ is σ-invariant. Let $\chi \in \mathrm{Irr}_{p'}(B_0(G))$. Then $\chi_N \in \mathrm{Irr}_{p'}(B_0(N))$ is P-invariant. Hence $\chi_N = \theta^{x_1} \times \cdots \times \theta^{x_t}$, for some P_1-invariant $\theta \in \mathrm{Irr}_{p'}(B_0(S_1))$. Since the determinantal order of θ is trivial, by Corollary 6.28 of [2] θ extends to $\hat{\theta} \in \mathrm{Irr}(S_1P_1)$. Since S_1P_1/P_1 is a p-group $\hat{\theta}$ lies in the principal block of S_1P_1. By hypothesis, $\hat{\theta}$ is σ-invariant, and hence also θ is σ-invariant. Consequently χ_N is σ-invariant. By Lemma 3.1, we have that χ is σ-invariant.

Assume finally that every odd degree $\mathrm{Irr}(B_0(G))$ is σ-invariant. Let $\tau \in \mathrm{Irr}_{p'}(B_0(P_1S_1))$. Hence $\theta = \tau_{S_1} \in \mathrm{Irr}_{p'}(B_0(S_1))$ is P_1-invariant. Write $\eta = \theta^{x_1} \times \cdots \times \theta^{x_t} \in \mathrm{Irr}_{p'}(B_0(N))$. Then η is P-invariant. Since the determinantal order of η is 1, again we have that η extends to some $\chi \in \mathrm{Irr}(G)$, which necessarily lies in $B_0(G)$. By hypothesis, χ is σ-invariant. This implies that χ_{S_1}, which is a multiple of θ, is σ-invariant. Therefore, θ is σ-invariant. □

5. Sylow normalizers with a normal 2-complement

In this section, we prove one half of Theorem B. We shall need the following results.

Lemma 5.1. *Let G be a group and let p be a prime number. Let $\chi, \psi \in \mathrm{Irr}(G)$ and $\beta \in \mathrm{Irr}_{p'}(G)$ be such that $\chi = \beta\psi$. If β and ψ lie in the principal block of G, then also does χ. If χ and ψ lie in the principal block, then also does β.*

Proof. The first part of the statement is Lemma 3.5 of [9]. The second part follows from a similar argument. □

Lemma 5.2. *Let G be a finite group, let p be a prime, let $P \in \mathrm{Syl}_p(G)$, and let $\chi \in \mathrm{Irr}_{p'}(G)$. Assume that $N \triangleleft G$. Then χ_N has a P-invariant irreducible constituent θ. Also, any two of them are $\mathbf{N}_G(P)$-conjugate.*

Proof. To show that χ_N has a P-invariant irreducible constituent, is enough to use that χ has p'-degree. For the conjugacy part, use the Frattini argument. □

Theorem 5.3. *Let G be a group, and let $P \in \mathrm{Syl}_p(G)$, where $p = 2$. Suppose that $\mathbf{N}_G(P)$ has a normal p-complement. Assume that, whenever H is an almost simple group involved in G with $H = \mathbf{F}^*(H)Q$, where $Q \in \mathrm{Syl}_p(H)$ and such that $\mathbf{N}_H(Q)$ has a normal p-complement, every $\psi \in \mathrm{Irr}_{p'}(B_0(H))$ is σ-invariant. Then every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ is σ-invariant.*

Proof. We proceed by induction on $|G|$. Let $\chi \in \mathrm{Irr}_{p'}(G)$ lie in the principal block of G. Let N be a minimal normal subgroup of G. By Lemma 5.2 let $\theta \in \mathrm{Irr}(N)$ be P-invariant under χ. Let $\psi \in \mathrm{Irr}(G_\theta|\theta)$ be the Clifford correspondent of χ. By Brauer's third main theorem and Corollary 6.2 of [6], we have that ψ lies in the principal block of G_θ. If $G_\theta < G$, then by induction ψ is σ-invariant, so also $\chi = \psi^G$ is σ-invariant. Hence, we may assume that $G = G_\theta$.

We claim that it is enough to show that θ is σ-invariant. In such case, by Theorem 3.2(b), we have that θ extends to G. By Corollary 3.5, we can choose a σ-invariant extension $\psi \in \mathrm{Irr}(G)$ of θ in the principal block of G. By Gallagher's theorem, $\chi = \beta\psi$ for some $\beta \in \mathrm{Irr}_{p'}(G/N)$. Moreover β lies in the principal block of G by Lemma 5.1. By Corollary 2.2, β lies in the principal block of G/N, and hence β is σ-invariant by induction. Consequently, χ is σ-invariant, and the claim follows.

Suppose now that $NP < G$. Since χ has p'-degree, some irreducible constituent τ of χ_{PN} has p'-degree, and hence τ extends θ. Notice that τ lies in the principal block of PN for PN/N is a p-group. Since $\mathbf{N}_{PN}(P)$ has a normal p-complement, by the inductive hypothesis, τ is σ-invariant. Hence, also θ is σ-invariant, and by the previous paragraph we are done.

Therefore, we may assume that $NP = G$. If N is a p-group, then G is a p-group, and there is nothing to prove, because σ fixes 2-power roots of unity. If N is a p'-group, then $N \subseteq \ker\chi$, and again we are done. Hence, we may assume that N is a direct product of non-abelian simple groups S_i which are transitively permuted by P. By the previous argument, we may also assume that N is the unique minimal normal subgroup of G. By

Lemma 4.2 and Lemma 4.3, we may assume that $G = SP$, where $S \triangleleft G$ is a non-abelian simple group of order divisible p, and it is the unique minimal normal subgroup of G. Thus G is almost simple, and by hypothesis $\chi^\sigma = \chi$ and we are done. □

6. Odd degree irreducible characters in the principal block

The aim of this section is to prove the other half of Theorem B. The following is an easy observation.

Lemma 6.1. *Let N and M be distinct normal subgroups of a group G. Let P be a Sylow p-subgroup of G. Suppose that $\mathbf{N}_{G/N}(PN/N)$ and $\mathbf{N}_{G/M}(PM/M)$ have a normal p-complement. If $N \cap M = 1$, then $\mathbf{N}_G(P)$ has a normal p-complement.*

Proof. By elementary group theory we have that $\mathbf{N}_{G/N}(PN/N) = \mathbf{N}_G(P)N/N$. By hypothesis, $\mathbf{N}_G(P)/\mathbf{N}_N(P) \cong \mathbf{N}_G(P)N/N$ has a normal p-complement. Similarly, $\mathbf{N}_G(P)/\mathbf{N}_M(P)$ has a normal p-complement. Hence

$$\mathbf{N}_G(P) = \mathbf{N}_G(P)/(\mathbf{N}_N(P) \cap \mathbf{N}_M(P))$$

has a normal p-complement. □

We will use the following which appears as Corollary 6.4 in [6].

Lemma 6.2. *Let $H \leq G$, let b be a p-block of H, and let $\psi \in \mathrm{Irr}(H)$. Assume that the block $B = b^G$ is defined. Then the p-part $f\,|G : H|\psi(1)$ is the p-part of $\sum_{\chi \in \mathrm{Irr}(B)}[\psi^G, \chi]\chi(1)$. In particular, if p does not divide $|G : H|\psi(1)$, then there exists $\chi \in \mathrm{Irr}(B)$ of p'-degree such that p does not divide $[\psi^G, \chi]$.*

The two results below are stated for the reader's convenience.

Lemma 6.3. *Let P be a p-subgroup of G. Let $N \triangleleft G$ be such that $N \subseteq P$. Write $\bar{G} = G/N$. Let b be a block of $\mathbf{N}_G(P)$ of defect P and assume that $\bar{b}$ is a block of $\mathbf{N}_G(P)/N$ contained in b. Then b^G is the unique block containing $\bar{b}^{\bar{G}}$.*

Proof. This is Lemma 3.2 of [7]. □

Theorem 6.4. *Let p be a prime. Let $N \triangleleft H$ be such that H/N is solvable of order not divisible by p. Assume that $H = \mathbf{C}_H(Q)N$, where Q is a Sylow p-subgroup of N. If γ is an irreducible character in the principal block of H, then γ_N is irreducible. In fact $\gamma \mapsto \gamma_N$ defines a bijection $\mathrm{Irr}(B_0(H)) \to \mathrm{Irr}(B_0(N))$.*

Proof. This is Lemma 1 of [1]. □

Lemma 6.5. *Let H be a group of odd order. If every irreducible character of H is σ-invariant, then $H = 1$.*

Proof. We have that $\chi(x) = \chi(x^2)$ for every $\chi \in \mathrm{Irr}(H)$, and therefore we have that every $x \in H$ is H-conjugate to x^2. Then apply Lemma 5.1 of [7]. $\square$

We can now prove the remaining half of Theorem B.

Theorem 6.6. *Let G be a finite group and let $p = 2$. Suppose that every $\mathrm{Irr}_{p'}(B_0(G))$ is σ-invariant. Assume that whenever H is an almost simple group involved in G with $H = F^*(H)Q$ where $Q \in \mathrm{Syl}_p(H)$ and such that every $\mathrm{Irr}_{p'}(B_0(H))$ is σ-invariant then $\mathbf{N}_H(Q)$ has a normal p-complement. If $P \in \mathrm{Syl}_p(G)$, then $\mathbf{N}_G(P)$ has a normal p-complement.*

Proof. Let G be a minimal counterexample to the statement. Recall that $\mathrm{Irr}(B_0(G/N)) \subseteq \mathrm{Irr}(B_0(G))$, by elementary block theory. By Lemma 6.1 and the minimality of G, we may assume that G has a unique minimal normal subgroup N, and that $\mathbf{N}_G(P)N/N$ has a normal p-complement. Since $\mathbf{O}_{p'}(G)$ is contained in the kernel of every character in $\mathrm{Irr}(B_0(G))$ (by Theorem 6.10 of [6]), we may assume that N is not a p'-group.

Step 1. We may assume that N is a direct product of non-abelian simple groups.

If N is a p-group, then by minimality of G as a counterexample, we have that $\mathbf{N}_G(P)/N = P/N \times V/N$, so $\mathbf{N}_G(P) = PV$. Let $L = \mathbf{O}_{p'}(PV)$. We want to show that $PV = PL$. Since PV/PL has odd order, by Lemma 6.5 it suffices to show that every $\mathrm{Irr}(PV/PL)$ is σ-invariant. Let $\tau \in \mathrm{Irr}(PV/PL)$. Now $\tau \in \mathrm{Irr}(P/N \times V/N)$, and we can write $\tau = 1_P \times \gamma$, where $\gamma \in \mathrm{Irr}(V/N)$, and γ contains L in its kernel. By Theorem 2.1(d), let $\chi \in \mathrm{Irr}_{p'}(G/N)$ be such that

$$\chi_V = e\gamma + p\Delta,$$

where p does not divide e. By Theorem 2.1(b), we know that χ belongs to the block $\bar{b}^{\bar{G}}$, where $\bar{G} = G/N$ and $\bar{b}$ is the unique block of PV/N covering γ. We have that $\bar{b}$ is contained in $B_0(PV)$ for $\tau \in \mathrm{Irr}(\bar{b})$ lies over 1_L (using Theorem 10.20 of [6]). By Lemma 6.3 and Brauer's third main theorem, $B_0(G)$ is the unique block of G containing $\bar{b}^{\bar{G}}$. Hence $\chi \in \mathrm{Irr}_{p'}(B_0(G))$, and by hypothesis, we deduce that χ is σ-invariant. Since

$$\chi_V = e\gamma^\sigma + p\Delta,$$

we conclude that $\gamma^\sigma = \gamma$, and so τ is also σ-invariant.

Step 2. We may assume that $G = KP$, where $K = N\mathbf{C}_G(Q)$ and $Q = P \cap N \in \mathrm{Syl}_p(N)$.

We have that $K \triangleleft G$ by the Frattini argument. Let $R = P \cap K \in \mathrm{Syl}_p(K)$, so that $Q \subseteq R$. Then $\mathbf{C}_G(R) \subseteq \mathbf{C}_G(Q) \subseteq K$. By Lemma 3.1 of [9] and Brauer's third main theorem, $B_0(G)$ is the unique block covering the principal block of K, so $\mathrm{Irr}(G/K) \subseteq \mathrm{Irr}(B_0(G))$. By minimality of G, we have that $\mathbf{N}_G(P)K/K = PK/K \times W/K$. We claim that $W = K$. Since W/K has odd order, it suffices to show that every $\gamma \in \mathrm{Irr}(W/K)$ is σ-invariant, by Lemma 6.5. By Theorem 2.1(d), let $\chi \in \mathrm{Irr}_{p'}(G)$ be such that

$$\chi_W = e\gamma + p\Delta,$$

for some integer e not divisible by p. Since χ lies in $B_0(G)$, then it is fixed by σ. This implies $\gamma^\sigma = \gamma$, as desired. Thus, $\mathbf{N}_G(P)K = PK$ and therefore $\mathbf{N}_G(P) \subseteq PK$. If $KP < G$, then it suffices to show that PK satisfies the hypothesis. Let τ be an odd degree irreducible character lying in $B_0(PK)$. Then $\theta = \tau_K \in \mathrm{Irr}_{p'}(B_0(K))$ is P-invariant. By Problem 4.2 of [6], and using that $\mathbf{N}_G(P) \subseteq PK$, notice that $B_0(KP)^G$ is defined and equals the principal block of G. By Lemma 6.2, let $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ lie over τ (and hence over θ). By Lemma 5.2, χ_K contains a unique P-invariant constituent, namely θ. Since $\chi^\sigma = \chi$, then we have that θ^σ is another P-invariant constituent of χ_K. By uniqueness, $\theta^\sigma = \theta$. Then $\tau^\sigma = \tau$ by Lemma 3.1.

Step 3. We may assume that $PN \triangleleft G$.

By minimality, it is enough to show that $\mathbf{N}_G(P)N$ satisfies the hypothesis of the theorem.

We know that $\mathbf{N}_G(P)N/N = PN/N \times V/N$. Notice that $V = N\mathbf{C}_V(Q)$, because $G = N\mathbf{C}_G(Q)$. Let $\tau \in \mathrm{Irr}_{p'}(B_0(PV))$ and let $\theta \in \mathrm{Irr}(N)$ be a P-invariant constituent of τ_N (by Lemma 5.2). Since θ lies in the principal block of N, by Theorem 6.4, we have that θ is V-invariant. Now, again $B_0(PV)^G = B_0(G)$, and we can find $\chi \in \mathrm{Irr}(B_0(G))$ of odd degree over τ, by Lemma 6.3. By hypothesis, $\chi^\sigma = \chi$, and therefore θ^σ and θ are $\mathbf{N}_G(P)$-conjugate. Since θ is $\mathbf{N}_G(P)$-invariant, we deduce that $\theta^\sigma = \theta$. Now, $V \triangleleft PV$ and every irreducible constituent of τ_V lies in the principal block of V. Since θ is PV-invariant, by Theorem 6.4, we deduce that $\tau_V = e\gamma$, where γ is the only character in the principal block over θ. By uniqueness, γ is σ-invariant, and τ is σ-invariant by Lemma 3.1.

Step 4. Let $M = N\mathbf{C}_G(P)$. Then $PM = G$.

We have that $M \triangleleft G$ by the Frattini argument, so $PM \triangleleft G$. By Lemma 3.1 of [9] and Brauer's first main theorem, $B_0(G)$ is the unique block covering the principal block of PM. Hence $\mathrm{Irr}(G/PM) \subseteq \mathrm{Irr}(B_0(G))$. By hypothesis, every irreducible character of the odd order group G/PM is σ-invariant. By Lemma 6.5, this forces $G = PM$.

Final step. Suppose that $PN < G$. Let $\tau \in \mathrm{Irr}_{p'}(B_0(PN))$ and let $\chi \in \mathrm{Irr}(B_0(G))$ lie over τ (using that $PN \triangleleft G$ and Theorem 9.2 of [6]). Then χ has odd degree and so it is σ-invariant. By Theorem 6.4, $\chi_{PN} = \tau$, and therefore τ is also σ-invariant. By minimality of G, we have that $\mathbf{N}_{PN}(P)$ has a normal p-complement, this is, $\mathbf{N}_{PN}(P) = \mathbf{C}_{PN}(P)P$. Hence

$$\mathbf{N}_G(P) = \mathbf{N}_G(P) \cap (\mathbf{C}_G(P)PN) = \mathbf{C}_G(P)\mathbf{N}_{PN}(P) = \mathbf{C}_G(P)P,$$

as wanted. Hence, we may assume that $NP = G$. Then N is the direct product of non-abelian simple groups $\{S_1, \ldots, S_t\}$ which are transitively permuted by P. By Lemma 4.2 and Lemma 4.3, we may assume that $G = SP$, where $S \triangleleft G$ is a non-abelian simple group of order divisible p. By the first paragraph of this proof S is the unique minimal normal subgroup of G. Thus G is almost simple, and by hypothesis $\mathbf{N}_G(P)$ has a normal p-complement. □

We conclude these notes by showing that Conjecture A implies the 2-self-normalizing conjecture in [7].

Theorem 6.7. *Let G be a finite group and $P \in \mathrm{Syl}_2(G)$. Assume that Conjecture A is true for G. Then $P = \mathbf{N}_G(P)$ if, and only if, every odd-degree irreducible character of G is σ-invariant.*

Proof. If $P = \mathbf{N}_G(P)$, then G has a unique block of maximal defect. Therefore, every irreducible odd-degree character of G is σ-invariant. Conversely, assume that every irreducible odd-degree character of G is σ-invariant. Since G satisfies Conjecture A, we have that $\mathbf{N}_G(P) = P \times V$. Now, let $\alpha \in \mathrm{Irr}(V)$. By Theorem 2.1, let $\chi \in \mathrm{Irr}(G)$ of odd degree such that $\hat{\chi} = \alpha$. Now,

$$\chi_V = e\alpha + 2\Delta\,,$$

where e is odd. Since χ is σ-invariant, it follows that $[\chi_V, \alpha^\sigma] = e$, and by uniqueness we deduce that $\alpha = \alpha^\sigma$. By Lemma 6.5, we conclude that $V = 1$, as desired. □

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