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Brauer characters and coprime action

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It is an open problem to show that under a coprime action, the number of invariant Brauer characters of a finite group is the number of the Brauer characters of the fixed point subgroup. We prove that this is true if the non-abelian simple groups satisfy a stronger condition.

Introduction

Let A and G be finite groups and assume that A acts on G via group automorphisms. In the case when the orders of A and G are coprime and A is solvable, G. Glauberman found in 1968 a canonical bijection between the set of irreducible A-invariant characters

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of G and the set of irreducible characters of the fixed point subgroup $\mathbf{C}_G(A)$. If A is not solvable, another canonical bijection was found by I.M. Isaacs in 1973. The mere fact that the sets of A-invariant irreducible characters of G and of irreducible characters of $\mathbf{C}_G(A)$ have the same cardinality has already important consequences. For instance, it proves that the actions of A on the irreducible characters and on the conjugacy classes of G are permutation isomorphic.

Let us now turn to the analogous question on (p-)Brauer characters (for a prime p), that is an open problem proposed by G. Navarro (see [12]). K. Uno [17] gave a natural map between the set of irreducible A-invariant p-Brauer characters of G and the set of irreducible p-Brauer characters of $\mathbf{C}_G(A)$, when G is a p-solvable group. If G is not p-solvable (and therefore, A is solvable by the Odd Order Theorem), no progress has been made since 1983. The case where A is cyclic is a well-known consequence of the so-called Brauer's argument on character tables. In fact, this also proves the case when G is a quasi-simple group, using the classification of finite simple groups (see [15]). The general case, however, seems out of reach with those methods.

In this paper, we present a reduction of this problem to a question on finite simple groups. We prove that if the finite simple non-abelian groups satisfy the *inductive Brauer–Glauberman condition* (see Definition 6.1), then the answer to Problem 5 of [12] is affirmative. This is our main result.

Theorem A. *Let G and A be finite groups. Suppose that A acts on G with $(|A|, |G|) = 1$. Suppose that all finite non-abelian simple groups involved in G satisfy the inductive Brauer Glauberman condition from Definition 6.1. Then the number of irreducible p-Brauer characters of G fixed by A is the number of irreducible p-Brauer characters of $\mathbf{C}_G(A)$. Consequently, the actions of A on the irreducible p-Brauer characters and on the p-regular conjugacy classes of G are permutation isomorphic.*

In [15] the inductive Brauer–Glauberman condition is checked for all simple groups not of Lie type, as well as for simple groups of Lie type in the defining characteristic case.

In contrast to other inductive conditions on simple groups coming from global/local conjectures, the inductive Brauer–Glauberman condition splits into two parts. One part can be seen as a stronger version of the main statement, since it requires the existence of some equivariant bijection with some Clifford-theoretic properties, that can be seen as analogous to those introduced in [8]. The second (and more surprising) part requires the existence of a bijection which we call *fake Galois action* (Definition 4.4). In certain specific situations, the Glauberman correspondence is only explained by the action of Galois automorphisms on ordinary characters. The fake Galois action required by the inductive Brauer–Glauberman condition provides a tool to overcome the difficulty caused by the fact that, in general, there is no action of the Galois group on the Brauer characters.

This article is structured in the following way: We introduce the notation on ordinary and Brauer characters in Section 1. Afterwards, we collect some well-known results about the Glauberman correspondence. In Section 3, we define a partial order relation on modular character triples. We introduce the notion of *fake Galois conjugate* character triples in Section 4. Afterwards we study coprime actions on the direct product of quasi-simple groups in Section 5. The aforementioned relations between character triples play a central role in the definition of the inductive Brauer–Glauberman condition (see Definition 6.1). In Section 6 we study some implications of the validity of the inductive Brauer–Glauberman condition for central products of covering groups of simple non-abelian groups. Finally, we conclude with the proof of Theorem A.

1. Notation

Unless otherwise stated, all groups considered are finite. We use the notation of [5] for ordinary characters and [13] for Brauer characters.

Let us fix some notation for the rest of the article. We fix a prime p. Choose a maximal ideal M in the ring R of algebraic integers with $p \in M$, so that $F = R/M$ is an algebraically closed field of characteristic p. Write ${}^*\colon R \to F$ to denote the natural ring homomorphism. This homomorphism can be extended to $S = \{r/s \mid r \in R, s \in R\backslash M\}$ by

$$(r/s)^* = r^*(s^*)^{-1},$$

for every $r \in R$ and $s \in R \setminus M$, see Chapter 2 of [12].

Let G be a group. The irreducible p-Brauer characters $\mathrm{IBr}(G)$ of G (with respect to M) are defined in Chapter 2 of [13].

Let $H \subseteq G$ and $\chi \in \mathrm{Irr}(G)$. We write $\mathrm{Irr}(\chi_H)$ to denote the set of irreducible constituents of the restriction of χ to H. Let $\psi \in \mathrm{Irr}(H)$. Then $\mathrm{Irr}(G|\psi)$ denotes the set of irreducible constituents of ψ^G, the induced character of ψ to G. We use the analogous notation for Brauer characters.

We write G^0 to denote the set of p-**regular** elements of G, i.e., the set of elements of G or order coprime to p. If $\chi \in \mathrm{Irr}(G)$, we denote by χ^0 the restriction of χ to G^0, which is a Brauer character by Corollary 2.9 of [13].

If a group A acts on a set Λ, we write A_λ to denote the stabilizer in A of $\lambda \in \Lambda$. If a group A acts on G, then A acts on $\mathrm{Irr}(G)$ and on $\mathrm{IBr}(G)$. We write $\mathrm{Irr}_A(G)$ to denote the set of A-invariant elements of $\mathrm{Irr}(G)$ and $\mathrm{IBr}_A(G)$ to denote the set of A-invariant elements of $\mathrm{IBr}(G)$.

Recall that the Glauberman-Isaacs correspondence is a uniquely defined natural bijection

$$\pi_{(G,A)}\colon \mathrm{Irr}_A(G) \to \mathrm{Irr}(\mathbf{C}_G(A)),$$

that exists whenever a group A acts coprimely on a group G. (As we have mentioned, the case where A is solvable was proved by Glauberman and the remaining case, where $|G|$ is odd, by Isaacs. T. Wolf proved in [18] that both correspondences agreed when both are defined.) In this paper, we are interested in the case where A is solvable, but we do prove some results in more generality for possible future use. According to Theorem 2.1 of [19] the Glauberman-Isaacs correspondence satisfies then

$$\pi_{(G,A)} = \pi_{(T,A/B)} \circ \pi_{(G,B)},$$

for any $B \lhd A$, where $T = \mathbf{C}_G(B)$.

2. Review on character counts above Glauberman-Isaacs correspondents

The main goal of this section is to count Brauer characters lying above characters of normal p'-subgroups and their Glauberman correspondents.

We shall use the following.

Lemma 2.1. *Assume that A acts on G, $K \lhd G$ is A-invariant, $(|G:K|, |A|) = 1$ and $\mathbf{C}_{G/K}(A) = G/K$. If $\eta \in \mathrm{IBr}_A(K)$, then every $\chi \in \mathrm{IBr}(G|\eta)$ is A-invariant.*

Proof. This follows from the same considerations as in Lemma 2.5 of [18]. □

Suppose that $K \lhd G$ and η is an irreducible G-invariant character of K. If η is an ordinary character, then $|\mathrm{Irr}(G|\eta)|$ can be determined by a purely group theoretical method due to P.X. Gallagher. There is an analogous result for Brauer characters. If η is a Brauer character we say that $Kg \in (G/K)^0$ (or g) is η-**good** if every extension $\varphi \in \mathrm{IBr}(\langle K, g\rangle)$ of η is U-invariant where $U/K = \mathbf{C}_{G/K}(Kg)$. Notice that η always extends to $\langle K, g\rangle$ since $\langle K, g\rangle/K$ is cyclic (Theorem 8.12 of [13]). Also, U acts on $\mathrm{IBr}(\langle K, g\rangle|\eta)$ as $\langle K, g\rangle \lhd U$. It is clear that if $Kg \in (G/K)^0$ is η-good, then every G-conjugate of Kg also is, so we can talk about p-regular η-good classes of G/K.

Theorem 2.2. *Suppose that $K \lhd G$ and that $\eta \in \mathrm{IBr}(K)$ is G-invariant. Then, $|\mathrm{IBr}(G|\eta)|$ is equal to the number of p-regular η-good classes of G/K.*

Proof. See Theorem 6.2 of [7]. □

The following is basically a particular case of Theorem 2.12 of [19].

Theorem 2.3. *Let A act coprimely on G. Suppose that $K \lhd G$ is A-invariant and $G = KC$, where $C = \mathbf{C}_G(A)$. Then $\eta \in \mathrm{Irr}_A(K)$ extends to G_η iff $\eta' \in \mathrm{Irr}(K \cap C)$ extends to $C_{\eta'}$, where $\eta' \in \mathrm{Irr}(\mathbf{C}_K(A))$ is the Glauberman-Isaacs correspondent of η.*

Proof. Notice that $G_\eta \cap C = C_{\eta'}$ by Lemma 2.5(b) of [19]. Hence, we may assume that η and η' are C-invariant.

Suppose that η extends to some $\chi \in \mathrm{Irr}(G)$. By [18, Lem. 2.5] we see that $\chi \in \mathrm{Irr}_A(G)$. Since $[G, A] \leq K$, by [19, Thm. 2.12]

$$\eta' = \pi_{(K,A)}(\eta) = \pi_{(K,A)}(\chi_K) = (\pi_{(G,A)}(\chi))_{K \cap C}.$$

Hence $\pi_{(G,A)}(\chi)$ is an extension of η'. Analogously we see that η extends to G, if η' extends to C. □

Next, we need similar results for Brauer characters.

Lemma 2.4. *Let $K \triangleleft G$, $\theta \in \mathrm{Irr}(K)$ be G-invariant and suppose that K is a p'-group. Then θ extends to an ordinary character of G if and only if θ extends to a Brauer character of G.*

Proof. Let $\widetilde{\theta} \in \mathrm{Irr}(G)$ be an extension of θ to G. Then $(\widetilde{\theta})^0$ is a Brauer character of G extending θ. Suppose that $\varphi \in \mathrm{IBr}(G)$ extends θ. Let Q/N be a Sylow q-subgroup of G/N for some prime q. If $q \neq p$, then φ_Q is an ordinary character extending θ. If $q = p$, then θ extends to Q by Corollary (6.28) of [5]. Hence θ extends to G by Corollary (11.31) of [5]. □

Theorem 2.5. *Suppose that A acts coprimely on G. Let $K \triangleleft G$ be A-invariant. Suppose that $p \nmid |K|$ and $G = KC$, where $C = \mathbf{C}_G(A)$. Let $\eta \in \mathrm{Irr}_A(K)$ be G-invariant and write $\eta' \in \mathrm{Irr}(K \cap C)$ to denote its Glauberman-Isaacs correspondent. Then*

$$|\mathrm{IBr}(G|\eta)| = |\mathrm{IBr}(C|\eta')|.$$

Proof. By Theorem 2.2 it suffices to show that for every $c \in C$, the element cK is η-good if and only if the element $cK \cap C$ is η'-good. According to [9, Thm. 4.7] it suffices to show that for every U with $K \leq U \leq G$ and abelian U/K, η extends to U as a Brauer character if and only if η' extends to $U \cap C$ as a Brauer character. We apply Theorem 2.3 and Lemma 2.4 in U. □

Corollary 2.6. *Suppose that A acts coprimely on G. Let $K \triangleleft G$ be A-invariant. Suppose that K is a p'-group and $G = KC$, where $C = \mathbf{C}_G(A)$. Let $N \triangleleft G$ be contained in $K \cap C$ and let $\theta \in \mathrm{Irr}(N)$. Then*

$$|\mathrm{IBr}_A(G|\theta)| = |\mathrm{IBr}(C|\theta)|.$$

Proof. Let $\mathcal{B}$ be a set of representatives of the C-orbits of $\mathrm{Irr}_A(K|\theta)$ and $\mathcal{B}' = \{\pi_{(K,A)}(\eta) \mid \eta \in \mathcal{B}\}$. Then $\mathcal{B}' \subseteq \mathrm{Irr}(K \cap C|\theta)$ by [19, Lem. 2.4] and $\mathcal{B}'$ is a set of representatives of the C-orbits of $\mathrm{Irr}(K \cap C|\theta)$. By Corollary 5.2 of [18], we deduce that the bijection $\pi_{(K,A)}$ is C-equivariant. Hence $\mathcal{B}'$ is a set of representatives of the C-orbits of $\mathrm{Irr}(K \cap C|\theta)$.

Every element of $\mathrm{IBr}_A(G|\theta)$ lies over a unique element of $\mathcal{B}$ and also every element of $\mathrm{IBr}(C|\theta)$ lies over a unique element of $\mathcal{B}'$. Thus

$$|\mathrm{IBr}_A(G|\theta)| = \sum_{\eta \in \mathcal{B}} |\mathrm{IBr}_A(G|\eta)| \quad \text{and} \quad |\mathrm{IBr}(C|\theta)| = \sum_{\eta \in \mathcal{B}} |\mathrm{IBr}(C|\pi_{(K,A)}(\eta))|.$$

By Lemma 2.1, for every $\eta \in \mathcal{B}$ we have that $|\mathrm{IBr}_A(G|\eta)| = |\mathrm{IBr}(G|\eta)|$. Hence, it suffices to show that $|\mathrm{IBr}(G|\eta)| = |\mathrm{IBr}(C|\pi_{(K,A)}(\eta))|$ for every $\eta \in \mathcal{B}$. By the Clifford correspondence on Brauer characters, we may assume that $\eta \in \mathcal{B}$ is G-invariant. Now, the result follows from Theorem 2.5. $\square$

3. Central isomorphisms of modular character triples

In this section we introduce a new order relation on modular character triples. It can be seen as a modular analogue of the notion of central isomorphic (ordinary) character triples introduced in [14]. Afterwards we record several results on the construction of modular character triples respecting this new relation. Those results mainly follow from an easy inspection of analogous results on the related order and equivalence relations given in [14] and [16] on ordinary character triples. Although the consideration follow the ideas in [14] and [16] we do not take into account p-blocks.

Let us start by recalling some facts on modular character triples. If $N \lhd G$ and $\theta \in \mathrm{IBr}(N)$ is G-invariant, then the triple (G, N, θ) is called a **modular character triple**. Isomorphisms of modular character triples (see Definition 8.25 of [13]) establish an equivalence relation on them. If $(\sigma, \tau) : (G, N, \theta) \to (\Gamma, M, \varphi)$ is an isomorphism of modular character triples, then τ is an isomorphism $G/N \to \Gamma/M$ and for every $N \leq J \leq G$ σ yields a bijection $\sigma_J : \mathrm{IBr}(J|\theta) \to \mathrm{IBr}(J^\tau|\varphi)$, where $\tau(J/N) = J^\tau/M$.

If $N \leq J \leq G$, $\psi \in \mathrm{IBr}(J|\theta)$ and $\overline{g} = gN \in G/N$, we define $\psi^{\overline{g}} \in \mathrm{IBr}(J^g|\theta)$ by

$$\psi^{\overline{g}}(x^g) = \psi(x) \text{ for every } x \in J.$$

Note that this is well-defined. We say that a modular character triple isomorphism

$$(\sigma, \tau)\colon (G, N, \theta) \to (\Gamma, M, \varphi)$$

is **strong** if

$$(\sigma_J(\psi))^{\tau(\overline{g})} = \sigma_{J^g}(\psi^{\overline{g}}),$$

for all $\overline{g} \in G/N$, all groups J with $N \leq J \leq G$ and all $\psi \in \mathrm{IBr}(J|\theta)$.

Let (G, N, θ) be a modular character triple. Let $\mathfrak{X}$ be an F-representation affording the Brauer character θ. According to Theorem 8.14 of [13], there exists a projective representation $\mathcal{P}$ of G, such that $\mathcal{P}_N = \mathfrak{X}$. Moreover, we can choose $\mathcal{P}$ such that its factor set α satisfies

$$\alpha(g, n) = 1 = \alpha(n, g),$$

for every $g \in G$ and $n \in N$. In this situation, we say that $\mathcal{P}$ is a **projective representation of G associated to θ**. Then α can be seen as a map on $G/N \times G/N$ (see the remarks after Theorem 8.14 of [13]). Furthermore, if $\mathrm{Proj}_F(J/N, \alpha^{-1})$ is a set of representatives of the similarity classes of irreducible projective F-representations of J/N with factor set α^{-1} where $N \leq J \leq G$, then

$$\mathrm{Rep}_F(J,\theta) = \{Q \otimes \mathcal{P}_J : Q \in \mathrm{Proj}_F(J/N, \alpha^{-1})\}$$

is a set of representatives of similarity classes of representations affording a Brauer character in $\mathrm{IBr}(J|\theta)$ (see for example [14, Thm. 3.1]).

Theorem 3.1. *Let (G,N,θ) and (H,M,θ') be modular character triples satisfying the following assumptions:*

(i) $G = NH$ *and* $M = N \cap H$*,*
(ii) *there exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ of G and H associated to θ and θ', respectively, whose factor sets α and α' coincide via the natural isomorphism $\tau\colon G/N \to H/M$.*

Now, for $N \leq J \leq G$, let $\sigma_J \in \mathrm{Ch}(J|\theta) \to \mathrm{Ch}(J\cap H|\theta')$ be the linear map given by

$$\mathrm{tr}(\mathcal{Q}\otimes\mathcal{P}_J) \mapsto \mathrm{tr}(\mathcal{Q}_{J\cap H}\otimes\mathcal{P}'_{J\cap H})$$

for any projective representation $\mathcal{Q}$ of J/N, whose factor set is inverse to the one of $\mathcal{P}_J$. Then

$$(\sigma,\tau)\colon (G,N,\theta) \to (H,M,\theta')$$

is a strong isomorphism of modular character triples.

Proof. See Theorem 3.2 of [14]. □

In the situation of Theorem 3.1, we say that (σ,τ) an **isomorphism of modular character triples given by $\mathcal{P}$ and $\mathcal{P}'$**.

The following gives examples of (strong) modular character triple isomorphisms.

Lemma 3.2. *Let (G,N,θ) be a modular character triple.*

(a) *If $\alpha\colon G\to H$ is an isomorphism of groups, then (G,N,θ) is strongly isomorphic to (H,M,φ) where $M=\alpha(N)$ and $\varphi\in\mathrm{IBr}(M)$ is the character defined by $\varphi(\alpha(n)) = \theta(n)$ for every $n\in N^0$.*
(b) *If $M \lhd G$ and $M\leq\ker(\theta)$, then (G,N,θ) and $(G/M,N/M,\overline{\theta})$ are strongly isomorphic, where $\overline{\theta}(nM)=\theta(n)$ for every $n\in N^0$.*
(c) *Suppose that $\mu\colon G\to H$ is an epimorphism and that $K=\ker(\mu)\leq\ker(\theta)$. Then (G,N,θ) and (H,M,φ) are strongly isomorphic, where $M=\mu(N)$ and $\varphi\in\mathrm{Irr}(M)$ is the unique character of M with $\varphi(\mu(n))=\theta(n)$ for every $n\in N^0$.*
(d) *Suppose that there exists some $\eta\in\mathrm{IBr}(G)$ such that $\eta_N\theta=\varphi\in\mathrm{IBr}(N)$. Then (G,N,θ) and (G,N,φ) are strongly isomorphic.*

Proof. The first two parts are straightforward from the definition, and the third is a direct consequence of (a) and (b). To prove (d), let $\mathcal{P}$ be a projective representation of G associated to θ. Write $\mathcal{P}' = \mathcal{P} \otimes \mathcal{D}$, where $\mathcal{D}$ is a representation of G affording η. It is straightforward to check that $\mathcal{P}'$ is a projective representation of G associated to φ and that the factor sets of $\mathcal{P}$ and $\mathcal{P}'$ agree, since $\mathcal{D}$ is an actual representation. Now part (d) follows by applying Theorem 3.1. $\square$

Definition 3.3. Let (G, N, θ) and (H, M, θ') be modular character triples satisfying the following conditions:

(i) $G = NH$, $M = N \cap H$ and $\mathbf{C}_G(N) \leq H$.
(ii) There exist a projective representation $\mathcal{P}$ of G associated to θ with factor set α and a projective representation $\mathcal{P}'$ of H associated to θ' with factor set α' such that
  (ii.1) $\alpha|_{H\times H} = \alpha'$, and
  (ii.2) for every $c \in \mathbf{C}_G(N)$ the scalar matrices $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated with the same scalar (notice that $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are scalar by Schur's Lemma).

Let (σ, τ) be the isomorphism of character triples given by $\mathcal{P}$ and $\mathcal{P}'$ as in Theorem 3.1. Then we call (σ, τ) a **central isomorphism of modular character triples**, and we write

$$(G, N, \theta) \succ_{Br,c} (H, M, \theta').$$

By Lemma 3.3 of [14], condition (ii.2) above is equivalent to

$$\mathrm{IBr}(\psi_{\mathbf{C}_J(N)}) = \mathrm{IBr}(\sigma_J(\psi)_{\mathbf{C}_J(N)}),$$

for every $\psi \in \mathrm{IBr}(J|\theta)$ and $N \leq J \leq G$. This is a modular analog of the relation $\sim_c$ defined in [14]. In particular, the fact that $\succ_{Br,c}$ defines an order relation on the set of modular character triples and is thereby transitive follows from Lemma 3.8 of [14].

We shall frequently use the following.

Lemma 3.4. *Let (G, N, θ) and (H, M, θ') be modular character triples with*

$$(G, N, \theta) \succ_{Br,c} (H, M, \theta').$$

(a) *If $\mathcal{P}$ is a projective representation of G associated to θ with factor set α, then there exists a projective representation $\mathcal{P}'$ of H associated to θ' with factor set α' such that*
  (a.i) *$\alpha|_{H\times H} = \alpha'$, and*
  (a.ii) *for every $c \in \mathbf{C}_G(N)$ the scalar matrices $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated with the same scalar.*
(b) *If $\mathcal{P}'$ is a projective representation of H associated to θ' with factor set α', then there exists a projective representation $\mathcal{P}$ of G associated to θ with factor set α such that*

(b.i) $\alpha|_{H\times H} = \alpha'$, *and*

(b.ii) *for every* $c \in \mathbf{C}_G(N)$ *the scalar matrices* $\mathcal{P}(c)$ *and* $\mathcal{P}'(c)$ *are associated with the same scalar.*

Proof. Let $\mathcal{Q}$ and $\mathcal{Q}'$ be projective representations as in 3.3. In the situation of (a), by Theorem 8.14 of [13], there exists a unique map $\xi\colon G/N \to F^\times$ such that $\mathcal{P}(g) = \mathcal{Q}(g)\xi(g)$ for every $g \in G$. Then $\mathcal{P}' := \mathcal{Q}'\xi$ is a projective representation of H having the required properties. Analogous arguments prove part (b). $\square$

The following result analyzes the behavior of central isomorphic modular character triples with respect to certain quotients.

Lemma 3.5. *Suppose that* $(G, N, \theta) \succ_{Br,c} (H, M, \theta')$. *Let* $\epsilon\colon G \to G_1$ *be an epimorphism. Write* $N_1 = \epsilon(N)$ *and* $H_1 = \epsilon(H)$. *Suppose that* $Z = \ker(\epsilon) \leq \mathbf{Z}(N) \cap \ker(\theta) \cap \ker(\theta')$ *and* $\epsilon(\mathbf{C}_G(N)/Z) = \mathbf{C}_{G_1}(N_1)$. *Then*

$$(G_1, N_1, \theta_1) \succ_{Br,c} (H_1, M_1, \theta_1'),$$

where $\theta_1 \in \mathrm{IBr}(N_1)$ *is such that* $\theta = \theta_1 \circ \epsilon$ *and* $\theta_1' \in \mathrm{IBr}(M_1)$ *is such that* $\theta' = \theta_1' \circ \epsilon$.

Proof. See the proof of Corollary 4.5 of [14]. Note that there the stronger assumption $Z \leq \mathbf{Z}(G)$ is only used for the block-theoretic statements that are not relevant in the context here. $\square$

Let G_i be finite groups for $i = 1, 2$. Recall that $\mathrm{IBr}(G_1 \times G_2) = \{\theta_1 \times \theta_2 \mid \theta_i \in \mathrm{IBr}(G_i)\}$. The following lemma tells us how to construct central isomorphic modular character triples using direct and semi-direct products.

Lemma 3.6. *Let* m *be a positive integer. Suppose that* $(G_i, N_i, \theta_i) \succ_{Br,c} (H_i, M_i, \theta_i')$ *for* $1 \leq i \leq m$. *Assume that all* G_i, H_i, N_i, M_i *are isomorphic. Write* $G = \times_{i=1}^m G_i$, $H = \times_{i=1}^m H_i$, $N = \times_{i=1}^m N_i$, $M = \times_{i=1}^m M_i$, $\theta = \theta_1 \times \cdots \times \theta_m \in \mathrm{IBr}(N)$ *and* $\theta' = \theta_1' \times \cdots \times \theta_m' \in \mathrm{IBr}(M)$. *Suppose that* $\mathfrak{S}_m$ *acts on* G *by permutation of the isomorphic factors* G_i, *inducing an action on* N, H *and* M *by permuting their factors. Assume further* $(\mathfrak{S}_m)_\theta = (\mathfrak{S}_m)_{\theta'}$. *Then*

$$(G \rtimes (\mathfrak{S}_m)_\theta, N, \theta) \succ_{Br,c} (H \rtimes (\mathfrak{S}_m)_\theta, M, \theta').$$

Proof. Follows from the same arguments as in Theorems 5.1 and 5.2 in [16]. $\square$

The following result is key, and lies deeper than the others already mentioned in this section.

Theorem 3.7. *Let* $(G, N, \theta) \succ_{Br,c} (H, M, \theta')$. *Suppose that* $N \lhd G_1$ *and* $G_1/\mathbf{C}_{G_1}(N)$ *is equal to* $G/\mathbf{C}_G(N)$ *as a subgroup of* $\mathrm{Aut}(N)$. *Let* $H_1 \leq G_1$ *such that* $H_1 \geq \mathbf{C}_{G_1}(N)$, *and*

$H_1/\mathbf{C}_{G_1}(N)$ and $H/\mathbf{C}_G(N)$ are equal as subgroups of $\mathrm{Aut}(N)$*. Then*

$$(G_1, N, \theta) \succ_{Br,c} (H_1, M, \theta').$$

Proof. This is, for Brauer characters, a particular case of Theorem 5.3 of [16]. $\square$

The following nearly trivial observation will be useful later.

Lemma 3.8. *Suppose that $(G, N, \theta) \succ_{Br,c} (H, M, \theta')$. Let Γ with $N \le G \le \Gamma$ and $x \in \Gamma$. Then $(G^x, N^x, \theta^x) \succ_{Br,c} (H^x, M^x, (\theta')^x)$.*

Proof. According to Definition 3.3 one can obtain projective representations giving $(G^x, N^x, \theta^x) \succ_{Br,c} (H^x, M^x, (\theta')^x)$ by conjugation with x from those projective representations giving $(G, N, \theta) \succ_{Br,c} (H, M, \theta')$. $\square$

Next, we discuss the Brauer characters of a central product of groups and their relation with Brauer central isomorphic character triples.

Lemma 3.9. *Let $N \lhd G$ and T_i with $N \le T_i \le G$ be such that $G/N = T_1/N \times \cdots \times T_k/N$. Suppose that $[T_i, T_j] = 1$ for every $i \ne j$. Given $\theta \in \mathrm{IBr}(N)$ and $\varphi_i \in \mathrm{IBr}(T_i|\theta)$, there is a unique $\chi = \varphi_1 \cdot \ldots \cdot \varphi_k \in \mathrm{IBr}(G|\theta)$ such that χ_{T_i} is a multiple of φ_i. Moreover, the map*

$$\begin{aligned} \mathrm{IBr}(T_1|\theta) \times \cdots \times \mathrm{IBr}(T_k|\theta) &\to \mathrm{IBr}(G|\theta) \\ (\varphi_1, \ldots, \varphi_k) &\mapsto \varphi_1 \cdot \ldots \cdot \varphi_k \end{aligned}$$

is a natural bijection.

Proof. This is a natural adaptation of Lemma 5.1 of [8] to Brauer characters. $\square$

Lemma 3.10. *Let $N \lhd H \le G$. Let $Z \lhd G$ be an abelian group such that $Z \le \mathbf{C}_G(N)$ and $Z \cap N = Z \cap M$. Suppose that $(H, N, \theta) \succ_{Br,c} (K, M, \theta')$. Then*

$$(HZ, NZ, \theta \cdot \nu) \succ_{Br,c} (KZ, MZ, \theta' \cdot \nu)$$

for every $\nu \in \mathrm{IBr}_H(Z|\lambda)$ where $\lambda \in \mathrm{IBr}(\theta_{Z \cap M})$.

Proof. See Proposition 3.9(b) of [14] together with Theorem 3.7. $\square$

Later in this paper, we will need to control the values of certain projective representations, see for instance Lemma 4.7. The following method for constructing projective representations from representations given in [14] will be useful.

We recall that if $\epsilon\colon \hat{G} \to G$ is an epimorphism with $\ker(\epsilon) = Z$, then a **Z-section** $\mathrm{rep}\colon G \to \hat{G}$ of ϵ is a map such that $\epsilon \circ \mathrm{rep} = \mathrm{id}_G$ and $\mathrm{rep}(1) = 1$.

Theorem 3.11. *Let (G, N, θ) be a character triple with $\theta \in \mathrm{Irr}(N)$. There exists a finite group $\hat{G}$, an epimorphism $\epsilon\colon \hat{G} \to G$ with cyclic kernel $Z \leq \mathbf{Z}(\hat{G})$ and a Z-section* $\mathrm{rep}\colon G \to \hat{G}$ *satisfying:*

(a) $\hat{N} = N_1 \times Z = \epsilon^{-1}(N)$, *$N_1 \cong N$ via $\epsilon|_{N_1}$ and $N_1 \lhd \hat{G}$. The action of $\hat{G}$ on $\hat{N}$ coincides with the action of G on N via ϵ.*

(b) *The character $\theta_1 = \theta \circ \epsilon|_{N_1} \in \mathrm{Irr}(N_1)$ extends to $\hat{G}$. The Z-section* $\mathrm{rep}\colon G \to \hat{G}$ *satisfies* $\mathrm{rep}(n) \in N_1$, $\mathrm{rep}(ng) = \mathrm{rep}(n)\mathrm{rep}(g)$ *and* $\mathrm{rep}(gn) = \mathrm{rep}(g)\mathrm{rep}(n)$ *for every $n \in N$ and $g \in G$.*

(c) $\epsilon(\mathbf{C}_{\hat{G}}(\hat{N})) = \mathbf{C}_G(N)$.

In particular, if $\mathcal{D}$ is a representation of $\hat{G}$ such that $\mathcal{D}|_{N_1}$ affords θ_1, then the map $\mathcal{P}$ defined for every $g \in G$ by

$$\mathcal{P}(g) = \mathcal{D}(\mathrm{rep}(g))$$

is a projective representation of G associated to θ.

Proof. See Theorem 4.1 of [14]. $\square$

Remark 3.12. The analog of Theorem 3.11 for modular character triples (G, N, φ) also holds.

4. Fake Galois action

We will start this section by reviewing how the Glauberman correspondence and a certain Galois action on ordinary characters are naturally related. We then describe the interplay between the projective representations associated with Galois conjugate characters. Afterwards we introduce a new relation between modular character triples which resembles Galois action. Finally, we investigate properties of this new relation.

Assume that a group $A \leq \mathfrak{S}_m$ is given, and let $G > 1$ be a group. Then A acts naturally on $\widetilde{G} = G \times \cdots \times G = G^m$, the external direct product of m copies of G. Furthermore, if A is transitive, then

$$\mathbf{C}_{\widetilde{G}}(A) = \{(g, \ldots, g) \in \widetilde{G} \mid g \in G\}$$

and an irreducible A-invariant character χ of $\widetilde{G}$ has the form

$$\chi = \theta \times \cdots \times \theta$$

for some $\theta \in \mathrm{Irr}(G)$. The group $\mathbf{C}_{\widetilde{G}}(A)$ is naturally isomorphic to $\widetilde{G}$. Furthermore, if $(|A|, |G|) = 1$, then the Glauberman correspondent of χ is some Galois conjugate of θ viewed as a character of $\mathbf{C}_{\widetilde{G}}(A)$. (See Proposition 4.2 below.)

Notation 4.1. Let $\Delta^m : G \to \widetilde{G}$ be the injective morphism defined by $g \mapsto (g, \ldots, g)$ for every $g \in G$. Then Δ^m defines natural bijections $\mathrm{Irr}(G) \to \mathrm{Irr}(\Delta^m G)$ and $\mathrm{IBr}(G) \to \mathrm{IBr}(\Delta^m G)$, where $\Delta^m G$ is the image $\Delta^m(G)$, that is the diagonal subgroup of $\widetilde{G}$. If from the context m is clear, we omit the superscript m and write Δ instead of Δ^m.

Whenever n is a natural number, we write $\mathbb{Q}_n$ to denote the cyclotomic extension of $\mathbb{Q}$ obtained by adjoining a primitive n-th root of unity to $\mathbb{Q}$. For a fixed positive integer m, let π be the set of primes dividing m. Let n be any positive integer, we denote by $\sigma_m \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ the Galois automorphism defined by

$$\sigma_m(\xi) = \xi^{\sigma_m} = \xi_\pi \xi_{\pi'}^m,$$

for every root of unity $\xi \in \mathbb{Q}_n$, where ξ_π and $\xi_{\pi'}$ are respectively the π-part and the π'-part of ξ. This defines a Galois automorphism of $\mathbb{Q}_n$ for any n.

If χ is any function taking values in a cyclotomic field, notice that χ^{σ_m} defined by

$$\chi^{\sigma_m}(x) = \chi(x)^{\sigma_m}$$

is well-defined. For instance, if χ is an irreducible character of a group G, then χ^{σ_m} is an irreducible character of G. However, this is not longer true if χ is an irreducible Brauer character of G.

Proposition 4.2. *Assume that a solvable subgroup $A \leq \mathfrak{S}_m$ is transitive. Let G be a finite group with $(|A|, |G|) = 1$. Let $\widetilde{G}$ be the m-th direct product of copies of G. Let $\theta \in \mathrm{Irr}(G)$ and let $\chi = \theta \times \cdots \times \theta \in \mathrm{Irr}_A(\widetilde{G})$. Then, the Glauberman correspondent of χ is the character $\chi' = \Delta(\theta^{\sigma_m})$, where θ^{σ_m} is the image of θ under the Galois automorphism σ_m.*

Proof. This is essentially the content of Exercise 13.11 of [5]. To do the case where A is cyclic of prime order, use Exercise 4.7 of [5]. The general case, follows by induction on $|A|$. □

It is clear that the Clifford theory of two ordinary irreducible Galois conjugate characters is related.

Proposition 4.3. *Let $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$. Let m be an integer coprime to $|N|$. Then θ and $\theta' = \theta^{\sigma_m}$ satisfy:*

(a) $G_\theta = G_{\theta'}$.

(b) *Assume $G = G_\theta$. There exist $\mathcal{P}$ and $\mathcal{P}'$ projective representations of G associated to θ and θ' such that:*

  (b.1) *the factor sets α and α' of $\mathcal{P}$ and $\mathcal{P}'$ satisfy*

$$\alpha(g, g')^{\sigma_m} = \alpha'(g, g')$$

  for every $g, g' \in G$, and

(b.2) *for every* $c \in \mathbf{C}_G(N)$ *the scalar matrices* $\mathcal{P}(c)$ *and* $\mathcal{P}'(c)$ *are associated with* ξ *and* ξ^{σ_m} *for some root of unity* $\xi \in \mathbb{Q}_{|G|}$.

Proof. By Theorem 3.11 and [5, Thm. 10.3], there exists a projective representation $\mathcal{P}$ of G associated to θ whose entries are in $\mathbb{Q}_k$ for some $k \geq 1$. Choose $\mathcal{P}' = \mathcal{P}^{\sigma_m}$. The result follows from straightforward calculations. □

Let U be the subgroup of p'-th roots of unity of $\mathbb{C}^\times$ (the roots of order coprime to p). Then the epimorphism $^*: S \to F$ restricts to a group isomorphism $^*: U \to F^\times$ by [13, Lem. 2.1]. Let $\omega_m: F^\times \to F^\times$ be the map that σ_m induces via $^*: U \to F^\times$. We denote by ζ^{ω_m} the image of $\zeta \in F^\times$ under ω_m.

As we have said, the Galois group $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ does not act in general on $\mathrm{IBr}(G)$ (see page 43 of [13]). The fact stated in Proposition 4.3 motivates the following relation between modular character triples, which characterizes when a pair of irreducible Brauer characters behaves like Galois conjugates via σ_m with respect to the Clifford theory.

Definition 4.4. Let (G, N, φ) and (G, N, φ') be modular character triples and let m be an integer coprime to $|N|$. We write

$$(G, N, \varphi)^{(m)} \approx (G, N, \varphi'),$$

if there exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ of G associated to φ and φ' such that:

(i) for every $x, y \in G$ the factor sets α and α' of $\mathcal{P}$ and $\mathcal{P}'$ satisfy

$$\alpha(x, y)^{\omega_m} = \alpha'(x, y),$$

and

(ii) for every $c \in \mathbf{C}_G(N)$ the scalar matrices $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated with scalars ζ and ζ^{ω_m}.

In the above situation we may say that φ' is a **fake m-th Galois conjugate** of φ with respect to $N \lhd G$.

We mention here some immediate consequences.

Remark 4.5. Let (G, N, φ) be a modular character triple, m an integer coprime to $|N|$.

(a) Then φ^{σ_m} satisfies $\varphi^{\sigma_m}(n) = \varphi(n^m)$ for every $n \in N$.
(b) If φ is linear, then $\varphi^{\sigma_m} = \varphi^m \in \mathrm{IBr}(N)$.
(c) Let $\varphi' \in \mathrm{IBr}(N)$ and $\nu \in \mathrm{IBr}(\varphi|_{\mathbf{Z}(N)})$. Then $(N, N, \varphi)^{(m)} \approx (N, N, \varphi')$ if and only if $\mathrm{IBr}(\varphi'|_{\mathbf{Z}(N)}) = \{\nu^m\}$.

We give an alternative reformulation for $(G, N, \varphi)^{(m)} \approx (G, N, \varphi')$ in Lemma 4.8. In a first step we show that for a Brauer character there always exists a projective representation associated to it with particular properties.

Lemma 4.6. *Let $N \lhd G$ and $\widetilde{\varphi} \in \mathrm{IBr}(G)$ with $\varphi := \widetilde{\varphi}_N \in \mathrm{IBr}(N)$. Then $\mathbf{C}_G(N)' \leq \ker(\widetilde{\varphi})$.*

Proof. Let $\mathcal{D}$ be a representation affording $\widetilde{\varphi}$ and $c \in \mathbf{C}_G(N)$. Then $\mathcal{D}(c)$ commutes with the irreducible representation $\mathcal{D}_N$. By Schur's Lemma, this implies that $\mathcal{D}(c)$ is scalar. The map $\lambda \colon \mathbf{C}_G(N) \to F^\times$ given by $\mathcal{D}(c) = \lambda(c)I$ is a homomorphism. Hence $\mathbf{C}_G(N)/\ker(\lambda)$ is an abelian p'-group. This proves the statement. $\square$

Lemma 4.7. *Let (G, N, φ) be a modular character triple, m an integer coprime to $|N|$ and $(F^\times)_{m'}$ the subgroup in $F^\times$ of elements of order coprime to m. Then, there exists a projective representation $\mathcal{P}$ of G associated to φ with factor set α such that:*

(i) *$\alpha(g, g') \in (F^\times)_{m'}$ for every $g, g' \in G$, and*
(ii) *for every $c \in \mathbf{C}_G(N)$, $\mathcal{P}(c)$ is the scalar matrix associated to some $\xi \in (F^\times)_{m'}$.*

Proof. Let π be the set of primes dividing m different from p, so that $p \notin \pi$.

First assume that φ extends to some $\chi \in \mathrm{IBr}(G)$. In this case it suffices to prove that there exists an extension of φ to G such that every π-element of $\mathbf{C}_G(N)$ lies in $\ker(\chi)$. By Lemma 4.6, we may assume that $\mathbf{C}_G(N)' = 1$. Hence $\mathbf{C}_G(N)$ is abelian and the Hall π-subgroup C of $\mathbf{C}_G(N)$ satisfies $C \lhd G$. Then $N \cap C = 1$ and $NC \cong N \times C$. We want to prove that there exists an extension of φ to G containing C in its kernel. It suffices to prove that the character $\overline{\varphi} \in \mathrm{IBr}(NC/C)$ given by φ extends to G/C.

Let q be any prime. If $q \in \pi$ and $Q/N \in \mathrm{Syl}_q(G/N)$, then $\varphi \in \mathrm{IBr}(NC/C)$ extends to QC/C because of $(q, |NC : C|) = 1$ by Theorem 8.13 of [12]. If $q \notin \pi$ and $Q/N \in \mathrm{Syl}_q(G/N)$ we have that $Q \cap C = 1$. Then $\chi_Q \in \mathrm{IBr}(Q)$ defines an irreducible Brauer character of $QC/C \cong Q$ that is an extension of $\overline{\varphi}$. According to Theorem 8.29 of [12], this implies that φ extends to G/C, and we are done in this case.

Now, we consider the general case. By Theorem 3.11, respectively Remark 3.12, there exists a central extension $\epsilon \colon \hat{G} \to G$ of G with finite cyclic kernel Z and a Z-section $\mathrm{rep} \colon G \to \hat{G}$ of ϵ such that:

(a) $\hat{N} = N_1 \times Z = \epsilon^{-1}(N)$, the groups N_1 and N are isomorphic via $\epsilon|_{N_1}$ and $N_1 \lhd \hat{G}$. Moreover, the action of $\hat{G}$ on $\hat{N}$ coincides with the action of G on N via ϵ.
(b) $\varphi_1 = \varphi \circ \epsilon|_{N_1} \in \mathrm{IBr}(N_1)$ extends to $\hat{G}$. The Z-section $\mathrm{rep} \colon G \to \hat{G}$ satisfies $\mathrm{rep}(n) \in N_1$, $\mathrm{rep}(ng) = \mathrm{rep}(n)\mathrm{rep}(g)$ and $\mathrm{rep}(g)\mathrm{rep}(n) = \mathrm{rep}(gn)$ for every $n \in N$ and $g \in G$.
(c) $\epsilon(\mathbf{C}_{\hat{G}}(\hat{N})) = \mathbf{C}_G(N)$.

According to (b) the character φ_1 extends to $\hat{G}$. By the first part of the proof, there is an extension $\widetilde{\varphi}_1 \in \mathrm{IBr}(\hat{G})$ such that every π-element of $\mathbf{C}_{\hat{G}}(\hat{N})$ lies in $\ker(\widetilde{\varphi}_1)$. Let $\mathcal{D}$ be a representation affording $\widetilde{\varphi}_1$ and let $\mathcal{P}\colon G \to \mathrm{GL}_{\varphi(1)}(F)$ be defined by

$$\mathcal{P}(g) = \mathcal{D}(\mathrm{rep}(g)) \text{ for every } g \in G.$$

For every $g, g' \in G$ we obtain

$$\mathcal{P}(g)\mathcal{P}(g') = \mathcal{D}(z_{g,g'})\mathcal{P}(gg'),$$

where $z_{g,g'} \in Z \leq \mathbf{Z}(\hat{G})$ is given by $\mathrm{rep}(g)\mathrm{rep}(g') = z_{g,g'}\mathrm{rep}(gg')$. Since $\mathcal{D}(z_{g,g'})$ is a scalar matrix, $\mathcal{P}$ is a projective representation of G associated to φ with factor set $\alpha\colon G \times G \to F^\times$ defined by $\alpha(g,g')I = \mathcal{D}(z_{g,g'})$ for every $g, g' \in G$. It is straightforward to check that $\mathcal{P}$ satisfies the required properties. $\square$

As a consequence of Lemma 4.7, we can reformulate Definition 4.4 in the following convenient way.

Lemma 4.8. *Let (G, N, φ) and (G, N, φ') be modular character triples, and let m be coprime to $|N|$. Then the following are equivalent:*

(a) $(G, N, \varphi)^{(m)} \approx (G, N, \varphi')$,

(b) *there exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ of G associated to φ and φ' both having the properties given in 4.7 and such that*

 (i') *the factor sets α and α' of $\mathcal{P}$ and $\mathcal{P}'$ satisfy*

$$\alpha(g,g')^m = \alpha'(g,g') \textit{ for every } g, g' \in G,$$

 and

 (ii') *for every $c \in \mathbf{C}_G(N)$, the scalar matrices $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated with scalars ζ and ζ^m respectively.*

Proof. Since $(G, N, \varphi)^{(m)} \approx (G, N, \varphi')$ there exist projective representations $\mathcal{Q}$ and $\mathcal{Q}'$ of G associated to φ and φ' having the properties listed in Definition 4.4. Let $\mathcal{P}$ be a projective representation of G associated to φ with the properties described in Lemma 4.7. Then there exists a map $\xi\colon G \to F^\times$ with $\mathcal{P} = \xi\mathcal{Q}$, see Theorem 3.1(b) of [14]. Hence ξ is constant on N-cosets in G. Let $\xi'\colon G \to F$ with $\xi'(g) := \xi(g)^{\omega_m}$ for every $g \in G$. Then $\mathcal{P}' := \xi'\mathcal{Q}'$ is again a projective representation of G associated to φ'.

In order to verify that $\mathcal{P}$ and $\mathcal{P}'$ satisfy the condition in (i') let $g, g' \in G$. According to Definition 4.4(i), the factor sets β and β' of $\mathcal{Q}$ and $\mathcal{Q}'$ satisfy $\beta(g,g')^{\omega_m} = \beta'(g,g')$. By the definitions, the factor sets α and α' of $\mathcal{P}$ and $\mathcal{P}'$ satisfy

$$\alpha(g,g') = \frac{\xi(g)\xi(g')}{\xi(gg')}\beta(g,g') \text{ and } \alpha'(g,g') = \frac{\xi'(g)\xi'(g')}{\xi'(gg')}\beta'(g,g').$$

This implies that $\alpha(g,g')^{\omega_m} = \alpha'(g,g')$. By the choice of $\mathcal{P}$, $\alpha(g,g')$ is an m'-root of unity in $F^\times$, hence $\alpha'(g,g') = \alpha(g,g')^m$ also is. This proves that $\mathcal{P}$ and $\mathcal{P}'$ satisfy the condition (i').

In order to verify that $\mathcal{P}$ and $\mathcal{P}'$ satisfy the condition in (ii') let $c \in \mathbf{C}_G(N)$ and $\zeta \in F^\times$ be the scalar associated to $\mathcal{Q}(c)$. According to Definition 4.4(ii), $\mathcal{Q}'(c)$ is the scalar matrix associated with ζ^{ω_m}. By definition $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are the scalar matrices associated with $\zeta\xi(c)$ and $(\zeta\xi(c))^{\omega_m}$. By the choice of $\mathcal{P}$, $\zeta\xi(c)$ is an m'-root of unity in $F^\times$, hence $(\zeta\xi(c))^{\omega_m} = (\zeta\xi(c))^m$ also is. Hence they satisfy the condition (ii').

Moreover we see that $\mathcal{P}'$ is a projective representation having the properties mentioned in 4.7.

To prove the converse, just notice that the projective representations $\mathcal{P}$ and $\mathcal{P}'$ in (b) have the properties described in Lemma 4.7 and recall that ω_m act on m'-th roots of unity of $F^\times$ by raising them to its m-th power. It is then immediate that $\mathcal{P}$ and $\mathcal{P}'$ give $(G,N,\theta)^{(m)} \approx (G,N,\theta')$. $\square$

The notion of fake Galois conjugate character triples is important in our later application, since it allows us to construct from Galois conjugate character triples new central isomorphic character triples, see Theorem 4.10 below.

Notation 4.9. Let m be a positive integer. For groups $H \leq G$ we denote by H^m the external direct product of m copies of H, whenever the context is clear. For groups $K, H \leq G$ with $K \leq \mathbf{N}_G(H)$, we denote by $\breve{\Delta}_H K$ the group $\langle H^m, \Delta K\rangle$. Note that $\langle H^m, \Delta K\rangle = H^m(\Delta K)$ in G^m because of $K \leq \mathbf{N}_G(H)$.

Theorem 4.10. *Let (G,N,φ) and (G,N,φ') be modular character triples. Let $Z = \mathbf{Z}(N)$, $\widetilde{N} := N^m$, $\breve{G} = \breve{\Delta}_Z G$ and $\breve{N} = \breve{\Delta}_Z N$. Further let $\nu \in \mathrm{IBr}(\varphi_Z)$, $\widetilde{\varphi} = \varphi \times \cdots \times \varphi \in \mathrm{IBr}(\widetilde{N})$, $\widetilde{\nu} = \nu \times \cdots \times \nu \in \mathrm{IBr}(\widetilde{Z})$ and $\breve{\varphi} = \Delta\varphi' \cdot \widetilde{\nu} \in \mathrm{IBr}(\breve{N})$. Then the following are equivalent:*

(i) $(G,N,\varphi)^{(m)} \approx (G,N,\varphi')$,
(ii) $(\widetilde{N}(\breve{G} \rtimes \mathfrak{S}_m), \widetilde{N}, \widetilde{\varphi}) \succ_{Br,c} (\breve{G} \rtimes \mathfrak{S}_m, \breve{N}, \breve{\varphi})$.

Proof. We first prove that (i) implies (ii). Let $\mathcal{Q}$ and $\mathcal{Q}'$ be projective representations of G giving

$$(G,N,\varphi)^{(m)} \approx (G,N,\varphi')$$

satisfying (b.i') and (b.ii') of Lemma 4.8.

We construct projective representations $\mathcal{P}_0$ and $\mathcal{P}'_0$ of $\widetilde{N}\breve{G}$ and $\breve{G}$ associated to $\widetilde{\varphi}$ and $\breve{\varphi}$ respectively. Note that $\widetilde{N}\breve{G} = \breve{\Delta}_N G$. Straightforward calculations show that the map $\mathcal{P}_0 \colon \widetilde{N}\breve{G} \to \mathrm{GL}_{\varphi(1)^m}(F)$ given by

$$\mathcal{P}_0((n_1,\ldots,n_m)\Delta g) = \mathcal{Q}(n_1 g) \otimes \cdots \otimes \mathcal{Q}(n_m g) \text{ for every } (n_1,\ldots,n_m) \in \widetilde{N} \text{ and } g \in G$$

defines a projective representation associated to $\widetilde{\varphi}$. The factor set α_0 of $\mathcal{P}_0$ satisfies

$$\alpha_0(\widetilde{n}\Delta g, \widetilde{n}'\Delta g') = \beta(g, g')^m \text{ for every } \widetilde{n}, \widetilde{n}' \in \widetilde{N} \text{ and } g, g' \in G,$$

where β denotes the factor set of $\mathcal{Q}$. Let τ be an F-representation of $\widetilde{Z}$ affording $\widetilde{\nu}$ where $\widetilde{Z} := Z^m$. The map $\mathcal{P}'_0 \colon \breve{G} \to \mathrm{GL}_{\breve{\varphi}(1)}(F)$ given by

$$\mathcal{P}'_0(\widetilde{z}\Delta g) = \tau(\widetilde{z})\mathcal{Q}'(g) \text{ for every } \widetilde{z} \in \widetilde{Z} \text{ and } g \in G$$

is a projective representation associated to $\breve{\varphi}$. (The map is well-defined since $\mathrm{IBr}(\widetilde{\nu}_{\Delta Z}) = \mathrm{IBr}((\Delta\varphi')_{\Delta Z})$.) The factor set α'_0 of $\mathcal{P}'_0$ satisfies

$$\alpha'_0(\widetilde{z}\Delta g, \widetilde{z}'\Delta g') = \beta'(g, g') = \beta(g, g')^m = \alpha_0(\widetilde{z}\Delta g, \widetilde{z}'\Delta g')$$

for every $g, g' \in G$ and $z, z' \in \widetilde{Z}$, where β' denotes the factor set of $\mathcal{Q}'$. (Recall that $\beta'(g, g') = \beta(g, g')^m$ since $\mathcal{Q}$ and $\mathcal{Q}'$ satisfy Lemma 4.8(b.i')).

In the next step we extend $\mathcal{P}_0$ and $\mathcal{P}'_0$ to projective representations $\mathcal{P}$ and $\mathcal{P}'$ of $(\widetilde{N}\breve{G}) \rtimes \mathfrak{S}_m$ and $\breve{G} \rtimes \mathfrak{S}_m$ with the required properties. Note that $\mathfrak{S}_m$ has a natural action on the tensor space $\bigotimes F^{\varphi(1)}$ by permuting the tensors. This induces a representation $\mathcal{R} \colon \mathfrak{S}_m \to \mathrm{GL}_{\varphi(1)^m}(F)$. The map $\mathcal{P} \colon \breve{\Delta}_N G \rtimes \mathfrak{S}_m \to \mathrm{GL}_{\varphi(1)^m}(F)$ given by

$$\mathcal{P}(x\sigma) = \mathcal{P}_0(x)\mathcal{R}(\sigma) \text{ for every } x \in \widetilde{N}\breve{G} \text{ and } \sigma \in \mathfrak{S}_m$$

is a projective representation of $\widetilde{N}\breve{G} \rtimes \mathfrak{S}_m = \widetilde{N}(\breve{G} \rtimes \mathfrak{S}_m)$. Note that $\mathcal{P}_{N \wr \mathfrak{S}_m}$ is a representation, as defined in [4, Thm. 25.6]. By straightforward calculations using the definition of $\mathcal{R}$ we see that the factor set α of $\mathcal{P}$ satisfies

$$\alpha(\widetilde{n}\Delta g\sigma, \widetilde{n}'\Delta g'\sigma') = \alpha_0(\widetilde{n}\Delta g, \widetilde{n}'\Delta g') \text{ for every } g, g' \in G, \widetilde{n}, \widetilde{n}' \in \widetilde{N} \text{ and } \sigma, \sigma' \in \mathfrak{S}_m.$$

In the next step we define $\mathcal{P}'$ by extending $\mathcal{P}'_0$. Note that $[\Delta G, \mathfrak{S}_m] = 1$. Hence $\widetilde{\nu}$ and τ are $\mathfrak{S}_m$-invariant and the map $\mathcal{P}' \colon \breve{G} \rtimes \mathfrak{S}_m \to \mathrm{GL}_{\varphi'(1)}(F)$ with $\mathcal{P}'(g\widetilde{z}\sigma) := \mathcal{P}'_0(g)\tau(\widetilde{z})$ for every $g \in \breve{G}$, $\widetilde{z} \in \widetilde{Z}$ and $\sigma \in \mathfrak{S}_m$ is a projective representation whose factor set α' satisfies

$$\alpha'(g\sigma, g'\sigma') = \alpha'_0(g, g') \text{ for every } g, g' \in \breve{G} \text{ and } \sigma, \sigma' \in \mathfrak{S}_m.$$

Altogether this proves that $\mathcal{P}$ and $\mathcal{P}'$ have the property described in Definition 3.3(ii.1)

In the last step we compare $\mathcal{P}(x)$ and $\mathcal{P}'(x)$ for $x \in \mathbf{C}_{(\widetilde{N}\breve{G}) \rtimes \mathfrak{S}_m}(\widetilde{N})$. We have that $\mathbf{C}_{(\widetilde{N}\breve{G}) \rtimes \mathfrak{S}_m}(\widetilde{N}) = \breve{\Delta}_Z \mathbf{C}_G(N)$. Then $x = \widetilde{z}\Delta c$ for some $\widetilde{z} \in \widetilde{Z}$ and $c \in \mathbf{C}_G(N)$. Since $\mathcal{Q}(c)$ and $\mathcal{Q}'(c)$ are scalar matrices associated with some ζ and ζ^m, respectively by 4.8(b.ii'). By definition $\mathcal{P}(\widetilde{z}\Delta c)$ and $\mathcal{P}'(\widetilde{z}\Delta c)$ are scalar matrices associated with $\tau(\widetilde{z})\zeta^m$. This implies that $\mathcal{P}$ and $\mathcal{P}'$ satisfy the property in Definition 3.3(ii.2).

Using the definitions of the various groups we see that the group theoretic conditions in Definition 3.3(i) are satisfied. This proves (i) implies (ii) according to Definition 3.3.

In the following we only sketch the considerations that prove that (ii) implies (i): we start by choosing a projective representation $\mathcal{Q}$ of G associated to φ as in Lemma 4.7. Then one can construct a projective representation $\mathcal{P}$ of $\widetilde{N}(\breve{G} \rtimes \mathfrak{S}_m)$ associated to $\widetilde{\varphi}$ as in the first part of the proof. Let $\mathcal{P}'$ be the projective representation of $\breve{G} \rtimes \mathfrak{S}_m$ associated to $\breve{\varphi}$ given by Lemma 3.4(a). Then $\mathcal{P}'|_{\Delta G}$ defines via the natural isomorphism $\Delta G \to G$ a projective representation $\mathcal{Q}'$ of G associated to φ' because $\mathcal{P}'|_{\Delta N}$ affords $\Delta\varphi'$. It is easy to check that $\mathcal{Q}$ and $\mathcal{Q}'$ give $(G, N, \varphi)^{(m)} \approx (G, N, \varphi')$ using Lemma 4.8. □

Corollary 4.11. *Assume the notation and the situation of Theorem 4.10. Let $H \lhd G$ with $H \leq \mathbf{C}_G(N)$ and write $\breve{G}_1 = \breve{\Delta}_{ZH}G$. Then*

$$(\widetilde{N}(\breve{G}_1 \rtimes \mathfrak{S}_m), \widetilde{N}, \widetilde{\varphi}) \succ_{Br,c} (\breve{G}_1 \rtimes \mathfrak{S}_m, \breve{N}, \breve{\varphi}).$$

Proof. Note that $\widetilde{N}\breve{G}_1 = \breve{\Delta}_{NH}G$ and $\breve{G}_1 = \breve{\Delta}_{HZ}G$ are well-defined. The group $\breve{\Delta}_{NH}G \rtimes \mathfrak{S}_m$ induces on $\widetilde{N}$ the same automorphisms as $\breve{\Delta}_N G \rtimes \mathfrak{S}_m$. Accordingly, the statement follows from Theorem 4.10 through an application of Theorem 3.7. □

Also, as a corollary of Theorem 4.10, we obtain that $(G, N, \varphi)^{(m)} \approx (G, N, \varphi')$ is a property that only depends on the characters φ and φ' as well as the automorphisms induced by G on N. The actual structure of G has no influence on the relation.

Corollary 4.12. *Let (G, N, φ) and (G, N, φ') be modular character triples. Suppose that for an integer m coprime to $|N|$,*

$$(G, N, \varphi)^{(m)} \approx (G, N, \varphi').$$

Let G_1 be a group such that $N \lhd G_1$ and $G_1/\mathbf{C}_{G_1}(N)$ is equal to $G/\mathbf{C}_G(N)$ as a subgroup of $\mathrm{Aut}(N)$*. Then*

$$(G_1, N, \varphi)^{(m)} \approx (G_1, N, \varphi').$$

Proof. This follows from combining Theorem 4.10 and Theorem 3.7. □

We have defined the notion of fake m-th Galois conjugate character triples. We conclude this section by introducing fake m-th Galois action and verifying their existence on p-solvable groups. Indeed, the fact that fake Galois actions do always exist on p-solvable groups will lead to a simpler verification of the inductive Brauer–Glauberman condition defined in the subsequent Section 6, see Remark 6.2.

Definition 4.13. Let $N \lhd G$. Let $\mathscr{S} \leq \mathrm{IBr}(N)$ be a G-invariant subset. Let m be an integer coprime to $|N|$. We say that there exists a **fake m-th Galois action on $\mathscr{S}$ with respect to G** if there exists a G-equivariant bijection

$$f_m : \mathscr{S} \to \mathscr{S}$$

such that

$$(G_\varphi, N, \varphi)^{(m)} \approx (G_\varphi, N, f_m(\varphi)) \text{ for every } \varphi \in \mathscr{S}.$$

Let N be a p-solvable group. Then the Galois group $\mathrm{Gal}(\mathbb{Q}_{|N|}/\mathbb{Q})$ acts on $\mathrm{IBr}(N)$, see the proof of Proposition 4.15 for more details. In this case, we show that there exists a fake m-th Galois action on $\mathrm{IBr}(N)$ for any integer m coprime to $|N|$ and with respect to any G with $N \triangleleft G$. We first need a lemma.

Lemma 4.14. *Let $N \triangleleft G$ and $\theta \in \mathrm{Irr}(N)$. Assume that θ is G-invariant and $\theta^0 \in \mathrm{IBr}(N)$. Let $k = exp(G)$. Write $L = \mathbb{Q}_k$ and $S_L = \{r/s \mid r \in R \cap L, s \in R \cap L - M\}$. There exists a projective representation of G associated to θ with matrix entries in S_L.*

Proof. We notice that by Brauer's Theorem [5, Thm. 10.3], L is a splitting field for G. By Problem 2.12 of [13], let $\mathcal{Y}$ be an S_L-representation of N affording θ. We check that there exists a projective representation of G associated to θ extending $\mathcal{Y}$ with entries in S_L. For every $\overline{g} \in G/N$, the representation $\mathcal{Y}$ extends to a representation $\mathcal{Y}_{\overline{g}}$ of $\langle N, g\rangle$ as in Theorem 8.12 of [13], where $g \in G$ with $gN = \overline{g}$. Define $\mathcal{D}(g) = \mathcal{Y}_{gN}(g)$ for every $g \in G$. Then $\mathcal{D}$ is a projective representation of G extending $\mathcal{Y}$, by Lemma 8.27 of [13]. Hence, it suffices to control the matrix entries of the representations $\mathcal{Y}_g$. In other words, we only need to check the case where θ extends to G.

Let $\widetilde{\theta} \in \mathrm{Irr}(G)$ be an extension of θ. Let $\mathcal{X}$ be a representation affording $\widetilde{\theta}$ with entries in S_L (again such $\mathcal{X}$ does exist by Problem 2.12 of [13]). We have that $\mathcal{X}_N$ affords θ. Hence, there is some $T \in GL_n(L)$ such that $\mathcal{X}_N = T^{-1}\mathcal{Y}T$. Write $T = (t_{ij})$, where $t_{ij} \in L$. By Lemma 2.5 of [13], since all t_{ij} are algebraic over $\mathbb{Q}$, there exists $\beta \in L$ such that all $\beta t_{ij} \in R$ but not all $\beta t_{ij} \in M$. Since $\mathcal{X}_N = (\beta T)^{-1}\mathcal{Y}(\beta T)$, we may assume that $t_{ij} \in R$ and $T^* \neq 0$ (replacing T by βT). By assumption, the F-representations $\mathcal{X}_N^*$ and $\mathcal{Y}^*$ are irreducible. Moreover $T^*\mathcal{X}_N^* = \mathcal{Y}^*T^*$. By Schur's Lemma, this implies $T^* \in GL_n(F)$. In particular, we have that $\det(T^*) = \det(T)^* \neq 0$ and so $\det(T) \notin M$. Thus $T \in GL_n(S_L)$ and the representation $T\mathcal{X}T^{-1}$ with entries in S_L extends $\mathcal{Y}$. □

Proposition 4.15. *Let N be a p-solvable group and m an integer coprime to $|N|$. If $N \triangleleft G$, then there exists a fake m-th Galois action on* $\mathrm{IBr}(N)$ *with respect to G.*

Proof. Let $B_{p'}(N)$ be the subset of $\mathrm{Irr}(N)$ defined in Section 5 of [6]. By Corollary 10.3 of [6], we have that $B_{p'}(N)$ provides a canonical lift of $\mathrm{IBr}(N)$. Moreover, $\mathrm{Aut}(N)$ and $\mathrm{Gal}(\mathbb{Q}_{|N|}/\mathbb{Q})$ act on the set $B_{p'}(N)$. Thus, the bijection $B_{p'}(N) \to \mathrm{IBr}(N)$ given by $\theta \mapsto \theta^0$ commutes with the action of $\mathrm{Aut}(N)$. Consider σ_m as defined at the beginning of this section. Let $\varphi \in \mathrm{IBr}(N)$ and $\theta \in B_{p'}(N)$ with $\varphi = \theta^0$. Since $\theta^{\sigma_m} \in B_{p'}(N)$, we have that $\varphi^{\sigma_m} = (\theta^{\sigma_m})^0 \in \mathrm{IBr}(N)$. Hence the map $f_m : \mathrm{IBr}(N) \to \mathrm{IBr}(N)$ defined by $\varphi \mapsto \varphi^{\sigma_m}$ is an $\mathrm{Aut}(N)$-equivariant bijection.

Let $\varphi \in \mathrm{IBr}(N)$. We want to prove that $(G_\varphi, N, \varphi)^{(m)} \approx (G_\varphi, N, \varphi^{\sigma_m})$. We may assume that $G = G_\varphi$. Let $\theta \in B_{p'}(N)$ be the canonical lift of φ. Then θ is G-invariant and θ^{σ_m} is the canonical lift of φ^{σ_m}. Let $k = \exp(G)$ and write $L = \mathbb{Q}_k$. By Lemma 4.14, there exists a projective representation $\mathcal{D}$ of G associated to θ with matrix entries in $S_L = \{r/s \mid r \in R \cap L, s \in R \cap L - M \cap L\}$. In particular, the map $\mathcal{D}^{\sigma_m}$ is a well-defined projective representation of G associated to θ^{σ_m} with matrix entries in S_L. It is straightforward to check that the F-projective representations $\mathcal{P} = \mathcal{D}^*$ and $\mathcal{P}' = (\mathcal{D}^{\sigma_m})^*$ associated to φ and φ^{σ_m} satisfy the required properties of Definition 4.4. $\square$

5. Coprime action on simple groups and their direct products

In this section we study the situation where a group A acts coprimely on the direct product of isomorphic non-abelian simple groups. We describe the structure of A and related groups.

The starting point of our considerations is the following consequence of the classification of finite simple groups.

Theorem 5.1. *Let S be a simple non-abelian group. Let B act on S faithfully with $(|B|, |S|) = 1$. Then B is cyclic, $\mathbf{N}_{\mathrm{Aut}(S)}(B) = \mathbf{C}_{\mathrm{Aut}(S)}(B)$ and*

$$\mathbf{Z}(\mathbf{C}_X(B)) = \mathbf{C}_X(B) \cap \mathbf{Z}(X), \tag{5.1}$$

where X is the universal covering group of S and B acts on X via the canonical identification $\mathrm{Aut}(S) = \mathrm{Aut}(X)$ from [1, Ex. 6, Chapt. 11].

Proof. We identify B with the corresponding subgroup of $\mathrm{Aut}(S)$. According to the classification of finite simple groups the group S has to be a simple group of Lie type and B is $\mathrm{Aut}(S)$-conjugate to some group of field automorphisms of S, see for example Section 2 of [10]. Accordingly B is cyclic. The structure of $\mathrm{Aut}(S)$ is described in Theorem 2.5.12 of [3]. Straightforward computations with $\mathrm{Aut}(S)$ prove that $\mathbf{N}_{\mathrm{Aut}(S)}(B) = \mathbf{C}_{\mathrm{Aut}(S)}(B)$.

Let X be the universal covering group of S. For the proof of $\mathbf{Z}(\mathbf{C}_X(B)) = \mathbf{C}_X(B) \cap \mathbf{Z}(X)$ we may assume that $B \neq 1$ and hence by the above that S is a simple group of Lie type. Arguing as in the beginning of Section 2 of [10] we see that the Schur multiplier of S is generic or $S = {}^2\mathrm{B}_2(8)$ and B is a cyclic group of order 3.

Let us consider first the case where $S = {}^2\mathrm{B}_2(8)$ and that B is a cyclic group of order 3. Then $X = 2^2.{}^2\mathrm{B}_2(8)$ and $\mathbf{C}_X(B) = {}^2\mathrm{B}_2(2)$. The group ${}^2\mathrm{B}_2(2)$ is a Frobenius group of order $5 \cdot 4$ and has trivial centre. Accordingly $\mathbf{Z}(X) \cap \mathbf{C}_X(B)$ is trivial as well.

Hence we can assume that $X = \mathbf{X}^F$, for some simply-connected simple algebraic group $\mathbf{X}$ and some Steinberg endomorphism $F\colon X \to X$. We can assume that B is generated by some automorphism that is induced by some Steinberg endomorphism $F_0\colon \mathbf{X} \to \mathbf{X}$. Without loss of generality we can assume that some power of F_0 coincides with F. According to Theorem 24.15 of [11] we see that

$$\mathbf{Z}(\mathbf{C}_X(B)) = \mathbf{Z}(\mathbf{X}^{F_0}) = \mathbf{Z}(\mathbf{X})^{F_0} = (\mathbf{Z}(\mathbf{X})^F)^{F_0} = (\mathbf{Z}(\mathbf{X}^F))^{F_0} = \mathbf{Z}(X) \cap \mathbf{C}_X(B).$$

This is the equation. □

Notation 5.2. Let S be a non-abelian finite simple group, let X be the universal covering group of S. Let r be a positive integer r and $\widetilde{X} := X^r$, the external direct product of r copies of X. Suppose $A \leq \mathrm{Aut}(\widetilde{X})$ with $(|A|, |X|) = 1$. Let $\widetilde{\Gamma} \leq \mathrm{Aut}(\widetilde{X})$ such that $\widetilde{\Gamma} \leq \mathbf{C}_{\mathrm{Aut}(\widetilde{X})}(A)$ and that $\widetilde{\Gamma}A$ acts transitively on the factors of $\widetilde{X}$.

In the following we identify $\mathrm{Aut}(\widetilde{X})$ with $\mathrm{Aut}(S) \wr \mathfrak{S}_r$, using that $\mathrm{Aut}(X) \cong \mathrm{Aut}(S)$ by Exercise 6 of Chapter 11 in [1] and $\mathrm{Aut}(\widetilde{X}) = \mathrm{Aut}(X) \wr \mathfrak{S}_r$ by straight-forward considerations. We use the notation introduced for elements of wreath products given in [2].

Note that $\widetilde{X}$ is also the internal direct product of $X_1, \ldots, X_r$, where for $i = 1, \ldots, r$, the group X_i is defined by

$$X_i = 1 \times \cdots \times X \times \cdots \times 1$$

with a non-trivial factor at the i-th position. The natural isomorphism $\mathrm{pr}_i \colon X_i \to X$ induces the epimorphism

$$\overline{\mathrm{pr}}_i \colon \mathrm{Aut}(\widetilde{X})_{X_i} \to \mathrm{Aut}(X) \text{ with } \alpha \mapsto \mathrm{pr}_i^{-1} \circ \alpha|_{X_i} \circ \mathrm{pr}_i\,. \tag{5.2}$$

Lemma 5.3. *Assume the notation in Notation 5.2. Write $B_i = \overline{\mathrm{pr}}_i(A_{X_i})$ for each $i \in \{1, \ldots, r\}$.*

(a) *The groups B_i are cyclic.*
(b) *If $\widetilde{\Gamma}A$ acts transitively on $\{X_1, \ldots, X_r\}$, then B_i and B_1 are $\mathrm{Aut}(X)$-conjugate and the A-orbits on the set $\{X_1, \ldots, X_r\}$ all have the same length.*

Proof. Notice that B_i acts on X with $(|X|, |B_i|) = 1$. By Theorem 5.1, we have that B_i is cyclic. This proves part (a).

Now, since $\widetilde{\Gamma}A$ acts transitively on $\{X_1, \ldots, X_r\}$, there is some $\alpha_i \in \widetilde{\Gamma}A$ such that $X_1^{\alpha_i} = X_i$. It is easy to check that $B_i = \overline{\mathrm{pr}}(A_{X_i}) = \overline{\mathrm{pr}}((A_{X_1})^{\alpha_i}) = B_1^{\beta_i}$, where $\beta_i \in \mathrm{Aut}(X)$ is given by $\beta_i \circ \mathrm{pr}_i = \mathrm{pr}_1 \circ \alpha_i^{-1}|_{X_i}$.

Furthermore $\widetilde{\Gamma}$ permutes transitively the A-orbits on $\{X_1, \ldots, X_r\}$. Hence these A-orbits all have the same length. This concludes the proof of part (b). □

According to the classification of finite simple groups Schreier's conjecture holds, i.e. $\mathrm{Out}(S)$ is always solvable. In particular, if X is the universal covering group of the non-abelian simple group S and π is the set of primes dividing $|X|$, then $\mathrm{Aut}(X)$ is π-separable, and hence there are Hall π'-subgroups in $\mathrm{Aut}(X)$.

Using this fact one can determine a *convenient* group containing A.

Proposition 5.4. *Assume the notation in Notation 5.2. Write $B_i = \overline{\mathrm{pr}}_i(A_{X_i})$ for each $i \in \{1, \ldots, r\}$. Suppose that $\widetilde{\Gamma} A$ acts transitively on $\{X_1, \ldots, X_r\}$. Let π be the set of prime divisors of $|X|$. Let H be a Hall π'-subgroup of $\mathrm{Aut}(X)$. Then A is $\mathrm{Aut}(X)^r$-conjugate to a subgroup of $H \wr \mathfrak{S}_r$. Also, $B_i = B_1$ for all $i = 1, \ldots, r$.*

Proof. By the discussion preceding the statement of this proposition $\mathrm{Aut}(X)$ is π-separable. Hence $\mathrm{Aut}(X)^r A$ is also π-separable. Let $K = \mathrm{Aut}(X)^r A \cap \mathfrak{S}_r$. Notice that K is a π'-subgroup of $\mathfrak{S}_r$. Moreover, $H^r \rtimes K$ is a Hall π'-subgroup of $\mathrm{Aut}(X)^r A$. Since A is a π'-subgroup of $\mathrm{Aut}(X)^r A$, there exist $a \in A$ and $\alpha \in \mathrm{Aut}(X)^r$, such that

$$A^{a\alpha} = A^\alpha \leq H^r \rtimes K \leq H \wr \mathfrak{S}_r.$$

For the latter part we may assume $A \leq H \wr \mathfrak{S}_r$. Now, $B_1, B_i \leq H$ are $\mathrm{Aut}(X)$-conjugate by Lemma 5.3(b). In particular $|B_i| = |B_1|$. Since H is cyclic, by Theorem 5.1, this implies $B_i = B_1$. $\square$

Proposition 5.5. *Assume the notation in Notation 5.2. Suppose that $A \leq H \wr \mathfrak{S}_r$. Write $B = \overline{\mathrm{pr}}(A_{X_1})$. Then A is H^r-conjugate to a subgroup of $B \wr \mathfrak{S}_r$.*

Proof. It is enough to prove the statement in the case where A acts transitively on $\{X_1, \cdots, X_r\}$ by working on A-orbits.

If $A \leq H \wr \mathfrak{S}_r$ acts transitively on $\{X_1, \ldots, X_r\}$, then for every $i \in \{1, \ldots, r\}$, there exist $a_i \in A$ such that

$$X_1^{a_i} = X_i,$$

and $h_i \in H$ such that for every $x \in X$

$$(x, 1, \ldots, 1)^{a_i} = (1, \ldots, x^{h_i}, \ldots, 1) \in X_i.$$

Let $h = (1_H, h_2^{-1}, \ldots, h_r^{-1}) \in H^r$. We claim that $A^h \leq B \wr \mathfrak{S}_r$. First notice that $\overline{\mathrm{pr}}_1((A^h)_{X_1}) = \overline{\mathrm{pr}}_1((A_{X_1})^h) = B$. Now, write $\widehat{a}_i = h^{-1} a_i h \in A^h \leq H \wr \mathfrak{S}_r$ for each $i \in \{1, \ldots, r\}$. Then

$$(x, 1, \ldots, 1)^{\widehat{a}_i} = (1, \ldots, x, \ldots, 1) \in X_i$$

for every $x \in X$.

Let $y \in A^h$. Then $y = (y_1, \ldots, y_r)\rho \in H \wr \mathfrak{S}_r$. Let $i \in \{1, \ldots, r\}$. Write $j = \rho^{-1}(i)$, so that

$$X_i^y = X_j.$$

Since

$$(x, 1 \ldots, 1)^{\widehat{a}_i y \widehat{a}_j^{-1}} = (x^{y_i}, 1 \ldots, 1) \in X_1,$$

for every $x \in X$, we have that $\widehat{a}_i y \widehat{a}_j^{-1} \in (A^h)_{X_1}$. Consequently $y_j \in B$. This argument applies for every $i \in \{1, \ldots, r\}$, hence the claim follows. $\square$

Recall the notation from the previous section. In the m-th direct product G^m of G and for $U \leq G$ we write $\Delta U \leq G^m$ for the diagonally embedded group U. We denote by $\breve{\Delta}_Z U$ the group generated in G^m by the m-th direct product Z^m of Z and ΔU, whenever $U \leq \mathbf{N}_G(Z)$. If we want to emphasize that $\breve{\Delta}_Z U$ is constructed in G^m we write $\breve{\Delta}_Z^m U$.

Proposition 5.6. *Assume the notation in Proposition 5.4. Let $B = B_1$ and let $\Gamma = \mathbf{C}_{\mathrm{Aut}(X)}(B)$. Suppose that $A \leq B \wr \mathfrak{S}_r$ and that A acts transitively on $\{X_1, \ldots, X_r\}$. Then*

$$\widetilde{\Gamma} A \leq (\breve{\Delta}_B \Gamma) \rtimes \mathfrak{S}_r.$$

Proof. Let $c \in \widetilde{\Gamma}$. Then $c = (c_1, \ldots, c_r)\rho$ with $c_i \in \mathrm{Aut}(X)$ and $\rho \in \mathfrak{S}_r$. Let $a \in A_{X_1}$. Then $a = (b_1, \ldots, b_r)\sigma$ with $b_i \in B$ and $\sigma \in \mathfrak{S}_r$. The equation $ac = ca$ implies that

$$b_1 = c_1^{-1} b_{\rho^{-1}(1)} c_1 \in B.$$

This holds for every $a \in A_{X_1}$. Hence $c_1 \in \mathbf{N}_{\mathrm{Aut}(X)}(B)$. By Theorem 5.1, we have that $\mathbf{N}_{\mathrm{Aut}(X)}(B) = \mathbf{C}_{\mathrm{Aut}(X)}(B) = \Gamma$. Proceeding like this for elements $a \in A_{X_i}$, we conclude $c_i \in \Gamma$ for every $i \in \{1, \ldots, r\}$. Hence $\widetilde{\Gamma} \leq \Gamma \wr \mathfrak{S}_r$.

Now, for $a \in A$ with $a = (b_1, \ldots, b_r)\sigma \in B \wr \mathfrak{S}_r$ and $c \in \widetilde{\Gamma}$ with $c = (c_1, \ldots, c_r)\rho \in \Gamma \wr \mathfrak{S}_r$ the equation $ac = ca$ implies that

$$b_i c_{\sigma^{-1}(i)} = c_i b_{\rho^{-1}(i)}.$$

Hence $c_{\sigma^{-1}(1)} c_i^{-1} \in B$ for every $i \in \{1, \ldots, r\}$. Since A acts transitively on $\{X_1, \ldots, X_r\}$, we have that

$$c_j c_1^{-1} \in B,$$

for every $j \in \{1, \ldots, r\}$. This proves that $(c_1, \ldots, c_r) \in B^r \Delta\Gamma = \breve{\Delta}_B \Gamma$. This proves that $\widetilde{\Gamma} \leq (\breve{\Delta}_B \Gamma) \rtimes \mathfrak{S}_r$. $\square$

For our later application we consider the case where $\widetilde{\Gamma} A$ acts transitively.

Proposition 5.7. *Assume the notation in Proposition 5.4. Let $B = B_1$ and let $\Gamma = \mathbf{C}_{\mathrm{Aut}(X)}(B)$. Suppose that $A \leq B \wr \mathfrak{S}_r$ and that $\widetilde{\Gamma} A$ acts transitively on $\{X_1, \ldots, X_r\}$. Let m be the length of an A-orbit in $\{X_1, \ldots, X_r\}$. Then for some $\tau \in \mathfrak{S}_r$ we have that*

$$A^\tau \leq (B \wr \mathfrak{S}_m)^{r/m} \quad \textit{and} \quad (\widetilde{\Gamma} A)^\tau \leq ((\breve{\Delta}_B^m \Gamma) \rtimes \mathfrak{S}_m) \wr \mathfrak{S}_{\frac{r}{m}}.$$

Proof. By Proposition 5.6 the statement holds when $m = r$.

Let $d = r/m$. We may assume that the A-orbits on $\{X_1, \ldots, X_r\}$ are exactly $\{X_1, \ldots, X_m\}, \ldots, \{X_{(d-1)m+1}, \ldots, X_{dm}\}$ after conjugating $\widetilde{\Gamma}A$ by some $\tau \in \mathfrak{S}_r$. (Notice that A^τ and $\widetilde{\Gamma}^\tau$ satisfy the same hypotheses as A and $\widetilde{\Gamma}$.) This proves the first statement.

Let $a = (b_1, \ldots, b_r)\sigma \in A^\tau \leq (B \wr \mathfrak{S}_m)^d$ and $c = (c_1, \ldots, c_r)\rho \in \widetilde{\Gamma}^\tau$. Note that $\sigma \in (\mathfrak{S}_m)^d$, hence we can write $\sigma = \sigma_1 \cdots \sigma_d$ where $\sigma_l \in \mathfrak{S}_m$ permutes the set $\{(l-1)m+1, \ldots, lm\}$. The equation $ac = ca$ implies that

$$b_i c_{\sigma^{-1}(i)} = c_i b_{\rho^{-1}(i)} \quad \text{and} \quad \sigma^\rho = \sigma.$$

Notice that $\sigma^\rho = \sigma$ implies that for every $l \in \{1, \ldots, d\}$, we have that $\sigma_l^\rho = \sigma_k$ for a unique $k \in \{1, \ldots, d\}$. Proceeding as in the first paragraph of the proof of Proposition 5.6 we can prove that $c_i \in \Gamma$. Also, arguing as in the second paragraph of the proof of Proposition 5.6 we see that

$$c_{\sigma^{-1}(i)} c_i^{-1} \in B$$

for every $i \in \{1, \ldots, r\}$. We can proceed like this for every $a \in A$. Since A is transitive on each $\{(l-1)m+1, \ldots, lm\}$ we conclude that

$$c_j c_l^{-1} \in B$$

for every $j \in \{(l-1)m+1, \ldots, lm\}$ and for every $l \in \{1, \ldots, d\}$. Hence $(c_1, \ldots, c_r) \in (B^m \Delta\Gamma)^d = (\overset{m}{\breve{\Delta}}_B \Gamma)^d$.

Finally, since $\sigma^\rho = \sigma$ for every σ coming from an element $a \in A$, we conclude that ρ permutes the set $\{\{(l-1)m+1, \ldots, lm\} \mid l = 1, \ldots, d\}$ and also permutes the elements of the set $\{(l-1)m+1, \ldots, lm\}$ for each $l = 1, \ldots, d$. Hence $\rho \in \mathfrak{S}_m \wr \mathfrak{S}_d$. We conclude that $c = (c_1, \ldots, c_r)\rho \in ((\overset{m}{\breve{\Delta}}_B \Gamma) \rtimes \mathfrak{S}_m) \wr \mathfrak{S}_d$. $\square$

6. The inductive Brauer–Glauberman condition

In this section we introduce the inductive Brauer–Glauberman condition. Then we outline some consequences for central products of covering groups of non-abelian simple groups.

Definition 6.1. Let S be a non-abelian simple group and X the universal covering group of S. We say that S satisfies the **inductive Brauer–Glauberman condition** if for every $B \leq \mathrm{Aut}(X)$ with $(|X|, |B|) = 1$ the following conditions are satisfied:

(i) For $Z := \mathbf{Z}(X)$, $\Gamma := \mathbf{C}_{\mathrm{Aut}(X)}(B)$, $C_0 := \mathbf{C}_X(B)$ and $C := C_0 Z$ there exists a Γ-equivariant bijection

$$\Omega_B \colon \mathrm{IBr}_B(X) \to \mathrm{IBr}_B(C),$$

such that for every $\theta \in \mathrm{IBr}_B(X)$

$$(X \rtimes \Gamma_\theta, X, \theta) \succ_{Br,c} (C \rtimes \Gamma_\theta, C, \Omega_B(\theta)).$$

(ii) For every non-negative integer m with $(|X|, m) = 1$, there exists a fake m-th Galois action on $\mathrm{IBr}(C_0)$ with respect to $C_0 \rtimes \Gamma$.

The above condition can be divided into two requirements. Each of them has to be satisfied for the universal covering group X and all groups $B \leq \mathrm{Aut}(X)$ with $(|X|, |B|) = 1$. In order to check the requirements in Definition 6.1 for all non-abelian simple groups some simplifications are available. First we only need to consider B up to $\mathrm{Aut}(X)$-conjugation. Moreover the condition 6.1(ii) is true in all required cases whenever 6.1(ii) is true for the universal covering group of every non-abelian simple group and $B = 1$.

Remark 6.2. Let S be a non-abelian simple group and X the universal covering group of S.

(a) The group S satisfies the inductive Brauer–Glauberman condition, if the conditions hold for some $\mathrm{Aut}(X)$-transversal of subgroups B of $\mathrm{Aut}(X)$ with $(|X|, |B|) = 1$.
(b) Let $B \leq \mathrm{Aut}(X)$ with $(|X|, |B|) = 1$ and $|B| \neq 1$ and assume that $\mathbf{C}_X(B)$ is quasi-simple. Then Condition 6.1(ii) holds for X and B, if for every integer m with $(m, |X|) = 1$ there exists a fake m-th Galois action on $\mathrm{IBr}(X_1)$ with respect to $X_1 \rtimes \mathrm{Aut}(X_1)$, where X_1 is the universal covering group of the unique non-abelian composition factor of $\mathbf{C}_X(B)$.
(c) Let $B \leq \mathrm{Aut}(X)$ with $(|X|, |B|) = 1$ and $|B| \neq 1$ and assume that $\mathbf{C}_X(B)$ is not quasi-simple. Then Condition 6.1(ii) holds for X and B.

Proof. For the proof of (a) we suppose that 6.1(i) and 6.1(ii) from Definition 6.1 are satisfied for the universal covering group X of some simple non-abelian group and for some $B \leq \mathrm{Aut}(X)$ with $(|X|, |B|) = 1$. Let $\alpha \in \mathrm{Aut}(X)$. Define $\Omega_{B^\alpha}(\chi^\alpha) = \Omega_B(\chi)^\alpha$ for every $\chi \in \mathrm{IBr}_B(X)$. Then Ω_{B^α} is a Γ^α-equivariant bijection. By Lemma 3.8 we have that condition 6.1(i) is satisfied for Ω_{B^α}. Let m be an integer with $(|X|, m) = 1$. Let $f_m \colon \mathrm{IBr}(\mathbf{C}_X(B)) \to \mathrm{IBr}(\mathbf{C}_X(B))$ give the fake m-th Galois action on $\mathrm{IBr}(\mathbf{C}_X(B))$ with respect to $\mathbf{C}_X(B) \rtimes \Gamma$, as in Definition 4.13. Define $f'_m(\varphi^\alpha) = f_m(\varphi)^\alpha$ for every $\varphi \in \mathrm{IBr}(\mathbf{C}_X(B))$. It is easy to prove an analogue of Lemma 3.8 for m-th Galois conjugate modular character triples via Lemma 4.8. This implies that f'_m gives a fake m-th Galois action on $\mathbf{C}_X(B^\alpha)$ with respect to $\mathbf{C}_X(B^\alpha) \rtimes \Gamma^\alpha$.

Now we prove parts (b) and (c). Let X be the universal covering group of some non-abelian simple group S. Let $B \leq \mathrm{Aut}(X)$ with $(|B|, |X|) = 1$. If $B \neq 1$, then the group S is a simple group of Lie type and B is $\mathrm{Aut}(X)$-conjugate to some subgroup of the field automorphisms of X, see for example Section 2 of [10]. By part (a) we may assume that B consists of field automorphisms. By Theorem 2.2.7 of [3], the group $\mathbf{C}_X(B)$ is either quasi-simple, solvable or $\mathbf{C}_X(B) \in \{\mathrm{B}_2(2), \mathrm{G}_2(2), {}^2\mathrm{F}_4(2), {}^2\mathrm{G}_2(3)\}$.

Suppose that $\mathbf{C}_X(B)$ is not quasi-simple. Write $C_0 = \mathbf{C}_X(B)$. If

$$C_0 \in \{\mathrm{B}_2(2), \mathrm{G}_2(2), {}^2\mathrm{F}_4(2), {}^2\mathrm{G}_2(3)\},$$

then $\mathrm{Out}(C_0)$ is cyclic and $\mathbf{Z}(C_0)$ is trivial. Hence, it is easy to show that the identity yields a fake m-th Galois action on $\mathrm{IBr}(C_0)$ with respect to $C_0 \rtimes \mathrm{Aut}(C_0)$, for every positive integer m with $(|X|, m) = 1$. If C_0 is a solvable group, then Theorem 4.15 guarantees that for every m with $(|X|, m) = 1$, there exists a fake m-th Galois action on $\mathrm{IBr}(C_0)$ with respect to any G in which C_0 is normal. This proves part (c).

In all other cases $\mathbf{C}_X(B)$ is quasi-simple and there exists some non-abelian simple group S_1 such that $\mathbf{C}_X(B)$ is a central quotient of the universal covering group X_1 of S_1. By assumption for every m with $(|X_1|, m) = 1$, there exists a fake m-th Galois action on $\mathrm{IBr}(X_1)$ with respect to $X_1 \rtimes \mathrm{Aut}(X_1)$. Since $\mathbf{C}_{\mathrm{Aut}(X)}(B) \leq \mathrm{Aut}(X_1)$ this gives the required fake m-th Galois action on $\mathrm{IBr}(\mathbf{C}_X(B))$ according to an analogue of Lemma 3.5 for fake Galois conjugate modular character triples. We use that if m is such that $(|\mathbf{C}_X(B)|, m) = 1$, then also $(|X_1|, m) = 1$ by [1, 33.12]. This proves (b). $\square$

Notation 6.3. Let S be a non-abelian simple group and X its universal covering. We write $Z := \mathbf{Z}(X)$. If $B \leq \mathrm{Aut}(X)$, then we write $C_0 := \mathbf{C}_X(B)$, $C := C_0 Z$ and $\Gamma := \mathbf{C}_{\mathrm{Aut}(X)}(B)$. For a positive integer r, let $\widetilde{X} := X^r$, $\widetilde{Z} := Z^r$, $\widetilde{C} := C^r$. Let $\Delta\colon X \to \widetilde{X}$ be the map defined as in 4.1. We also continue using the constructions introduced in Notation 4.9 and write $\breve{C} := \breve{\Delta}_Z C_0$. For $1 \leq i \leq r$ let X_i and $\overline{\mathrm{pr}}_i$ be defined as in 5.2.

Our aim in this section is to prove the following.

Theorem 6.4. *Let S be a non-abelian simple group satisfying the inductive Brauer–Glauberman condition. Let X be the universal covering group of S, r a positive integer and $A \leq \mathrm{Aut}(\widetilde{X})$ with $(|A|, |X|) = 1$. Write $\widetilde{\Gamma} := \mathbf{C}_{\mathrm{Aut}(\widetilde{X})}(A)$. Suppose that $\widetilde{\Gamma}A$ acts transitively on the factors $\{X_1, \ldots, X_r\}$ of $\widetilde{X}$. Then there exists a $\widetilde{\Gamma}$-equivariant bijection*

$$\widetilde{\Omega}_{\widetilde{X},A}\colon \mathrm{IBr}_A(\widetilde{X}) \to \mathrm{IBr}_A(\mathbf{C}_{\widetilde{X}}(A)\widetilde{Z}),$$

such that for every $\chi \in \mathrm{IBr}_A(\widetilde{X})$ and $\chi' := \widetilde{\Omega}_{\widetilde{X},A}(\chi)$

$$(\widetilde{X} \rtimes \widetilde{\Gamma}_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\mathbf{C}_{\widetilde{X}}(A)\widetilde{Z} \rtimes \widetilde{\Gamma}_\chi, \mathbf{C}_{\widetilde{X}}(A)\widetilde{Z}, \chi'). \tag{6.1}$$

We prove this result by first establishing a particular case and subsequently generalizing to the above statement. In the following we use the identification $\mathrm{Aut}(\widetilde{X}) = \mathrm{Aut}(X) \wr \mathfrak{S}_r$. For $A \leq \mathrm{Aut}(\widetilde{X})$, write $B := \overline{\mathrm{pr}}_1(A_{X_1}) \leq \mathrm{Aut}(X)$.

Proposition 6.5. *If A acts transitively on $\{X_1, \dots, X_r\}$ and $A \le B \wr \mathfrak{S}_r$, then Theorem 6.4 holds.*

First we determine the group $\mathbf{C}_{\widetilde{X}}(A)\widetilde{Z}$ and some related sets of characters.

Lemma 6.6. *If A acts transitively on $\{X_1, \dots, X_r\}$ and $A \le B \wr \mathfrak{S}_r$, then*

(a) $\mathbf{C}_{\widetilde{X}}(A) = \Delta C_0 = \mathbf{C}_{\widetilde{X}}(B \wr \mathfrak{S}_r)$.
(b) $\mathrm{IBr}_A(\widetilde{X}) = \{\theta \times \dots \times \theta \mid \theta \in \mathrm{IBr}_B(X)\}$,
(c) $\mathrm{IBr}_A(\widetilde{Z}) = \{\nu \times \dots \times \nu \mid \nu \in \mathrm{IBr}_B(Z)\}$,
(d) $\mathrm{IBr}_B(C) = \{\varphi \cdot \nu \mid \varphi \in \mathrm{IBr}(C_0)$ *and* $\nu \in \mathrm{IBr}_B(Z)$ *with* $\mathrm{IBr}(\varphi_{\mathbf{C}_Z(B)}) = \mathrm{IBr}(\nu_{\mathbf{C}_Z(B)})\}$,
(e) $\mathrm{IBr}_A(\widetilde{C}) = \{(\varphi \times \dots \times \varphi) \cdot (\nu \times \dots \times \nu) \mid \varphi \in \mathrm{IBr}(C_0)$ *and* $\nu \in \mathrm{IBr}_B(Z)\}$,
(f) $\mathrm{IBr}_A(\breve{C}) = \{(\Delta\varphi) \cdot \mu \mid \varphi \in \mathrm{IBr}(C_0)$ *and* $\mu \in \mathrm{IBr}_A(\widetilde{Z})\}$.

Proof. Part (a) easily follows from Lemma 2.2 of [9]. The rest follows from straightforward considerations (use Lemma 3.9 for parts (d), (e) and (f)). □

Note that by Lemma 6.6(a), we have that $\breve{C} = \breve{\Delta}_Z C_0 = \mathbf{C}_{\widetilde{X}}(A)\widetilde{Z}$. Before defining the map $\widetilde{\Omega}_{\widetilde{X},A}$ we introduce the bijection $\widetilde{f}_r \colon \mathrm{IBr}_A(\widetilde{C}) \to \mathrm{IBr}_A(\breve{C})$.

Proposition 6.7. *In the situation of Proposition 6.5, let $\breve{\Gamma} := \breve{\Delta}_B \Gamma$ and $\breve{Y} := \breve{C} \rtimes (\breve{\Gamma} \rtimes \mathfrak{S}_r)$. Then there exists a $\Delta\Gamma$-equivariant bijection*

$$\widetilde{f}_r \colon \mathrm{IBr}_A(\widetilde{C}) \to \mathrm{IBr}_A(\breve{C})$$

such that for every $\widetilde{\psi} \in \mathrm{IBr}_A(\widetilde{C})$ and $\breve{\psi} := \widetilde{f}_r(\widetilde{\psi})$ we have $(\widetilde{C}\breve{Y}_{\widetilde{\psi}}, \widetilde{C}, \widetilde{\psi}) \succ_{Br,c} (\breve{Y}_{\widetilde{\psi}}, \breve{C}, \breve{\psi})$.

Proof. Write $Z_0 = \mathbf{Z}(C_0)$, $\widetilde{Z}_0 = Z_0^r$ and $\widetilde{C}_0 = C_0^r$. Note that the assumption that A acts transitively on $\{X_1, \dots, X_r\}$ with $(|X|, |A|) = 1$ implies $(r, |X|) = 1$. Since S satisfies the inductive Brauer–Glauberman condition, there exists a fake Galois r-th action on $\mathrm{IBr}(C_0)$ with respect to $C_0 \rtimes \Gamma$. Let f_r give a fake r-th Galois action as in Definition 4.13.

Let $\widetilde{\psi} \in \mathrm{IBr}_A(\widetilde{C})$. By Lemma 6.6(f), we have that $\widetilde{\psi} = \widetilde{\varphi} \cdot \widetilde{\nu}$ for some $\widetilde{\varphi} = \varphi \times \dots \times \varphi \in \mathrm{IBr}_A(\widetilde{C}_0)$ and $\widetilde{\nu} = \nu \times \dots \times \nu \in \mathrm{IBr}_A(\widetilde{Z})$ with $\mathrm{IBr}(\varphi|_{C_0 \cap Z}) = \mathrm{IBr}(\nu|_{C_0 \cap Z})$. We define $\widetilde{f}_r(\widetilde{\psi}) := \Delta(f_r(\varphi)) \cdot \widetilde{\nu}$. By this definition $\widetilde{f}_r$ is a bijection. Note that $\widetilde{f}_r$ is $\Delta\Gamma$-equivariant as f_r is Γ-equivariant. In particular, for $\breve{\psi} = \widetilde{f}_r(\widetilde{\psi})$ we have that $\breve{Y}_{\widetilde{\psi}} = \breve{Y}_{\breve{\psi}}$, we have that $[Z, B] \subseteq \ker(\nu)$.

Let $\varphi' := f_r(\varphi)$ and $Y_0 := C_0 \rtimes \Gamma_\varphi$. Since f_r yields a fake Galois r-th action on $\mathrm{IBr}(C_0)$ with respect to Y_0, we have that

$$(Y_0, C_0, \varphi)^{(r)} \approx (Y_0, C_0, \varphi'). \tag{6.2}$$

Let $\breve{C}_0 := \breve{\Delta}_{Z_0} C_0$, $\lambda \in \mathrm{IBr}(\varphi_{Z_0})$, $\widetilde{\lambda} = \lambda \times \cdots \times \lambda \in \mathrm{IBr}(\widetilde{Z}_0)$ and $\breve{\varphi} = \Delta\varphi' \cdot \widetilde{\lambda} \in \mathrm{IBr}(\breve{C}_0)$. Write $\breve{Y}_0 := \breve{\Delta}_{Z_0 B} Y_0$. Since $Z_0 B \leq \mathbf{C}_{Y_0}(C_0)$, by Corollary 4.11 we have that Equation (6.2) yields

$$(\widetilde{C}_0(\breve{Y}_0 \rtimes \mathfrak{S}_r), \widetilde{C}_0, \widetilde{\varphi}) \succ_{Br,c} (\breve{Y}_0 \rtimes \mathfrak{S}_r, \breve{C}_0, \breve{\varphi}). \tag{6.3}$$

According to Theorem 5.1, we have $Z_0 = Z \cap C_0$ and consequently $\breve{C} = \widetilde{Z}\breve{C}_0$. Then Equation (6.3) together with Lemma 3.10 implies

$$(\widetilde{C}\breve{Y}_{\widetilde{\psi}}, \widetilde{C}, \widetilde{\psi}) \succ_{Br,c} (\breve{Y}_{\breve{\psi}}, \breve{C}, \breve{\psi}). \quad \square$$

We can finally prove Proposition 6.5.

Proof of Proposition 6.5. We define $\widetilde{\Omega} := \widetilde{\Omega}_{\widetilde{X},A}\colon \mathrm{IBr}_A(\widetilde{X}) \to \mathrm{IBr}_B(\breve{C})$ by

$$\widetilde{\Omega}(\theta \times \cdots \times \theta) = \widetilde{f}_r(\Omega_B(\theta) \times \cdots \times \Omega_B(\theta)),$$

for every $\theta \in \mathrm{IBr}_B(X)$, where Ω_B is given by Definition 6.1(i) and $\widetilde{f}_r$ is given by Proposition 6.7. This map is well-defined according to Lemma 6.6. Also, by Lemma 6.6(e) this map is a bijection since $\widetilde{f}_r$ is a bijection.

Recall $\widetilde{\Gamma} = \mathbf{C}_{\mathrm{Aut}(\widetilde{X})}(A)$. Write $\Upsilon = \breve{\Delta}_B \Gamma \rtimes \mathfrak{S}_r = \Delta\Gamma(B \wr \mathfrak{S}_r)$. By Proposition 5.6

$$\widetilde{\Gamma} \leq \widetilde{\Gamma}A \leq \Upsilon.$$

In order to prove that $\widetilde{\Omega}$ is $\widetilde{\Gamma}$-equivariant we show that $\widetilde{\Omega}$ is actually Υ-equivariant. In view of the description of $\mathrm{IBr}_A(\widetilde{X})$ and $\mathrm{IBr}_A(\widetilde{C})$ given in Lemma 6.6, it follows that $B \wr \mathfrak{S}_r$ acts trivially on those sets. Hence $\widetilde{\Omega}$ is $B \wr \mathfrak{S}_r$-equivariant. By definition, Ω_B is Γ-equivariant and by Proposition 6.7, $\widetilde{f}_r$ is $\Delta\Gamma$-equivariant. Hence $\widetilde{\Omega}$ is $\Delta\Gamma$-equivariant, so $\widetilde{\Gamma}$-equivariant.

It remains to prove that for every $\chi \in \mathrm{IBr}_A(\widetilde{X})$

$$(\widetilde{X} \rtimes \widetilde{\Gamma}_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\widetilde{C} \rtimes \widetilde{\Gamma}_\chi, \widetilde{C}, \widetilde{\Omega}(\chi)).$$

Let $\chi \in \mathrm{IBr}_A(\widetilde{X})$ and $\theta \in \mathrm{IBr}_B(X)$ with $\chi = \theta \times \cdots \times \theta$. According to Definition 6.1(i), $\theta' := \Omega_B(\theta)$ satisfies

$$(X \rtimes \Gamma_\theta, X, \theta) \succ_{Br,c} (C \rtimes \Gamma_\theta, C, \theta').$$

For $\widetilde{\psi} = \theta' \times \cdots \times \theta' \in \mathrm{IBr}(\widetilde{C})$, the equation above and Theorem 3.6 imply

$$(\widetilde{X} \rtimes (\Gamma_\theta \wr \mathfrak{S}_r), \widetilde{X}, \chi) \succ_{Br,c} (\widetilde{C} \rtimes (\Gamma_\theta \wr \mathfrak{S}_r), \widetilde{C}, \widetilde{\psi}).$$

Using $\Upsilon \leq \Gamma \wr \mathfrak{S}_r$ and $\Upsilon_\chi \leq \Gamma_\theta \wr \mathfrak{S}_r$ we deduce

$$(\widetilde{X} \rtimes \Upsilon_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\widetilde{C} \rtimes \Upsilon_\chi, \widetilde{C}, \widetilde{\psi}). \tag{6.4}$$

Let $\breve{Y} := \breve{C} \rtimes \Upsilon$, $\breve{\psi} := \widetilde{f}_r(\widetilde{\psi})$. Then $\breve{\psi} = \widetilde{\Omega}(\chi)$. By Proposition 6.7 we know

$$(\widetilde{C}\breve{Y}_{\widetilde{\psi}}, \widetilde{C}, \widetilde{\psi}) \succ_{Br,c} (\breve{Y}_{\widetilde{\psi}}, \breve{C}, \breve{\psi}).$$

Because of $\widetilde{C} \rtimes \Upsilon_\chi = \widetilde{C}\breve{Y}_{\widetilde{\psi}}$ and $\breve{C} \rtimes \Upsilon_{\breve{\psi}} = \breve{Y}_{\breve{\psi}}$, then the equation above is exactly

$$(\widetilde{C} \rtimes \Upsilon_\chi, \widetilde{C}, \widetilde{\psi}) \succ_{Br,c} (\breve{C} \rtimes \Upsilon_{\widetilde{\psi}}, \breve{C}, \breve{\psi}). \tag{6.5}$$

Since $\succ_{Br,c}$ is a partial order relation, $\succ_{Br,c}$ is transitive, so Equations (6.4) and (6.5) imply

$$(\widetilde{X} \rtimes \Upsilon_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\breve{C} \rtimes \Upsilon_{\widetilde{\psi}}, \breve{C}, \breve{\psi}).$$

Since $\widetilde{\Gamma}_\chi \leq \Upsilon_\chi = \Upsilon_{\widetilde{\psi}}$ and $\breve{\psi} = \widetilde{\Omega}(\chi)$ this proves the statement. $\square$

Remark 6.8. Assume the situation described in Proposition 6.5 as well as the notation of Theorem 6.4. The proof of Proposition 6.5 actually shows that the conclusions of Theorem 6.4 also hold with $\Upsilon = \breve{\Delta}_B\Gamma \rtimes \mathfrak{S}_r \supseteq \widetilde{\Gamma}$ in place of $\widetilde{\Gamma}$.

Proposition 6.9. *Let m be the length of some A-orbit on $\{X_1, \dots, X_r\}$. If $\widetilde{\Gamma}A$ acts transitively on $\{X_1, \dots, X_r\}$ and $A \leq (B \wr \mathfrak{S}_m)^{\frac{r}{m}}$, then Theorem 6.4 holds.*

Proof. Let $\Lambda = \{X_1, \dots, X_r\}$ and $\Lambda_1, \dots, \Lambda_d$ be the A-orbits on Λ. Notice that $\widetilde{\Gamma}$ permutes the A-orbits transitively by hypothesis. Hence $r = dm$, where m is the length of any A-orbit. We may assume $\Lambda_j = \{X_{(j-1)m+1}, \dots, X_{jm}\}$ for every $1 \leq j \leq d$. Since $A \leq (B \wr \mathfrak{S}_m)^d$, we know from Proposition 5.7 that

$$A\mathbf{C}_{\mathrm{Aut}(\widetilde{X})}(A) = A\widetilde{\Gamma} \leq \Upsilon,$$

where $\Upsilon := (\breve{\Delta}_B^m\Gamma \rtimes \mathfrak{S}_m) \wr \mathfrak{S}_d$.

For every $1 \leq j \leq d$ let $X_{\Lambda_j} := \langle Y \mid Y \in \Lambda_j \rangle$ and $Z_{\Lambda_j} := \langle \mathbf{Z}(Y) \mid Y \in \Lambda_j \rangle$. Clearly A acts on X_{Λ_j} with $(|A|, |X_{\Lambda_j}|) = 1$. Let Υ_j be the projection of $\mathrm{Stab}_\Upsilon(X_{\Lambda_j})$ into $\mathrm{Aut}(X_{\Lambda_j})$. Then Υ_j is isomorphic to $\breve{\Delta}_B^m\Gamma \rtimes \mathfrak{S}_m$. By Lemma 6.5 (and using Remark 6.8), there is a Υ_1-equivariant bijection

$$\widetilde{\Omega}_{\Lambda_1,A} \colon \mathrm{IBr}_A(X_{\Lambda_1}) \to \mathrm{IBr}_A(\breve{C}_{\Lambda_1}),$$

where $\breve{C}_{\Lambda_j} := \mathbf{C}_{X_{\Lambda_j}}(A)Z_{\Lambda_1}$. Furthermore for every $\chi_1 \in \mathrm{IBr}_A(X_{\Lambda_1})$ and $\breve{\chi}_1 := \widetilde{\Omega}_{\Lambda_1,A}(\chi_1)$ we have

$$(X_{\Lambda_1} \rtimes (\Upsilon_1)_{\chi_1}, X_{\Lambda_1}, \chi_1) \succ_{Br,c} (\breve{C}_{\Lambda_1} \rtimes (\Upsilon_1)_{\chi_1}, \breve{C}_{\Lambda_1}, \breve{\chi}_1). \tag{6.6}$$

For $2 \leq j \leq d$ we define Υ_j-equivariant bijections

$$\widetilde{\Omega}_{\Lambda_j,A}\colon \mathrm{IBr}_A(X_{\Lambda_j}) \to \mathrm{IBr}_A(\breve{C}_{\Lambda_j})$$

from $\widetilde{\Omega}_{X_{\Lambda_1},A}$ via the permutation action of $\mathfrak{S}_d$. For every $\chi_j \in \mathrm{IBr}_A(X_{\Lambda_j})$ and $\breve{\chi}_j := \widetilde{\Omega}_{\Lambda_j,A}(\chi_j)$ we have

$$(X_{\Lambda_j} \rtimes (\Upsilon_j)_{\chi_j}, X_{\Lambda_j}, \chi_j) \succ_{Br,c} (\breve{C}_{\Lambda_j} \rtimes (\Upsilon_j)_{\chi_j}, \breve{C}_{\Lambda_j}, \breve{\chi}_j) \tag{6.7}$$

by a transfer of Equation (6.6) via Lemma 3.8. Note that $\mathrm{IBr}_A(\widetilde{X})$ is in natural correspondence with $\mathrm{IBr}_A(X_{\Lambda_1}) \times \cdots \times \mathrm{IBr}_A(X_{\Lambda_d})$ and $\breve{C} = \mathbf{C}_{\widetilde{X}}(A)\widetilde{Z} = \breve{C}_{\Lambda_1} \times \cdots \times \breve{C}_{\Lambda_d}$. For $\chi_j \in \mathrm{IBr}_A(X_{\Lambda_j})$, the map $\widetilde{\Omega}_{\widetilde{X},A}\colon \mathrm{IBr}_A(\widetilde{X}) \to \mathrm{IBr}(\breve{C})$ given by $\chi_1 \times \cdots \times \chi_d \mapsto \widetilde{\Omega}_{\Lambda_1,A}(\chi_1) \times \cdots \times \widetilde{\Omega}_{\Lambda_d,A}(\chi_d)$ is a well-defined $(\Upsilon_1 \times \cdots \times \Upsilon_d)$-equivariant bijection. By definition $\widetilde{\Omega}_{\widetilde{X},A}$ is $\mathfrak{S}_d$-equivariant, where $\mathfrak{S}_d$ is identified with the subgroup of Υ acting on the groups X_{Λ_i} by permutation. Hence it is Υ-equivariant.

A straightforward argument proves that every character in $\mathrm{IBr}_A(\widetilde{X})$ is $(\Upsilon_1 \times \cdots \times \Upsilon_d)$-conjugate to some $\chi = \chi_1 \times \cdots \times \chi_d$ where either χ_i and χ_j are $\mathfrak{S}_d$-conjugate or χ_i and χ_j are not Υ-conjugate (again $\mathfrak{S}_d$ is identified with the subgroup of Υ acting on the X_{Λ_i} groups by permutation). This implies that the stabilizer Υ_χ of χ in Υ satisfies

$$\Upsilon_\chi = ((\Upsilon_1)_{\chi_1} \times \cdots (\Upsilon_d)_{\chi_d}) \rtimes (\mathfrak{S}_d)_\chi.$$

The Equations (6.6) and (6.7) for $2 \leq i \leq d$ together with Theorem 3.6 imply that the character χ satisfies

$$(\widetilde{X} \rtimes \Upsilon_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\breve{C} \rtimes \Upsilon_\chi, \breve{C}, \chi'),$$

where $\chi' := \widetilde{\Omega}(\chi) = \breve{\chi}_1 \times \cdots \times \breve{\chi}_d$. Of course, since $\widetilde{\Gamma} \leq \Upsilon$, we deduce

$$(\widetilde{X} \rtimes \widetilde{\Gamma}_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\breve{C} \rtimes \widetilde{\Gamma}_\chi, \breve{C}, \chi').$$

By Lemma 3.8 this proves the statement. $\square$

Proof of Theorem 6.4 . By Proposition 5.4 there exists some $\alpha \in \mathrm{Aut}(X)^r$ such that $A^\alpha \leq B \wr \mathfrak{S}_r$, where B is the projection of A_{X_1} on $\mathrm{Aut}(X)$. Let m be the length of an A-orbit on $\{X_1, \ldots, X_r\}$. Since $\widetilde{\Gamma}$ acts transitively on the A-orbits, $d = r/m$ is the number of A-orbits. Let $\tau \in \mathfrak{S}_d$ be as given in Proposition 5.6. Then $A^{\alpha\tau} \leq (B \wr \mathfrak{S}_m)^d$. Let $\widetilde{\Omega}_{\widetilde{X},A^{\alpha\tau}}$ be the $\widetilde{\Gamma}^{\alpha\tau}$-equivariant bijection given by Proposition 6.9. Define $\widetilde{\Omega}_{\widetilde{X},A}$ by $\chi \mapsto \widetilde{\Omega}_{\widetilde{X},A^{\alpha\tau}}(\chi^{\alpha\tau})^{(\alpha\tau)^{-1}}$ for every $\chi \in \mathrm{IBr}_A(\widetilde{X})$. It is easy to check that $\widetilde{\Omega}_{\widetilde{X},A}$ is a $\widetilde{\Gamma}$-equivariant bijection. Use Lemma 3.8 to check the central character triple isomorphism condition with respect to $\widetilde{\Omega}_{\widetilde{X},A}$. $\square$

The above statement is relevant by implying the following two results.

Theorem 6.10. *Let A act coprimely on G. Let $K \lhd G$ be an A-invariant perfect subgroup. Suppose that $G = K\mathbf{C}_G(A)$. Write $C = \mathbf{C}_G(A)$ and $M = K \cap C$. Suppose further that A acts trivially on $N = \mathbf{Z}(G) \leq K$, $K/N = S_1 \times \cdots \times S_r \cong S^r$, where S is a non-abelian simple group, and CA permutes transitively $\{S_1, \dots, S_r\}$. If S satisfies the inductive Brauer–Glauberman condition, then there exists a C-equivariant bijection*

$$\Omega' \colon \mathrm{IBr}_A(K) \to \mathrm{IBr}(M),$$

such that $(G_\chi, K, \chi) \succ_{Br,c} (C_\chi, M, \chi')$ for every $\chi \in \mathrm{IBr}_A(K)$ and $\chi' = \Omega'(\chi)$.

Proof. Because of $G = K\mathbf{C}_G(A)$ the group $\mathbf{C}_A(K)$ coincides with $\mathbf{C}_A(G)$ and we may assume that A acts faithfully on K, i.e. $A \leq \mathrm{Aut}(K)$. Let $\widetilde{S} := S^r$. Let X be the universal covering group of S. Then, $\widetilde{X}$ is the universal covering group of $\widetilde{S}$. Write $\widetilde{Z} := \mathbf{Z}(\widetilde{X})$. Since K is a covering of $\widetilde{S}$, there exists an epimorphism $\pi \colon \widetilde{X} \to K$ with $L := \ker(\pi) \leq \widetilde{Z}$, where $\widetilde{Z} := \mathbf{Z}(\widetilde{X})$. In fact, $\widetilde{X}$ is the universal covering of K.

The map π induces an isomorphism $\mathrm{Aut}(\widetilde{X})_L \to \mathrm{Aut}(K)$. Hence, the groups A and $\overline{C} = C\mathbf{C}_G(K)/\mathbf{C}_G(K)$ can be seen as groups of automorphisms of $\widetilde{X}$. In fact, under this identification, the group $A\overline{C} \leq A\mathbf{C}_{\mathrm{Aut}(\widetilde{X})}(A)$ acts transitively on the factors of $\widetilde{X}$. Because of $(|A|, |K|) = 1$ we have $(|A|, |\widetilde{X}|) = 1$ by [1, 3.3.12]. Write $\widetilde{\Gamma} := \mathbf{C}_{\mathrm{Aut}(\widetilde{X})}(A)$ and $\breve{C} := \mathbf{C}_{\widetilde{X}}(A)\widetilde{Z}$. By Theorem 6.4, there exists a $\widetilde{\Gamma}$-equivariant bijection $\widetilde{\Omega} := \widetilde{\Omega}_{\widetilde{X},A} \colon \mathrm{IBr}_A(\widetilde{X}) \to \mathrm{IBr}_A(\breve{C})$ such that

$$(\widetilde{X} \rtimes \widetilde{\Gamma}_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\breve{C} \rtimes \widetilde{\Gamma}_\chi, \breve{C}, \widetilde{\Omega}(\chi))$$

for every $\chi \in \mathrm{IBr}_A(\widetilde{X})$. Since $\overline{C} \leq \widetilde{\Gamma}$, we have that $\widetilde{\Omega}$ is $\overline{C}$-equivariant and

$$(\widetilde{X} \rtimes \overline{C}_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\breve{C} \rtimes \overline{C}_\chi, \breve{C}, \chi') \tag{6.8}$$

for every $\chi \in \mathrm{IBr}_A(\widetilde{X})$ and $\chi' := \widetilde{\Omega}(\chi)$. According to Equation (6.8) and Definition 3.3(ii)

$$\widetilde{\Omega}(\mathrm{IBr}_A(\widetilde{X} \mid 1_L)) = \mathrm{IBr}_A(\breve{C} \mid 1_L).$$

Note that $\pi(\breve{C}) = \pi(\mathbf{C}_{\widetilde{X}}(A)\widetilde{Z}) = \mathbf{C}_K(A) = M$ and $\mathrm{IBr}_A(M) = \mathrm{IBr}(M)$. Hence $\widetilde{\Omega}$ defines, via π, a C-equivariant bijection

$$\Omega' \colon \mathrm{IBr}_A(K) \to \mathrm{IBr}(M).$$

Notice that $\mathbf{C}_{\widetilde{X} \rtimes \overline{C}}(\widetilde{X}) = \widetilde{Z}$, $\pi(\widetilde{Z}/L) = N = \mathbf{Z}(K) = \mathbf{C}_{K \rtimes \overline{C}}(K)$ and $L \leq \widetilde{Z} \cap \ker(\chi) \cap \ker(\chi')$ for every $\chi \in \mathrm{IBr}_A(\widetilde{X}/L)$ and $\chi' := \widetilde{\Omega}(\chi) \in \mathrm{IBr}_A(\breve{C}/L)$. By Lemma 3.5, Equation (6.8) implies that

$$(K \rtimes \overline{C}_\chi, K, \chi) \succ_{Br,c} (M \rtimes \overline{C}_\chi, M, \chi')$$

for every $\chi \in \mathrm{IBr}_A(K)$ and $\chi' := \Omega'(\chi)$. Finally, a direct application of Theorem 3.7 yields

$$(G_\chi, K, \chi) \succ_{Br,c} (C_\chi, M, \chi'). \quad \square$$

Corollary 6.11. *Let A act coprimely on G. Let $K \lhd G$ be A-invariant. Suppose that $G = K\mathbf{C}_G(A)$. Write $C = \mathbf{C}_G(A)$ and $M = K \cap C$. Suppose further that A acts trivially on $N = \mathbf{Z}(G) \leq K$, $K/N = S_1 \times \cdots \times S_r \cong S^r$, where S is a non-abelian simple group, and CA permutes transitively $\{S_1, \ldots, S_r\}$. If S satisfies the inductive Brauer–Glauberman condition, then there exists a bijection*

$$\Omega' \colon \mathrm{IBr}_A(K) \to \mathrm{IBr}(M),$$

satisfying $(G_\chi, K, \chi) \succ_{Br,c} (C_\chi, M, \chi')$ for every $\chi \in \mathrm{IBr}_A(K)$ and $\chi' = \Omega'(\chi)$.

Proof. Let $K_1 = [K, K]$. Since K/N is a direct product of simple non-abelian groups, it follows that K_1 is perfect and $K = K_1 N$. Let $N_1 = N \cap K_1$. Notice that $\mathbf{C}_G(K_1) = \mathbf{C}_G(K)$. Also K is the central product of K_1 and N. Write $M_1 = M \cap K_1$.

Let $\Omega'_1 \colon \mathrm{IBr}_A(K_1) \to \mathrm{IBr}(M_1)$ be the bijection given by Theorem 6.10. Every $\chi \in \mathrm{IBr}_A(K)$ has the form $\chi_1 \cdot \mu$, where $\chi_1 \in \mathrm{IBr}_A(K_1)$, $\mu \in \mathrm{IBr}(N)$ and both characters lie over the same Brauer character of N_1. Define $\Omega' \colon \mathrm{IBr}_A(K) \to \mathrm{IBr}(M)$ by

$$\chi_1 \cdot \mu \mapsto \Omega'_1(\chi_1) \cdot \mu.$$

It is clear that Ω' is a C-equivariant bijection. Let $\chi = \chi_1 \cdot \mu \in \mathrm{IBr}_A(K)$. By Theorem 6.10 we have $(G_{\chi_1}, K_1, \chi_1) \succ_{Br,c} (C_{\chi_1}, M_1, \Omega'_1(\chi_1))$. A direct application of Lemma 3.10 implies

$$(G_\chi, K, \chi) \succ_{Br,c} (C_\chi, M, \Omega'(\chi)). \quad \square$$

7. A reduction theorem

We prove a relative version of Theorem A (relative to a normal subgroup) that allows us to use a key inductive argument.

Theorem 7.1. *Let A act coprimely on G. Let $N \lhd G$ be stabilized by A and write $C = \mathbf{C}_G(A)$. Let $\theta \in \mathrm{IBr}_A(N)$. Suppose that the non-abelian simple groups involved in G/N*

satisfy the inductive Brauer–Glauberman condition. Then

$$|\mathrm{IBr}_A(G|\theta)| = |\mathrm{IBr}(CN|\theta)|.$$

Proof. We proceed by induction on $|G : N|$.

Step 1. We may assume θ is G-invariant.
Let $T = G_\theta$. Then $CN \cap T = (CN)_\theta$. By the Clifford correspondence for Brauer characters [13, Thm 8.9]

$$|\mathrm{IBr}_A(G|\theta)| = |\mathrm{IBr}_A(T|\theta)| \text{ and } |\mathrm{IBr}(CN|\theta)| = |\mathrm{IBr}(CN \cap T|\theta)|.$$

If $T < G$, then by induction hypothesis with respect to $|T : N| < |G : N|$

$$|\mathrm{IBr}_A(T|\theta)| = |\mathrm{IBr}(\mathbf{C}_T(A)N|\theta)| = |\mathrm{IBr}(CN \cap T|\theta)|.$$

Step 2. We may assume that $N \leq \mathbf{Z}(GA)$ is a p'-group.
By Theorem 8.28 of [13], there exists a strong isomorphism of modular character triples

$$(\sigma, \tau) \colon (GA, N, \theta) \to (\Gamma, M, \varphi)$$

such that $M \leq \mathbf{Z}(\Gamma)$ is a p'-group. Whenever $N \leq H \leq GA$, we write H^τ to denote the subgroup of Γ such that $\tau(H/N) = H^\tau/M$. Then $A \cong \tau(AN/N) = (AN)^\tau/M$, so that $(AN)^\tau/M$ acts on G^τ/M as A acts on G/N and $(AN)^\tau/M$ acts trivially on M. Therefore

$$(CN)^\tau/M = \tau(CN/N) = \tau(\mathbf{C}_{G/N}(A)) = \mathbf{C}_{G^\tau/M}((AN)^\tau/M) = \mathbf{C}_{G^\tau}((AN)^\tau)/M.$$

By Theorem 8.13 of [13], θ extends to AN, and hence φ extends to $(AN)^\tau$. Recall that φ is a linear character since $M \leq \mathbf{Z}(\Gamma)$. Let π be the set of primes dividing $|A|$. Write $\varphi = \varphi_\pi \varphi_{\pi'}$. Recall that φ_π and $\varphi_{\pi'}$, the π-part and π'-part of φ, are powers of φ. In particular, φ_π extends to $(AN)^\tau$. If $q \notin \pi$, then by Theorem 8.13 of [13] we have that φ_π extends to Q for every $Q/M \in \mathrm{Syl}_q(\Gamma/M)$. Thus φ_π extends to Γ according to [13, Thm 8.29]. By parts (d) and (b) of Lemma 3.2, the modular character triple (Γ, M, φ) is strongly isomorphic to $(\Gamma, M, \varphi_{\pi'})$ and we may assume that $\varphi_{\pi'}$ is faithful. Write $\varphi' = \varphi_{\pi'}$. We have that $|M| = o(\varphi')$ is a π'-number. Hence M has a complement B in $(AN)^\tau$ by Schur–Zassenhaus' Lemma. Thus B acts coprimely on G^τ and $(CN)^\tau/M = \mathbf{C}_{G^\tau}(B)/M$. Since (GA, N, θ) is strongly isomorphic to (Γ, M, φ') we have that

$$|\mathrm{IBr}_A(G|\theta)| = |\mathrm{IBr}_B(G^\tau|\varphi')| \text{ and } |\mathrm{IBr}(CN|\theta)| = |\mathrm{IBr}(\mathbf{C}_{G^\tau}(B)|\varphi')|$$

and the claim follows.

Step 3. We may assume $G = KC$ for every A-invariant K with $N < K \lhd G$.
Let $N < K \lhd G$ be A-invariant. We have that C acts on $\mathrm{IBr}_A(K|\theta)$. Let $\mathcal{B}$ be a complete set of representatives of C-orbits on $\mathrm{IBr}_A(K|\theta)$. Let $K \leq H \leq G$ and $\psi \in \mathrm{IBr}_A(H|\theta)$. We have that H/K acts transitively on $\mathrm{IBr}(\psi_K)$ by [13, Cor. 8.7]. Also A acts on $\mathrm{IBr}(\psi_K)$. Since $(|A|, |H/K|) = 1$, by Glauberman's Lemma ([5, Lem. 13.8] and [5, Cor. 13.9]) there is some A-invariant character in $\mathrm{IBr}(\psi_K)$ and any two of them are C-conjugate. This proves that every $\psi \in \mathrm{IBr}_A(H|\theta)$ lies over a unique element of $\mathcal{B}$. By the previous argument for $H = G$ and $H = CK$ we have that

$$|\mathrm{IBr}_A(G|\theta)| = \sum_{\eta \in \mathcal{B}} |\mathrm{IBr}_A(G|\eta)| \quad \text{and} \quad |\mathrm{IBr}_A(CK|\theta)| = \sum_{\eta \in \mathcal{B}} |\mathrm{IBr}_A(CK|\eta)|.$$

By the inductive hypothesis $|\mathrm{IBr}_A(G|\eta)| = |\mathrm{IBr}(CK|\eta)|$ for every $\eta \in \mathcal{B}$. Since A acts coprimely on CK/K and $\mathbf{C}_{CK/K}(A) = CK/K$, we have that $\mathrm{IBr}_A(CK|\eta) = \mathrm{IBr}(CK|\eta)$ by Lemma 2.1. Hence $|\mathrm{IBr}_A(G|\theta)| = |\mathrm{IBr}_A(CK|\theta)|$. If $CK < G$, then by induction $|\mathrm{IBr}_A(CK|\theta)| = |\mathrm{IBr}(C|\theta)|$, and the claim follows.

Step 4. We may assume $\mathbf{O}_p(G) = 1$.
Write $O = \mathbf{O}_p(G)$. If $O > 1$, then $|G/O : NO/O| < |G : N|$. To prove the claim use that $\mathbf{C}_{G/O}(A) = CO/O$ by coprime action, the fact that $O \leq \ker(\varphi)$ for every $\varphi \in \mathrm{IBr}(G)$ [13, Lem. 2.32] and the inductive hypothesis.

Step 5. Every chief factor K/N of GA with $K \leq G$ is a direct product of isomorphic non-abelian simple groups and $N = \mathbf{Z}(G)$.
Let K/N be a chief factor of GA with $K \leq G$. We may assume that $G = KC$ by *Step 3* and that K/N is not a p-group by *Step 4*. If K/N is a p'-group, then $|\mathrm{IBr}_A(G|\theta)| = |\mathrm{IBr}(C|\theta)|$ according to Corollary 2.6. Hence we can assume that GA has no abelian chief factor of the form K/N with $K \leq G$. In particular $N = \mathbf{Z}(G)$.

Final Step. Let K/N be a chief factor of GA with $K \leq G$. By *Step 4* we may assume $G = KC$. By *Step 5* we have that $K/N \cong S_1 \times \cdots \times S_r$, where the S_i are simple non-abelian groups. Notice that CA permutes transitively the groups S_i in K/N and hence they are all isomorphic. Since $S := S_1$ is involved in G/N, then S satisfies the inductive Brauer–Glauberman condition. Write $M = C \cap K$. By Corollary 6.11 there is a C-equivariant bijection

$$\Omega' \colon \mathrm{IBr}_A(K) \to \mathrm{IBr}(M),$$

such that $(G_\eta, K, \eta) \succ_{Br,c} (C_\eta, M, \Omega'(\eta))$ for every $\eta \in \mathrm{IBr}_A(K)$. Since $N \leq \mathbf{C}_G(K)$, then Ω' actually yields a bijection $\mathrm{IBr}_A(K|\theta) \to \mathrm{IBr}(M|\theta)$. Let $\mathcal{B}$ be a set of representatives of C-orbits on $\mathrm{IBr}_A(K|\theta)$. Every element of $\mathrm{IBr}_A(G|\theta)$ lies over a unique element of $\mathcal{B}$ as in *Step 3*. Since Ω' is C-equivariant, we have that $\mathcal{B}' = \{\Omega'(\eta) \mid \eta \in \mathcal{B}\}$ is a set of representatives of C-orbits on $\mathrm{IBr}(M|\theta)$. Hence

$$|\mathrm{IBr}_A(G|\theta)| = \sum_{\eta \in \mathcal{B}} |\mathrm{IBr}_A(G|\eta)| \quad \text{and} \quad |\mathrm{IBr}(C|\theta)| = \sum_{\eta \in \mathcal{B}} |\mathrm{IBr}(C|\Omega'(\eta))|.$$

For every $\eta \in \mathcal{B}$, we have $\mathrm{IBr}_A(G|\eta) = \mathrm{IBr}(G|\eta)$ by Lemma 2.1 and $(G_\eta, K, \eta) \succ_{Br,c} (C_\eta, M, \Omega'(\eta))$ by Corollary 6.11. Thus $|\mathrm{IBr}(G_\eta|\eta)| = |\mathrm{IBr}(C_\eta|\Omega'(\eta))|$ for every $\eta \in \mathcal{B}$. The result follows then by using the Clifford correspondence [13, Thm. 8.9]. □

Corollary 7.2. *Let Γ be a group that acts coprimely on a group G. Suppose that every simple non-abelian group involved in G satisfies the inductive Brauer–Glauberman condition. Then the actions of Γ on the Brauer characters of G and on the p-regular classes of G are permutation isomorphic*

Proof. For every $A \leq \Gamma$, Theorem 7.1 with $N = 1$ guarantees that $|\mathrm{IBr}_A(G)| = |\mathrm{IBr}(\mathbf{C}_G(A))|$. The map $K \mapsto K \cap \mathbf{C}_G(A)$ is a well-defined bijection between the set of A-invariant p-regular classes of G and the set of p-regular conjugacy classes of $\mathbf{C}_G(A)$. Hence the number of A-invariant irreducible Brauer characters of G equals the number of A-invariant p-regular conjugacy classes of G. By Lemma 13.23 of [5], this proves the statement. □

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