CHARACTER CORRESPONDENCES IN SOLVABLE GROUPS WITH A SELF-NORMALIZING SYLOW SUBGROUP

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Abstract. Let G be a finite solvable group and let $P \in \mathrm{Syl}_p(G)$ for some prime p. Whenever $|G : \mathbf{N}_G(P)|$ is odd, I. M. Isaacs described a correspondence between irreducible characters of degree not divisible by p of G and $\mathbf{N}_G(P)$. This correspondence is *natural* in the sense that an algorithm is provided to compute it, and the result of the application of the algorithm does not depend on choices made. In the case where $\mathbf{N}_G(P) = P$, G. Navarro showed that every irreducible character χ of degree not divisible by p has a unique linear constituent χ^* when restricted to P, and that the map $\chi \mapsto \chi^*$ defines a bijection. Navarro's bijection is obviously natural in the sense described above. We show that these two correspondences are the same under the intersection of the hypotheses.

Introduction

In 1972, J. McKay foresaw that, for a finite simple group G, the number of odd-degree irreducible characters in G is the same as the number of odd-degree irreducible characters in the normalizer of a Sylow 2-subgroup of G. This astonishing conjecture (which has been recently confirmed in [MS16] for every finite group) was generalized by various authors to other families of groups and other primes until it took the form we know nowadays.

The McKay Conjecture. *Let G be a finite group and let $P \in \mathrm{Syl}_p(G)$ for any prime p. Then*

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|,$$

where $\mathrm{Irr}_{p'}(G)$ denotes the set of irreducible characters of G of degree coprime to p.

By saying a McKay bijection for G and p (where p will be generally clear from the context) we mean a bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$.

In 1973, I. M. Isaacs created an algorithm to compute a McKay bijection whenever G is solvable and $|G : \mathbf{N}_G(P)|$ is odd. Such algorithm is rather involved. In fact, if ξ corresponds to χ under the algorithm, then ξ is not always a constituent of the restriction $\chi_{\mathbf{N}_G(P)}$ of χ to $\mathbf{N}_G(P)$. (However, Isaacs showed that ξ is a constituent of $\chi_{\mathbf{N}_G(P)}$ if G has odd-order). Later on, precisely 30 years later, G. Navarro proved that a much simpler algorithm could be used to compute a McKay bijection whenever G was a p-solvable group with a self-normalizing Sylow p-subgroup. He proved that for every $\chi \in \mathrm{Irr}_{p'}(G)$,

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the restriction χ_P contains a unique linear character $\lambda_\chi \in \mathrm{Irr}(P)$. The algorithm is clear in this situation, $\chi \mapsto \lambda_\chi$.

The question whether the application of these two algorithms gives the same result when the hypotheses meet has been raised by I. M. Isaacs and G. Navarro during the "Group Representation Theory and Applications" program at the MSRI. In this short note, we positively answer this question in Theorem 3.6. The key is a character extension result, namely Theorem 3.5.

This is good news because natural bijections, as described in the abstract, are hoped to be unique.

Throughout this note we follow the notation of [Isa76] for ordinary characters. By a constituent of a character, we will mean an irreducible constituent. Indeed, if Δ is a character of some finite group H, then

$$\Delta = \sum_{\psi \in \mathrm{Irr}(H)} a_\psi \psi,$$

for $a_\psi \in \mathbb{Z}_{\geqslant 0}$. We say that ψ is a constituent of Δ if $a_\psi \neq 0$.

If $N \lhd G$ and $\chi \in \mathrm{Irr}(G)$, then we say that θ lies under χ if θ is a constituent of χ_N. We also say that χ lies over θ in this situation and write $\chi \in \mathrm{Irr}(G|\theta)$.

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1. Isaacs' 1973 correspondece

In 1973, only one year after the McKay conjecture was proposed, I. M. Isaacs described McKay bijections for solvable groups under a certain condition. In the beginning, he was interested in constructing a Glauberman-type character correspondence in the case of coprime actions on groups of odd-order. Not only did he succeed on that initial goal, but also he established the ideal way of solving the McKay conjecture.

Theorem 1.1 (Isaacs, 1973)**.** *Let G be solvable and let $P \in \mathrm{Syl}_p(G)$. Assume $|G : \mathbf{N}_G(P)|$ is odd. Then there exists a natural bijection between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$.*

The two main features of the above bijection are that it is equivariant with respect to the action of $\mathrm{Aut}(G)_P$ on characters as well as equivariant with respect to the action of $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ on characters. (Although these are non-trivial facts, we do not intend to prove them here, since that would unnecessarily lengthen this exposition.)

The $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$-equivariance implies that the multisets $\{\{\mathbb{Q}(\chi) \mid \chi \in \mathrm{Irr}_{p'}(G)\}\}$ and $\{\{\mathbb{Q}(\xi) \mid \xi \in \mathrm{Irr}_{p'}(\mathbf{N}_G(P))\}\}$ of field extensions of $\mathbb{Q}$ are the same. For a character ψ of some group H, the field of values of ψ over $\mathbb{Q}$ is defined as $\mathbb{Q}(\psi) = \mathbb{Q}(\psi(h) \mid h \in H)$. This property does not hold in general, not even in solvable groups, as shown by $\mathrm{GL}(2,3)$ and $p = 3$. However, G. Navarro proposed in 2004 that there should exist McKay bijections preserving fields of values over $\mathbb{Q}_p$, the field of p-adic numbers. This is known as the Galois-McKay conjecture, see [Nav04].

In general, the existence of McKay bijections equivariant with respect to the action of group automorphisms stabilizing the Sylow p-subgroup was conjectured by E. C. Dade (see Conjecture 9.13 of [Nav18]).

There is a huge machinery underlying Theorem 1.1 that culminated in the complete description of the character theory of fully ramified sections. This is done in Theorem 9.1 of [Isa73], and the description depends on a special character ψ arising in such context. Note that ψ does not generally contain the trivial character as a constituent. This fact can be already observed in the group $\mathrm{SL}(2,3)$.

Remark. As we have already mentioned in the Introduction, if $\chi \in \mathrm{Irr}_{p'}(G)$ and $\xi \in \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ correspond under the bijection in Theorem 1.1, it is not true that ξ is a constituent of $\chi_{\mathbf{N}_G(P)}$ (see the last paragraph of Section 9 of [Isa73]). This situation is related to the character theory of fully ramified section briefly discussed above. We give next a specific example.

Let K be extraspecial of order 3^3 and exponent 3 and let $H = \mathrm{SL}(2,3)$. The group H acts naturally on K by fixing $L = \mathbf{Z}(K)$ and acting faithfully on K/L. Let $P \in \mathrm{Syl}_2(H)$, so that $\mathbf{N}_G(P) = L \times H$. If $1_L \neq \theta \in \mathrm{Irr}(L)$, then θ is G-invariant and fully-ramified with respect to K/L. Let $\varphi \in \mathrm{Irr}(K)$ be over θ. Since φ is G-invariant, then φ extends canonically to $\tilde{\varphi} \in \mathrm{Irr}(PK)$ by Corollary 6.28 of [Isa76]. The extension $\tilde{\varphi}$ is also G-invariant and thus extends to G by Corollary 11.22 of [Isa76]. In particular, φ extends to G. Then $\mathscr{C} = (G, K, \mathbf{N}_G(P), L, \varphi, \theta)$ is a constellation (with prime $\ell = 3$) as in Definition 4.5 of [Isa73] (the element $x \in \mathbf{Z}(P)$ of order 2 gives the desired automorphism of G). By Theorem 5.6 of [Isa73], the canonical character ψ arising from $\mathscr{C}$ is a faithful, non-necessarily irreducible, character of $G/K \cong H$. By Theorem 3.5 [Isa73], ψ has degree 3. In particular, ψ must be the sum of a linear character and an irreducible character of degree 2 of $G/K \cong H$. Applying Theorem 5.7, and since $\mathbf{C}_{K/L}(x) = 1$ for every $x \in P$, we get that $\psi(x) = -1$ whenever $x \in H$ has order 2, and $\psi(x) = 1$ whenever $x \in H$ has order 4. By Corollary 6.3 of [Isa73], $\psi(x) = \pm\sqrt{-3}$ whenever $x \in H$ has order 3. This latter equality forces the linear constituent of ψ to be non-trivial. Two characters $\chi \in \mathrm{Irr}(G|\varphi)$ and $\xi \in \mathrm{Irr}(\mathbf{N}_G(P)|\theta)$ of odd degree correspond under the bijection of Theorem 1.1 if, and only if, $\chi_{\mathbf{N}_G(P)} = \psi_{\mathbf{N}_G(P)}\xi$ (as in Theorem 7.1 of [Isa73]).

Another example of this situation was constructed in Section 3 of [FN95]. There the authors were interested in characters correspondences arising under coprime actions of groups.

2. Navarro's 2003 correspondence

In 2003, G. Navarro studied McKay bijections under the key hypothesis that groups had a self-normalizing Sylow p-subgroup.

Note that if a Sylow p-subgroup P is self-normalizing in G, then

$$\mathrm{Irr}_{p'}(\mathbf{N}_G(P)) = \mathrm{Lin}(P) = \{\lambda \in \mathrm{Irr}(P) \mid \lambda(1) = 1\}.$$

Theorem 2.1 (Navarro, 2003)**.** *Let G be p-solvable and let $P \in \mathrm{Syl}_p(G)$. Assume that $\mathbf{N}_G(P) = P$. If $\chi \in \mathrm{Irr}_{p'}(G)$, then*

$$\chi_P = \chi^* + \Delta,$$

where $\chi^* \in \mathrm{Lin}(P)$ and Δ is either zero or every constituent of Δ has degree divisible by p. Furthermore the map defined by $\chi \mapsto \chi^*$ yields a bijection.

Navarro's bijection is natural. Moreover, it is straight-forward to verify that it is both $\mathrm{Aut}(G)_P$ and $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$-equivariant.

It is worth mentioning that, already in 2003, Navarro suspected that the p-solvability condition above could be replaced by the condition that p is odd. This prediction was confirmed in [NTV14], where the authors also provided a new method to prove Theorem 2.1. The following extension result is the key of the new method. Notice that, for p-solvable groups, the conditions below on simple groups are superfluous.

Theorem 2.2 (Navarro, Tiep, Vallejo, 2014). *Let G be a finite group, p any prime, $P \in \mathrm{Syl}_p(G)$, and assume that $P = \mathbf{N}_G(P)$. Let $L \lhd G$. Let $\chi \in \mathrm{Irr}_{p'}(G)$, and let $\theta \in \mathrm{Irr}(L)$ be P-invariant under χ. Assume that all non-abelian simple groups of order divisible by p involved in L satisfy the* inductive Alperin-McKay condition *for p. Then θ extends to G_θ.*

The inductive Alperin-McKay condition was defined in [Spä13]. Fortunately, the inductive Alperin-McKay condition is not really necessary in this context. In [NT16], the authors prove the above extension result without conditions on simple groups for $p = 2$. Indeed, by adapting the proof of Theorem 3.3 of [NT16] for odd primes, the hypotheses on simple groups in Theorem 2.2 can be removed in general (see Theorem 3.6 of [Val16]).

3. Solvable groups with self-normalizing Sylow p-subgroups

Suppose that G is solvable and $\mathbf{N}_G(P) = P$ for some prime p, and either $p = 2$ or G has odd order (i.e. $|G : P|$ is odd). By Theorem 2.1, if $\chi \in \mathrm{Irr}_{p'}(G)$, then $\chi_P = \chi^* + \Delta$, where $\chi^*(1) = 1$ and either Δ is zero or all its constituents have degree divisible by p. By Theorem 1.1, there is some $\xi \in \mathrm{Irr}(P)$ linear, such that, χ and ξ correspond. Are χ^* and ξ the same character? This is the same as asking if ξ is a constituent of χ_P. Following the algorithm for constructing ξ in Theorem 9.1 and Theorem 10.6 of [Isa73], one may suspect that problems can arise only if there are fully ramified situations. We show below (in Theorem 3.5) that fully ramified situations do not occur when computing ξ.

We first list some elementary results that will be often used from now on.

Lemma 3.1. *Let $K \lhd G$ and $H \leqslant G$ be such that $G = KH$ and $N = K \cap H$. Let $\varphi \in \mathrm{Irr}(K)$ be G-invariant and suppose that $\varphi_N = \theta$. Then restriction defines a bijection*

$$\mathrm{Irr}(G|\varphi) \to \mathrm{Irr}(H|\theta).$$

Proof. See Lemma 2.7 of [Isa82]. □

Lemma 3.2. *Let $N \lhd G$ and let $\chi \in \mathrm{Irr}(G)$. Assume that χ_N has some P-invariant constituent, where $P \in \mathrm{Syl}_p(G)$. Then every two of them are $\mathbf{N}_G(P)$-conjugate. In particular, this happens if χ has degree coprime to p.*

Proof. Let $\theta \in \mathrm{Irr}(N)$ be the P-invariant constituent of χ_N by hypothesis. Suppose that θ^x is P-invariant for some $x \in G$. Then $P, P^{x^{-1}} \subseteq G_\theta$. Hence there exists some $y \in G_\theta$ such that $P^{yx} = P$, note that $\theta^{yx} = \theta^x$ and $yx \in \mathbf{N}_G(P)$.

If $\chi \in \mathrm{Irr}_{p'}(G)$ and θ lies under χ, then $|G : G_\theta|$ is a p'-number, hence $P^x \subseteq G_\theta$ for some $x \in G$ so $\theta^{x^{-1}}$ is P-invariant. □

We will also make use of the following consequences of the Glauberman correspondence.

Lemma 3.3. *Let P be a p-group that acts on a group K stabilizing $N \lhd K$. Assume that K/N is a p'-group. Let $\theta \in \mathrm{Irr}(N)$ be P-invariant. Then θ^K has a P-invariant constituent, which is unique if $\mathbf{C}_{K/N}(P) = 1$.*

Proof. Follows from Theorem 13.31 and Problem 13.10 of [Isa76]. □

Having self-normalizing Sylow p-subgroups is related to the hypothesis on the centralizer in the above Lemma in the following way.

Lemma 3.4. *Suppose that $K \lhd G$ is complemented by H. Then H is self-normalizing in G if, and only if, $\mathbf{C}_K(H) = 1$.*

Proof. Note that $\mathbf{N}_G(H) \cap K = \mathbf{N}_K(H) = \mathbf{C}_K(H)$ since $[\mathbf{N}_K(H), H] \subseteq H \cap K = 1$. □

For the sake of completeness, we give an elementary proof of the p-solvable case of Theorem 2.2 following [NT16].

Theorem 3.5. *Let G be p-solvable and let $N \lhd G$. Suppose that $\mathbf{N}_G(P) = P$, where $P \in \mathrm{Syl}_p(G)$. If $\chi \in \mathrm{Irr}_{p'}(G)$ and $\theta \in \mathrm{Irr}(N)$ lies under χ, then θ extends to G_θ.*

Proof. Since χ has p'-degree, some Sylow p-subgroup of G is contained in G_θ. By conjugating θ by some element of G, we may assume that $P \subseteq G_\theta$. Hence we may assume that θ is G-invariant. First note that by the Frattini argument, whenever $M \lhd G$, we have that

$$\mathbf{N}_{G/M}(PM/M) = \mathbf{N}_G(P)M/M.$$

Hence the hypothesis on the normalizer of the Sylow is inherited by quotients of G. We proceed by induction on $|N|$.

Suppose that $M \lhd G$ and $1 < M < N$. Let $\tau \in \mathrm{Irr}(M)$ be P-invariant under χ. In particular, τ lies under θ. Moreover $G = NG_\tau$, since for every $g \in G$ we have that $\tau^g = \tau^n$ for some $n \in N$. First, if $G = G_\tau$ then by induction τ extends to $\eta \in \mathrm{Irr}(G)$. By Gallagher's theorem $\chi = \alpha\eta$ for some $\alpha \in \mathrm{Irr}_{p'}(G/M)$ and $\theta = \beta\eta_N$ for some $\beta \in \mathrm{Irr}(N/M)$. Note that β is G-invariant, again by Gallagher's theorem. Since β lies under α of p'-degree and G/M has self-normalizing Sylow p-subgroups, by induction β extends to $\gamma \in \mathrm{Irr}(G/M)$. Then $\gamma\eta \in \mathrm{Irr}(G)$ satisfies $\gamma_N\eta_N = \beta\eta_N = \theta$, as wanted. Thus, we may assume that $G_\tau < G$, and so $N_\tau < N$. Let $\varphi \in \mathrm{Irr}(N_\tau|\tau)$ be the Clifford correspondent of θ and let $\psi \in \mathrm{Irr}(G_\tau)$ be under χ and over φ. In particular, ψ is the Clifford correspondent of χ and τ. Hence ψ has p'-degree. By induction, φ extends to $G_\tau = G_\varphi$. Let $\eta \in \mathrm{Irr}(G_\tau)$ extend φ. Then $(\eta^G)_N = (\eta_{N_\tau})^N = \varphi^N = \theta$ extends θ, as wanted.

By the previous paragraph, we may assume that N is a minimal normal subgroup of G. If N is a p-group, then $N \leqslant P$. Since χ has p'-degree, some constituent μ of χ_P has p'-degree. In particular, μ is linear and lies over θ, hence $\mu_N = \theta$. If $Q \in \mathrm{Syl}_q(G)$ for some prime $q \neq p$, we have that θ extends to QN by Corollary 6.20 of [Isa76]. According to Corollary 11.31 of [Isa76], θ extends to G as wanted. Otherwise, N is a p'-group. Then the hypothesis $\mathbf{N}_G(P) = P$ implies that $\mathbf{C}_N(P) = 1$ by Lemma 3.4. By the Glauberman correspondence θ must be 1_N, which trivially extends to G. □

We are now ready to prove the main result of this note.

Theorem 3.6. *Let G be solvable and let $P \in \mathrm{Syl}_p(G)$. Assume that $\mathbf{N}_G(P) = P$ and either that $p = 2$ or G has odd order. Then the character correspondences given by Theorems 1.1 and 2.1 are the same.*

Proof. If G is either a p-group or a p'-group, then there is nothing to prove, so $1 < P < G$.

Given $\chi \in \mathrm{Irr}_{p'}(G)$, the restriction χ_P has a unique linear constituent χ^* by Theorem 2.1. Hence, in order to show that the two bijections are equal, it is enough to show that the character $\xi \in \mathrm{Lin}(P)$ corresponding to χ via Theorem 1.1 is a constituent of χ_P.

We compute ξ by following the algorithm in Theorem 10.9 of [Isa73]. Since $\mathbf{N}_G(P) = P$, we have that $\mathbf{O}^{p'}(G) = G$ by the Frattini argument. Take $K = \mathbf{O}^p(G)$ and $L = K'$, so that $L < K < G$, $G = KP$ and K/L is an abelian p'-group. Write $H = PL$ so that $K \cap H = L$. Let $\theta \in \mathrm{Irr}(L)$ be the unique P-invariant constituent of χ_L (by Lemma 3.2 together with the self-normalizing Sylow hypothesis) and let $\eta \in \mathrm{Irr}(G_\theta|\theta)$ be the Clifford correspondent of χ. Note that $\chi_H = \eta_H + \Xi$, by MacKey's formula, where the constituents of Ξ do not lie over θ.

By Theorem 3.5, θ extends to G_θ. Note that, by Gallagher's theorem, every character in $\mathrm{Irr}(K_\theta|\theta)$ extends θ. Note that P acts on K_θ/L with $\mathbf{C}_{K_\theta/L}(P) = 1$, by Lemma 3.4. By Lemma 3.3, let φ be the unique P-invariant extension of θ to K_θ, so that actually φ is G_θ-invariant (using $G_\theta = K_\theta P$). Hence restriction defines a bijection $\mathrm{Irr}(G_\theta|\varphi) \to \mathrm{Irr}(H|\theta)$.

Since η has p'-degree, then $\eta_{K_\theta} \in \mathrm{Irr}(K_\theta|\theta)$ by Corollary 11.29 of [Isa76]. In particular, $\eta \in \mathrm{Irr}(G_\theta|\varphi)$, so $\eta_H \in \mathrm{Irr}(H|\theta)$. Notice that we have proven that $\chi_H = \eta_H + \Xi$, where $\eta_H \in \mathrm{Irr}_{p'}(H|\theta)$ and the constituents of Ξ do not lie over θ.

Now H is a group with a self-normalizing Sylow p-subgroup and we can repeat the above process with respect to $\eta_H \in \mathrm{Irr}_{p'}(H)$. We can keep proceeding like this until we reach a linear character ξ of P. At every step, we are choosing a constituent of the restriction of χ to some subgroup containing P, so in the end ξ is a constituent of χ_P, as wanted. □

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