Magnetospheric electrodynamics

A generalized Grad-Shafranov solver

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CoCoNut Meeting
Valencia / December 16, 2016

No clear picture from merger simulations

Failing of jet launching despite favorable conditions

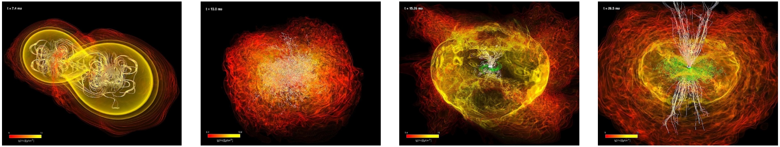


Figure: Simulation of the merger of two neutron stars (Rezzolla et al., 2011) with gravitational mass of 1.5 solar masses each during a time of 26.5 ms.

- Despite favorable conditions (e.g., magnetic fields) no jets clearly emerge after the BH formation (Rezzolla et al., 2011; Kiuchi et al., 2014). Simulations by Ruiz et al. (2016) did, however, discover jet launching.
- Possible explanations for missing jets: Short simulation time or *field reversals* observed over the low density funnel.

Current set of 'standard' magnetospheric field topologies (e.g., split-monopole, paraboloidal) may not be sufficient for time evolution simulations of the electromagnetic fields anymore.

Blandford/Znajek explain jet powering I

Creating a force-free black hole magnetosphere

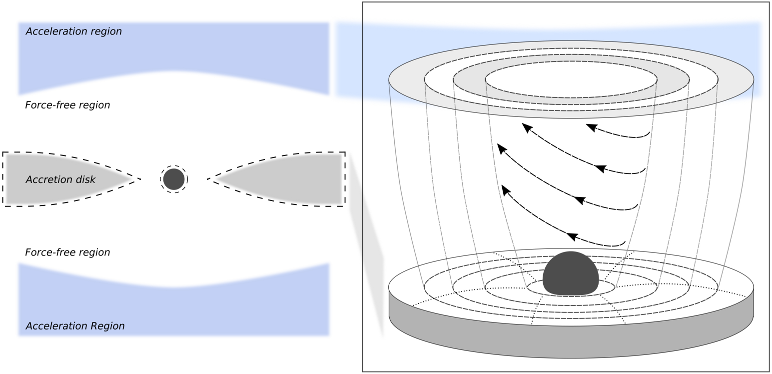


Figure: Schematic visualization of the Blandford/Znajek model (cf. MacDonald and Thorne, 1982). The black hole is embedded in a force-free magnetosphere. Magnetic fields are supported by a thin disc in the $\theta = \pi/2$ equatorial plane. The acceleration region which involves a break-down of the idealized conditions is set up at infinity and not considered for the derivations. A non-degenerate plasma generation region is schematically represented by the dashed lines.

Blandford/Znajek explain jet powering II

Spacetime magnetospheric electrodynamics

Blandford and Znajek (1977) intensively exploit the *covariant* form of the Maxwell equations in Kerr spacetime.

$$(*F^{\mu\nu})_{;\nu} = 0$$
$$\varepsilon_0^{-1} J^\mu = F^{\mu\nu}_{;\nu} = g^{-1/2}\left(g^{1/2}F^{\mu\nu}\right)_{,\nu}$$
$$F_{\mu\nu} = \mathcal{A}_{\nu,\mu} - \mathcal{A}_{\mu,\nu}$$

The existence of time-like and axial-like *symmetries* help to reduce the complexity of the resulting equations.

$$\mathcal{A}_{\mu,t} = \mathcal{A}_{\mu,\phi} = 0$$
$$\Longrightarrow F_{t\phi} = F_{\phi t} = 0$$

The *force-free condition* ultimately reduces to a differential equation governing the magnetosphere.

$$F_{\mu\nu}J^\nu = 0$$

$$\begin{aligned}4\frac{\Sigma}{\Delta}II' = &-\left(\frac{\Sigma-2Mr}{\Sigma\sin\theta}\Psi_{,r}\right)_{,r} - \left(\frac{\Sigma-2Mr}{\Delta\Sigma\sin\theta}\Psi_{,\theta}\right)_{,\theta}\\ &+\omega^2\left\{\sin\theta\left(\frac{A}{\Sigma}\Psi_{,r}\right)_{,r} + \frac{1}{\Delta}\left(\frac{A\sin\theta}{\Sigma}\Psi_{,\theta}\right)_{,\theta}\right\}\\ &-4Ma\omega\left\{\sin\theta\left(\frac{r\Psi_{,r}}{\Sigma}\right)_{,r} + \frac{r}{\Delta}\left(\frac{\sin\theta}{\Sigma}\Psi_{,\theta}\right)_{,\theta}\right\}\\ &+\frac{\sin\theta}{\Sigma\Delta}(A\omega-2Mar)\left(\Delta\left(\Psi_{,r}\right)^2+\left(\Psi_{,\theta}\right)^2\right)\omega'\end{aligned}$$

- Second order non-linear elliptic PDE
- Singular surfaces (so called *light cylinders*)
- Mathematical treatment differs from the (analytical) approach in the neutron star case

Solving GS as an elliptic PDE

Numerical PDE solving routine with SOR scheme

The Grad-Shafranov equation is resolved into a *linear* part $\mathcal{G}_l$ and the *non-linear* source terms $\mathcal{G}_s$.

$$
\begin{aligned}
\mathcal{G}_l = {} & \omega^2 \left\{ \sin\theta \left(\frac{A}{\Sigma} \Psi_{,r} \right)_{,r} + \frac{1}{\Delta} \left(\frac{A \sin\theta}{\Sigma} \Psi_{,\theta} \right)_{,\theta} \right\} \\
& - 4Ma\omega \left\{ \sin\theta \left(\frac{r \Psi_{,r}}{\Sigma} \right)_{,r} + \frac{r}{\Delta} \left(\frac{\sin\theta}{\Sigma} \Psi_{,\theta} \right)_{,\theta} \right\} \\
& - \left(\frac{\Sigma - 2Mr}{\Sigma \sin\theta} \Psi_{,r} \right)_{,r} - \left(\frac{\Sigma - 2Mr}{\Delta \Sigma \sin\theta} \Psi_{,\theta} \right)_{,\theta} \\
\mathcal{G}_s = {} & 4 \frac{\Sigma}{\Delta} I I' - \frac{\sin\theta}{\Sigma \Delta} \left(A\omega - 2Mar \right) \left(\Delta \left(\Psi_{,r} \right)^2 + \left(\Psi_{,\theta} \right)^2 \right) \omega'
\end{aligned}
$$

- After discretization throughout the numerical grid, $\mathcal{G}_l$ may be used as the linear operator in a numerical *PDE solving scheme* with non-linear sources $\mathcal{G}_s$.
- Following Contopoulos et al. (2013), a *SOR (successive overrelaxation)* method is employed for the relaxation procedure.

Grad-Shafranov is not easily solved

The Contopoulos et al. (2013) strategy I

Contopoulos et al. (2013) suggest a simultaneous numerical treatment for the three functions Ψ, $\omega\left(\Psi\right)$ and $I\left(\Psi\right)$ to solve the relativistic Grad-Shafranov equation:

- Discretize initial guesses for all physical quantities on a 256×64 numerical grid:

$$\begin{aligned} \Psi\left(R,\theta\right) &= 1 - \cos\theta && \text{(Pulsar potential)} \\ \omega\left(\Psi\right) &= 0.5\,\Omega_{BH} && \text{(Ideal condition)} \\ I\left(\Psi\right) &= -0.5\,\omega\left(\Psi\right)\Psi\left(2-\Psi\right) && \text{(Pulsar potential)} \end{aligned}$$

  $\omega\left(\Psi\right)$ and $I\left(\Psi\right)$ are stored in functional tables accessed through interpolation.

- Use a *successive overrelaxation (SOR)* method to update Ψ. Boundaries set to Dirichlet in θ and Neumann in r.

Numerics at the light cylinders I

Close-up: Understanding the singular surfaces

Field quantities of the *3+1* decomposition (as measured by the *ZAMOs*) are required to stay finite. Lee et al. (2000) derive the following:

$$\rho = \left(\frac{\Omega - \omega}{4\pi^2 \alpha}\right) \frac{\frac{8\pi^2 I}{\alpha^2}\frac{\mathrm{d}I}{\mathrm{d}\Psi} - \boldsymbol{G}\cdot\nabla\Psi}{D}$$

$$\boldsymbol{j}_T = \left(\frac{1}{4\pi^2 \varpi}\right) \frac{\frac{8\pi^2 I}{\alpha^2}\frac{\mathrm{d}I}{\mathrm{d}\Psi} - \left(\frac{(\Omega-\omega)\varpi}{\alpha}\right)^2 \boldsymbol{G}\cdot\nabla\Psi}{D}$$

where D denotes the light cylinder condition

$$D = 1 - \frac{(\omega - \Omega)^2\,\varpi^2}{\alpha^2}.$$

Smoothness of Ψ throughout the magnetosphere is imposed as a regularity condition (as also used in, e.g., Contopoulos et al., 2013).

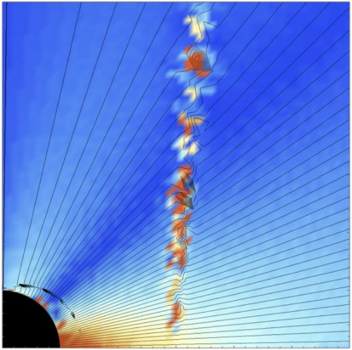


Figure: Numerical artifacts develop at the singular surfaces of the Grad-Shafranov equation (exaggerated). These breakings of field lines may cause the numerical solution to blow up.

Strategy outline: Ensure smooth passing through the light cylinders and reconstruct potential functions consistently.

Grad-Shafranov is not easily solved

The Contopoulos et al. (2013) strategy II

- Every other iteration, impose *weighted corrections* to ω and I according to the non-smoothness of Ψ at the light cylinders (singular surfaces of the GS-equation). Update scheme according to:

$$\Psi_{new} = 0.5 \cdot \left[\Psi^+_{LC} + \Psi^-_{LC}\right]$$
$$\omega\left(\Psi_{new}\right) = \omega\left(\Psi_{old}\right) + \mu_\omega \cdot \left[\Psi^+_{LC} - \Psi^-_{LC}\right]$$
$$I\left(\Psi_{new}\right) = I\left(\Psi_{old}\right) + \mu_I \cdot \left[\Psi^+_{LC} - \Psi^-_{LC}\right]$$

  Ψ^+_{LC} and Ψ^-_{LC} refer to the extrapolated potential lines at the light cylinder. The numerical parameters are derived empirically and depend on the setup.

- Additional *polynomial data fitting* (order determined empirically) is applied to the functions $\omega\left(\Psi\right)$ and $I\left(\Psi\right)$.
- A relaxed state is said to be reached after ~ 5000 iterations.

Numerics at the light cylinders II

Close-up: Relaxation and smoothing procedures

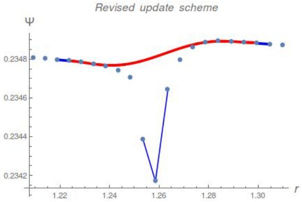


Figure: Visualization of the smoothing scheme applied at the light cylinders.

The Grad-Shafranov equation may be studied on the singular surfaces after the smoothing procedure:

$$4\frac{\Sigma}{\Delta}II' = \mathcal{GS}_{LC}$$

$$+\,\mathcal{GS}_D \cdot \left[\frac{\omega^2 A \sin^2\theta}{\Sigma} - \frac{4Mar\omega \sin^2\theta}{\Sigma} - 1 + \frac{2Mr}{\Sigma}\right]$$

At the location of the light cylinders, a simplified equation can be solved (cf. Uzdensky, 2004) in order to relate the defining functions $\mathcal{A}_\phi$, ω and I.

Simultaneous relaxation of ω and II'

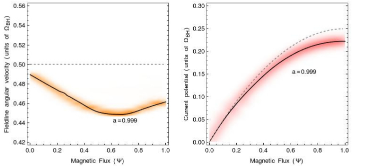


Figure: Numerical solution of the Grad-Shafranov (spin parameter $a = 0.9999$). *Colored shading*: Location of the functions throughout the numerical procedure.

Separate relaxation of ω or II'

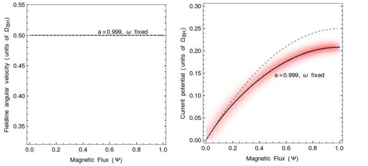


Figure: Numerical solution of the Grad-Shafranov (spin parameter $a = 0.9999$). *Colored shading*: Location of the functions throughout the numerical procedure.

Make use of Grad-Shafranov solutions

Construction of initial data for time evolution

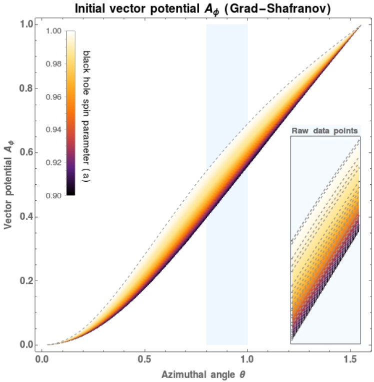


Figure: Visualization of the *vector potential* as a solution of the Grad-Shafranov equation for a *field line angular velocity* ω fixed to $\Omega_{BH}/2$ and black hole *spin parameters* between $a = 0.9$ and $a = 0.9999$. The data points used for interpolation are depicted in the bottom left. (Grad-Shafranov solver, $2 \cdot 10^6$ iterations)

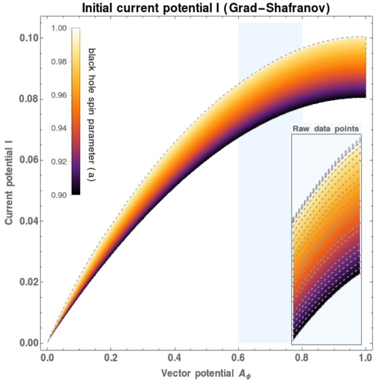


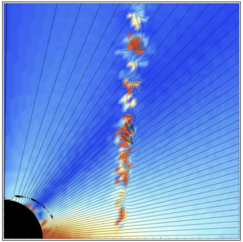
Figure: Visualization of the *current potential* as a solution of the Grad-Shafranov equation for a *field line angular velocity* ω fixed to $\Omega_{BH}/2$ and black hole *spin parameters* between $a = 0.9$ and $a = 0.9999$. The data points used for interpolation are depicted in the bottom left. (Grad-Shafranov solver, $2 \cdot 10^6$ iterations)

Overview: Research stages and methods

Numerical simulations as astrophysical experiments

Stage I: **Theory/Numerics**

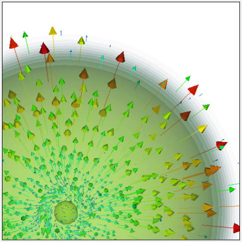
Grad-Shafranov initial data



- Implementation and testing of a numerical solving procedure for the GS equation.
- *Expand solving scheme towards more complicated field topologies.*

Stage II: **Simulations**

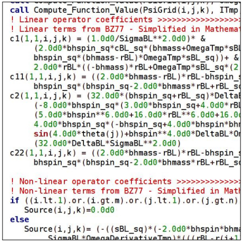
Einstein Toolkit setup



- Preparation of GS solutions as initial data for time evolution setups.
- *Adaptation of a suitable evolution scheme (employing 2D and 3D codes).*

Stage III: **Evaluation/Feedback**

Fortran code segment



```fortran
call Compute_Function_Value(PsiGrid(i,j,k), ITmp
! Linear operator coefficients >>>>>>>>>>>>>>>>>
! Linear terms from BZ77 - Simplified in Mathema
c1(1,1,i,j,k) = (1.0d0/SigmaBL**2.0d0)* &
     (2.0d0*bhspin_sq*cBL_sq*(bhmass+OmegaTmp*sBL
     bhspin_sq*(bhmass-rBL)*OmegaTmp*sBL_sq))+ &
     2.0d0*rBL*((-bhmass)*rBL+OmegaTmp*sBL_sq*(2
c11(1,1,i,j,k) = ((2.0d0*bhmass-rBL)*rBL-bhspin_
     bhspin_sq*(bhspin_sq-2.0d0*bhmass*rBL+rBL_s
c2(1,1,i,j,k) = (32.0d0*(bhspin_sq+rBL_sq)*Delta
     (-8.0d0*bhspin_sq*(3.0d0*bhspin_sq+4.0d0*rBL
     (5.0d0*bhspin**6.0d0+16.0d0*rBL**6.0d0+16.0
     4.0d0*bhspin_sq*(-bhspin_sq+4.0d0*bhspin*bh
     sin(4.0d0*theta(j))+bhspin**4.0d0*DeltaBL*O
     (32.0d0*DeltaBL*SigmaBL**2.0d0)
c22(1,1,i,j,k) = ((2.0d0*bhmass-rBL)*rBL-bhspin_
     bhspin_sq*(bhspin_sq-2.0d0*bhmass*rBL+rBL_s

! Non-linear operator coefficients >>>>>>>>>>>>>
! Non-linear terms from BZ77 - Simplified in Math
if ((i.lt.1).or.(i.gt.m).or.(j.lt.1).or.(j.gt.n)
   Source(i,j,k)=0.0d0
else
   Source(i,j,k)= (-((sBL_sq)*(-2.0d0*bhspin*bhm
```

- *Classify GS initial data in terms of stability and the observation of jet launching.*
- *Understand the role of force-free evolution vs. ideal MHD implementations.*

Open forum: Let's discuss

Questions. Answers. Remarks. Discussion.

Thank you.

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erc European Research Council Established by the European Commission

Studienstiftung des deutschen Volkes

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