FAITHFUL AND TRANSITIVE MONOID ACTIONS

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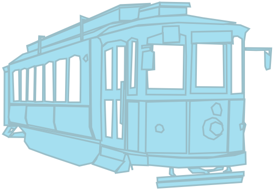

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ACTIONS

An action of a monoid M on a nonempty set Ω is a monoid homomorphism

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- Generalisation of group actions
- Automata Theory

ACTIONS

The action is faithful if the corresponding monoid homomorphism is injective. We refer to the pair (M, Ω) as transformation monoid.

The action is transitive if for each $\alpha, \beta \in \Omega$, the cyclic sets coincide $\alpha M = \beta M$.

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Objective

Give a description of transitive transformation monoids

STEINBERG

Related Work

[Ste10] B. Steinberg. A Theory of Transformation Monoids: Combinatorics and Representation Theory. Electron. J. Combin., 17(1): Research Paper 164, 56 pp., 2010.

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It is based on

- Module theory - Green
- Semigroup representation - Clifford, Munn, Ponizovsky

PRELIMINARIES

Proposition

Any transitive and faithful action of a monoid M on a set Ω is equivalent to the action of M on $eM/{\equiv}$ where e is an idempotent in $\mathrm{I}(M)$ and $\equiv$ is a right-congruence on eM.

PRELIMINARIES

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$$eM = \biguplus_{e_j \in \mathsf{E}(eM)} G_{e_j} = \biguplus_{x_i \in T} \biguplus_{e_j \in \mathsf{E}(eM)} H x_i e_j$$

PRELIMINARIES

M embeds in $G_e \wr_{\mathsf{n}} \overline{M}$ via the map $m \mapsto (\rho_m, \overline{m})$, where

$$(j)\rho_m = g \qquad \text{such that} \qquad e_j m = g e_{j'}$$

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For each element x_i in a right-transversal T, we define an equivalence relation $\simeq^i$ in the set $\mathsf{n} = \{1, \cdots, n\}$ as follows

$$j \simeq^i j' \quad \Leftrightarrow \quad Hx_i(j)\rho_m = Hx_i(j')\rho_m, \quad \text{for all } m \in M$$

ON TRANSITIVE TRANSFORMATION MONOIDS

Theorem

If (M, Ω) is a transitive transformation monoid, there exists a core-free subgroup H of G_e such that every member of the partition $\{Hx_i e_j \mid x_i \in T, e_j \in \mathrm{E}(eM)\}$ is contained in a $\equiv$-class. Furthermore,

i) If $Hx_i e_j \equiv Hx_{i'} e_{j'}$ then $i = i'$ and $j \simeq^i j'$;
ii) If $Hx_i e_j \equiv Hx_i e_{j'}$ then $(Hx_i e_j)m \equiv (Hx_i e_{j'})m$ for all $m \in M$;
iii) For different idempotents $e_j, e_{j'}$ there exists some $x_i \in T$ with $Hx_i e_j$ and $Hx_i e_{j'}$ not $\equiv$-related.

The converse of this Theorem also holds.

ON TRANSITIVE TRANSFORMATION MONOIDS

Consequences

- We can construct all transitive and faithful actions of a monoid that restrict to a given transitive and faithful group action.
- There exist non-equivalent transitive and faithful monoid actions that restrict to the same group action.
- These theorems extend the classical theorems on the description of transitive permutation groups.

ON PRIMITIVE TRANSFORMATION MONOIDS

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A Theory of Transformations Monoids [Ste10]

"We hope that the theory of primitive permutation groups can be used to understand primitive transformation monoids in the case the maximal subgroups of I(M) are non-trivial"

ON PRIMITIVE TRANSFORMATION MONOIDS

Theorem

Let M be a transitive transformation monoid such that G_e is non-trivial. Then M is primitive if and only if there exists a maximal core-free subgroup H of G_e such that the relations $\simeq$ defined by H in E(eM) satisfy the following conditions

i) For different idempotents $e_j, e_{j'}$ there exist finite sequences $(x_{i_t})_{t=1}^{s}$ and $(e_{j_t})_{t=1}^{s+1}$ of elements of T and E(eM), respectively, such that

$$j = j_1 \simeq^{i_1} j_2 \simeq^{i_2} \cdots \simeq^{i_{s-1}} j_s \simeq^{i_s} j_{s+1} = j'$$

ii) For different idempotents $e_j, e_{j'}$ there exists some $x_i \in T$ with $j \not\simeq^{i} j'$.

ON PRIMITIVE TRANSFORMATION MONOIDS

Let (M, Ω) be a transitive transformation monoid.

Consequences

- If G_e has a primitive action, there exists a unique primitive action of M that restricts to the primitive action of G_e.
- If G_e is a non-trivial nilpotent group. Then M has a primitive action if and only if M is a cyclic group of prime order.
- If $|\Omega e| = 2$, then the action of M on Ω cannot be primitive.

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