From higher-order rewriting systems to higher-order categorial algebras and higher-order Curry-Howard isomorphisms

Morphisms of rewriting systems

Juan Climent Vidal[1], Enric Cosme Llópez[2,3][0000−0001−8618−7328], and Raúl Ruiz Mora[2][0009−0004−3739−7811]

[1] Universitat de València, Departament de Lògica i Filosofia de la Ciència, Spain.
[2] Universitat de València, Departament de Matemàtiques, Spain.
[3] Nantong University, School of Mathematics and Statistics, China.

Abstract. We develop a tiered definition for morphisms between rewriting systems. At layer 0, which encompasses the underlying algebraic basis—operations, and variables—we employ the concept of a derivor to formalize the notion of a zeroth-order morphism. We introduce the extension of this definition to terms, subsequently constructing the category of zeroth-order rewriting systems. This framework is extended to layer 1 by incorporating rewriting rules; here, a morphism is defined via the preceding zeroth-order morphism coupled with a mapping that assigns the rewriting rules in the domain to paths within the codomain. We show that this mapping can be extended to paths and that this extension is compatible with path classes, leading to the construction of the category of rewriting systems. Finally, this inductive process is iterated at layer 2 to account for second-order rewriting rules. A second-order morphism is characterized by a morphism of rewriting systems and a mapping that assigns second-order rewriting rules in the domain to second-order paths in the codomain. By extending this to second-order paths and ensuring compatibility with second-order path classes, we construct the category of second-order rewriting systems.

Keywords: Rewriting systems · Morphisms · Categorial algebras · Curry-Howard isomorphisms.

1 Introduction

In the first part of this work [9] we introduced $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ the set of paths associated with a rewriting system $\boldsymbol{\mathcal{A}} = (\Sigma, X, \mathcal{A})$ and equipped it with a structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra, a structure of category, and a structure of Artinian ordered set. Moreover, we identified $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$, a subquotient of $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$, that is isomorphic to the algebraic, categorical, and ordered structures on $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$. This constituted a Curry-Howard type isomorphism. Additionally, we proved that these two structures are isomorphic to $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}})$, the free partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra in the variety of partial algebras $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}})$. Similarly, in the second part of this work [9] we introduced $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ the set of second-order paths associated with a second-order rewriting system $\boldsymbol{\mathcal{A}}^{(2)} = (\boldsymbol{\mathcal{A}}, \mathcal{A}^{(2)})$ and equipped it with a structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra, a structure of category, and a structure of Artinian ordered set. Moreover, we identified $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$, a subquotient of $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$, that is isomorphic to the algebraic, categorical, and ordered structures on $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$. This constituted a Curry-Howard type isomorphism. Additionally, we proved that these two structures are isomorphic to $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}})$, the free partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra in the variety of partial algebras $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}})$.

In this part, we start by defining $\mathsf{Rws}_{\mathfrak{d}}^{(0)}$, the category of zeroth order rewriting systems, which is at the basis of all the work. Let $\mathsf{Sig}_{\mathfrak{d}}$ be the category whose objects are the signatures, i.e., families $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ of operation symbols, and whose morphisms from the signature Σ to the signature Λ are the derivors $c = (c_n)_{n\in\mathbb{N}}$ (see [8]). Let $\mathsf{Alg}_{\mathfrak{d}}$ be the category whose objects are the algebras, i.e., the ordered pairs $(\Sigma, \mathbf{A})$, where Σ is a signature and $\mathbf{A}$ a Σ-algebra, and whose morphisms from the algebra $(\Sigma, \mathbf{A})$ to the algebra $(\Lambda, \mathbf{B})$ are the ordered pairs (c, f), with c a derivor from Σ to Λ and f a homomorphism of Σ-algebras from $\mathbf{A}$ to $\mathbf{c}_{\mathfrak{d}}^{*}(\mathbf{B})$, where $\mathbf{c}_{\mathfrak{d}}^{*}$ is the functor from $\mathsf{Alg}(\Lambda)$ to $\mathsf{Alg}(\Sigma)$ determined by the derivor c (see [8]).

Then $\mathsf{Rws}_{\mathfrak{d}}^{(0)}$ is the category whose objects are the ordered pairs (Σ, X), where Σ is a signature and X a set, which we will call zeroth order rewriting systems, and whose morphisms from the zeroth order rewriting system $\boldsymbol{\mathcal{A}}^{(0)} = (\Sigma, X)$ to the zeroth order rewriting system $\boldsymbol{\mathcal{B}}^{(0)} = (\Lambda, Y)$ are the ordered triples $((\Sigma, X), \mathbf{f}^{(0)}, (\Lambda, Y))$, denoted by $\mathbf{f}^{(0)}\colon \boldsymbol{\mathcal{A}}^{(0)} \longrightarrow \boldsymbol{\mathcal{B}}^{(0)}$ for short, in which $\mathbf{f}^{(0)} = (c, f^{(0)})$, where

1. c is a derivor from Σ to Λ and
2. $f^{(0)}$ is a mapping from X to $\mathrm{T}_{\Lambda}(Y)$.

Note that $\mathrm{T}_{\Lambda}(Y)$ is the underlying set of the Σ-algebra $\mathbf{c}_{\mathfrak{d}}^{*}(\mathbf{T}_{\Lambda}(Y))$. Let us also note that there exists a full embedding of $\mathsf{Rws}_{\mathfrak{d}}^{(0)}$ into $\mathsf{Alg}_{\mathfrak{d}}$, because to give a mapping $f^{(0)}$ from X to $\mathrm{T}_{\Lambda}(Y)$ is, naturally, equivalent to give a Σ-homomorphism $f^{(0)\sharp}$ from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{c}_{\mathfrak{d}}^{*}(\mathbf{T}_{\Lambda}(Y))$, obtained by the universal property of $\mathrm{T}_{\Sigma}(X)$. Actually, we prove that $\mathsf{Rws}_{\mathfrak{d}}^{(0)}$ is isomorphic to $\mathsf{Tw}_{\mathfrak{d}}^{(0)}$, the full subcategory of $\mathsf{Alg}_{\mathfrak{d}}$ determined by the algebras of the form $(\boldsymbol{\mathcal{A}}^{(0)}, \mathbf{T}_{\Sigma}(X))$, with $\boldsymbol{\mathcal{A}}^{(0)}$ varying over the zeroth-order rewriting systems, what we have called in this work zeroth-order towers.

We next consider rewriting systems and morphisms between them. Let $\boldsymbol{\mathcal{A}}^{(1)} = (\Sigma, X, \mathcal{A}^{(1)}) = (\boldsymbol{\mathcal{A}}^{(0)}, \mathcal{A}^{(1)})$ and $\boldsymbol{\mathcal{B}}^{(1)} = (\Lambda, Y, \mathcal{B}^{(1)}) = (\boldsymbol{\mathcal{B}}^{(0)}, \mathcal{B}^{(1)})$ be rewriting systems. A morphism from $\boldsymbol{\mathcal{A}}^{(1)}$ to $\boldsymbol{\mathcal{B}}^{(1)}$ will be an ordered triple $(\boldsymbol{\mathcal{A}}^{(1)}, \mathbf{f}^{(1)}, \boldsymbol{\mathcal{B}}^{(1)})$, denoted by $\mathbf{f}^{(1)}\colon \boldsymbol{\mathcal{A}}^{(1)} \longrightarrow \boldsymbol{\mathcal{B}}^{(1)}$ or $\mathbf{f}^{(1)}$ for short, in which $\mathbf{f}^{(1)}$ is an ordered pair $(\mathbf{f}^{(0)}, f^{(1)})$ where

1. $\mathbf{f}^{(0)} = (c, f^{(0)})$ is a morphism from $\boldsymbol{\mathcal{A}}^{(0)}$ to $\boldsymbol{\mathcal{B}}^{(0)}$ and

2. $f^{(1)}$ is a mapping from $\mathcal{A}^{(1)}$ to $\mathrm{Pth}_{\boldsymbol{\mathcal{B}}^{(1)}}$ such that, for every $\mathfrak{p} = (M, N) \in \mathcal{A}^{(1)}$, we have that $f^{(1)}(\mathfrak{p}) \in \mathrm{Pth}_{\boldsymbol{\mathcal{B}}^{(1)}}(f^{(0)\sharp}(M), f^{(0)\sharp}(N))$.

From this definition we obtain, by recursion on $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(1)}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(1)}}})$, a Σ-homomorphism $f^{(1)\flat}$ from $\mathbf{Pth}^{(0,1)}_{\boldsymbol{\mathcal{A}}^{(1)}}$ to $\mathbf{c}^{*}_{\mathfrak{d}}(\mathbf{Pth}^{(0,1)}_{\boldsymbol{\mathcal{B}}^{(1)}})$ such that $f^{(1)\flat}$ coincides with $f^{(0)\sharp}$ when restricted to the $(1,0)$-identity paths and with $f^{(1)}$ when restricted to the echelons, i.e., to the one-step paths canonically associated to the rewrite rules on $\mathcal{A}^{(1)}$. We next provide $[\mathrm{Pth}_{\boldsymbol{\mathcal{B}}^{(1)}}]$, i.e., the underlying set of the Σ-algebra $\mathbf{c}^{*}_{\mathfrak{d}}([\mathbf{Pth}^{(0,1)}_{\boldsymbol{\mathcal{B}}^{(1)}}])$, with a structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(1)}}$-algebra, that we denote by $[\mathbf{Pth}^{\mathbf{f}^{(1)}}_{\boldsymbol{\mathcal{B}}^{(1)}}]$, and we prove that it belongs to the QE-variety determined by $\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(1)}}$. Since the partial $\Lambda^{\boldsymbol{\mathcal{B}}^{(1)}}$-algebras $[\mathbf{PT}_{\boldsymbol{\mathcal{B}}^{(1)}}]$ and $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{B}}^{(1)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(1)}})$ are isomorphic to $[\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(1)}}]$, an analogous statement holds for the respective partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(1)}}$-algebras $[\mathbf{PT}^{\mathbf{f}^{(1)}}_{\boldsymbol{\mathcal{B}}^{(1)}}]$ and $\mathbf{T}^{\mathbf{f}^{(1)}}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{B}}^{(1)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(1)}})$, built following a similar approach. Thus, since we have proved that $[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(1)}}]$ is isomorphic to $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(1)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}})$ (or to $[\mathbf{PT}_{\boldsymbol{\mathcal{A}}^{(1)}}]$), we obtain, by the universal property of it, a unique $\Sigma^{\boldsymbol{\mathcal{A}}^{(1)}}$-homomorphism from $[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(1)}}]$ to $[\mathbf{Pth}^{\mathbf{f}^{(1)}}_{\boldsymbol{\mathcal{B}}^{(1)}}]$ (from $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(1)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(1)}})$ to $\mathbf{T}^{\mathbf{f}^{(1)}}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{B}}^{(1)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(1)}})$ or from $[\mathbf{PT}_{\boldsymbol{\mathcal{A}}^{(1)}}]$ to $[\mathbf{PT}^{\mathbf{f}^{(1)}}_{\boldsymbol{\mathcal{B}}^{(1)}}]$), which we denote by $f^{[1]@}$, satisfying

$$f^{[1]@} \circ \mathrm{pr}^{\mathrm{Ker}(\mathrm{CH}^{(1)})}_{\boldsymbol{\mathcal{A}}^{(1)}} = \mathrm{pr}^{\mathrm{Ker}(\mathrm{CH}^{(1)})}_{\boldsymbol{\mathcal{B}}^{(1)},\varphi} \circ f^{(1)\flat}.$$

Now, with respect to rewriting systems and morphisms on it, we prove that it does not form a category since, among other things, the natural composition of morphisms is not associative. To overcome this problem, we introduce an equivalence relation on the morphisms. Given two morphisms $\mathbf{f}^{(1)} = (\mathbf{f}^{(0)}, f^{(1)})$ and $\mathbf{g}^{(1)} = (\mathbf{g}^{(0)}, g^{(1)})$, from $\boldsymbol{\mathcal{A}}^{(1)}$ to $\boldsymbol{\mathcal{B}}^{(1)}$, we say that $\mathbf{f}^{(1)}$ is equivalent to $\mathbf{g}^{(1)}$, written $\mathbf{f}^{(1)} \cong^{(1)} \mathbf{g}^{(1)}$, if and only if, $\mathbf{f}^{(0)} = \mathbf{g}^{(0)}$ and $\mathrm{pr}^{\mathrm{Ker}(\mathrm{CH}^{(1)})}_{\boldsymbol{\mathcal{B}}^{(1)}} \circ f^{(1)\flat} = \mathrm{pr}^{\mathrm{Ker}(\mathrm{CH}^{(1)})}_{\boldsymbol{\mathcal{B}}^{(1)}} \circ g^{(1)\flat}$. We prove that, for equivalent morphisms $\mathbf{f}^{(1)} \cong^{(1)} \mathbf{g}^{(1)}$, it holds that $f^{[1]@} = g^{[1]@}$ and the respective partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(1)}}$-algebras $[\mathbf{Pth}^{\mathbf{f}^{(1)}}_{\boldsymbol{\mathcal{B}}^{(1)}}]$ and $[\mathbf{Pth}^{\mathbf{g}^{(1)}}_{\boldsymbol{\mathcal{B}}^{(1)}}]$ are equal. This allows us to construct the category $\mathsf{Rws}^{[1]}_{\mathfrak{d}}$, whose objects are rewriting systems and morphisms are $\cong^{(1)}$-classes of morphisms of rewriting systems. Actually, we prove that $\mathsf{Rws}^{[1]}_{\mathfrak{d}}$ is a category and is isomorphic to the category of towers, denoted by $\mathsf{Tw}^{[1]}_{\mathfrak{d}}$, determined by the partial algebras of the form $(\boldsymbol{\mathcal{A}}^{(1)}, \mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(1)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(1)}}))$, with $\boldsymbol{\mathcal{A}}^{(1)}$ varying over the rewriting systems.

Finally, we next consider second-order rewriting systems and second-order morphisms between them. Let $\boldsymbol{\mathcal{A}}^{(2)} = (\Sigma, X, \mathcal{A}^{(1)}, \mathcal{A}^{(2)}) = (\boldsymbol{\mathcal{A}}^{(1)}, \mathcal{A}^{(2)})$ and $\boldsymbol{\mathcal{B}}^{(2)} = (\Lambda, Y, \mathcal{B}^{(1)}, \mathcal{B}^{(2)}) = (\boldsymbol{\mathcal{B}}^{(1)}, \mathcal{B}^{(2)})$ be second-order rewriting systems. A morphism from $\boldsymbol{\mathcal{A}}^{(2)}$ to $\boldsymbol{\mathcal{B}}^{(2)}$ will be an ordered triple $(\boldsymbol{\mathcal{A}}^{(2)}, \mathbf{f}^{(2)}, \boldsymbol{\mathcal{B}}^{(2)})$, denoted by $\mathbf{f}^{(2)} \colon \boldsymbol{\mathcal{A}}^{(2)} \longrightarrow \boldsymbol{\mathcal{B}}^{(2)}$ or $\mathbf{f}^{(2)}$ for short, in which $\mathbf{f}^{(2)}$ is an ordered pair $(\mathbf{f}^{(1)}, f^{(2)})$ where

1. $\mathbf{f}^{(1)} = (\mathbf{f}^{(0)}, f^{(1)})$ is a morphism from $\boldsymbol{\mathcal{A}}^{(1)}$ to $\boldsymbol{\mathcal{B}}^{(1)}$ and

2. $f^{(2)}$ is a mapping from $\mathcal{A}^{(2)}$ to $\mathrm{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}$ such that, for every $\mathfrak{p}^{(2)} = ([M],[N]) \in \mathcal{A}^{(2)}$, we have that $f^{(2)}(\mathfrak{p}^{(2)}) \in \mathrm{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}(f^{[1]@}[M]), f^{[1]@}([N]))$.

From this definition we obtain, in a similar manner, all the results introduced so far, thus obtaining the category $\mathsf{Rws}_{\mathfrak{d}}^{[\![2]\!]}$, whose objects are second-order rewriting systems and morphisms are $\cong^{(2)}$-classes of morphisms of second-order rewriting systems. We also prove that $\mathsf{Rws}_{\mathfrak{d}}^{[\![2]\!]}$ is a category and is isomorphic to the category of second-order towers, denoted by $\mathsf{Tw}_{\mathfrak{d}}^{[\![2]\!]}$, determined by the partial algebras of the form $(\boldsymbol{\mathcal{A}}^{(2)}, \mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}))$, with $\boldsymbol{\mathcal{A}}^{(2)}$ varying over the second-order rewriting systems.

In this paper, next to each result, the reader will find the corresponding reference to the complete proof in [9]. Our work is framed in the study of morphisms of rewriting systems in the context of many-sorted algebras. Nevertheless, to facilitate understanding, in this paper we have opted to present the single-sorted version of our findings.

The only prerequisites for reading this work are familiarity with category theory [15,16], universal algebra [1,4,5,6,7,13,14,17,18,19], the theory of ordered sets [2,11] and set theory [3,12]. The reader is advised to consult in [9], the first and second parts of this project.

2 Preliminaries

In this paper we will use the notion of derivor (see [8]) as ground for our definition of morphism of rewriting systems. We begin this section by introducing the concept of clone.

Definition 1. *A* clone *is a triple* $\mathbf{C} = ((C_n)_{n\in\mathbb{N}}, ((\pi_{k,n})_{k\in n})_{n\in\mathbb{N}}, (\circ_{m,n})_{m,n\in\mathbb{N}})$ *where*

– *For every* $n \in \mathbb{N}$, C_n *is a set.*
– *For every* $k, n \in \mathbb{N}$ *with* $k \in n$, $\pi_{k,n} \in C_n$.
– *For every* $m, n \in \mathbb{N}$, $\circ_{m,n} \colon C_m \times (C_n)^m \longrightarrow C_n$. *We agree to use only* $\circ$ *leaving the index* m *and* n *implicit.*

satisfying that

1. *For every* $n \in \mathbb{N}$ *and every* $c \in C_n$, $c \circ (\pi_{i,n})_{i\in n} = c$;
2. *For every* $k, n, m \in \mathbb{N}$ *with* $k \in n$ *and every* $(c_i)_{i\in n} \in C_m^n$, $\pi_{k,n} \circ (c_i)_{i\in n} = c_k$; *and*
3. *For every* $m, n, p \in \mathbb{N}$, *every* $c \in C_m$, *every* $(d_i)_{i\in m} \in C_n^m$ *and every* $(e_j)_{j\in n} \in C_p^n$, $c \circ ((d_i \circ (e_j)_{j\in n})_{i\in m}) = (c \circ (d_i)_{i\in m}) \circ (e_j)_{j\in n}$.

A morphism *from the clone* $\mathbf{C}$ *to the clone* $\mathbf{C}'$ *is a family* $f = (f_n)_{n\in\mathbb{N}}$ *where, for every* $n \in \mathbb{N}$, $f_n \colon C_n \longrightarrow C'_n$ *such that*

1. *For every* $k, n, m \in \mathbb{N}$ *with* $k \in n$ $f_n(\pi_{k,n}) = \pi'_{k,n}$; *and*
2. *For every* $m, n \in \mathbb{N}$, *every* $c \in C_m$, *every* $(d_i)_{i\in m} \in C_n^m$, $f_n(c \circ (d_i)_{i\in m}) = f_m(c) \circ (f_n(d_i))_{i\in m}$.

In the next two propositions we introduce a representation of the free clone.

Proposition 1. *[9, Proposition 2.10.6] Let Σ be a signature and let $V = \{v_i \mid i \in \mathbb{N}\}$ be a denumerable set of variables. Then*

- *The family $\mathrm{T}_\Sigma(V) = (\mathrm{T}_\Sigma(V_n))_{n\in\mathbb{N}}$ where $V_n = \{v_i \mid i \in n\}$,*
- *the family $((v_k)_{k\in n})_{n\in\mathbb{N}}$, and*
- *the operation $P \circ (Q_i)_{i\in m} = \left(\binom{v_i}{Q_i}_{i\in m}\right)^\sharp (P) \in \mathrm{T}_\Sigma(V_n)$,*

is a clone that we will denote by $\mathbf{T}_\Sigma(V)$.

Proposition 2. *[9, Proposition 2.10.11] Let $V = \{v_i \mid i \in \mathbb{N}\}$. For every signature Σ, the pair $(\eta^\Sigma, \mathbf{T}_\Sigma(V))$, where η^Σ is the canonical inclusion of Σ into the clone $\mathbf{T}_\Sigma(V)$ which assigns, to every operation symbol σ in Σ_n, the term $\sigma((v_i)_{i\in n})$, has the following universal property: for every clone $\mathbf{C}$ and every family of mappings $f = (f_n)_{n\in\mathbb{N}} \colon \Sigma \longrightarrow C$, there exists a unique morphism of clones $f^\sharp \colon \mathbf{T}_\Sigma(V) \longrightarrow \mathbf{C}$ such that $f^\sharp \circ \eta^\Sigma = f$, that is, for every $n \in \mathbb{N}$, $f^\sharp_n \circ \eta^\Sigma_n = f_n$.*

We next introduce the concept of derivor from one signature to another.

Definition 2. *Let Σ and Λ be two signatures. A* derivor *from Σ to Λ is a family $c = (c_n)_{n\in\mathbb{N}}$ where, for every $n \in \mathbb{N}$,*

$$c_n \colon \Sigma_n \longrightarrow \mathrm{T}_\Lambda(V_n)$$

and $V_n = \{v_i \mid i \in n\}$. Hence, for every $n \in \mathbb{N}$ and every $\sigma \in \Sigma_n$, $c_n(\sigma) \in \mathrm{T}_\Lambda(V_n)$, i.e., $c_n(\sigma)$ is a term with variables in V_n and operation symbols in Λ.

Remark 1. While derivors are not the most general type of morphism that might be considered between signatures—for instance, one could consider polyderivors, see [10]—, they are an important class of such morphisms. One reason for its relevance is its formal properties (see below), another that there are many mathematical examples of them which are of interest (see [10]).

Moreover, we show that signatures and derivors constitute a category.

Definition 3. *Let $c \colon \Sigma \longrightarrow \Lambda$ and $d \colon \Lambda \longrightarrow \Omega$ be derivors. Then $d \circ c = (d_n)_{n\in\mathbb{N}} \circ (c_n)_{n\in\mathbb{N}}$, the* composition *of c and d, is the derivor $(d^\sharp_n \circ c_n)_{n\in\mathbb{N}}$, where $d^\sharp_n \circ c_n$ is the mapping from Σ_n to $\mathrm{T}_\Omega(V_n)$, being $d^\sharp_n$ the n-th component of the canonical extension of d obtained in Proposition 2.*

For every signature Σ, the identity *derivor at Σ is $\eta^\Sigma = (\eta^\Sigma_n)_{n\in\mathbb{N}}$, the canonical inclusion of Σ into the clone $\mathbf{T}_\Sigma(X)$.*

Proposition 3. *[9, Proposition 2.10.18] Signatures and derivors constitute a category denoted by $\mathsf{Sig}_\mathfrak{d}$.*

Finally, we assign to every derivor a functor which transforms Λ-algebras intro Σ-algebras. This assignment is the morphism part of a functor $\mathrm{Alg}_\mathfrak{d}$ from $\mathsf{Sig}_\mathfrak{d}$ to Cat. We have not defined this functor explicitly because it is not strictly necessary in the presentation of this work.

Proposition 4. *[9, Proposition 2.10.21] Let $c\colon \Sigma \longrightarrow \Lambda$ be a derivor. Then $\mathbf{c}_{\mathfrak{d}}^{*}$ is the functor from $\mathsf{Alg}(\Lambda)$ to $\mathsf{Alg}(\Sigma)$ that sends*

1. *every Λ-algebra $\mathbf{B} = (B, G)$ to $\mathbf{c}_{\mathfrak{d}}^{*}(\mathbf{B}) = (B, G^{\sharp} \circ d)$ where $G^{\sharp}$ is the canonical extension of G obtained from Proposition 2; and*
2. *every Λ-homomorphism f from $\mathbf{B}$ to $\mathbf{B}'$ to the homomorphism f from $\mathbf{c}_{\mathfrak{d}}^{*}(\mathbf{B})$ to $\mathbf{c}_{\mathfrak{d}}^{*}(\mathbf{B}')$.*

3 Zeroth-order morphisms

In this section we begin by introducing the notions of zeroth-order rewriting systems and that of zeroth-order morphism based on the notion of derivor.

Definition 4. *A* zeroth-order rewriting system *is an ordered pair $\boldsymbol{\mathcal{A}}^{(0)} = (\Sigma, X)$ where Σ is a signature and X is a set. Let $\boldsymbol{\mathcal{A}}^{(0)} = (\Sigma, X)$ and $\boldsymbol{\mathcal{B}}^{(0)} = (\Lambda, Y)$ be zeroth-order rewriting sytems. A* zeroth-order morphism *from $\boldsymbol{\mathcal{A}}^{(0)}$ to $\boldsymbol{\mathcal{B}}^{(0)}$ is an ordered pair $\mathbf{f}^{(0)} = (c, f^{(0)})$, denoted by $\mathbf{f}^{(0)}\colon \boldsymbol{\mathcal{A}}^{(0)} \longrightarrow \boldsymbol{\mathcal{B}}^{(0)}$, where*

- *$c = (c_n)_{n\in\mathbb{N}}$ is a derivor from Σ to Λ.*
- *$f^{(0)}$ is a mapping from X to $\mathrm{T}_{\Lambda}(Y)$.*

Following a similar fashion, we define the notions of identity zeroth-order morphism and composition of zeroth-order morphisms.

Definition 5. *Let $\boldsymbol{\mathcal{A}}^{(0)} = (\Sigma, X)$, $\boldsymbol{\mathcal{B}}^{(0)} = (\Lambda, Y)$ and $\boldsymbol{\mathcal{C}}^{(0)} = (\Omega, Z)$ be three zeroth-order rewriting systems. Let $\mathbf{f}^{(0)}\colon \boldsymbol{\mathcal{A}}^{(0)} \longrightarrow \boldsymbol{\mathcal{B}}^{(0)}$ and $\mathbf{g}^{(0)}\colon \boldsymbol{\mathcal{B}}^{(0)} \longrightarrow \boldsymbol{\mathcal{C}}^{(0)}$ be zeroth-order morphisms where $\mathbf{f}^{(0)} = (c, f^{(0)})$ and $\mathbf{g}^{(0)} = (d, g^{(0)})$. Then $\mathbf{g}^{(0)} \circ \mathbf{f}^{(0)}$, the* composition *of $\mathbf{f}^{(0)}$ and $\mathbf{g}^{(0)}$, is the zeroth-order morphism $(d \circ c, g^{(0)\sharp} \circ f^{(0)})$, where $d \circ c$ is the composition of derivors, introduced in Definition 3, and $g^{(0)\sharp} \circ f^{(0)}$ is the mapping from X to $\mathrm{T}_{\Omega}(Z)$ being $g^{(0)\sharp}$ the canonical extension of $g^{(0)}$ to the free Λ-algebra $\mathbf{T}_{\Lambda}(Y)$.*

For every zeroth-order rewriting system $\boldsymbol{\mathcal{A}}^{(0)} = (\Sigma, X)$, the identity *zeroth-order morphism at (Σ, X) is $(\eta^{\Sigma}, \eta^{X})$.*

Finally, we show that they constitute a category.

Proposition 5. *[9, Proposition 2.11.5] Zeroth-order rewriting systems and zeroth-order morphisms constitute a category, that we denote by $\mathsf{Rws}_{\mathfrak{d}}^{(0)}$.*

4 First-order morphisms

The generalization to rewriting systems is based on the study of rewriting systems done in parts one and two of [9]. Let us begin by defining the notion of morphism between rewriting systems.

Definition 6. *Let $\boldsymbol{\mathcal{A}} = (\Sigma, X, \mathcal{A})$ and $\boldsymbol{\mathcal{B}} = (\Lambda, Y, \mathcal{B})$ be rewriting systems. A* morphism *from $\boldsymbol{\mathcal{A}}$ to $\boldsymbol{\mathcal{B}}$ is an ordered pair $\mathbf{f}^{(1)} = (c, (f^{(i)})_{i\in 2})$, denoted by $\mathbf{f}^{(1)}\colon \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{B}}$, where*

1. $\mathbf{f}^{(0)} = (c, f^{(0)})$, *the* underlying *zeroth-order morphism of* $\mathbf{f}^{(1)}$, *is a zeroth-order morphism from* (Σ, X) *to* (Λ, Y), *introduced in Definition 4; and*
2. $f^{(1)}\colon \mathcal{A} \longrightarrow \mathrm{Pth}_{\boldsymbol{\mathcal{B}}}$ *is a mapping satisfying that, for every rewrite rule* $\mathfrak{p} = (M, N) \in \mathcal{A}$, *we have that*

$$f^{(1)}(\mathfrak{p}) \in \mathrm{Pth}_{\boldsymbol{\mathcal{B}}}\left(f^{(0)\sharp}(M), f^{(0)\sharp}(N)\right).$$

The alternative notation $\mathbf{f}^{(1)} = (\mathbf{f}^{(0)}, f^{(1)})$ *will also be used.*

We now show that, given $\mathbf{f}^{(1)} = (\mathbf{f}^{(0)}, f^{(1)})$ a morphism of rewriting systems, we can extend the mapping $f^{(1)}$ to the set of paths in $\boldsymbol{\mathcal{A}}$, i.e., $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$.

Proposition 6. *[9, Proposition 34.1.1] Let* $\mathbf{f}^{(1)} = (c, (f^{(i)})_{i\in 2})\colon \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{B}}$ *be a morphism. Then there exists a mapping* $f^{(1)\flat}$ *from* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ *to* $\mathrm{Pth}_{\boldsymbol{\mathcal{B}}}$, *which we call the* path extension mapping of $f^{(1)}$, *satisfying that*

1. $\mathrm{sc}^{(0,1)}_{\boldsymbol{\mathcal{B}}} \circ f^{(1)\flat} = f^{(0)\sharp} \circ \mathrm{sc}^{(0,1)}_{\boldsymbol{\mathcal{A}}}$.
2. $\mathrm{tg}^{(0,1)}_{\boldsymbol{\mathcal{B}}} \circ f^{(1)\flat} = f^{(0)\sharp} \circ \mathrm{tg}^{(0,1)}_{\boldsymbol{\mathcal{A}}}$.
3. $f^{(1)\flat} \circ \mathrm{ip}^{(1,0)\sharp}_{\boldsymbol{\mathcal{A}}} = \mathrm{ip}^{(1,0)\sharp}_{\boldsymbol{\mathcal{B}}} \circ f^{(0)\sharp}$;
4. $f^{(1)\flat} \circ \mathrm{ech}^{(1,\mathcal{A})}_{\boldsymbol{\mathcal{A}}} = f^{(1)}$.

Proof. Let us define $f^{(1)\flat}$ by Artinian recursion on $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}})$ as follows.

Base step of the Artinian recursion.

Let $\mathfrak{P}$ be a minimal element of $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}})$. Then $\mathfrak{P}$ is either (1) an $(1,0)$-identity path or (2) an echelon. If (1), then $\mathfrak{P} = \mathrm{ip}^{(1,0)\sharp}_{\boldsymbol{\mathcal{A}}}(P)$ for some term $P \in \mathrm{T}_{\Sigma}(X)$. We define $f^{(1)\flat}(\mathfrak{P})$ to be the $(1,0)$-identity path at $f^{(0)\sharp}(P)$ which is a term in $\mathrm{T}_{\Lambda}(Y)$, i.e., $f^{(1)\flat}(\mathfrak{P}) = \mathrm{ip}^{(1,0)\sharp}_{\boldsymbol{\mathcal{B}}}\left(f^{(0)\sharp}(P)\right)$. If (2), i.e., if $\mathfrak{P}$ is an echelon associated with $\mathfrak{p} \in \mathcal{A}$ then we define $f^{(1)\flat}(\mathfrak{P}) = f^{(1)}(\mathfrak{p})$.

Inductive step of the Artinian recursion.

Let $\mathfrak{P}$ be a non-minimal element of $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}})$. We can assume that $\mathfrak{P}$ is a not a $(1,0)$-identity path, since those paths already have an image for the path extension mapping. Let us suppose that, for every path $\mathfrak{Q} \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$, if $\mathfrak{Q} <_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}} \mathfrak{P}$, then the value of the path extension mapping at $\mathfrak{Q}$ has already been defined.

We have that $\mathfrak{P}$ is either (1) a path of length m strictly greater than one containing at least one echelon or (2) an echelonless path.

If (1), let $i \in m$ be the first index for which the one-step subpath $\mathfrak{P}^{i,i}$ of $\mathfrak{P}$ is an echelon. We consider different cases for i according to the definition of $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$.

If $i = 0$, we have that the paths $\mathfrak{P}^{0,0}$ and $\mathfrak{P}^{1,m-1}$ $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$-precede the path $\mathfrak{P}$. In this case, we set $f^{(1)\flat}(\mathfrak{P}) = f^{(1)\flat}(\mathfrak{P}^{1,m-1}) \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{B}}}} f^{(1)\flat}(\mathfrak{P}^{0,0})$.

If $i \neq 0$, we have that the paths $\mathfrak{P}^{0,i-1}$ and $\mathfrak{P}^{i,m-1}$ $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$-precede the path $\mathfrak{P}$. In this case, we set $f^{(1)\flat}(\mathfrak{P}) = f^{(1)\flat}(\mathfrak{P}^{i,m-1}) \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{B}}}} f^{(1)\flat}(\mathfrak{P}^{0,i-1})$.

If (2), i.e., if $\mathfrak{P}$ is an echelonless path in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}},s}$, then the conditions for the path extraction algorithm are met. Then, there exists a unique n-ary operation symbol $\sigma \in \Sigma_n$ associated to $\mathfrak{P}$. Let $(\mathfrak{P}_j)_{j\in n}$ be the family of paths in $\mathrm{Pth}^n_{\boldsymbol{\mathcal{A}}}$ which we can extract from $\mathfrak{P}$. In this case, we set $f^{(1)\flat}(\mathfrak{P}) = \sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{B}}}}((f^{(1)\flat}(\mathfrak{P}_j))_{j\in n})$.

Proposition 7. *[9, Propositions 35.1.1, 35.1.3, 35.2.1, 35.2.2 and 35.3.1] Let* $\mathbf{f}^{(1)} = (c, (f^{(i)})_{i\in 2})$ *be a morphism from* $\boldsymbol{\mathcal{A}}$ *to* $\boldsymbol{\mathcal{B}}$*. Then the sets* $\mathrm{Pth}_{\boldsymbol{\mathcal{B}}}$ *and* $[\mathrm{Pth}_{\boldsymbol{\mathcal{B}}}]$ *are equipped, in a natural way, with a structure of partial* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-algebra that we denote by* $\mathbf{Pth}_{\boldsymbol{\mathcal{B}}}^{\mathbf{f}^{(1)}}$ *and* $[\mathbf{Pth}_{\boldsymbol{\mathcal{B}}}^{\mathbf{f}^{(1)}}]$*, respectively. The sets* $\mathrm{PT}_{\boldsymbol{\mathcal{B}}}$ *and* $[\mathrm{PT}_{\boldsymbol{\mathcal{B}}}]$ *are equipped, in a natural way, with a structure of partial* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-algebra that we denote by* $\mathbf{PT}_{\boldsymbol{\mathcal{B}}}^{\mathbf{f}^{(1)}}$ *and* $[\mathbf{PT}_{\boldsymbol{\mathcal{B}}}^{\mathbf{f}^{(1)}}]$*, respectively. The set* $\mathrm{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{B}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{B}}})$ *is equipped, in a natural way, with a structure of partial* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-algebra that we denote by* $\mathrm{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{B}}}}^{\mathbf{f}^{(1)}}(\mathbf{Pth}_{\boldsymbol{\mathcal{B}}})$*.*

Proof. By Proposition 4.

Proposition 8. *[9, Theorems 35.2.6 and 35.3.4] The partial* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-algebras* $[\mathbf{Pth}_{\boldsymbol{\mathcal{B}}}^{\mathbf{f}^{(1)}}]$, $[\mathbf{PT}_{\boldsymbol{\mathcal{B}}}^{\mathbf{f}^{(1)}}]$ *and* $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{B}}}}^{\mathbf{f}^{(1)}}(\mathbf{Pth}_{\boldsymbol{\mathcal{B}}})$ *are isomorphic and belong to* $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}})$*.*

4.1 Quotient path-extension mapping

In order to complete the study of morphisms between rewriting systems, we show that the path-extension mapping is compatible with the kernels of the Curry-Howard mappings of $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$. Thus defining a mapping from $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$ to $[\mathrm{Pth}_{\boldsymbol{\mathcal{B}}}]$.

Proposition 9. *[9, Proposition 36.0.2] The mapping* $\mathrm{pr}_{\boldsymbol{\mathcal{B}}}^{\mathrm{Ker}(\mathrm{CH}^{(1)})} \circ f^{(1)\flat}$*, from* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ *to* $[\mathrm{Pth}_{\boldsymbol{\mathcal{B}}}]$*, is a* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-homomorphism from* $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}$ *to* $[\mathbf{Pth}_{\boldsymbol{\mathcal{B}}}^{\mathbf{f}^{(1)}}]$ *satisfying that*

$$\mathrm{Ker}(\mathrm{CH}^{(1)}_{\boldsymbol{\mathcal{A}}}) \subseteq \mathrm{Ker}(\mathrm{pr}_{\boldsymbol{\mathcal{B}}}^{\mathrm{Ker}(\mathrm{CH}^{(1)})} \circ f^{(1)\flat}).$$

Definition 7. *Following Proposition 9 and taking into account the Universal Property of the Quotient, there exists a unique* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-homomorphism, that we will denote by* $f^{[1]@}$*, i.e.,* $f^{[1]@}\colon [\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}] \longrightarrow [\mathbf{Pth}_{\boldsymbol{\mathcal{B}}}^{\mathbf{f}^{(1)}}]$*, satisfying that*

$$f^{[1]@} \circ \mathrm{pr}_{\boldsymbol{\mathcal{A}}}^{\mathrm{Ker}(\mathrm{CH}^{(1)})} = \mathrm{pr}_{\boldsymbol{\mathcal{B}}}^{\mathrm{Ker}(\mathrm{CH}^{(1)})} \circ f^{(1)\flat},$$

namely $f^{[1]@} = (\mathrm{pr}_{\boldsymbol{\mathcal{B}}}^{\mathrm{Ker}(\mathrm{CH}^{(1)})} \circ f^{(1)\flat})^{\natural}$*. We will call this mapping the* quotient path extension mapping *of* $\mathbf{f}^{(1)}$*. Formally, for every path class* $[\mathfrak{P}]$ *in* $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$*,* $f^{[1]@}([\mathfrak{P}]) = \left[f^{(1)\flat}(\mathfrak{P})\right]$*.*

Remark 2. Let us recall that the partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebras $[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]$ and $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}})$, which is the free partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra in the category $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}})$ determined by $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}$, are isomorphic. Thus, the construction of the quotient path-extension mapping could be done taking into account that the partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra $[\mathbf{Pth}_{\boldsymbol{\mathcal{B}}}^{\mathbf{f}^{(1)}}]$ belongs to $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}})$ and use the universal property of $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}})$. Both constructions define the same mapping.

We now present the different relations of the quotient path-extension mapping with several defined mappings.

Proposition 10. *[9, Propositions 36.1.1, 36.1.2, 36.1.3 and 36.1.4] Let* $\mathbf{f}^{(1)} = (c, (f^{(i)})_{i\in 2})$ *be a morphism from* $\boldsymbol{\mathcal{A}}$ *to* $\boldsymbol{\mathcal{B}}$*. Then the following equalities hold*

- $\mathrm{CH}^{[1]}_{\boldsymbol{\mathcal{B}}} \circ f^{[1]@} = f^{[1]@} \circ \mathrm{CH}^{[1]}_{\boldsymbol{\mathcal{A}}}$.
- $\mathrm{ip}^{([1],Y)@}_{\boldsymbol{\mathcal{B}}} \circ f^{[1]@} = f^{[1]@} \circ \mathrm{ip}^{([1],X)@}_{\boldsymbol{\mathcal{A}}}$.
- $\mathrm{sc}^{(0,[1])}_{\boldsymbol{\mathcal{B}}} \circ f^{[1]@} = f^{(0)\sharp} \circ \mathrm{sc}^{(0,[1])}_{\boldsymbol{\mathcal{A}}}$.
- $\mathrm{tg}^{(0,[1])}_{\boldsymbol{\mathcal{B}}} \circ f^{[1]@} = f^{(0)\sharp} \circ \mathrm{tg}^{(0,[1])}_{\boldsymbol{\mathcal{A}}}$.
- $\mathrm{ip}^{([1],0)\sharp}_{\boldsymbol{\mathcal{B}}} \circ f^{(0)\sharp} = f^{[1]@} \circ \mathrm{ip}^{([1],0)\sharp}_{\boldsymbol{\mathcal{A}}}$.

4.2 The category $\mathsf{Rws}^{[1]}_{\mathfrak{d}}$

In order to define a category, we begin by defining the notion of identity morphism and that of composition of morphisms.

Definition 8. *Let* $\mathbf{f}^{(1)}\colon \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{B}}$ *and* $\mathbf{g}^{(1)}\colon \boldsymbol{\mathcal{B}} \longrightarrow \boldsymbol{\mathcal{C}}$ *be morphisms. Then* $\mathbf{g}^{(1)} \circ \mathbf{f}^{(1)}$*, the composition of* $\mathbf{f}^{(1)}$ *and* $\mathbf{g}^{(1)}$*, is the morphism* $(\mathbf{g}^{(0)} \circ \mathbf{f}^{(0)}, g^{(1)\flat} \circ f^{(1)})$*, where* $\mathbf{g}^{(0)} \circ \mathbf{f}^{(0)}$ *is the composition of the underlying zeroth-order morphisms, introduced in Definition 5, and* $g^{(1)\flat} \circ f^{(1)}$ *is the mapping from* $\mathcal{A}$ *to* $\mathrm{Pth}_{\boldsymbol{\mathcal{C}}}$ *being* $g^{(1)\flat}$ *the path extension mapping of* $g^{(1)}$ *introduced in Proposition 6.*

Let $\boldsymbol{\mathcal{A}} = (\Sigma, X, \mathcal{A})$ *be a rewriting system. The* identity *morphism at* $\boldsymbol{\mathcal{A}}$ *is given by* $(\eta^{\Sigma}, \eta^{X}, \mathrm{ech}^{(1,\mathcal{A})}_{\boldsymbol{\mathcal{A}}})$*, where* $\mathrm{ech}^{(1,\mathcal{A})}_{\boldsymbol{\mathcal{A}}}$ *is the echelon mapping associated with the rewriting system* $\boldsymbol{\mathcal{A}}$*.*

Proposition 11. *[9, Propositions 37.1.3 and 37.1.9] Let* $\mathbf{f}^{(1)}\colon \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{B}}$ *and* $\mathbf{g}^{(1)}\colon \boldsymbol{\mathcal{B}} \longrightarrow \boldsymbol{\mathcal{C}}$ *be morphisms. Thus,*

1. $\mathrm{ech}^{(1,\mathcal{A})@}_{\boldsymbol{\mathcal{A}}} = \mathrm{id}^{[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]}$.
2. $(g^{(1)\flat} \circ f^{(1)})^{@} = g^{[1]@} \circ f^{[1]@}$.

Rewriting systems and morphisms between them do not form a category since, among other things, the composition of morphisms is not associative. To overcome this problem, we introduce an equivalence relation on the morphisms.

Definition 9. *Let* $\mathbf{f}^{(1)}, \mathbf{g}^{(1)}\colon \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{B}}$ *be morphisms. We will say that* $\mathbf{f}^{(1)}$ *and* $\mathbf{g}^{(1)}$ *are* equivalent*, written* $\mathbf{f}^{(1)} \cong^{(1)} \mathbf{g}^{(1)}$*, if*

1. $(c, f^{(0)}) = (d, g^{(0)})$*, i.e.,* $c = d$ *and* $f^{(0)} = g^{(0)}$*; and*
2. $\mathrm{pr}^{\mathrm{Ker}(\mathrm{CH}^{(1)})}_{\boldsymbol{\mathcal{B}}} \circ f^{(1)\flat} = \mathrm{pr}^{\mathrm{Ker}(\mathrm{CH}^{(1)})}_{\boldsymbol{\mathcal{B}}} \circ g^{(1)\flat}$*. That is, for every path* $\mathfrak{P}$ *in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$*,* $[f^{(1)\flat}(\mathfrak{P})] = [g^{(1)\flat}(\mathfrak{P})]$.

Note that $\cong^{(1)}$ *is an equivalence relation. Therefore, to simplify notation, we will denote by* $[\mathbf{f}^{(1)}]$ *the equivalence class* $[\mathbf{f}^{(1)}]_{\cong^{(1)}}$*.*

We now show that the composition of morphism classes does not depend on the representative of the $\cong^{(1)}$-classes.

Proposition 12. *[9, Proposition 37.1.10] Let* $\mathbf{f}^{(1)}$*,* $\mathbf{f}'^{(1)}\colon \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{B}}$ *and* $\mathbf{g}^{(1)}$*,* $\mathbf{g}'^{(1)}\colon \boldsymbol{\mathcal{B}} \longrightarrow \boldsymbol{\mathcal{C}}$ *be morphisms. If* $\mathbf{f}^{(1)} \cong^{(1)} \mathbf{f}'^{(1)}$ *and* $\mathbf{g}^{(1)} \cong^{(1)} \mathbf{g}'^{(1)}$*, then* $\mathbf{g}^{(1)} \circ \mathbf{f}^{(1)} \cong^{(1)} \mathbf{g}'^{(1)} \circ \mathbf{f}'^{(1)}$*. Thus, the composition of their equivalent classes* $[\mathbf{g}^{(1)}] \circ [\mathbf{f}^{(1)}] = [\mathbf{g}^{(1)} \circ \mathbf{f}^{(1)}]$ *is well-defined and does not depend on the representatives.*

Proposition 13. *[9, Proposition 37.1.11] Rewriting systems together with the equivalence classes of morphisms constitute a category, that we denote by* $\mathsf{Rws}_{\mathfrak{d}}^{[1]}$.

Proposition 14. *[9, Proposition 37.2.1] Let* $\mathbf{f}^{(1)}, \mathbf{g}^{(1)} \colon \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{B}}$ *be morphisms such that* $\mathbf{f}^{(1)} \cong^{(1)} \mathbf{g}^{(1)}$. *Then* $[\mathbf{Pth}_{\boldsymbol{\mathcal{B}}}^{\mathbf{f}^{(1)}}] = [\mathbf{Pth}_{\boldsymbol{\mathcal{B}}}^{\mathbf{g}^{(1)}}]$ *and* $f^{[1]@} = g^{[1]@}$.

We next introduce the notion of tower and morphisms between them.

Definition 10. *The* tower *associated to the rewriting system* $\boldsymbol{\mathcal{A}}$ *is the ordered pair* $\mathbb{A} = (\boldsymbol{\mathcal{A}}, \mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}))$.

A tower morphism *from* $\mathbb{A} = (\boldsymbol{\mathcal{A}}, \mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}))$ *to* $\mathbb{B} = (\boldsymbol{\mathcal{B}}, \mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{B}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{B}}}))$ *is an ordered pair* $\mathbf{f}^{[1]@} = ([\mathbf{f}^{(1)}], f^{[1]@})$, *denoted by* $\mathbf{f}^{[1]@} \colon \mathbb{A} \longrightarrow \mathbb{B}$, *where* $[\mathbf{f}^{(1)}]$ *is the equivalence class of a morphism* $\mathbf{f}^{(1)} \colon \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{B}}$ *and* $f^{[1]@}$ *is its quotient path extension mapping.*

Definition 11. *Let* $\mathbf{f}^{[1]@} \colon \mathbb{A} \longrightarrow \mathbb{B}$ *and* $\mathbf{g}^{[1]@} \colon \mathbb{B} \longrightarrow \mathbb{C}$ *be tower morphisms. Then* $\mathbf{g}^{[1]@} \circ \mathbf{f}^{[1]@}$, *the* composition *of* $\mathbf{f}^{[1]@}$ *and* $\mathbf{g}^{[1]@}$ *is the tower morphism* $([\mathbf{g}^{(1)}] \circ [\mathbf{f}^{(1)}], g^{[1]@} \circ f^{[1]@})$, *where, according to Proposition 12 the composition* $[\mathbf{g}^{(1)}] \circ [\mathbf{f}^{(1)}] = [\mathbf{g}^{(1)} \circ \mathbf{f}^{(1)}]$ *is well-defined and, according to Proposition 11,* $g^{[1]@} \circ f^{[1]@}$ *is its quotient path extension mapping.*

Let $\mathbb{A} = (\boldsymbol{\mathcal{A}}, \mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}))$ *be a tower. The* identity *tower morphism at* $\mathbb{A}$ *is the ordered pair* $([\mathrm{id}^{\boldsymbol{\mathcal{A}}}], \mathrm{id}^{\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}})})$ *where,* $[\mathrm{id}^{\boldsymbol{\mathcal{A}}}]$ *is the equivalence class of the identity morphism at* $\boldsymbol{\mathcal{A}}$, *introduced in Definition 8, and, according to Proposition 11,* $\mathrm{id}^{\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}})}$ *is its quotient path extension mapping.*

Proposition 15. *[9, Proposition 37.2.5] Towers together with tower morphisms constitute a category, that we denote by* $\mathsf{Tw}_{\mathfrak{d}}^{[1]}$.

Proposition 16. *[9, Proposition 37.2.10] The categories* $\mathsf{Rws}_{\mathfrak{d}}^{[1]}$ *and* $\mathsf{Tw}_{\mathfrak{d}}^{[1]}$ *are isomorphic.*

5 Second-order morphisms

Let us begin by defining the notion of second-order morphism between second-order rewriting systems.

Definition 12. *Let* $\boldsymbol{\mathcal{A}}^{(2)} = (\boldsymbol{\mathcal{A}}, \mathcal{A}^{(2)})$ *and* $\boldsymbol{\mathcal{B}}^{(2)} = (\boldsymbol{\mathcal{B}}, \mathcal{B}^{(2)})$ *be second-order rewriting systems. A* second-order morphism *from* $\boldsymbol{\mathcal{A}}^{(2)}$ *to* $\boldsymbol{\mathcal{B}}^{(2)}$ *is an ordered pair* $\mathbf{f}^{(2)} = (c, (f^{(i)})_{i \in 3})$, *denoted by* $\mathbf{f}^{(2)} \colon \boldsymbol{\mathcal{A}}^{(2)} \longrightarrow \boldsymbol{\mathcal{B}}^{(2)}$, *where*

1. $\mathbf{f}^{(1)} = (c, (f^{(i)})_{i \in 2})$, *the* underlying morphism *of* $\mathbf{f}^{(2)}$, *is a morphism from* $\boldsymbol{\mathcal{A}}$ *to* $\boldsymbol{\mathcal{B}}$, *introduced in Definition 6; and*
2. $f^{(2)} \colon \mathcal{A}^{(2)} \longrightarrow \mathrm{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}$ *is a mapping satisfying that, for every second-order rewrite rule* $\mathfrak{p}^{(2)} = ([M], [N]) \in \mathcal{A}^{(2)}$, *we have that*

$$f^{(2)}\left(\mathfrak{p}^{(2)}\right) \in \mathrm{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}\left(f^{[1]@}([M]), f^{[1]@}([N])\right).$$

The alternative notation $\mathbf{f}^{(2)} = (\mathbf{f}^{(1)}, f^{(2)})$ *will also be used.*

We now show that, given $\mathbf{f}^{(2)} = (\mathbf{f}^{(1)}, f^{(2)})$ a second-order morphism of second-orderrewriting systems, we can extend the mapping $f^{(2)}$ to the set of second-order paths in $\boldsymbol{\mathcal{A}}^{(2)}$, i.e., $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$.

Proposition 17. *[9, Proposition 38.1.1] Let* $\mathbf{f}^{(2)} = (c, (f^{(i)})_{i\in 3})\colon \boldsymbol{\mathcal{A}}^{(2)} \longrightarrow \boldsymbol{\mathcal{B}}^{(2)}$ *be a morphism. Then there exists a mapping* $f^{(2)\flat}$ *from* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *to* $\mathrm{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}$*, which we call the* second-order path extension mapping of $f^{(2)}$*, satisfying that*

1. $\mathrm{sc}^{([1],2)}_{\boldsymbol{\mathcal{B}}} \circ f^{(2)\flat} = f^{[1]@} \circ \mathrm{sc}^{([1],2)}_{\boldsymbol{\mathcal{A}}}$.
2. $\mathrm{tg}^{([1],2)}_{\boldsymbol{\mathcal{B}}} \circ f^{(2)\flat} = f^{[1]@} \circ \mathrm{tg}^{([1],2)}_{\boldsymbol{\mathcal{A}}}$.
3. $f^{(2)\flat} \circ \mathrm{ip}^{([2],1)\sharp}_{\boldsymbol{\mathcal{A}}} = \mathrm{ip}^{(2,[1])\sharp}_{\boldsymbol{\mathcal{B}}} \circ f^{[1]@}$;
4. $f^{(2)\flat} \circ \mathrm{ech}^{(2,\mathcal{A}^{(2)})}_{\boldsymbol{\mathcal{A}}^{(2)}} = f^{(2)}$.

Proof. Let us define $f^{(2)\flat}$ by Artinian recursion on $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}})$ as follows.

Base step of the Artinian recursion.

Let $\mathfrak{P}^{(2)}$ be a minimal element of $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}})$. Then $\mathfrak{P}^{(2)}$ is either (1) an $(2,[1])$-identity second-order path or (2) a second-order echelon.

If (1), then $\mathfrak{P}^{(2)} = \mathrm{ip}^{(2,[1])\sharp}([P])$ for some path term class $[P]$ in $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$. we define $f^{(2)\flat}(\mathfrak{P}^{(2)})$ to be the $(2,[1])$-identity path at $f^{[1]@}([P])$ which is a path term class in $[\mathrm{PT}_{\boldsymbol{\mathcal{B}}}]$, i.e., $f^{(2)\flat}(\mathfrak{P}^{(2)}) = \mathrm{ip}^{(2,[1])\sharp}_{\boldsymbol{\mathcal{B}}^{(2)}}(f^{[1]@}([P]))$.

If (2), if $\mathfrak{P}^{(2)}$ is a second-order echelon associated to a second-order rewrite rule $\mathfrak{p}^{(2)}$ then we define $f^{(2)\flat}(\mathfrak{P}^{(2)}) = f^{(2)}(\mathfrak{p})$.

Inductive step of the Artinian recursion.

Let $\mathfrak{P}^{(2)}$ be a non-minimal element of $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}})$ We can assume that $\mathfrak{P}^{(2)}$ is not a $(2,[1])$-identity second-order path, since those second-order paths already have an image for the second-order path extension mapping. Let us suppose that, for every second-order path $\mathfrak{Q}^{(2)} \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$, if $\mathfrak{Q}^{(2)} <_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{P}^{(2)}$, then the value of the second-order path extension mapping at $\mathfrak{Q}^{(2)}$ has already been defined.

We have that $\mathfrak{P}^{(2)}$ is either (1) a second-order path of length m strictly greater than one containing at least one second-order echelon or (2) an echelonless second-order m-path.

If (1), let $i \in m$ be the first index for which the one-step subpath $\mathfrak{P}^{(2),i,i}$ of $\mathfrak{P}^{(2)}$ is a second-order echelon. We consider different cases for i according to the definition of $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}$.

If $i = 0$, we have that the second-order paths $\mathfrak{P}^{(2),0,0}$ and $\mathfrak{P}^{(2),1,m-1}$ $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}$-precede the second-order path $\mathfrak{P}^{(2)}$. In this case, we set

$$f^{(2)\flat}(\mathfrak{P}^{(2)}) = f^{(2)\flat}(\mathfrak{P}^{(2),1,m-1}) \circ^{1\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}} f^{(2)\flat}(\mathfrak{P}^{(2),0,0}).$$

If $i \neq 0$, we have that the second-order paths $\mathfrak{P}^{(2),0,i-1}$ and $\mathfrak{P}^{(2),i,m-1}$ $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}$-precede the second-order path $\mathfrak{P}^{(2)}$. In this case, we set

$$f^{(2)\flat}(\mathfrak{P}^{(2)}) = f^{(2)\flat}(\mathfrak{P}^{(2),i,m-1}) \circ^{1\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}} f^{(2)\flat}(\mathfrak{P}^{(2),0,i-1}).$$

If (2), i.e., if $\mathfrak{P}^{(2)}$ is an echelonless second-order path in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$. It could be the case that (2.1) $\mathfrak{P}^{(2)}$ is not head-constant. Then let $i \in m$ be the maximum index for which the subpath $\mathfrak{P}^{(2),0,i}$ of $\mathfrak{P}^{(2)}$ is a head-constant, echelonless second-order path. Note that the second-order pairs $\mathfrak{P}^{(2),0,i}$ and $\mathfrak{P}^{(2),i+1,m-1}$ $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}$-precede the second-order path $\mathfrak{P}^{(2)}$. In this case, we set

$$f^{(2)\flat}(\mathfrak{P}^{(2)}) = f^{(2)\flat}(\mathfrak{P}^{(2),i+1,m-1}) \circ^{1\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}} f^{(2)\flat}(\mathfrak{P}^{(2),0,i}).$$

Therefore, we are left with the case of $\mathfrak{P}^{(2)}$ being a head-constant echelonless second-order path. It could be the case that (2.2) $\mathfrak{P}^{(2)}$ is not coherent. Then let $i \in m$ be the maximum index for which the subpath $\mathfrak{P}^{(2),0,i}$ of $\mathfrak{P}^{(2)}$ is a coherent head-constant echelonless second-order path. Note that the pairs $\mathfrak{P}^{(2),0,i}$ and $\mathfrak{P}^{(2),i+1,m-1}$ $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}$-precede the second-order path $\mathfrak{P}^{(2)}$. We set

$$f^{(2)\flat}(\mathfrak{P}^{(2)}) = f^{(2)\flat}(\mathfrak{P}^{(2),i+1,m-1}) \circ^{1\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}} f^{(2)\flat}(\mathfrak{P}^{(2),0,i}).$$

Therefore, we are left with the case (2.3) of $\mathfrak{P}^{(2)}$ being a coherent head-constant echelonless second-order path. In this setting, the conditions for the second-order extraction algorithm are met. Then there exist a unique $n \in \mathbb{N}$ and a unique n-ary operation symbol $\tau \in \Sigma^{\boldsymbol{\mathcal{A}}}_{n}$ associated with $\mathfrak{P}^{(2)}$. Let $(\mathfrak{P}^{(2)}_{j})_{j\in n}$ be the family of second-order paths in $\mathrm{Pth}^{n}_{\boldsymbol{\mathcal{A}}^{(2)}}$ which we can extract from $\mathfrak{P}^{(2)}$. It could be the case that (2.3.1) $\tau = \sigma$, for some $\sigma \in \Sigma$. In this case, we set

$$f^{(2)\flat}(\mathfrak{P}^{(2)}) = \sigma^{\mathbf{c}^{*}_{\mathfrak{d}}(\mathbf{Pth}^{(0,2)}_{\boldsymbol{\mathcal{B}}^{(2)}})}((f^{(2)\flat}(\mathfrak{P}^{(2)}_{j}))_{j\in n}).$$

Therefore, we are left with the case (2.3.2) of τ being the operation symbol $\circ^{0}$. In this case, we set

$$f^{(2)\flat}(\mathfrak{P}^{(2)}) = f^{(2)\flat}(\mathfrak{P}^{(2)}_{1}) \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}} f^{(2)\flat}(\mathfrak{P}^{(2)}_{0}).$$

Proposition 18. *[9, Propositions 39.1.1, 39.1.3, 39.2.1, 39.2.2 and 39.3.1] Let* $\mathbf{f}^{(2)} = (c, (f^{(i)})_{i\in 3})$ *be a morphism from* $\boldsymbol{\mathcal{A}}^{(2)}$ *to* $\boldsymbol{\mathcal{B}}^{(2)}$*. Then the sets* $\mathrm{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}$ *and* $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}]\!]$ *are equipped, in a natural way, with a structure of partial* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-algebra that we denote by* $\mathbf{Pth}^{\mathbf{f}^{(2)}}_{\boldsymbol{\mathcal{B}}}$ *and* $[\![\mathbf{Pth}^{\mathbf{f}^{(2)}}_{\boldsymbol{\mathcal{B}}}]\!]$*, respectively. The sets* $\mathrm{PT}_{\boldsymbol{\mathcal{B}}^{(2)}}$ *and* $[\![\mathrm{PT}_{\boldsymbol{\mathcal{B}}^{(2)}}]\!]$ *are equipped, in a natural way, with a structure of partial* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-algebra that we denote by* $\mathbf{PT}^{\mathbf{f}^{(2)}}_{\boldsymbol{\mathcal{B}}^{(2)}}$ *and* $[\![\mathbf{PT}^{\mathbf{f}^{(2)}}_{\boldsymbol{\mathcal{B}}^{(2)}}]\!]$*, respectively. The set* $\mathrm{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{B}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}})$ *is equipped, in a natural way, with a structure of partial* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-algebra that we denote by* $\mathrm{T}^{\mathbf{f}^{(2)}}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{B}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}})$*.*

Proof. By Proposition 4.

Proposition 19. *[9, Theorems 39.2.6 and 39.3.4] The partial* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-algebras* $[\![\mathbf{Pth}^{\mathbf{f}^{(2)}}_{\boldsymbol{\mathcal{B}}^{(2)}}]\!]$, $[\![\mathbf{PT}^{\mathbf{f}^{(2)}}_{\boldsymbol{\mathcal{B}}^{(2)}}]\!]$ *and* $\mathbf{T}^{\mathbf{f}^{(2)}}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{B}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}})$ *are isomorphic and belong to* $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}})$*.*

5.1 Second-order quotient path extension mapping

In order to complete the study of second-order morphisms between second-order rewriting systems, we show that the second-order path-extension mapping is compatible with the supremum of the kernels of the second-order Curry-Howard mappings and the congruences $\Upsilon^{[1]}$ of $\boldsymbol{\mathcal{A}}^{(2)}$ and $\boldsymbol{\mathcal{B}}^{(2)}$. Thus defining a mapping from $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ to $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}]\!]$.

Proposition 20. *[9, Proposition 40.0.3] The mapping* $\mathrm{pr}^{[\![\cdot]\!]}_{\boldsymbol{\mathcal{B}}^{(2)}} \circ f^{(2)\flat}$ *from* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *to* $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}]\!]$ *is a* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-homomorphism from* $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *to* $[\![\mathbf{Pth}^{\mathbf{f}^{(2)}}_{\boldsymbol{\mathcal{B}}^{(2)}}]\!]$ *satisfying that*

$$\mathrm{Ker}(\mathrm{CH}^{(2)}_{\boldsymbol{\mathcal{A}}^{(2)}}) \vee \Upsilon^{[1]} \subseteq \mathrm{Ker}(\mathrm{pr}^{[\![\cdot]\!]}_{\boldsymbol{\mathcal{B}}^{(2)}} \circ f^{(2)\flat}).$$

Definition 13. *Following Proposition 20 and taking into account the Universal Property of the Quotient, there exists a unique* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-homomorphism, that we will denote by* $f^{[\![2]\!]@}$*, i.e.,* $f^{[\![2]\!]@}\colon [\![\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!] \longrightarrow [\![\mathbf{Pth}^{\mathbf{f}^{(2)}}_{\boldsymbol{\mathcal{B}}^{(2)}}]\!]$*, satisfying that*

$$f^{[\![2]\!]@} \circ \mathrm{pr}^{[\![\cdot]\!]}_{\boldsymbol{\mathcal{A}}^{(2)}} = \mathrm{pr}^{[\![\cdot]\!]}_{\boldsymbol{\mathcal{B}}^{(2)}} \circ f^{(2)\flat},$$

namely $f^{[\![2]\!]@} = (\mathrm{pr}^{[\![\cdot]\!]}_{\boldsymbol{\mathcal{B}}^{(2)}} \circ f^{(2)\flat})^{\natural}$*. We will call this mapping the* second-order quotient path extension mapping *of* $\mathbf{f}^{(2)}$*. Formally, for every second-order path class* $[\![\mathfrak{P}^{(2)}]\!]$ *in* $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$*,* $f^{[\![2]\!]@}\left([\![\mathfrak{P}^{(2)}]\!]\right) = [\![f^{(2)\flat}\left(\mathfrak{P}^{(2)}\right)]\!]$.

Remark 3. Let us recall that the partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebras $[\![\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ and $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}})$, which is the free partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra in the category $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}})$ determined by $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$, are isomorphic. Thus, the construction of the quotient path-extension mapping could be done taking into account that the partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra $[\![\mathbf{Pth}^{\mathbf{f}^{(2)}}_{\boldsymbol{\mathcal{B}}^{(2)}}]\!]$ belongs to $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}})$ and use the universal property of $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}})$. Both constructions define the same mapping.

We now present the different relations of the second-order quotient path-extension mapping with several defined mappings.

Proposition 21. *[9, Propositions 40.1.1, 40.1.2, 40.1.3, 40.1.4, 40.1.5 and 40.1.6] Let* $\mathbf{f}^{(2)} = (c, (f^{(i)})_{i\in 3})$ *be a second-order morphism from* $\boldsymbol{\mathcal{A}}^{(2)}$ *to* $\boldsymbol{\mathcal{B}}^{(2)}$*. Then the following equalities hold*

- $\mathrm{CH}^{[\![2]\!]}_{\boldsymbol{\mathcal{B}}} \circ f^{[\![2]\!]@} = f^{[\![2]\!]@} \circ \mathrm{CH}^{[\![2]\!]}_{\boldsymbol{\mathcal{A}}}$.
- $\mathrm{ip}^{([\![2]\!],Y)@}_{\boldsymbol{\mathcal{B}}} \circ f^{[\![2]\!]@} = f^{[\![2]\!]@} \circ \mathrm{ip}^{([\![2]\!],X)@}_{\boldsymbol{\mathcal{A}}}$.
- $\mathrm{sc}^{([1],[\![2]\!])}_{\boldsymbol{\mathcal{B}}} \circ f^{[\![2]\!]@} = f^{[1]@} \circ \mathrm{sc}^{([1],[\![2]\!])}_{\boldsymbol{\mathcal{A}}}$.
- $\mathrm{tg}^{([1],[\![2]\!])}_{\boldsymbol{\mathcal{B}}} \circ f^{[\![2]\!]@} = f^{[1]@} \circ \mathrm{tg}^{([1],[\![2]\!])}_{\boldsymbol{\mathcal{A}}}$.
- $\mathrm{sc}^{(0,[\![2]\!])}_{\boldsymbol{\mathcal{B}}} \circ f^{[\![2]\!]@} = f^{(0)\sharp} \circ \mathrm{sc}^{(0,[\![2]\!])}_{\boldsymbol{\mathcal{A}}}$.
- $\mathrm{tg}^{(0,[\![2]\!])}_{\boldsymbol{\mathcal{B}}} \circ f^{[\![2]\!]@} = f^{(0)\sharp} \circ \mathrm{tg}^{(0,[\![2]\!])}_{\boldsymbol{\mathcal{A}}}$.

– $\mathrm{ip}_{\boldsymbol{\mathcal{B}}}^{([\![2]\!],[1])\sharp} \circ f^{[1]@} = f^{[\![2]\!]@} \circ \mathrm{ip}_{\boldsymbol{\mathcal{A}}}^{([\![2]\!],[1])\sharp}$. – $\mathrm{ip}_{\boldsymbol{\mathcal{B}}}^{([\![2]\!],0)\sharp} \circ f^{(0)\sharp} = f^{[\![2]\!]@} \circ \mathrm{ip}_{\boldsymbol{\mathcal{A}}}^{([\![2]\!],0)\sharp}$.

5.2 The category $\mathsf{Rws}_{\mathfrak{d}}^{[\![2]\!]}$

In order to define a category, we begin by defining the notion of identity second-order order morphism and that of composition of second-order morphisms.

Definition 14. *Let $\mathbf{f}^{(2)}\colon \boldsymbol{\mathcal{A}}^{(2)} \longrightarrow \boldsymbol{\mathcal{B}}^{(2)}$ and $\mathbf{g}^{(2)}\colon \boldsymbol{\mathcal{B}}^{(2)} \longrightarrow \boldsymbol{\mathcal{C}}^{(2)}$ be second-order morphisms. Then $\mathbf{g}^{(2)} \circ \mathbf{f}^{(2)}$, the* composition *of $\mathbf{f}^{(2)}$ and $\mathbf{g}^{(2)}$, is the second-order morphism $(\mathbf{g}^{(1)} \circ \mathbf{f}^{(1)}, g^{(2)\flat} \circ f^{(2)})$, where $\mathbf{g}^{(1)} \circ \mathbf{f}^{(1)}$ is the composition of the underlying first- morphisms, introduced in Definition 8, and $g^{(2)\flat} \circ f^{(2)}$ is the mapping from $\mathcal{A}^{(2)}$ to $\mathrm{Pth}_{\boldsymbol{\mathcal{C}}^{(2)}}$ being $g^{(2)\flat}$ the second-order path extension mapping of $g^{(2)}$ introduced in Proposition 17.*

Let $\boldsymbol{\mathcal{A}}^{(2)}$ be a second-order rewriting system. The identity *second-order morphism at $\boldsymbol{\mathcal{A}}$ is given by $(\mathrm{id}^{\boldsymbol{\mathcal{A}}}, \mathrm{ech}_{\boldsymbol{\mathcal{A}}^{(2)}}^{(2,\mathcal{A}^{(2)})})$, where $\mathrm{ech}_{\boldsymbol{\mathcal{A}}^{(2)}}^{(2,\mathcal{A}^{(2)})}$ is the second-order echelon mapping associated with the second-order rewriting system $\boldsymbol{\mathcal{A}}^{(2)}$.*

Proposition 22. *[9, Propositions 41.1.3 and 41.1.9] Let $\mathbf{f}^{(2)}\colon \boldsymbol{\mathcal{A}}^{(2)} \longrightarrow \boldsymbol{\mathcal{B}}^{(2)}$ and $\mathbf{g}^{(2)}\colon \boldsymbol{\mathcal{B}}^{(2)} \longrightarrow \boldsymbol{\mathcal{C}}^{(2)}$ be second-order morphisms. Thus,*

1. $\mathrm{ech}_{\boldsymbol{\mathcal{A}}^{(2)}}^{(2,\mathcal{A}^{(2)})@} = \mathrm{id}^{[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]}$.
2. $(g^{(2)\flat} \circ f^{(2)})^{@} = g^{[\![2]\!]@} \circ f^{[\![2]\!]@}$.

Second-order rewriting systems and morphisms between them do not form a category since, among other things, the composition of morphisms is not associative. To overcome this problem, we introduce an equivalence relation on the morphisms.

Definition 15. *Let $\mathbf{f}^{(2)}, \mathbf{g}^{(2)}\colon \boldsymbol{\mathcal{A}}^{(2)} \longrightarrow \boldsymbol{\mathcal{B}}^{(2)}$ be second-order morphisms. We will say that $\mathbf{f}^{(2)}$ and $\mathbf{g}^{(2)}$ are* second-order equivalent*, written $\mathbf{f}^{(2)} \cong^{(2)} \mathbf{g}^{(2)}$, if*

1. *$f^{(1)} \cong^{(1)} g^{(1)}$; and*
2. *$\mathrm{pr}_{\boldsymbol{\mathcal{B}}^{(2)}}^{[\![\cdot]\!]} \circ f^{(2)\flat} = \mathrm{pr}_{\boldsymbol{\mathcal{B}}^{(2)}}^{[\![\cdot]\!]} \circ g^{(2)\flat}$. That is, for every second-order path $\mathfrak{P}^{(2)}$ in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$, $[\![f^{(2)\flat}(\mathfrak{P}^{(2)})]\!] = [\![g^{(2)\flat}(\mathfrak{P}^{(2)})]\!]$.*

Note that $\cong^{(2)}$ is an equivalence relation. Therefore, to simplify notation, we will denote by $[\![\mathbf{f}^{(2)}]\!]$ the equivalence class $[\![\mathbf{f}^{(2)}]\!]_{\cong^{(2)}}$.

We now show that the composition of second-order morphisms classes does not depend on the representative of the $\cong^{(2)}$-classes.

Proposition 23. *[9, Proposition 41.1.10] Let $\mathbf{f}^{(2)}$, $\mathbf{f}'^{(2)}\colon \boldsymbol{\mathcal{A}}^{(2)} \longrightarrow \boldsymbol{\mathcal{B}}^{(2)}$ and $\mathbf{g}^{(2)}$, $\mathbf{g}'^{(2)}\colon \boldsymbol{\mathcal{B}}^{(2)} \longrightarrow \boldsymbol{\mathcal{C}}^{(2)}$ be second-order morphisms. If $\mathbf{f}^{(2)} \cong^{(2)} \mathbf{f}'^{(2)}$ and $\mathbf{g}^{(2)} \cong^{(2)} \mathbf{g}'^{(2)}$, then it holds that $\mathbf{g}^{(2)} \circ \mathbf{f}^{(2)} \cong^{(2)} \mathbf{g}'^{(2)} \circ \mathbf{f}'^{(2)}$. Thus, the composition of their equivalence classes $[\![\mathbf{g}^{(2)}]\!] \circ [\![\mathbf{f}^{(2)}]\!] = [\![\mathbf{g}^{(2)} \circ \mathbf{f}^{(2)}]\!]$ is well-defined and does not depend on the representatives.*

Proposition 24. *[9, Proposition 41.1.11] Second-order rewriting systems together with the equivalence classes of second-order morphisms constitute a category, that we denote by* $\mathsf{Rws}_{\mathfrak{d}}^{[\![2]\!]}$.

Proposition 25. *[9, Proposition 41.2.1] Let* $\mathbf{f}^{(2)}, \mathbf{g}^{(2)} \colon \boldsymbol{\mathcal{A}}^{(2)} \longrightarrow \boldsymbol{\mathcal{B}}^{(2)}$ *be second-order morphisms such that* $\mathbf{f}^{(2)} \cong^{(2)} \mathbf{g}^{(2)}$. *Then* $[\![\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}^{\mathbf{f}^{(2)}}]\!] = [\![\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}^{\mathbf{g}^{(2)}}]\!]$ *and* $f^{[\![2]\!]@} = g^{[\![2]\!]@}$.

We next introduce the notion of second-order towers and their morphisms.

Definition 16. *The* second-order tower *associated to the second-order rewriting system* $\boldsymbol{\mathcal{A}}^{(2)}$ *is* $\mathbb{A}^{(2)} = (\boldsymbol{\mathcal{A}}^{(2)}, \mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}))$.

A second-order tower morphism *from* $\mathbb{A}^{(2)} = (\boldsymbol{\mathcal{A}}^{(2)}, \mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}))$ *to* $\mathbb{B}^{(2)} = (\boldsymbol{\mathcal{B}}^{(2)}, \mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{B}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{B}}^{(2)}}))$ *is an ordered pair* $\mathbf{f}^{[\![2]\!]@} = ([\![\mathbf{f}^{(2)}]\!], f^{[\![2]\!]@})$, *denoted by* $\mathbf{f}^{[\![2]\!]@} \colon \mathbb{A}^{(2)} \longrightarrow \mathbb{B}^{(2)}$, *where* $[\![\mathbf{f}^{(2)}]\!]$ *is the equivalence class of a second-order morphism* $\mathbf{f}^{(2)} \colon \boldsymbol{\mathcal{A}}^{(2)} \longrightarrow \boldsymbol{\mathcal{B}}^{(2)}$ *and* $f^{[\![2]\!]@}$ *is its second-order quotient path extension mapping introduced in Definition 13.*

Definition 17. *Let* $\mathbf{f}^{[\![2]\!]@} \colon \mathbb{A}^{(2)} \longrightarrow \mathbb{B}^{(2)}$ *and* $\mathbf{g}^{[\![2]\!]@} \colon \mathbb{B}^{(2)} \longrightarrow \mathbb{C}^{(2)}$ *be second-order tower morphisms. Then, the* composition *of* $\mathbf{f}^{[\![2]\!]@}$ *and* $\mathbf{g}^{[\![2]\!]@}$ *is the second-order tower morphism*

$$([\![\mathbf{g}^{(2)}]\!] \circ [\![\mathbf{f}^{(2)}]\!], g^{[\![2]\!]@} \circ f^{[\![2]\!]@}),$$

where, according to Proposition 23, the composition $[\![\mathbf{g}^{(2)}]\!] \circ [\![\mathbf{f}^{(2)}]\!] = [\![\mathbf{g}^{(2)} \circ \mathbf{f}^{(2)}]\!]$ *is well-defined and, according to Proposition 22,* $g^{[\![2]\!]@} \circ f^{[\![2]\!]@}$ *is its second-order quotient path extension mapping.*

Let $\mathbb{A}^{(2)} = (\boldsymbol{\mathcal{A}}^{(2)}, \mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}))$ *be a second-order tower. The* identity *second-order tower morphism at* $\mathbb{A}^{(2)}$ *is the ordered pair* $([\![\mathrm{id}^{\boldsymbol{\mathcal{A}}^{(2)}}]\!], \mathrm{id}^{\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}})})$, *where,* $[\![\mathrm{id}^{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *is the equivalence class of the identity second-order morphism at* $\boldsymbol{\mathcal{A}}^{(2)}$, *introduced in Definition 14, and, according to Proposition 22,* $\mathrm{id}^{\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}})}$ *is its second-order quotient path extension mapping.*

Proposition 26. *[9, Proposition 41.2.5] Second-order towers together with second-order tower morphisms constitute a category, that we denote by by* $\mathsf{Tw}_{\mathfrak{d}}^{[\![2]\!]}$.

Proposition 27. *[9, Proposition 41.2.10] The categories* $\mathsf{Rws}_{\mathfrak{d}}^{[\![2]\!]}$ *and* $\mathsf{Tw}_{\mathfrak{d}}^{[\![2]\!]}$ *are isomorphic.*

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