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THE SHAPLEY-SOLIDARITY VALUE FOR GAMES WITH A COALITION STRUCTURE

EMILIO CALVO

Department of Economic Analysis and ERI-CES
Universitat de Valencia
Av. dels Tarongers, s/n
46022 Valencia, Spain
Emilio.calvo@uv.es

ESTHER GUTIÉRREZ

Departamento de Economía Aplicada IV.
Universidad del País Vasco U.P.V./E.H.U
Av. Lehendakari aguirre, 83
48015 Bilbao, Spain
Mariaester.gutierrez@ehu.es

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A value for games with a coalition structure is introduced, where the rules guiding cooperation among the members of the same coalition are different from the interaction rules among coalitions. In particular, players inside a coalition exhibit a greater degree of solidarity than they are willing to use with players outside their coalition. The Shapley value is therefore used to compute the aggregate payoffs for the coalitions, and the solidarity value to obtain the payoffs for the players inside each coalition.

Keywords: Coalitional value; Shapley value; Owen value; Solidarity value.

JEL Classification: C71

1. Introduction

There are many settings in cooperative games where players naturally organize themselves into groups for the purpose of negotiating payoffs. This action can be modeled by incorporating their *coalition structure* into the game, which consists of an exogenous partition of players into a set of groups or unions. These unions sometimes arise for natural reasons. Players join together into groups of similar interests and characteristics in the case of trade unions, political parties, cartels, lobbies, etc. Another typical reason is due to geographical location, as in the case of cities, states and countries.

When groups are formed, the agents interact at two levels: first, bargaining takes place among the unions, and then bargaining occurs inside each union in order to share what the union has obtained. Owen [1977] was the first to follow this approach. In his coalitional value, unions play a quotient game among themselves and then each union receives a payoff that is shared among its players in an internal game. The payoffs at each level are given by the Shapley value [Shapley, 1953]. Thus, the same properties (axioms) that govern the interaction between groups also operate among the players of each group. The basic principle behind the Shapley value is to pay players according to their productivity. It can be expressed formally by the *marginality* axiom [see Young, 1985], that is, if the marginal contributions of a player in two games are the same, then his value should be the same. Alternatively, it can be expressed by the *null player* axiom, that is, if all the marginal contributions of a player in a game are zero, then the player should obtain zero. The Owen value applies this productivity principle when sharing rewards at both levels, among unions and within unions.

A direct consequence of applying the productivity principle redistributing rewards within unions is that null players always receive zero, irrespective of whether they are alone or inside a union. Nevertheless, it could be questioned as to whether a coalitional value having to follow the same behavior at both levels of bargaining is a legitimate point of view. A greater degree of solidarity among members of the same group than in the interaction among players of different groups also seems natural. It is very easy to find real-life examples where the groups formed seek to protect their weaker members by giving them a share of the gains obtained by the group.

Our goal is to develop a coalitional value where the rule followed to share the payoffs within each union is less competitive than the rule used in bargaining among unions.

We therefore follow the same approach as Owen. In the first step, unions play a quotient game among themselves and each union receives a payoff given by the Shapley value. This value expresses the competing principle of paying unions according to their productivity. In the second step, the same rule is applied for any subcoalition of a union, where the union is replaced by this subcoalition in the coalition structure. We thus establish the payoff that the subcoalition could obtain if its remaining members in the union withdrew from the game. This is the internal game that Owen uses to reward players within the union by applying the Shapley value to the internal game. Here we leave Owen's approach and replace the Shapley value in the internal game with another value which takes into account not only productivity principles but also some degree of cohesion, or solidarity, among the members.

Many values can be chosen at this point: The Kernel [Davis and Maschler, 1965], the nucleolus [Schmeidler, 1969], the equal division solution,[a] the egalitarian

[a]This value shares the payoffs equally among the members of the coalition.

Shapley values[b] [Joosten, 1996; van den Brink *et al.*, 2011], the consensus value[c] [Ju *et al.*, 2007], and the weighted coalitional Lorenz solutions [Arin and Feltkamp, 2002] among others.

Our choice is the *solidarity value*, introduced by Sprumont [1990] as an example of a *population monotonic allocation scheme* and characterized axiomatically by Nowak and Radzik [1994]. This value is a good compromise between productivity and solidarity principles: it takes into account the productivity principle, as the own player's marginal contributions are used in the calculation. However, it also exhibits a redistribution effect, as it not only takes into account the player's own marginal contribution, but also the marginal contributions of the remaining players. This means that his own marginal contribution is replaced with the average of the marginal contributions of all players in the coalition when computing the value. The value is thus obtained in two steps using this approach. First, unions play a quotient game among themselves and each union receives a payoff given by the Shapley value; and second, the outcome obtained by the union is shared among its members by paying the solidarity value in the internal game. We refer to this as the *Shapley-solidarity value*.

We start by offering new axiomatic support to the solidarity value. We take as reference the Myerson [1980] characterization of the Shapley value by means of the *balanced contributions* axiom. This property states that for any two players, the amount that each player would gain or lose by the other player's withdrawal from the game should be equal. This expresses the competing principle that each pair of players is balanced, as the loss in the payoff that each player can inflict on the other by withdrawing from the game is the same (consequently, if they are equally productive they receive the same payoffs). Myerson shows that the Shapley value is the only value which is efficient and satisfies balanced contributions.

We present a way to formulate the solidarity idea that all players are "in the same boat" as follows: Suppose that every player has the same opportunity to leave the game and compute the average variation of a player's payoff when every remaining player can leave the game. We then say that a value satisfies the *equal average gains axiom* if these expected payoff variations are the same for all players. Theorem 3 proves that the solidarity value is the unique value that satisfies efficiency and equal average gains.

This axiom enables us to offer the axiomatic characterization of the Shapley-solidarity value on the family of games with a coalition structure. The competing principle of interaction among unions is expressed by an axiom of balanced contributions between unions, and the solidarity among the members inside a union by an axiom of equal average gains between the members of the same union. In Theorem 5, we prove that the Shapley-solidarity value is the only value for games

[b]Convex combinations of the Shapley value and the equal division solution.
[c]A convex combination of the Shapley value and the CIS value [Driessen and Funaki, 1991]. The CIS value (center of imputation set) is defined by $\mathrm{CSI}_i(N, v) = v(i) + (1/n)(v(N) - \sum_{j \in N} v(j))$.

with coalition structures that satisfies efficiency, balanced contributions between unions and equal average gains between the members of the same union.

This result allows an easy and direct comparison with the Owen value. The competing principle in this value guides the interaction among unions and also the interaction among the members of the same union. Accordingly, the Owen value is the only coalitional value that satisfies efficiency, balanced contributions between unions and balanced contributions between the members of the same union (see Calvo *et al.*, 1996; Amer and Carreras, 1995).

The rest of the paper is organized as follows. Section 2 is devoted to definitions and notation. Section 3 introduces the new coalitional value. We provide the axiomatic characterization of this value in Sec. 4. Section 5 is devoted to comparison with other coalitional values existing in the literature. This is carried out with the help of an example and looking for the differences between the axioms which characterize these values. The conclusions are presented in Sec. 6.

2. Notation and Definitions

Let $U = \{1, 2, \ldots\}$ be the (infinite) set of potential players. A *cooperative game* with transferable utility (TU-game) is a pair (N, v) where $N \subseteq U$ is a nonempty and finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. An element i of N is called a *player* and every nonempty subset S of N a *coalition*. The real number $v(S)$ is called the *worth* of coalition S, and is interpreted as the total payoff that the coalition S, if it is formed, can obtain for its members. Let $\mathcal{G}^N$ denote the set of all cooperative TU-games with player set N.

For each two games (N, v) and $(N, w) \in \mathcal{G}^N$, the game $(N, v + w)$ is defined as $(v+w)(S) = v(S)+w(S)$ for each $S \subseteq N$. For all $S \subseteq N$, we denote the restriction of (N, v) to S as (S, v). For simplicity, we write $S \cup i$ instead of $S \cup \{i\}$, $N \backslash i$ instead of $N \backslash \{i\}$, and $v(i)$ instead of $v(\{i\})$.

Two players $\{i, j\} \subseteq N$ are *symmetric* in (N, v) if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \backslash \{i, j\}$.

Player $i \in N$ is a *null player* in (N, v) if $v(S \cup i) = v(S)$ for all $S \subseteq N \backslash i$.

A *value* is a function γ which assigns a real number $\gamma_i(N, v)$ to every TU-game (N, v) and every player $i \in N$, and represents an assessment made by i of his gains from participating in the game.

Let (N, v) be a game. For all $S \subseteq N$ and all $i \in S$, define

$$\Delta^i(v, S) := v(S) - v(S \backslash i).$$

We call $\Delta^i(v, S)$ the *marginal contribution of player i to coalition S in the TU-game* (N, v). The *Shapley value* [Shapley, 1953] of the game (N, v) is the payoff vector $\mathrm{Sh}(N, v) \in \mathbb{R}^N$ defined by

$$\mathrm{Sh}_i(N, v) = \sum_{S \subseteq N : i \in S} \frac{(n - s)!(s - 1)!}{n!} \Delta^i(v, S), \quad \text{for all } i \in N,$$

where $s = |S|$ and $n = |N|$.

For all $T \subseteq N$, *the unanimity game of the coalition* T, (N, u_T), is defined by

$$u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that the family of games $\{(N, u_T)\}_{T \subseteq N}$ is a basis for $\mathcal{G}^N$. This allows an alternative definition of the Shapley value as the linear map $\mathrm{Sh} : \mathcal{G}^N \to \mathbb{R}^N$, which is defined for all unanimity games (N, u_T) as follows

$$\mathrm{Sh}_i(N, u_T) = \begin{cases} \dfrac{1}{|T|} & \text{for all } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

For all finite sets $N \subseteq U$, a *coalition structure* over N is a partition of N, that is, $B = \{B_1, B_2, \ldots, B_m\}$ is a coalition structure if it satisfies that $\bigcup_{1 \leq k \leq m} B_k = N$ and $B_k \cap B_l = \varnothing$ when $k \neq l$. We also assume $B_k \neq \emptyset$ for all k. The sets $B_k \in B$ are called "unions" or "blocks". There are two trivial coalition structures: the first, which we denote by B^N, where only the grand coalition forms, that is, $B^N = \{N\}$; and the second is the coalition structure where each union is a singleton and is denoted by B^n, that is, $B^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$. Denote by $\mathcal{B}(N)$ the set of all coalition structures over N. A game (N, v) with a coalition structure $B \in \mathcal{B}(N)$ is denoted by (B, N, v). Let $\mathcal{CSG}^N$ denote the family of all TU-games with coalition structure with a player set N, and let $\mathcal{CSG}$ denote the set of all TU-games with a coalition structure.

For all games $(B, N, v) \in \mathcal{CSG}^N$, with $B = \{B_1, B_2, \ldots, B_m\}$, the game among unions, called the *quotient game*, is the TU-game $(M, v_B) \in \mathcal{G}^M$ where $M = \{1, 2, \ldots, m\}$ and $v_B(T) := v(\bigcup_{i \in T} B_i)$ for all $T \subseteq M$. That is, (M, v_B) is the game induced by (B, N, v) by considering the unions of B as players. Notice that for the trivial coalition structure B^n we have $(M, v_{B^n}) \equiv (N, v)$. For all $\{k, l\} \subseteq M$, we say that B_k and B_l are *symmetric coalitions* in (B, N, v) if players k and l are symmetric in the game (M, v_B). For all $k \in M$, we say that $B_k \in B$ is a *null coalition* if player $k \in M$ is a null player in the game (M, v_B).

Let $B \in \mathcal{B}(N)$. For all $k \in M$ and all $S \subseteq B_k$, denote by $B\,|_S$ the new coalition structure defined on $(\bigcup_{j \neq k} B_j) \cup S$, which appears when the remaining members of S in B_k leaves the game. That is,

$$B\,|_S = \{B_1, \ldots, B_{k-1}, S, B_{k+1}, \ldots, B_m\}.$$

A *coalitional value* is a function Φ that assigns a vector in $\mathbb{R}^N$ to each game with a coalition structure $(B, N, v) \in \mathcal{CSG}^N$. One of the most important coalitional values is the *Owen value* [Owen, 1977]. His approach resolves the problems of intercoalitional and intracoalitional bargaining by the same procedure. Let $(B, N, v) \in \mathcal{CSG}^N$. First, for all $k \in M$ and all $S \subseteq B_k$, Owen defined the game $(M, v_{B|_S})$ that describes what would happen in the quotient game if union B_k were replaced by S, that is,

$$v_{B|_S}(T) = v\left(\bigcup_{j\in T} B_j \backslash S'\right) \quad \text{for all } T \subseteq M,$$

where $S' = B_k \backslash S$.

Second, Owen defined an *internal game* (B_k, v_k) by setting $v_k(S) = \mathrm{Sh}_k(M, v_{B|_S})$ for all $S \subseteq B_k$. Thus, $v_k(S)$ is the payoff to S in $v_{B|_S}$. The *Owen value* of the game (B, N, v) is the payoff vector $\mathrm{Ow}(B, N, v) \in \mathbb{R}^N$ defined by

$$\mathrm{Ow}_i(B, N, v) := \mathrm{Sh}_i(B_k, v_k), \quad \text{for all } k \in M \text{ and all } i \in B_k. \tag{1}$$

Thus, each union $S \subseteq B_k$ plays the quotient game $(M, v_{B|_S})$ among the unions, and the payoff obtained, $\mathrm{Sh}_k(M, v_{B|_S})$, determines the reward of coalition S in the internal game (B_k, v_k). The total reward of union B_k is $\mathrm{Sh}_k(M, v_B)$ and is shared among its members, $i \in B_k$, again applying the Shapley value in the internal game (B_k, v_k), that is $\mathrm{Ow}_i(B, N, v) = \mathrm{Sh}_i(B_k, v_k)$. In that sense, we can denote the Owen value as $\mathrm{Ow} \equiv \Gamma^{(\mathrm{Sh},\mathrm{Sh})}$.

Note that the Owen value satisfies the *quotient game property*:

$$\sum_{i\in B_k} \mathrm{Ow}_i(B, N, v) = \mathrm{Sh}_k(M, v_B), \quad \text{for all } k \in M,$$

and for the trivial coalition structures B^n and B^N, $\mathrm{Ow}(B^N, N, v) = \mathrm{Ow}(B^n, N, v) = \mathrm{Sh}(N, v)$.

3. Definition of the Shapley-Solidarity Value

The standard motivation for incorporating a coalition structure into a game is that players are interested in joining a union in order to improve their bargaining position in the game. Hence, when a union is formed, all its members commit themselves to bargaining with the others as a unit. A critical question here is how to share the gains (or losses) obtained by the players in a union.

At this point, we need to recall that the Owen approach does not determine the value and the coalition structure simultaneously. On the contrary, the coalition structure is given *a priori* and fixed before starting to compute the value. The reasons for the existence of a coalition structure are varied and depend on the context at hand. Although we can agree that unions try to obtain as much as possible by not letting the others exploit their (individual) weaknesses when they are separated, that does not necessarily imply that the members of the union are only interested to be formed by productive players. For example, imagine that the union consists of a family with a child, who can be considered as a null member of the family during his childhood. In this context, a positive reward for the child seems quite natural.

This question is illustrated with the help of the next example.

Example 1. Consider the player set $N = \{1, 2, 3, 4\}$ with the coalition structure $B = \{B_r = \{1\}, B_k = \{2, 3\}, B_t = \{4\}\}$, where players 2 and 3 form the union $\mathrm{B}_k = \{2, 3\}$ and players 1 and 4 remain isolated. And let the unanimity game u_T, with $T = \{1, 2\}$.

In the game among unions $(M = \{r, k, t\}, (u_T)_B)$, unions B_r and B_k are symmetric players, that is, both contribute the same in the quotient game, so the Shapley value yields half each, and union B_t is a null player in the quotient game and therefore obtains zero. Inside union $B_k = \{2, 3\}$, the internal game $(B_k, (u_T)_k)$ is the unanimity game given by

$$(u_T)_k(\{2, 3\}) = 1/2, \quad (u_T)_k(\{2\}) = 1/2, \quad (u_T)_k(\{3\}) = 0.$$

Player 3 is again a null player in the internal game $(B_k, (u_T)_k)$, hence his Shapley value is zero. Therefore, the payoffs associated with the Owen value in (B, N, u_T) are

$$\mathrm{Ow}_1 = \frac{1}{2}, \quad \mathrm{Ow}_2 = \frac{1}{2}, \quad \mathrm{Ow}_3 = 0, \quad \mathrm{Ow}_4 = 0.$$

Therefore, for player 3 there is no difference between belonging to the union $\{2, 3\}$ or being isolated, as in $B^n = \{\{1\}, \{2\}, \{3\}, \{4\}\}$, as in this case $\mathrm{Ow}_3(B^n, N, u_T) = \mathrm{Sh}_3(N, u_T) = 0$. This is because the Owen value also rewards players in the internal game according to their productivity, and the productivity of player 3 is zero.

Our purpose is to consider coalitional values with some degree of solidarity in the interaction among players of the same union, in contrast to a competitive interaction among different unions. We therefore stick with the Shapley value at the first level of interaction among unions. However, we wish to apply a value with a greater degree of cohesion among players than the Shapley value at the interaction level between players within the same union.

There are several candidates for sharing the value $\mathrm{Sh}_k(M, v_B)$ among the players in B_k. An extreme option could be the egalitarian rule, which gives $\mathrm{Sh}_k(M, v_B)/|B_k|$ to each player i in B_k. It yields 1/4 to each player 2 and 3 in Example 1. Thus, players 2 and 3 are treated symmetrically, but this seems rather unfair as 2 is the only player that contributes to the rewards of the union $\{2, 3\}$. Although the redistribution of the gains obtained by the union is a good cohesion property, it also seems desirable to maintain the productivity principle as a reference. Can we make both principles compatible?

In this paper, we propose a new coalitional value that applies the *solidarity value*, introduced by Nowak and Radzik [1994], in the internal game (B_k, v_k). This value takes into account both principles (productivity and redistribution) in its definition. We will first recall the definition of the solidarity value in $\mathcal{G}$.

Let (N, v) be a game. For all $S \subseteq N$, define

$$\Delta^{\mathrm{av}}(v, S) := \frac{1}{s} \sum_{i \in S} \Delta^i(v, S).$$

$\Delta^{\text{av}}(v, S)$ is the *average of the marginal contributions* of players within coalition S in the game (N, v).

Definition 1. The solidarity value of the game (N, v) is the payoff vector $\text{Sl}(N, v) \in \mathbb{R}^N$ defined by

$$\text{Sl}_i(N, v) = \sum_{S \subseteq N: i \epsilon S} \frac{(n-s)!(s-1)!}{n!} \Delta^{\text{av}}(v, S), \quad \text{for all } i \in N.$$

The productivity principle is taken into account, as the players' marginal contributions are used in the calculation. Moreover, it also exhibits a redistribution effect, as it not only takes into account the player's own marginal contribution to the coalition that he belongs to, but also the marginal contributions of the remaining players in the coalition. In that way, even a null player, whose $\Delta^i(S, v)$ are all zero, can still obtain positive rewards if the associated $\Delta^{\text{av}}(S, v)$ are positive.

We now define the coalitional value in $\mathcal{CSG}$.

Definition 2. For all games $(B, N, v) \in \mathcal{CSG}^N$, for all $k \in M$ and all $i \in B_k$, the Shapley-solidarity value of (B, N, v) is the payoff vector $\xi(B, N, v) \in \mathbb{R}^N$ defined by:

$$\xi_i(B, N, v) := \text{Sl}_i(B_k, v_k), \tag{2}$$

where $v_k(S) = \text{Sh}_k(M, v_{B|_S})$ for all $S \subseteq B_k$.

First, union k plays the quotient game (M, v_B) among the unions, and the payoff obtained (by the Shapley value) is shared among its members by computing the solidarity value in the internal game (B_k, v_k). In this sense, we also denote the value ξ as $\Gamma^{(\text{Sh},\text{Sl})}$.

As the solidarity value satisfies efficiency in the internal game (B_k, v_k), it follows that ξ also satisfies the *quotient game property*:

$$\sum_{i \in B_k} \xi_i(B, N, v) = \text{Sh}_k(M, v_B), \quad \text{for all } k \in M.$$

Note that the Shapley-solidarity value applies different principles in the trivial coalition structures B^n and B^N: As in (B^N, N, v), all players are in the same union, ξ applies the cohesion principle and then $\xi(B^N, N, v) = \text{Sl}(N, v)$; as in (B^n, N, v) all players are isolated, ξ applies the productivity principle and then $\xi(B^n, N, v) = \text{Sh}(N, v)$. This is in contrast to the Owen value which applies the same competing principle to reward unions and players inside unions: $\text{Ow}(B^N, N, v) = \text{Ow}(B^n, N, v) = \text{Sh}(N, v)$.

In the game (B, N, u_T) in Example 1, the payoffs obtained with the Shapley-solidarity value are

$$\xi_1(B, N, u_T) = \frac{1}{2}, \quad \xi_2(B, N, u_T) = \frac{3}{8}, \quad \xi_3(B, N, u_T) = \frac{1}{8}, \quad \xi_4(B, N, u_T) = 0.$$

The principle of joining productivity and cohesion to reward players inside the union $\{2, 3\}$ is expressed by the transfer of $1/8$ from player 2 to the null player 3.

4. Axiomatic Characterization

This section provides an axiomatic characterization of the Shapley-solidarity value. This approach helps us to clarify the differences between and similarities with other coalitional values by looking at the differences and similarities in the properties which characterize these values.

We first look at the characterization of the solidarity value given by Novak and Radzik [1994], where a variation of the null player axiom is introduced as follows: Player $i \in N$ is an *A-null player* in (N, v) if $\Delta^{\text{av}}(v, S) = 0$ for all coalitions $S \subseteq N$ containing i. The solidarity value satisfies the following axiom in $\mathcal{G}$:

A-Null player axiom: For all (N, v) and all $i \in N$, if i is an A-null player, then $\gamma_i(N, v) = 0$.

Consider the following properties of a value γ in $\mathcal{G}^N$:

Efficiency: For all (N, v), $\sum_{i \in N} \gamma_i(N, v) = v(N)$.

Additivity: For all (N, v) and (N, v'), $\gamma(N, v + v') = \gamma(N, v) + \gamma(N, v')$.

Symmetry: For all (N, v) and all $\{i, j\} \subseteq N$, if i and j are symmetric players in (N, v), then $\gamma_i(N, v) = \gamma_j(N, v)$.

Null player axiom: For all (N, v) and all $i \in N$, if i is a null player in (N, v), then $\gamma_i(N, v) = 0$.

The following theorem is by Nowak and Radzik [1994].

Theorem 1 [Nowak and Radzik, 1994]. *A value γ on $\mathcal{G}^N$ satisfies* efficiency, additivity, symmetry *and* A-null player axiom *if, and only if, γ is the solidarity value.*

If we compare this theorem with the standard characterization of the Shapley value:

Theorem 2 [Shapley, 1953]. *A value γ on $\mathcal{G}^N$ satisfies* efficiency, additivity, symmetry *and* null player axiom *if, and only if, γ is the Shapley value.*

It is clear that both values differ only in the treatment of the null players. The null player axiom says that if all the marginal contributions of a player in a game are zero (hence he is not a productive player), then he should obtain zero. The interpretation of the A-null player is less evident. Notice that $\Delta^{\text{av}}(v, S) = 0$ means that the expected productivity of the players in coalition S is zero, as

$$\Delta^{\text{av}}(v, S) = \frac{1}{s} \sum_{i \in S} (v(S) - v(S \backslash i)),$$

is the expected variation in the worth of coalition S when every player in S has the same opportunity $1/s$ of withdrawing from the game. The A-null player axiom says that when the average productivity of all coalitions to which the player belongs is zero, then he *must receive zero*. But notice that the spirit behind the solidarity value is based on a type of cohesion principle which is difficult to express in individual productivity terms only. We therefore present an alternative way of formulating the idea that all players are "in the same boat".

Suppose that every player has the same opportunity of participating in the game. In that case we can interpret expression

$$E[\Delta\gamma_i(N,v)] := \frac{1}{n}\sum_{k\in N}(\gamma_i(N,v) - \gamma_i(N\backslash k, v)),$$

as the *expected variation in the payoff of player i when each of the players in coalition N has the same opportunity of withdrawing from the game.*[d] When player i leaves the game, we define $\gamma_i(N\backslash i, v) := 0$, as i is not in the game $(N\backslash i, v)$. We seek then a cohesion-type rule expressed by the equality in these expected payoff variations:

Equal average gains. For all (N,v) and all $\{i,j\} \subseteq N$,

$$E[\Delta\gamma_i(N,v)] = E[\Delta\gamma_j(N,v)].$$

We now offer a new characterization of the solidarity value with the help of this axiom.

Theorem 3. *A value γ on $\mathcal{G}$ satisfies* efficiency *and equal averaged gains if, and only if, γ is the solidarity value.*

Proof. *Existence.* It is well known that the solidarity value satisfies *efficiency.* Moreover, the solidarity value can be obtained recursively [Calvo, 2008] by

$$\mathrm{Sl}_i(S,v) = \frac{1}{s}\Delta^{\mathrm{av}}(v,S) + \sum_{j\in S\backslash i}\frac{1}{s}\mathrm{Sl}_i(S\backslash j, v), \quad \text{for all } S\subseteq N \text{ and all } i\in S,$$

starting with

$$\mathrm{Sl}_i(\{i\},v) = v(i), \quad \text{for all } i\in N.$$

Therefore, we have that for all $\{i,j\}\subseteq N$:

$$\mathrm{Sl}_i(N,v) - \frac{1}{n}\sum_{k\in N\backslash i}\mathrm{Sl}_i(N\backslash k, v) = \mathrm{Sl}_j(N,v) - \frac{1}{n}\sum_{k\in N\backslash j}\mathrm{Sl}_j(N\backslash k, v),$$

and this can be written as

$$\frac{1}{n}\sum_{k\in N}(\mathrm{Sl}_i(N,v) - \mathrm{Sl}_i(N\backslash k,v)) = \frac{1}{n}\sum_{k\in N}(\mathrm{Sl}_j(N,v) - \mathrm{Sl}_j(N\backslash k,v)),$$

where $\mathrm{Sl}_i(N\backslash i, v) := 0$.

Thus, the solidarity value satisfies *equal averaged gains.*

Uniqueness. Let γ be a value satisfying the above axioms and let $(N,v)\in\mathcal{G}^N$. We prove $\gamma = \mathrm{Sl}$ by induction over the number of players n. If $n=1$, by *efficiency,* $\gamma(\{i\},v) = \mathrm{Sl}(\{i\},v) = v(i)$ and hence the result holds. Assume that it is true for

[d]We are obviously in the context of a transferable utility game where it is assumed that players are risk neutral.

fewer than n players. We now prove it for n players. By *equal averaged gains*, we have that for all $\{i, j\} \subseteq N$:

$$\frac{1}{n}\sum_{k\in N}(\gamma_i(N,v)-\gamma_i(N\backslash k,v)) = \frac{1}{n}\sum_{k\in N}(\gamma_j(N,v)-\gamma_j(N\backslash k,v)). \quad (3)$$

By the induction hypothesis, $\gamma_i(N\backslash k, v) = \mathrm{Sl}_i(N\backslash k, v)$, for all $\{i, k\} \subseteq N$. Therefore, following (3):

$$\gamma_i(N,v)-\gamma_j(N,v) = \frac{1}{n}\left[\sum_{k\in N}(\mathrm{Sl}_i(N\backslash k,v)-\mathrm{Sl}_j(N\backslash k,v))\right].$$

This expression yields $(n-1)$ linearly independent equations which, jointly with the *efficiency*,

$$\sum_{i\in N}\gamma_i(N,v) = v(N),$$

form an $n \times n$ linear equations system. The matrix of this system is:

$$A_n = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -1 \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix}.$$

We now prove that $|A_n| = n$. Indeed, we proceed by induction. For $n = 2$, we have $|A_2| = 2$. Assume that it is true for less than n. We now prove it for n. We develop $|A_n|$ with the elements of the first column:

$$|A_n| = |A_{n-1}| + (-1)^{n-1}\begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{vmatrix}$$

$$= |A_{n-1}| + (-1)^{n-1}(-1)^{n-1} = n - 1 + 1 = n.$$

Therefore, $|A_n| \neq 0$, which implies that the system has only one solution. Thus, we conclude that $\gamma(N, v) = \mathrm{Sl}(N, v)$. □

The above characterization follows a similar approach to the Myerson [1980] characterization of the Shapley value by means of the *balanced contributions axiom*. Myerson [1980] introduced this property as a way of expressing the principle that the contributions the players make to the game must be balanced. Formally:

Balanced contributions. For all (N, v) and all $\{i, j\} \subseteq N$,

$$\gamma_i(N, v) - \gamma_i(N\backslash j, v) = \gamma_j(N, v) - \gamma_j(N\backslash i, v).$$

This property states that for any two players, the amount that each player would gain or lose by the other player's withdrawal from the game should be equal. In other terms, in the bargaining over the surplus, every pair of players $\{i, j\}$ are balanced because the loss in the payoff for player j that a player i can inflict by withdrawing from the game is the same as j can inflict on i.

Then we have:

Theorem 4 [Myerson, 1980]. *A value γ on $\mathcal{G}$ satisfies* efficiency *and* balanced contributions *if, and only if, γ is the Shapley value.*

Note that by applying *balanced contributions* to all players, we have that

$$\frac{1}{n-1}\sum_{k\in N\backslash i}(\gamma_i(N, v) - \gamma_i(N\backslash k, v)) = \frac{1}{n-1}\sum_{k\in N\backslash i}(\gamma_k(N, v) - \gamma_k(N\backslash i, v)),$$

which, assuming that $\gamma_i(N\backslash i, v) = 0$, is equivalent to

$$\frac{1}{n}\sum_{k\in N}(\gamma_i(N, v) - \gamma_i(N\backslash k, v)) = \frac{1}{n}\sum_{k\in N}(\gamma_k(N, v) - \gamma_k(N\backslash i, v)).$$

Hence, *balanced contributions* says that the average variation in the payoffs for player i when every remaining player can leave the game is the same as the average variation in the payoffs for the remaining players when i leaves the game. This makes the differences between the competing principle behind the Shapley value and the cohesion principle of the solidarity value more transparent.[e]

We are now ready to offer the axiomatic characterization of the Shapley-solidarity value on the family of games with a coalition structure. The competing principle of interaction among unions is expressed by an axiom of balanced contributions between unions, and the solidarity among members within a union, by an axiom of equal average gains between members of the same union.

For all coalitional values Φ and all $S \subseteq N$, let $\Phi(B, N, v)[S] := \sum_{i\in s}\Phi_i(B, N, v)$. For all $k \in M$, and all $i \in B_k$, define $B_{-i} := (B_1, \ldots, B_k\backslash i, \ldots, B_m)$, that is, B_{-i} is the new coalition structure when player i leaves the game.

E *Efficiency.* For all $(B, N, v) \in \mathcal{CSG}^N$, $\Phi(B, N, v)[N] = v(N)$.

CBC *Coalitional balanced contributions.* For all $(B, N, v) \in \mathcal{CSG}^N$ and all $\{k, l\} \subseteq M$,

$$\Phi(B, N, v)[B_k] - \Phi(B\backslash B_l, N\backslash B_l, v)[B_k]$$
$$= \Phi(B, N, v)[B_l] - \Phi(B\backslash B_k, N\backslash B_k, v)[B_l].$$

[e]An alternative characterization of the solidarity value can also be found in Kamijo and Kongo [2012]. There a weaker version of the balanced contributions axiom is used, and an invariance property of the value under the deletion of a particular type of player.

The *CBC* property states that for all $\{k, l\} \subseteq M$, the contribution of B_k to the total payoff of the members in B_l must be equal to the contribution of B_l to the total payoff of the members in B_k.

IEAG *Intracoalitional equal averaged gains.* For all $(B, N, v) \in \mathcal{CSG}^N$, all $k \in M$ and all $\{i, j\} \subseteq B_k$,

$$\frac{1}{|B_k|}\sum_{t\in B_k}(\Phi_i(B,N,v)-\Phi_i(B_{-t},N\backslash t,v)) = \frac{1}{|B_k|}\sum_{t\in B_k}(\Phi_j(B,N,v)-\Phi_j(B_{-t},N\backslash t,v)),$$

where $\Phi_i(B_{-i}, N\backslash i, v) := 0$ for all $i \in B_k$ and all $k \in M$.

The *IEAG* property states that the expected payoff variation for a player in a union B_k, when every player in this union has the same opportunity of withdrawing from the game, is equal for all players in B_k.

The characterization theorem is:

Theorem 5. *The Shapley-solidarity value ξ is the only value in $\mathcal{CSG}$ that satisfies* efficiency, coalitional balanced contributions, *and* intracoalitional equal averaged gains.

Proof. *Existence*. Let $(B, N, v) \in \mathcal{CSG}^N$ be a game. Since the Shapley value and the solidarity value satisfy *efficiency*, for all $k \in M$ we have that $\sum_{i\in B_k}\xi_i(B,N,v) = v_k(B_k) = \mathrm{Sh}_k(M,v_B)$, and then $\sum_{i\in N}\xi_i(B,N,v) = \sum_{k\in M}\sum_{i\in B_k}\xi_i(B,N,v) = \sum_{k\in M}\mathrm{Sh}_k(M,v_B) = v(N)$. Thus, ξ satisfies *efficiency*. Moreover, since $\xi(B,N,v)[B_k] = \mathrm{Sh}_k(M,v_B)$ for all $k \in M$, then ξ satisfies *CBC* if and only if

$$\mathrm{Sh}_k(M,v_B)-\mathrm{Sh}_k(M\backslash l,v_B) = \mathrm{Sh}_l(M,v_B)-\mathrm{Sh}_l(M\backslash k,v_B), \quad \text{for all } \{k,l\}\subseteq M.$$

This is true because the Shapley value satisfies *balanced contributions*.

Let $k \in M$. Taking into account that $\xi_i(B,N,v) = \mathrm{Sl}_i(B_k,v_k)$ for each $i \in B_k$, where $v_k(S) = \mathrm{Sh}_k(M, v_{B|_S})$ for each $S \subseteq B_k$, then ξ satisfies *IEAG* if and only if

$$\frac{1}{|B_k|}\sum_{t\in B_k}(\mathrm{Sl}_i(B_k,v_k)-\mathrm{Sl}_i(B_k\backslash t,v_k)) = \frac{1}{|B_k|}\sum_{t\in B_k}(\mathrm{Sl}_j(B_k,v_k)-\mathrm{Sl}_j(B_k\backslash t,v_k)), \quad \text{for each } \{i,j\}\subseteq B_k.$$

This is true because the solidarity value satisfies *equal average gains*.

Uniqueness. Let Φ be a coalitional value satisfying the above axioms. Let $(N, v) \in \mathcal{G}^N$ be a game, applying *IEAG* for $B = B^N$, so we have that for all

$\{i,j\} \subseteq N$:

$$\frac{1}{n}\sum_{t\in N}(\Phi_i(B^N,N,v)-\Phi_i(B^{N\backslash t},N\backslash t,v))$$
$$=\frac{1}{n}\sum_{t\in N}(\Phi_j(B^N,N,v)-\Phi_j(B^{N\backslash t},N\backslash t,v)).$$

And by Theorem 3, this expression jointly with *efficiency* imply that $\Phi(B^N,N,v) = \mathrm{Sl}(N,v)$ for all games $(N,v)\in\mathcal{G}^N$. Thus, Φ is uniquely determined when $|B|=1$.

We now use induction on $|B|$. Let us assume that this uniqueness is established for $|B|\le m-1$ and let $(B,N,v)\in\mathcal{CSG}^N$ be a game such that $|B|=m$. By *CBC*, for all $\{k,l\}\subseteq M$:

$$\Phi(B,N,v)[B_k]-\Phi(B,N,v)[B_l] \tag{4}$$
$$=\Phi(B\backslash B_l,N\backslash B_l,v)[B_k]-\Phi(B\backslash B_k,N\backslash B_k,v)[B_l]. \tag{5}$$

The induction hypothesis yields

$$\begin{cases}\Phi(B\backslash B_l,N\backslash B_l,v)[B_k]=\xi(B\backslash B_l,N\backslash B_l,v)[B_k]\\ \Phi(B\backslash B_k,N\backslash B_k,v)[B_l]=\xi(B\backslash B_k,N\backslash B_k,v)[B_l],\end{cases}$$

and because ξ satisfies *CBC*, we have

$$\xi(B\backslash B_l,N\backslash B_l,v)[B_k]-\xi(B\backslash B_k,N\backslash B_k,v)[B_l]=\xi(B,N,v)[B_k]-\xi(B,N,v)[B_l].$$

Therefore, using (4):

$$\Phi(B,N,v)[B_k]-\Phi(B,N,v)[B_l]=\xi(B,N,v)[B_k]-\xi(B,N,v)[B_l],$$

implies that

$$\Phi(B,N,v)[B_k]-\xi(B,N,v)[B_k]=\Phi(B,N,v)[B_l]-\xi(B,N,v)[B_l],$$

for all $\{k,l\}\subseteq M$. Then, by *efficiency*,

$$\Phi(B,N,v)[B_k]=\xi(B,N,v)[B_k],\quad \text{for all } k\in M. \tag{6}$$

Let $k\in M$. We now prove that $\Phi_i(B,N,v)=\xi_i(B,N,v)$ for all $i\in B_k$, by induction over the number of players in B_k. If $|B_k|=1$, expression (6) means that $\Phi_j(B,N,v)=\xi_j(B,N,v)$ for $\{j\}=B_k$. Suppose that $|B_k|\ge 2$. By *IEAG*, we have for all $\{i,j\}\subseteq B_k$:

$$\sum_{t\in B_k}(\Phi_i(B,N,v)-\Phi_i(B_{-t},N\backslash t,v))=\sum_{t\in B_k}(\Phi_j(B,N,v)-\Phi_j(B_{-t},N\backslash t,v)). \tag{7}$$

By the induction hypothesis:

$$\begin{cases}\Phi_i(B_{-t},N\backslash t,v)=\xi_i(B_{-t},N\backslash t,v)\\ \Phi_j(B_{-t},N\backslash t,v)=\xi_j(B_{-t},N\backslash t,v).\end{cases}$$

Hence, using (7):

$$\sum_{t\in B_k}(\Phi_i(B,N,v)-\Phi_j(B,N,v)) = \sum_{t\in B_k}(\xi_i(B_{-t},N\backslash t,v)-\xi_j(B_{-t},N\backslash t,v))$$
$$= \sum_{t\in B_k}(\xi_i(B,N,v)-\xi_j(B,N,v)).$$

This implies:

$$\Phi_i(B,N,v)-\Phi_j(B,N,v)=\xi_i(B,N,v)-\xi_j(B,N,v) \Rightarrow$$
$$\Phi_i(B,N,v)-\xi_i(B,N,v)=\Phi_j(B,N,v)-\xi_j(B,N,v).$$

Taking (6) into account, we conclude that $\Phi_i(B,N,v) = \xi_i(B,N,v)$, for all $i \in B_k$. □

We have used the domain $\mathcal{CSG}$ as the player set N varies when the *CBC* and *IAEG* axioms are applied. Notice the advantage of this characterization over others that use the additivity axiom and a fixed player set N, as it can be applied to any subdomain, provided only that such a domain is closed under restrictions in the player set. On the contrary, there are subdomains that are not closed under the addition of games, like simple games for example, and then axiom systems with additivity can fail to yield a unique value.

Remark 1. The axiom system in Theorem 5 is independent. Indeed:

(1) Let the coalitional value G be defined as $G_i(B,N,v)=0$ for all $(B,N,v)\in\mathcal{CSG}$ and all $i\in N$. It satisfies all axioms except efficiency.
(2) The Owen value satisfies all axioms, except *IEAG*.
(3) The coalitional value $\Gamma^{(\mathrm{Sl},\mathrm{Sl})}$, satisfies all axioms, except *CBC*.

5. Comparison With Other Coalitional Values

Several other coalitional values have been defined in the literature. We will now provide a brief overview of them.

5.1. *The Owen value*

As we have already seen, the Owen value was the starting point of coalitional values.[f] The main difference with the Shapley-solidarity value lies in that the competing principle is applied not only among unions, but also among the members of the same union. This is expressed in the following axiom:

[f] Recall that we restrict our attention to coalitional values that satisfy efficiency. Hence, the coalition structure is an additional element which influences the way in which the worth of the grand coalition is shared among its members. This means that we left component-wise efficient values out of our analysis, that is, values which satisfy $\sum_{i\in B_k}\Phi_i(B,N,v)=v(B_k)$ for all $B_k\in B$ like, for example, the Aumann-Dreze [1974] value.

IBC *Intracoalitional Balanced Contributions.* For all $(B, N, v) \in \mathcal{CSG}^N$, all $k \in M$ and all $\{i, j\} \subseteq B_k$,

$$\Phi_i(B, N, v) - \Phi_i(B_{-j}, N\backslash j, v) = \Phi_j(B, N, v) - \Phi_j(B_{-i}, N\backslash i, v).$$

Hence, in the Owen value every null player always receives zero, irrespective of whether he is alone or inside a union.

We can compare Ow and ξ axiomatically with the following characterization [Calvo *et al.*, 1996; Amer and Carreras, 1995].

Theorem 6. *The Owen value* Ow *is the only value in* $\mathcal{CSG}$ *that satisfies* efficiency, coalitional balanced contributions, *and* intracoalitional balanced contributions.

We should also mention two weighted versions of the Owen value. One is by Levy and McLean [1989] and the other by Gómez-Rúa and Vidal-Puga [2010]. In both versions the weighted Shapley value is applied in the game among unions with weights which are proportional to the size of the unions. Both values differ in the definition of the internal game.

For all $w \in \mathbb{R}^N_{++}$, the *weighted Shapley value* Sh^w is the linear map $\text{Sh}^w : \mathcal{G}^N \to \mathbb{R}^N$, which is defined for each unanimity game (N, u_T) as follows

$$\text{Sh}^w_i(N, u_T) = \begin{cases} \dfrac{w_i}{\sum_{j \in T} w_j} & \text{for all } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

In Levy and McLean [1989], the internal game (B_k, v^*_k) is defined by setting $v^*_k(S) = \text{Sh}^w_k(M, v_{B|_S})$ for all $S \subseteq B_k$, with weights $w_r = |B_r|$ for all $r \in M$. The coalitional value $\Gamma^{(\text{Sh}^w,\text{Sh})}$ is defined by

$$\Gamma^{(\text{Sh}^w,\text{Sh})}_i(B, N, v) := \text{Sh}_i(B_k, v^*_k), \quad \text{for all } k \in M \text{ and all } i \in B_k.$$

In Gómez-Rúa and Vidal-Puga [2010], the internal game (B_k, v'_k) is defined by setting $v'_k(S) = \text{Sh}^{w'}_k(M, v_{B|_S})$ for all $S \subseteq B_k$, with weights $w'_r = |B_r|$ for all $r \in M\backslash k$, and $w'_k = |S|$. The coalitional value $\Gamma^{(\text{Sh}^{w'},\text{Sh})}$ is defined by

$$\Gamma^{(\text{Sh}^{w'},\text{Sh})}_i(B, N, v) := \text{Sh}_i(B_k, v'_k), \quad \text{for all } k \in M \text{ and all } i \in B_k.$$

In $\Gamma^{(\text{Sh}^w,\text{Sh})}$ the weight of every subcoalition $S \subseteq B_k$ is always $|B_k|$ and in $\Gamma^{(\text{Sh}^{w'},\text{Sh})}$ this weight decreases with the size of S.

These values try to prevent what is called the Harsanyi paradox. As Harsanyi [1977] points out, Owen's approach assumes a symmetrical treatment for each union and this procedure implies that, in unanimity games, players would be better off bargaining by themselves than joining forces. For example, consider $N = \{1, 2, 3, 4\}$ and the unanimity game u_N. By symmetry it holds that $\text{Ow}_i(B^n, N, u_N) = 1/4$ for all $i \in N$. If players 2 and 3 join a union $\{2, 3\}$ as in Example 1, $B = \{B_r = \{1\}, B_k = \{2, 3\}, B_l = \{4\}\}$, it holds that all unions are symmetric in the quotient game and then the aggregated payoff corresponding to union $\{2,3\}$ is $1/3$. Therefore, the payoff for each player 2 and 3 will be $1/6$, lower than their initial payoffs. On the contrary, if we apply the weighted Shapley value Sh^w in the quotient game $(M,(u_N)_B)$ with weights $(w_r = 1, w_k = 2, w_l = 1)$, we obtain

$$\mathrm{Sh}^w_r(M,(u_N)_B) = \mathrm{Sh}^w_l(M,(u_N)_B) = 1/4, \quad \mathrm{Sh}^w_k(M,(u_N)_B) = 1/2,$$

and by symmetry between 2 and 3, we are back to the initial payoffs of $1/4$ for each player.

Unfortunately, this paradox cannot always be prevented. Consider the following symmetric[g] monotonic game.

Example 2. Let (N,v) be the game where $N = \{1,2,3,4\}$ and

$$v(S) = \begin{cases} 9, & \text{if } |S| = 4, \\ 5, & \text{if } |S| = 3, \\ 4, & \text{if } |S| = 2, \\ 2, & \text{if } |S| = 1. \end{cases}$$

When all players act as singletons, by symmetry, they each obtain 2.25: $\mathrm{Ow}_i(B^n, N, v) = 2.25$, $i \in N$. If players 2 and 3 join union $B_k = \{2,3\}$, as in Example 1, their payoffs decrease with the Owen value:

$$\mathrm{Ow}_i(B,N,v) = 2, \ i \in \{2,3\}.$$

The same happens with the weighted versions:

$$\Gamma_i^{(\mathrm{Sh}^w,\mathrm{Sh})}(B,N,v) = \Gamma_i^{(\mathrm{Sh}^{w'},\mathrm{Sh})}(B,N,v) = 2.083, \ i \in \{2,3\}.$$

It is also possible to find games in which the Owen value coincides with these two weighted versions: Let (N,v') be the game where

$$v'(S) = \begin{cases} 8, & \text{if } |S| = 4, \\ 4, & \text{if } |S| = 3, \\ 4, & \text{if } |S| = 2, \\ 1, & \text{if } |S| = 1. \end{cases}$$

Here $\mathrm{Ow}_i(B^n, N, v') = 2$, $i \in N$, and when 2 and 3 join union $B_k = \{2,3\}$, it holds that

$$\mathrm{Ow}_i(B,N,v) = \Gamma_i^{(\mathrm{Sh}^w,\mathrm{Sh})}(B,N,v) = \Gamma_i^{(\mathrm{Sh}^{w'},\mathrm{Sh})}(B,N,v) = 1.833, \quad i \in \{2,3\}.$$

Hence, this type of paradox can only be solved in particular classes of games, like convex games (see Proposition 3.1 in Vidal-Puga [2012]) or unanimity games; but not in general.

[g]A symmetric game is a game in which the worth of a coalition is a function of its size.

Moreover, even in unanimity games, applying weighted versions yields problematical consequences. For example, in Example 1, in the unanimity game u_T with $T = \{1, 2\}$, it holds that

$$\mathrm{Ow}_1(B^n, N, u_T) = 1/2,$$

and when union $B_k = \{2, 3\}$ is formed, it holds that

$$\Gamma_1^{(\mathrm{Sh}^w,\mathrm{Sh})}(B, N, u_T) = \Gamma_1^{(\mathrm{Sh}^{w'},\mathrm{Sh})}(B, N, u_T) = 1/3,$$

and if we add the null player 4 to the union B_k, $B' = \{B_r = \{1\}, B_k = \{2, 3, 4\}\}$, it holds that

$$\Gamma_1^{(\mathrm{Sh}^w,\mathrm{Sh})}(B, N, u_T) = \Gamma_1^{(\mathrm{Sh}^{w'},\mathrm{Sh})}(B, N, u_T) = 1/4.$$

Player 1 decreases his bargaining power only because player 2 adds null players to his union, whereas player 1 remains as a symmetric player in the quotient game among unions. When unions bargain over the surplus by applying the productivity principle, what is relevant should be the quotient game, which informs us about the *worth of the coalition of unions*, but nothing else, the size of the unions that form such a coalition being irrelevant. It is in the second stage, when the reward that a union has obtained must be shared among its members, when the size of the union obviously matters. If the size of a union is considered relevant in the quotient game, because we wished to prevent players belonging to large unions receiving very little, then perhaps applying a different value in the quotient game would be better, so that this type of ethical consideration could be incorporated into its definition.

Notice that the payoff behavior inside each union differs between $\Gamma^{(\mathrm{Sh}^w,\mathrm{Sh})}$ and $\Gamma^{(\mathrm{Sh}^{w'},\mathrm{Sh})}$, because $\Gamma^{(\mathrm{Sh}^w,\mathrm{Sh})}$ satisfies the null player axiom and $\Gamma^{(\mathrm{Sh}^{w'},\mathrm{Sh})}$ does not.

$$\Gamma_2^{(\mathrm{Sh}^w,\mathrm{Sh})}(B, N, u_T) = 2/3,\ \Gamma_3^{(\mathrm{Sh}^w,\mathrm{Sh})}(B, N, u_T) = 0, \quad \text{and}$$

$$\Gamma_2^{(\mathrm{Sh}^{w'},\mathrm{Sh})}(B, N, u_T) = 7/12,\ \Gamma_3^{(\mathrm{Sh}^{w'},\mathrm{Sh})}(B, N, u_T) = 1/12.$$

The reason why $\Gamma^{(\mathrm{Sh}^{w'},\mathrm{Sh})}$ yields a positive payoff for the null player 3 is because the bargaining power of a coalition changes when its size changes. This type of consideration is completely different from the cohesion principle among the members of the same union that inspires the Shapley-solidarity value.

From an axiomatic viewpoint, both values satisfy the same weighted version of the *coalitional balanced contributions axiom*, as each union receives the weighted Shapley value in the quotient game.

CPBC *Coalitional per capita balanced contributions.* For all $(B, N, v) \in \mathcal{CSG}^N$ and all $\{k, l\} \subseteq M$,

$$\frac{1}{|B_k|}[\Phi(B, N, v)[B_k] - \Phi(B\backslash B_l, N\backslash B_l, v)[B_k]]$$

$$= \frac{1}{|B_l|}[\Phi(B, N, v)[B_l] - \Phi(B\backslash B_k, N\backslash B_k, v)[B_l]].$$

This property says that the average amount that players in each union would gain or lose by the other union's withdrawal from the game should be equal. The average is taken over the number of players in each union [the property was introduced with this name in Gómez-Rúa and Vidal-Puga, 2011].

The values differ in the internal game, therefore the competing principle is expressed in a different way.

For the Levy and McLean value, a slight modification of *intracoalitional balanced contributions* is used, making a player a null player instead of leaving him out of the game: Given (N, v) and $i \in N$, define (N, v^{-i}) as $v^{-i}(S) = v(S \cap (N \backslash i))$ for all $S \subseteq N$.

INBC *Intracoalitional null balanced contributions.* For all $(B, N, v) \in \mathcal{CSG}^N$, all $k \in M$ and all $\{i, j\} \subseteq B_k$,

$$\Phi_i(B, N, v) - \Phi_i(B, N, v^{-j}) = \Phi_j(B, N, v) - \Phi_j(B, N, v^{-i}).$$

Moreover, symmetry within a union is needed.

IS *Intracoalitional symmetry*: For all $(B, N, v) \in \mathcal{CSG}^N$, all $k \in M$ and all $\{i, j\} \subseteq B_k$, if i and j are symmetric players in (N, v), then $\Phi_i(B, N, v) = \Phi_j(B, N, v)$.

Then we have two characterizations:

Theorem 7. *The value* $\Gamma^{(\mathrm{Sh}^w, \mathrm{Sh})}$ *is the only value on* $\mathcal{CSG}$ *that satisfies* efficiency, coalitional per capita balanced contributions, intracoalitional symmetry *and* intracoalitional null balanced contributions.

Theorem 8. *The value* $\Gamma^{(\mathrm{Sh}^{w'}, \mathrm{Sh})}$ *is the only value on* $\mathcal{CSG}$ *that satisfies* efficiency, coalitional per capita balanced contributions *and* intracoalitional balanced contributions.[h]

Remark 2. In any event, if applying a weighted Shapley value in the quotient game was considered compulsory, then a weighted Shapley-solidarity value could also be defined by using one of the two weighted versions of the internal game: either v_k^* or v_k'. That is, either $\Gamma^{(\mathrm{Sh}^w, \mathrm{Sl})}(B, N, v)$ or $\Gamma^{(\mathrm{Sh}^{w'}, \mathrm{Sl})}(B, N, v)$ can be defined.

5.2. *Two-step Shapley value and collective value*

Kamijo defined two new coalitional values called the *two-step Shapley value* (K) [Kamijo, 2009] and the *collective value* (K^w) [Kamijo, 2011]. At the first level, the Shapley value (respectively the weighted Shapley value) is used to determine the aggregate reward for each union in the quotient game. At the second level, within each union B_k, players take the Shapley value of the game restricted to the union, that is $\mathrm{Sh}(B_k, v)$ (the productivity component of the rule) as the status

[h]The proof of Theorem 8 can be found in Gómez-Rúa and Vidal-Puga [2011].

quo point; and the bargaining surplus of the union, $\mathrm{Sh}_k(M, v_B) - v(B_k)$ (respectively $\mathrm{Sh}_k^w(M, v_B) - v(B_k)$), is shared equally among the members (the solidarity component).

For all games $(B, N, v) \in \mathcal{CSG}^N$, the two-step Shapley value of (B, N, v) is given by the formula:

$$K_i(B, N, v) = \mathrm{Sh}_i(B_k, v) + \frac{1}{|B_k|}[\mathrm{Sh}_k(M, v_B) - v(B_k)], \quad \text{for all } k \in M \text{ and all } i \in B_k, \tag{8}$$

and the collective value is given by

$$K_i^w(B, N, v) = \mathrm{Sh}_i(B_k, v) + \frac{1}{|B_k|}[\mathrm{Sh}_k^w(M, v_B) - v(B_k)], \quad \text{for all } k \in M \text{ and all } i \in B_k,$$

where $w_k = |B_k|$ for all $k \in M$.

In Example 1, the two-step Shapley value, K, yields

$$K_1 = \frac{1}{2}, \quad K_2 = \frac{1}{4}, \quad K_3 = \frac{1}{4}, \quad K_4 = 0,$$

and the collective value, K^w, yields

$$K_1^w = \frac{1}{3}, \quad K_2^w = \frac{1}{3}, \quad K_3^w = \frac{1}{3}, \quad K_4^w = 0.$$

Player 3 now obtains the same as player 2 in both values. However, this egalitarian way of sharing the aggregated gains of a union seems rather unfair from the productivity point of view, as 3 is a null player that does not contribute to the rewards of the union $\{2, 3\}$.

The following axioms can be used to characterize these values. First, the *balanced contribution axiom* is used but applied only to the trivial coalition structure B^N.

BC{N} *Balanced contributions in* B^N. For all $(B^N, N, v) \in \mathcal{CSG}^N$, and all $\{i, j\} \subseteq N$,

$$\Phi_i(B^N, N, v) - \Phi_i(B^N_{-j}, N\backslash j, v) = \Phi_j(B^N, N, v) - \Phi_j(B^N_{-i}, N\backslash i, v).$$

According to the next axiom, two players in different unions are affected equally by the deletion of the union associated with the other player.

ClBC *Collective Balanced Contributions*. For all $(B, N, v) \in \mathcal{CSG}^N$ with $|B| \geq 2$, all $\{k, h\} \subseteq M$ $(k \neq h)$, all $i \in B_k$ and all $j \in B_h$,

$$\Phi_i(B, N, v) - \Phi_i(B\backslash B_h, N\backslash B_h, v) = \Phi_j(B, N, v) - \Phi_j(B\backslash B_k, N\backslash B_k, v).$$

An aggregated version of the above axiom is

ABC *Aggregate Balanced Contributions*. For all $(B, N, v) \in \mathcal{CSG}^N$ with $|B| \geq 2$, all $\{k, h\} \subseteq M$ $(k \neq h)$, all $i \in B_k$ and all $j \in B_h$,

$$|B_k|[\Phi_i(B, N, v) - \Phi_i(B \backslash B_h, N \backslash B_h, v)] = |B_h|[\Phi_j(B, N, v) - \Phi_j(B \backslash B_k, N \backslash B_k, v)].$$

We then have:

Theorem 9. *The two-step Shapley value K is the only value on $\mathcal{CSG}$ that satisfies* efficiency, balanced contributions in B^N *and* aggregated balanced contributions.

Theorem 10. *The collective value K^w is the only value on $\mathcal{CSG}$ that satisfies* efficiency, balanced contributions in B^N *and* collective balanced contributions.

The proof of Theorem 10 can be found in Kamijo [2011] and the proof of Theorem 9 follows the same lines and is left to the reader.[i]

5.3. *The Hamiache value*

In Hamiache [2006], a coalitional value is considered which in unanimity games allocates a large share of the total worth to larger unions. What is relevant for our discussion is that this value yields a zero payoff for all null players. Moreover, the value satisfies what Hamiache called the *independence of irrelevant players*: The value does not change if we withdraw null players from the game. This implies that the payoffs in our Example 1 are

$$H_1(B, N, u_T) = H_2(B, N, u_T) = 1/2, \quad H_3(B, N, u_T) = H_4(B, N, u_T) = 0.$$

Hence, we can here apply the same criticism of a lack of solidarity with the null players in the union as in the Owen and the Levy and McLean values.

We summarize this section with two tables.

In the first table, we compare the payoffs that the coalitional values yield in the unanimity game in Example 1.

(B, N, u_T)	Φ_1	Φ_2	Φ_3	Φ_4
ξ	1/2	3/8	1/8	0
Ow	1/2	1/2	0	0
$\Gamma^{(\text{Sh}^w, \text{Sh})}$	1/3	2/3	0	0
$\Gamma^{(\text{Sh}^{w'}, \text{Sh})}$	1/3	7/12	1/12	0
K	1/2	1/4	1/4	0
K^w	1/3	1/3	1/3	0
H	1/2	1/2	0	0

[i]An alternative characterization of K can be found in Calvo and Gutiérrez [2010].

In the second table, we show the properties that these coalitional values satisfy ($\mathbf{x}^*$ means that the property is used in the characterization of the value).[j]

	E	CBC	CPBC	IEAG	IBC	INBC	IS	BC{N}	ABC	CllBC
ξ	x*	x*	—	x*	—	—	x	—	—	—
Ow	x*	x*	—	—	x*	x	x	x	—	—
$\Gamma^{(\mathrm{Sh}^w,\mathrm{Sh})}$	x*	—	x*	—	—	x*	x*	x	—	—
$\Gamma^{(\mathrm{Sh}^{w'},\mathrm{Sh})}$	x*	—	x*	—	x*	—	x	x	—	—
K	x*	x	—	—	—	—	x	x*	x*	—
K^w	x*	—	x	—	—	—	x	x*	—	x*

6. Conclusion

In this paper, we have presented a new value for cooperative games with coalition structures. We have taken the Owen value as the starting point, in which the competing principle of rewarding players by their productivity is applied among unions and among the members of the same union. In the rewards within the same union, we have replaced the productivity principle with a new one, which exhibits a greater degree of solidarity among the players. This principle is expressed formally by an axiom called *intracoalitional equal average gains*. It says that the expected payoff variation for a player in a union, when every player in this union has the same opportunity of withdrawing from the game, is equal for all players of the union.

We have seen that this implies the use of the solidarity value in the internal game if we want to compute a coalitional value which satisfies *efficiency*, *coalitional balanced contributions* and *intracoalitional equal averaged gains*.

We argue that this value is a good compromise between productivity and solidarity principles: it takes into account the productivity principle, as the players' individual marginal contributions are used in the calculation. Hence, if a player increases his productivity he will increase his payoff. However, it also exhibits a redistribution effect, as it not only takes into account the player's own marginal contribution, but also the marginal contributions of the remaining players. This means that his own marginal contribution is replaced by the average of the marginal contributions of all players in the coalition when computing the value.

The redistribution effect inherent to the *IEAG* axiom is shown by the fact that null players can receive positive payoffs, as in the unanimity game considered in Example 1. This is in contrast to values that yield zero payoffs for null players, as the Owen [1977] value, the weighted version $\Gamma^{(\mathrm{Sh}^w,\mathrm{Sh})}$ of Levy and McLean [1989] and Hamiache [2006] value do. On the other hand, the differences in the players'

[j] There is not an equivalent characterization of the Hamiache value with variations of the balanced contributions axiom.

productivity are still taken into account, as null and non-null players within the same union are rewarded differently. This is in contrast to the two-step Shapley value [Kamijo, 2009] and the collective value [Kamijo, 2011], in which both types of players receive the same in this example. Only the value considered in Gómez-Rúa and Vidal-Puga [2010] yields a positive payoff for the null player, but less than the payoff for the productive player. However, the reason for yielding a reward for the null player in Example 1 is of a different nature: in the Shapley-solidarity value, it comes from solidarity behavior between the members of the union; in the Gómez-Rúa and Vidal-Puga value, it comes from the fact that adding the null player increases the size of the union, which increases the bargaining power of the union. Whether or not the size of a union should be relevant for bargaining in the quotient game is somewhat controversial.

As has been mentioned in the Introduction, the possibility is open to considering alternative values in the *internal game*, where the null player axiom is not satisfied. For example, the kernel [Davis and Maschler, 1965], the nucleolus [Schmeidler, 1969], the egalitarian Shapley values [Joosten, 1996], the consensus value [Ju *et al.*, 2007], and the weighted coalitional Lorenz solutions [Arin and Feltkamp, 2002]. This approach could be the object of further research.

Finally, we wish to mention that an alternative axiomatic characterization of the Shapley-solidarity value can be found in a previous version of this paper [Calvo and Gutiérrez, 2011]. Additivity and a consistency property appear in the set of axioms. It is then proved that the only difference between the Owen and the Shapley-solidarity value is that of replacing the null player with the A-null player axiom in the axiom system. Moreover, how to compute the value by using the random order approach, in a similar way to the Owen value, is also shown.

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