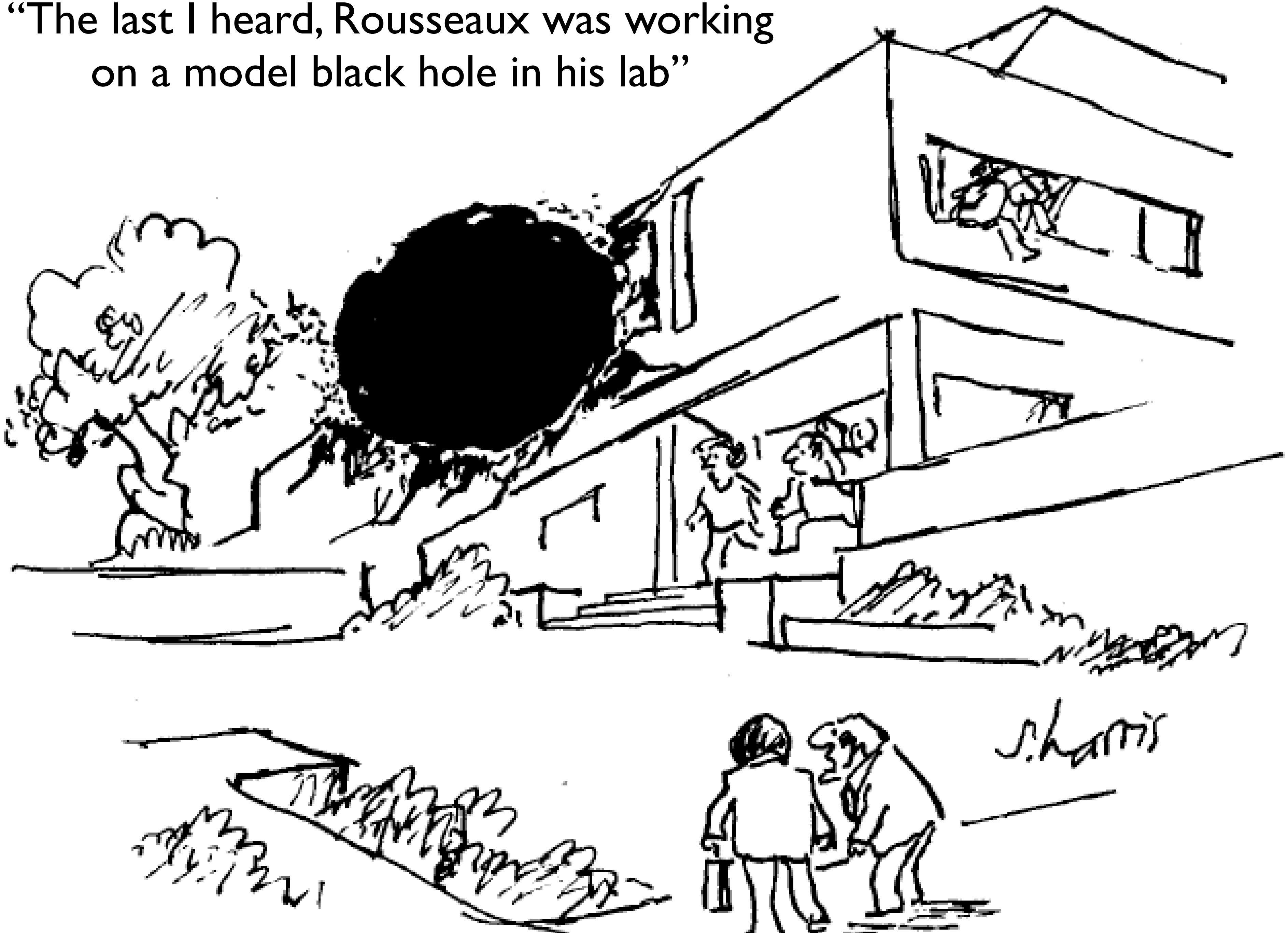
"The last I heard, Rousseaux was working on a model black hole in his lab"



WHAT CLASSICAL FLUID MECHANICS CAN TEACH US ON HAWKING RADIATION

Germain Rousseaux, Christian Mathis,
Philippe Maïssa, Jean-Charles Nardin,
CNRS, Dieudonné Laboratory,
Nice University, France.

Thomas Philbin, Ulf Leonhardt,
Saint Andrews University, Scotland.

Towards the Observation of Hawking Radiation
in Condensed Matter Systems
Valencia, February 1-7, 2009

Outline of the Talk

Maxwell, White, Hawking, Unruh...

A Review of Wave-Current Interaction

Nice Experiments in Nice

Fluid, Optical and Gravitational Horizons :
a Catastrophic Story of Caustics

Perspectives

Maxwell, White, Hawking, Unruh...

Once upon a time...

First identification of acoustics rays with null geodesics

Acoustic ray tracing in moving inhomogeneous fluids

Richard W. White

Naval Undersea Research and Development Center, Pasadena Laboratory, Pasadena, California 91107

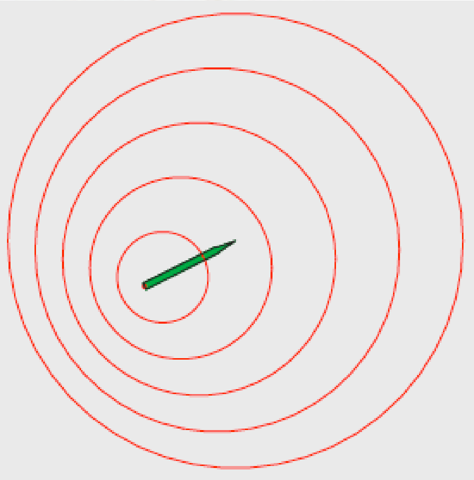
(Received 30 September 1971)

Null geodesics of the metric tensor formed from the coefficients of the second-order terms in the partial differential equation for sound are interpreted as the space–time path histories of sound pulses in a geometric ray trace theory for sound propagation in moving inhomogeneous inviscid fluids.

$$g_{ij}=\begin{pmatrix} (c^2-v^\alpha v^\alpha) & v^1 & v^2 & v^3 \\ v^1 & -1 & 0 & 0 \\ v^2 & 0 & -1 & 0 \\ v^3 & 0 & 0 & -1 \end{pmatrix}. \qquad (16)$$

Along null geodesics $(ds)^2=0$ or $g_{ij}dx^i dx^j=0$, which can be written, in terms of the substitute variables,

$$c^2=\left[v^\alpha-\frac{dx^\alpha}{dx^0}\right]\left[v^\alpha-\frac{dx^\alpha}{dx^0}\right], \qquad (17)$$



Geometrical Acoustics

In a flowing fluid, if sound moves a distance $\mathrm{d}\vec{x}$ in time $\mathrm{d}t$ then

$$||\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t|| = c_s\,\mathrm{d}t.$$

Write this as

$$(\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t) \cdot (\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t) = c_s^2 \mathrm{d}t^2.$$

Now rearrange a little: (Quadratic!)

$$-(c_s^2 - v^2)\,\mathrm{d}t^2 - 2\,\vec{v} \cdot \mathrm{d}\vec{x}\ \ \mathrm{d}t + \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x} = 0.$$

Courtesy Matt Visser

Acoustic Metric

Notation — four-dimensional coordinates:

$$x^\mu = (x^0; x^i) = (t; \vec{x}).$$

Then you can write this as

$$g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = 0.$$

With an effective acoustic metric

$$g_{\mu\nu}(t, \vec{x}) \propto \left[\begin{array}{ccc} -(c_s^2 - v^2) & \vdots & -\vec{v} \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -\vec{v} & \vdots & I \end{array} \right].$$

Sound cones!

Courtesy Matt Visser

Analogy between Fluid Mechanics and Classical Electromagnetism (v≠0 : curved space-time)

The basic idea: consider fluid flow – Unruh (1981, 1995), Visser (1998)

Continuity: $$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0$$

Euler's equation: $$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{F}$$

Assume fluid is irrotational ($\mathbf{v} = \nabla\phi$), inviscid and barotropic ($p = p(\rho)$) and linearize:

$$\rho \to \rho_0 + \rho_1 \qquad \phi \to \phi_0 + \phi_1 \qquad p \to p_0 + p_1$$

Relativistic wave equation:

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g}\, g^{\mu\nu} \frac{\partial}{\partial x^\nu} \phi_1 \right) = 0$$

with acoustic metric for massless scalar field:

where $$g_{\mu\nu} = \frac{\rho_0}{c} \left(\begin{array}{c|c} -(c^2 - v^2) & -\mathbf{v}^T \\ \hline -\mathbf{v} & \mathbf{I} \end{array} \right)$$ $$g = [\det(g^{\mu\nu})]^{-1}$$

Analogy between Fluid Mechanics and Classical Electromagnetism (v=0 : flat space-time)

Hydrodynamical quantities	Electromagnetic quantities
Specific enthalpy p/ρ	Scalar potential V
Velocity $\mathbf{u}$	Vector potential $\mathbf{A}$
Vorticity $\mathbf{w}$	Magnetic induction $\mathbf{B}$
Lamb vector $\mathbf{l}$	Electric field $\mathbf{E}$
Hydrodynamic charge q_H	Electric charge q_E

Maxwell 1861

$$\nabla.\mathbf{u} = 0 \qquad \nabla.\mathbf{A} = 0$$

$$\nabla^2 V - \frac{1}{c_L^2}\frac{\partial^2 V}{\partial t^2} = 0 \; ; \; \nabla^2 \mathbf{A} - \frac{1}{c_L^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mathbf{0} \; ; \; \nabla.\mathbf{A} + \frac{1}{c_L^2}\frac{\partial V}{\partial t} = 0$$

$$\nabla^2 \left(\frac{\delta p}{\rho_0}\right) - \frac{1}{c_S^2}\frac{\partial^2 \left(\frac{\delta p}{\rho_0}\right)}{\partial t^2} = 0 \; ; \; \nabla^2 \delta\mathbf{u} - \frac{1}{c_S^2}\frac{\partial^2 \delta\mathbf{u}}{\partial t^2} = 0 \; ; \; \nabla.\delta\mathbf{u} + \frac{1}{c_S^2}\frac{\partial \left(\frac{\delta p}{\rho_0}\right)}{\partial t} = 0$$

Rousseaux 2002

Curved space-time acts as a river flow

"Acoustic Metric"

$$d\tilde{s}^2 = -(1-\frac{v^2}{c_s^2})\,c_s^2 dT^2 + (1-\frac{v^2}{c_s^2})^{-1} dx^2$$

Schwarzschild Metric

$$d\tilde{s}^2 = -(1-\frac{v_{\rm ff}^2}{c^2})\,c^2 dt_S^2 + (1-\frac{v_{\rm ff}^2}{c^2})^{-1} dr^2$$

c_s : speed of sound

v : fluid velocity

$T = t + \int \frac{v}{c_s^2 - v^2} dx$

x : axial coordinate

$\Longleftrightarrow$

c : speed of light

$v_{\rm ff} = -c(r/r_g)^{1/2}$: free-fall velocity

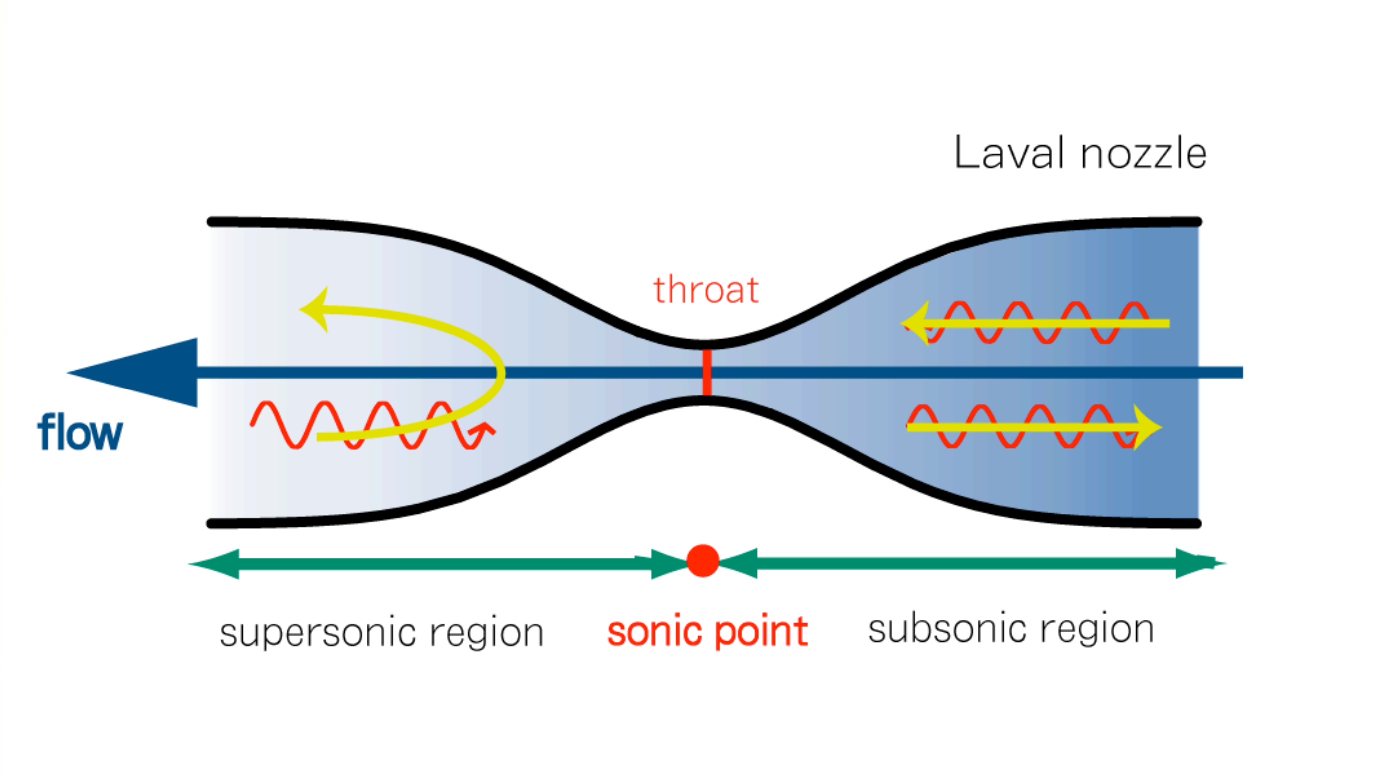
t_S : Schwarzschild time

r : radial coordinate

sonic point $\Longleftrightarrow$ horizon

Courtesy Okuzumi & Sakagami

Laval Nozzle = Dumb Hole



c_s : sound velocity
v : fluid velocity

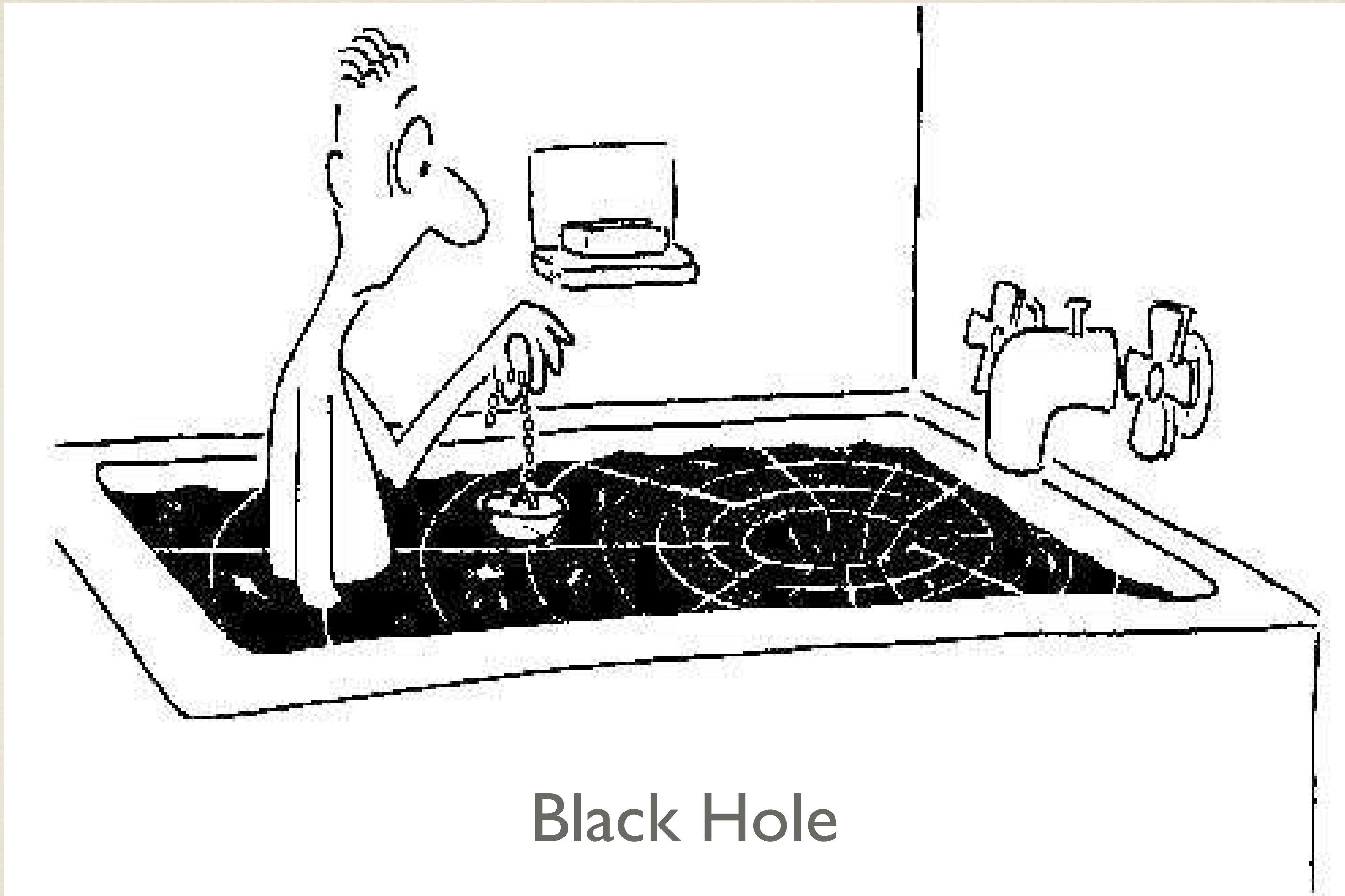
➡ "effective" sound velocity in the lab

$$c_s^{\text{eff}} = v \pm c_s = c_s(M \pm 1)$$

In the supersonic region,
sound waves cannot propagate against the flow
→ "Acoustic Black Hole"

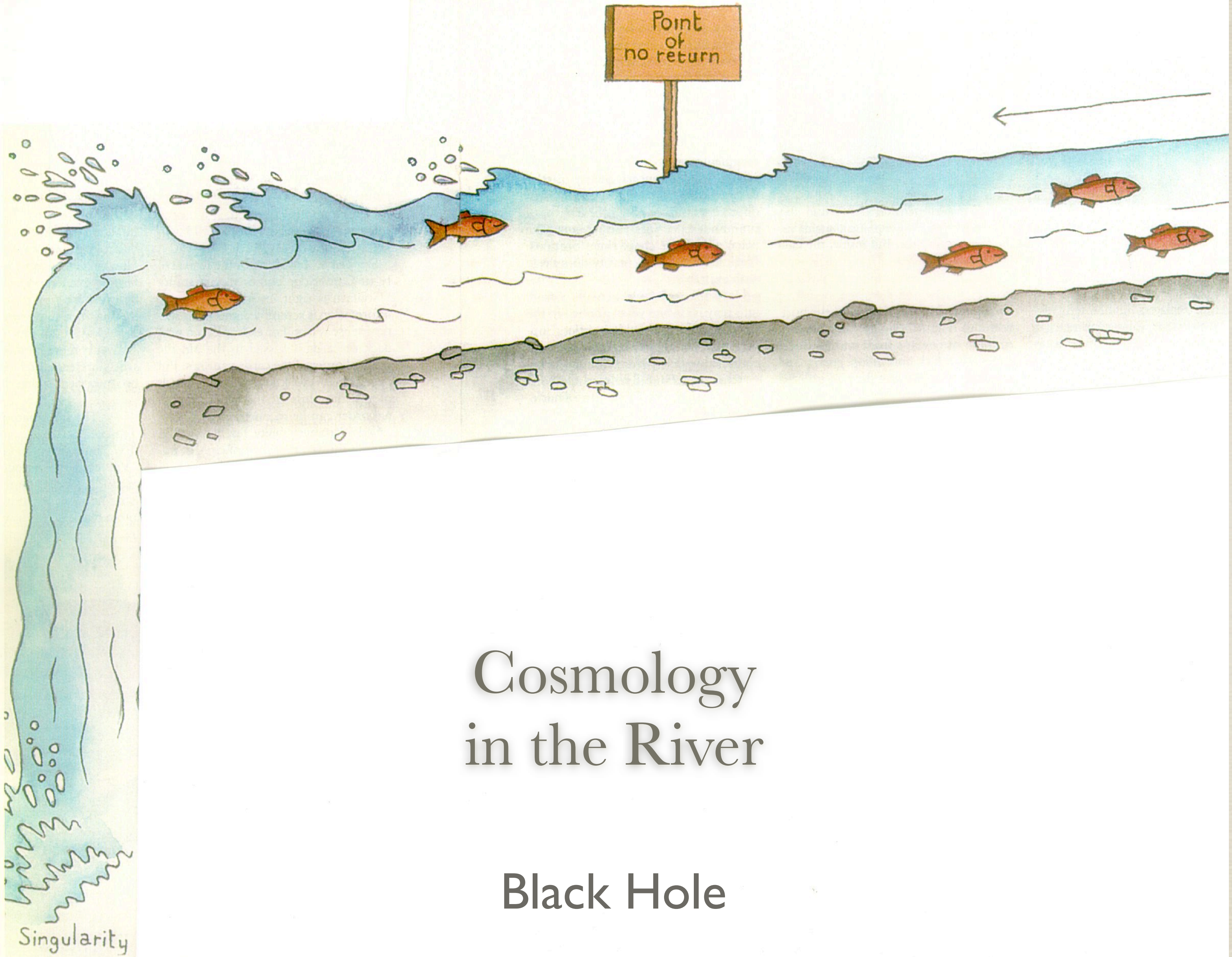
Courtesy Okuzumi & Sakagami

Cosmology
in the Bathroom



Black Hole

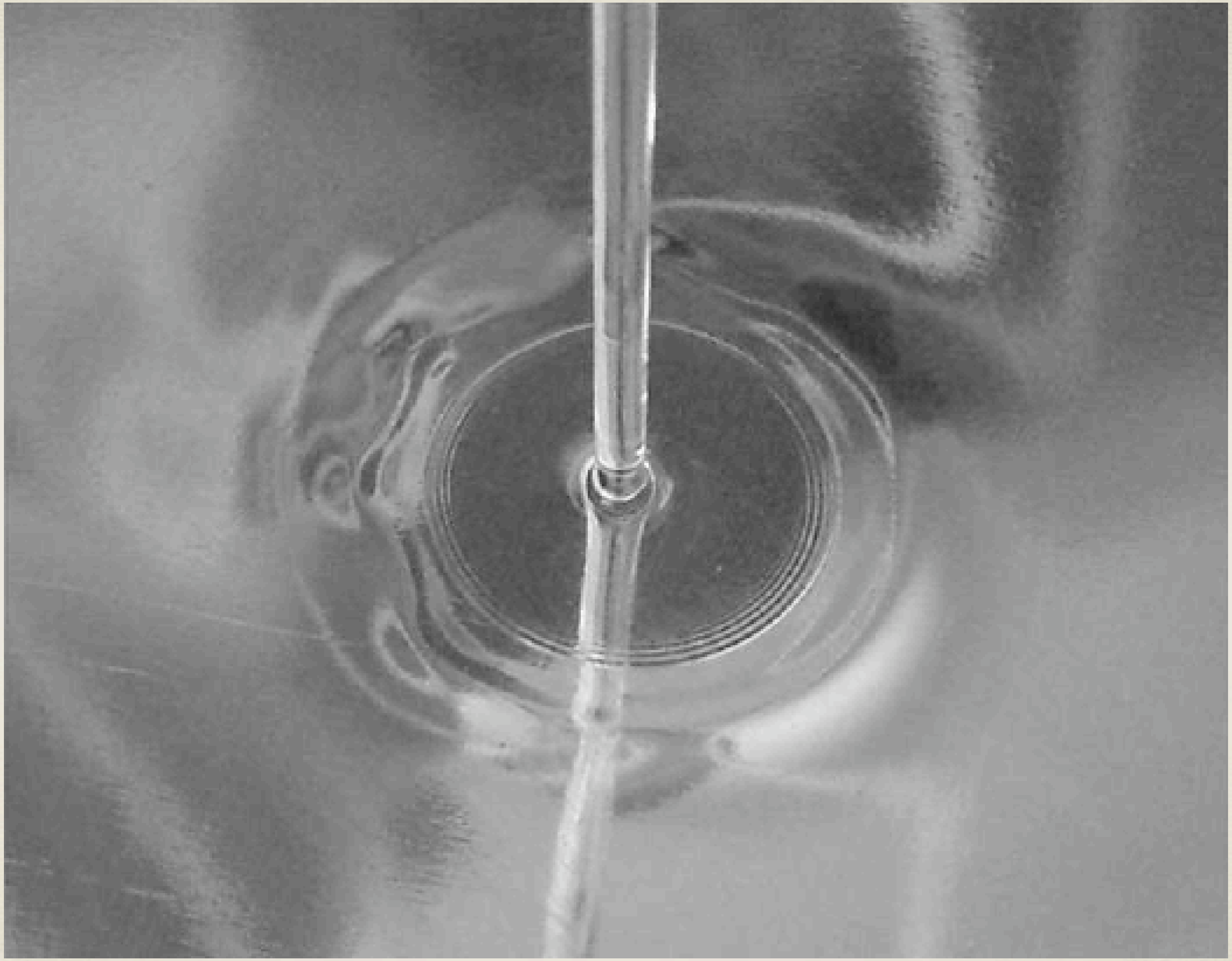
If the fluid is moving faster than waves (sound, gravity, light), then the waves are swept along with the flow, and cannot escape from that region.



Cosmology in the River

Black Hole

Cosmology in the Kitchen



White Hole

Hawking Temperature for a Condensed Matter Artificial Black Hole

$$T_H = \frac{\hbar}{2\pi k_B c_S} a_S = \frac{\hbar}{2\pi k_B c_S} \frac{1}{2} \left| \frac{\partial (c_S^2 - v^2)}{\partial x} \right|_{x=x_H}$$

The artificial "surface gravity" a_S is essentially the "spatial acceleration" of the fluid as it crosses the horizon.

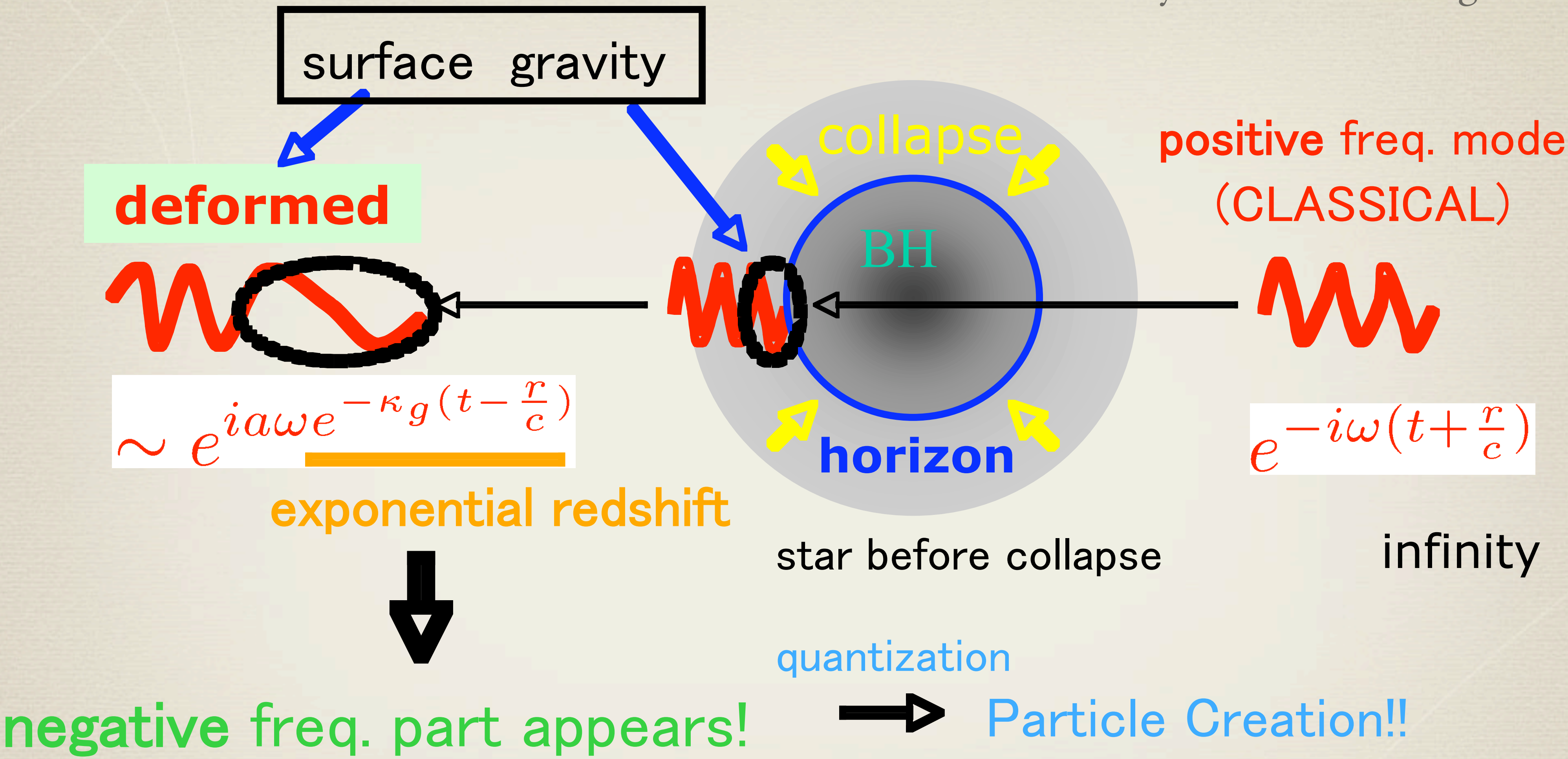
$$T_H \simeq 10^{-8} K$$

Hawking temperature is too feeble to be detectable in water !!!

=> Need for quantum fluids

Positive and Negative Frequencies
Mode Mixing

Courtesy Okuzumi & Sakagami



=> Classical Ingredient of Hawking Radiation :

Mode Conversion via Spontaneous

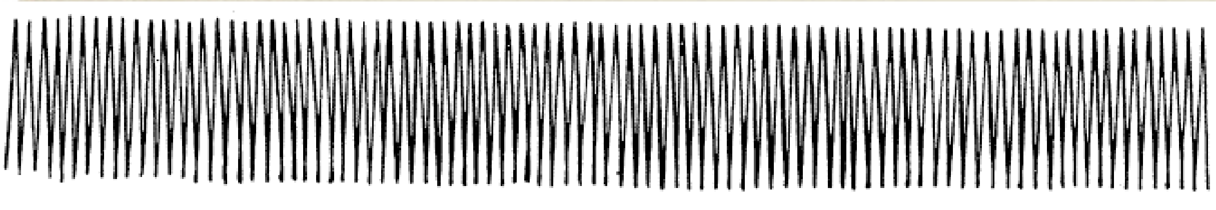
or Stimulated Emission

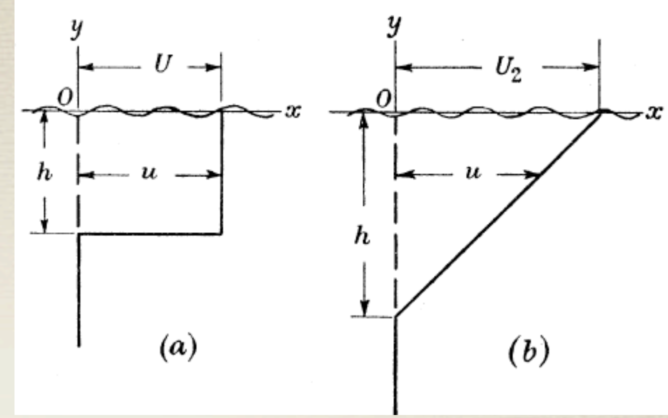
A Review of Wave-Current Interaction

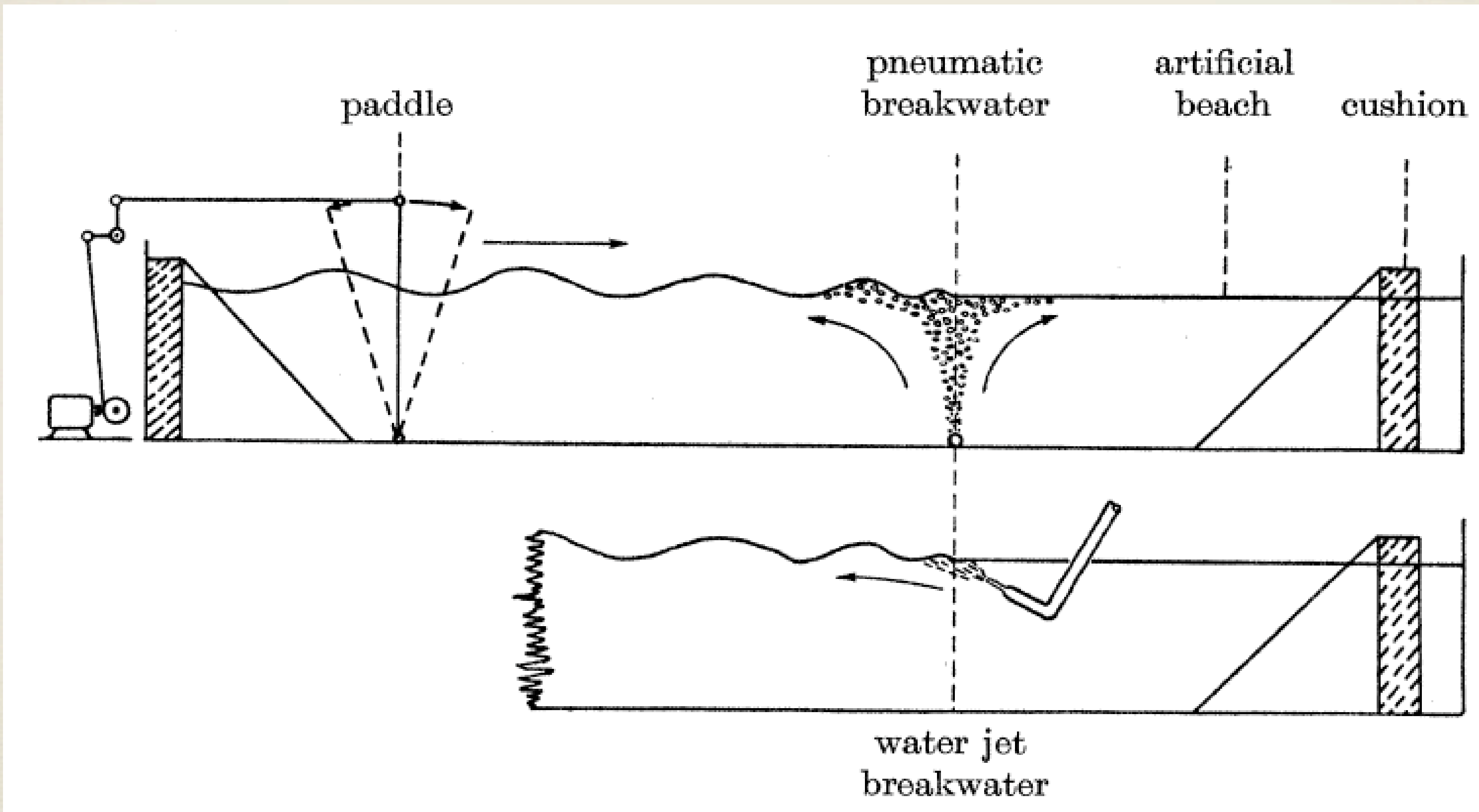
River Outlet and Wave-Breaking

Chawla & Kirby, 2002

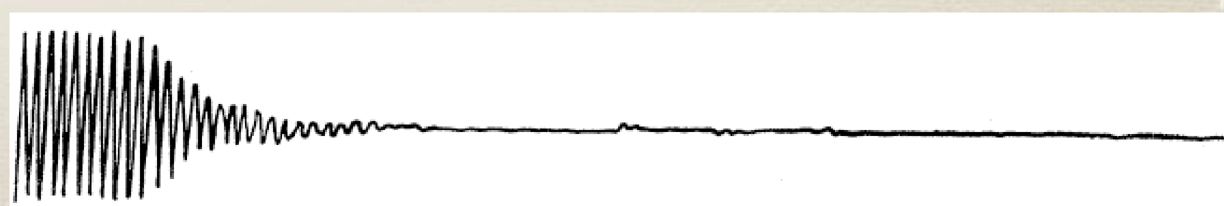
Pneumatic Breakwater (1955)







J.T. Evans (Exp.)
G.I. Taylor (Theo.)



Rayleigh Equation and Dispersion Relation

$$\frac{d^2w}{dz^2} - \left(k^2 + \frac{1}{U-c}\frac{d^2U}{dz^2}\right)w = 0 \quad -h \leq z \leq 0$$

$$(U-c)^2\frac{dw}{dz} = \left[g + (U-c)\frac{dU}{dz}\right]w \qquad z = 0$$

$$w = 0 \qquad z = -h$$

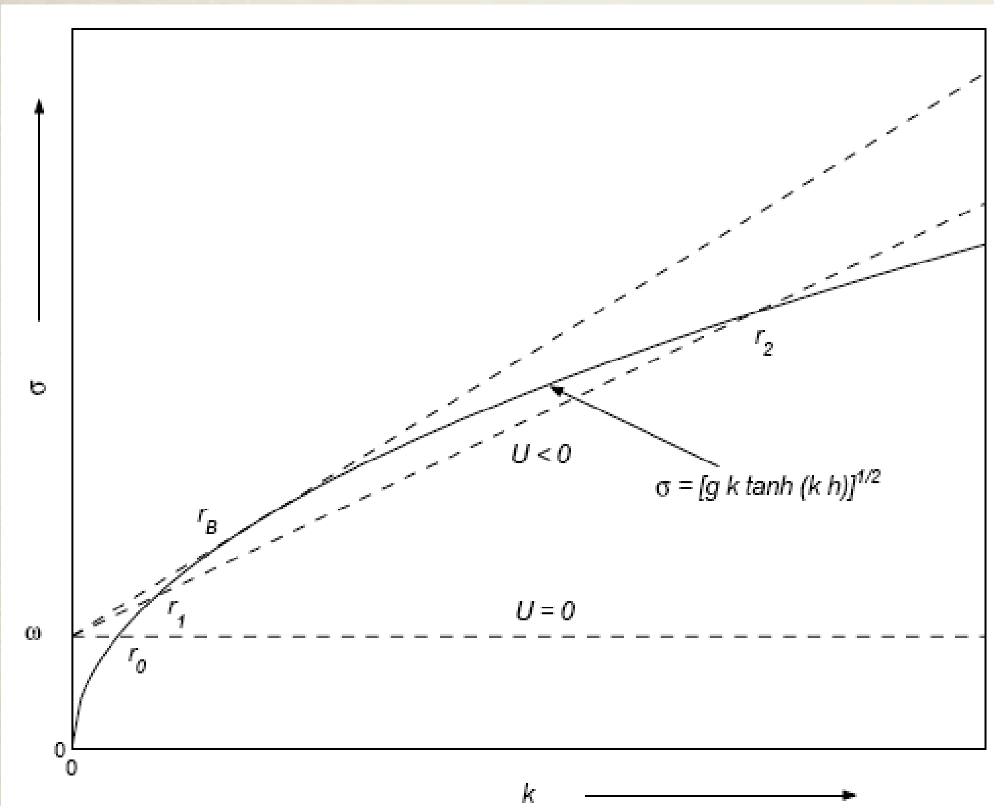
$$U = constante$$
$$(\omega - kU)^2 = gk\tanh(kh)$$

$$U = U_0 + \Omega z$$
$$(\omega - kU_0)^2 = [gk - \Omega(\omega - kU_0)]\tanh(kh)$$

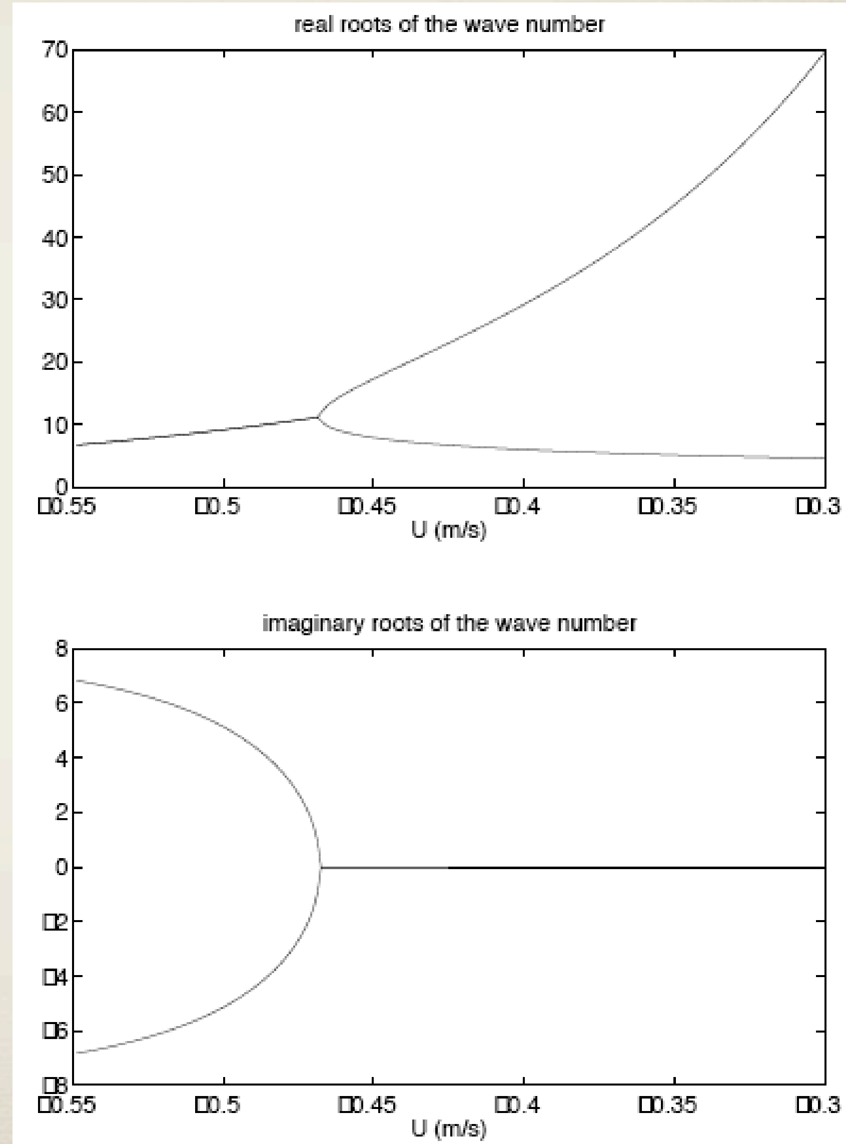
Classical View of the Fluid Mechanics Community

Chawla & Kirby, 2002

$$(\omega - kU)^2 = gk \tanh kh$$



$$\sigma^2 = gk \tanh q$$

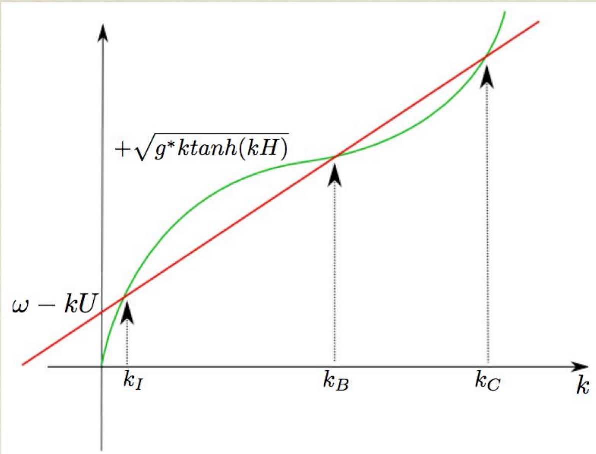


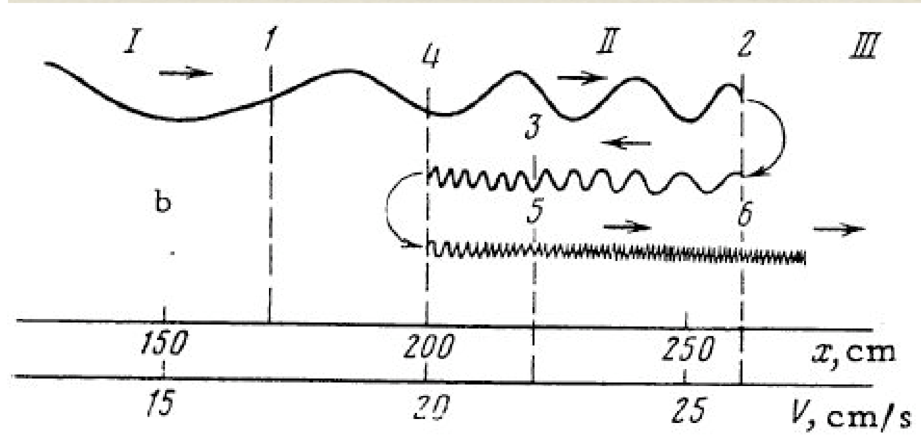
$$k = \frac{2U\omega + g \pm g\sqrt{1 + \frac{4U\omega}{g}}}{2U^2}$$

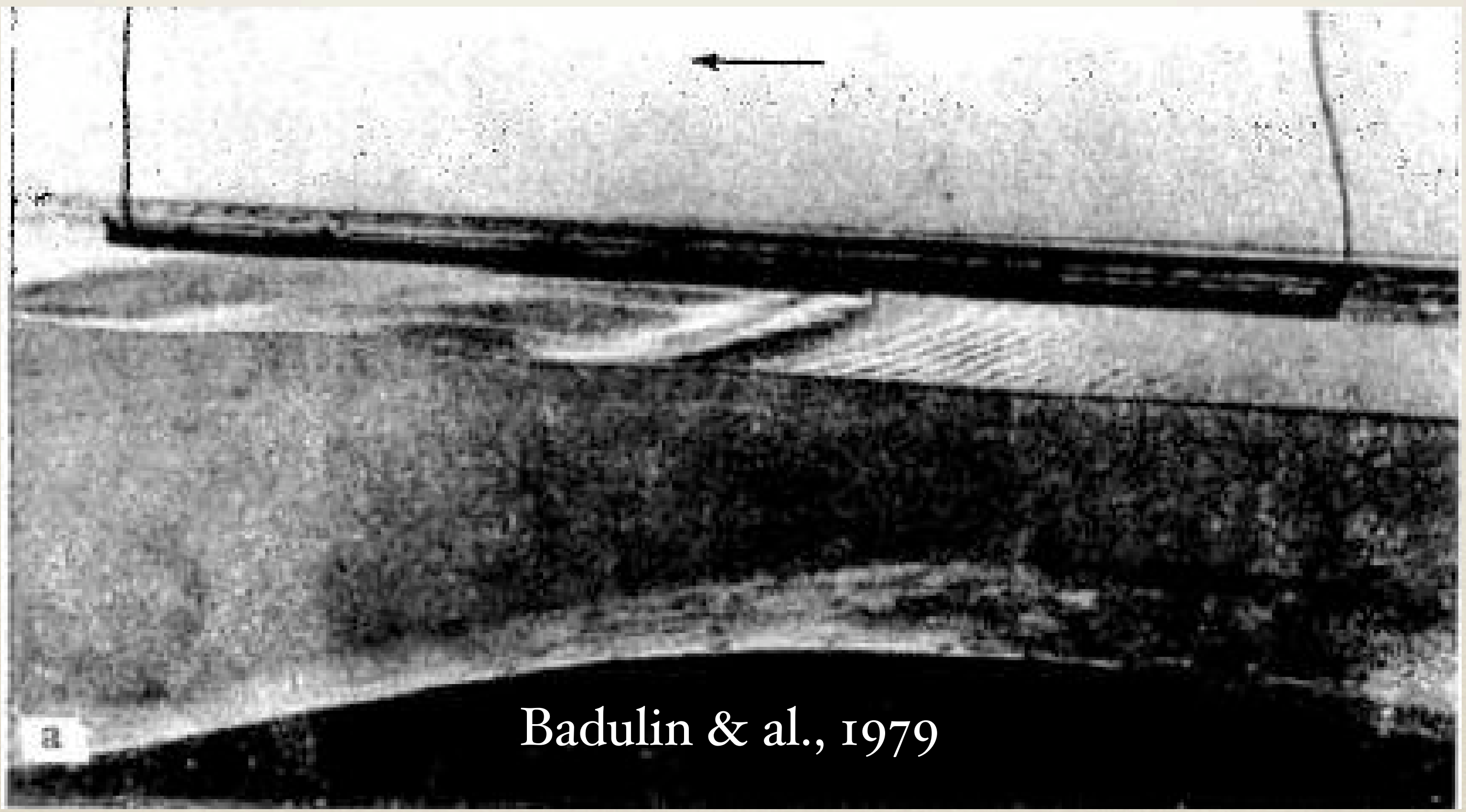
Deep Water

$$U < -\frac{g}{4\omega}$$

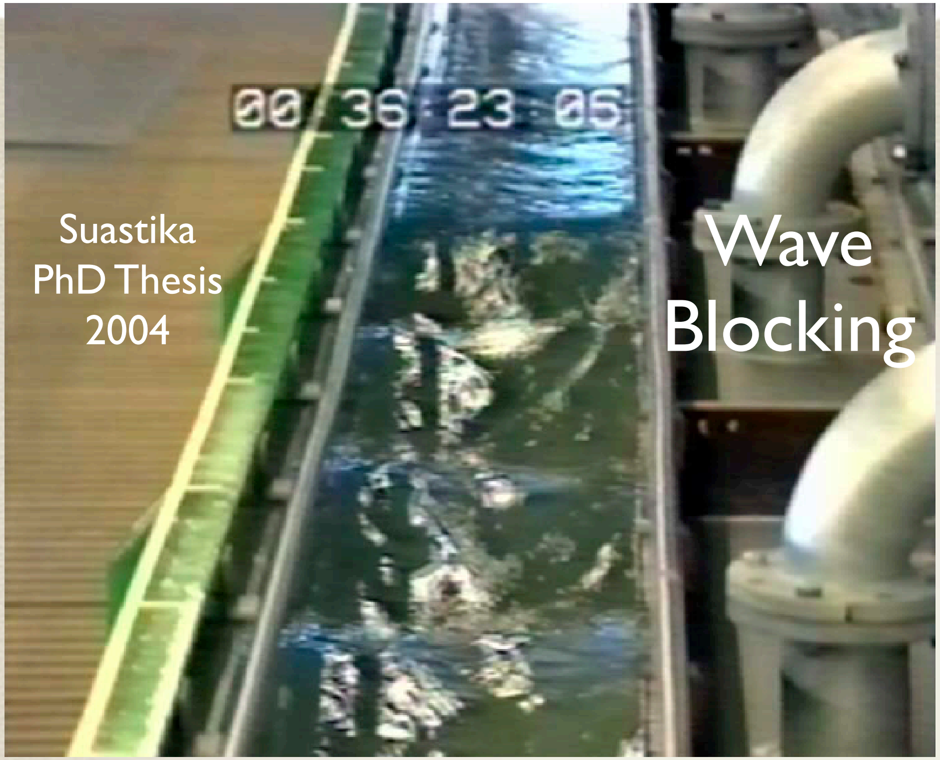
Capillary effects

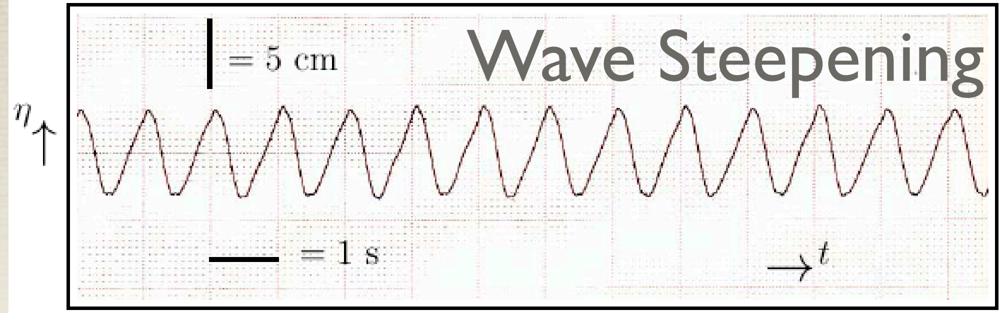




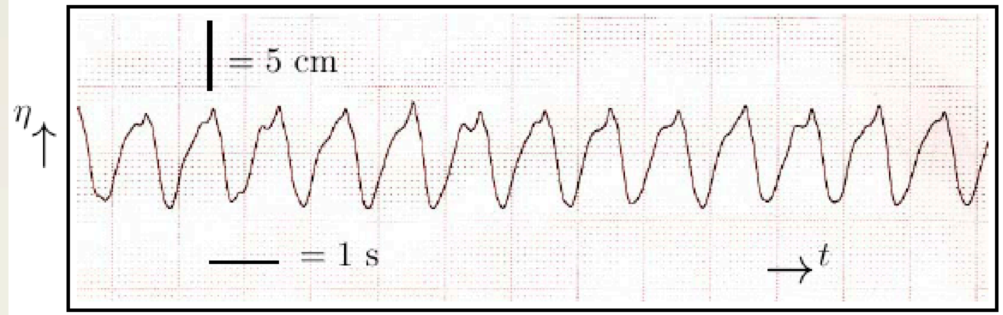


Badulin & al., 1979

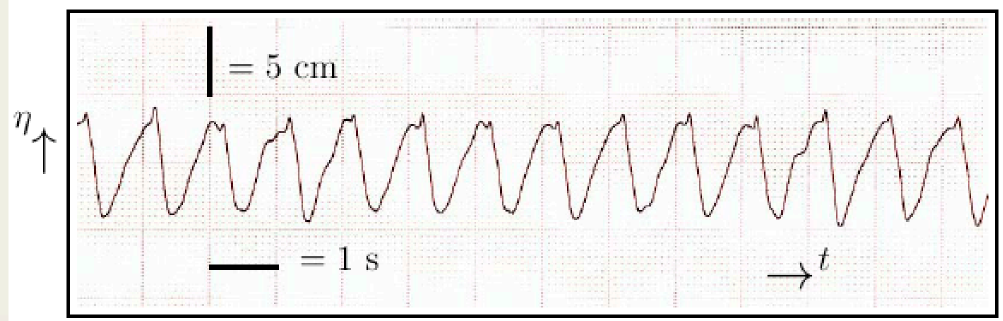




$x = 9.0$ m

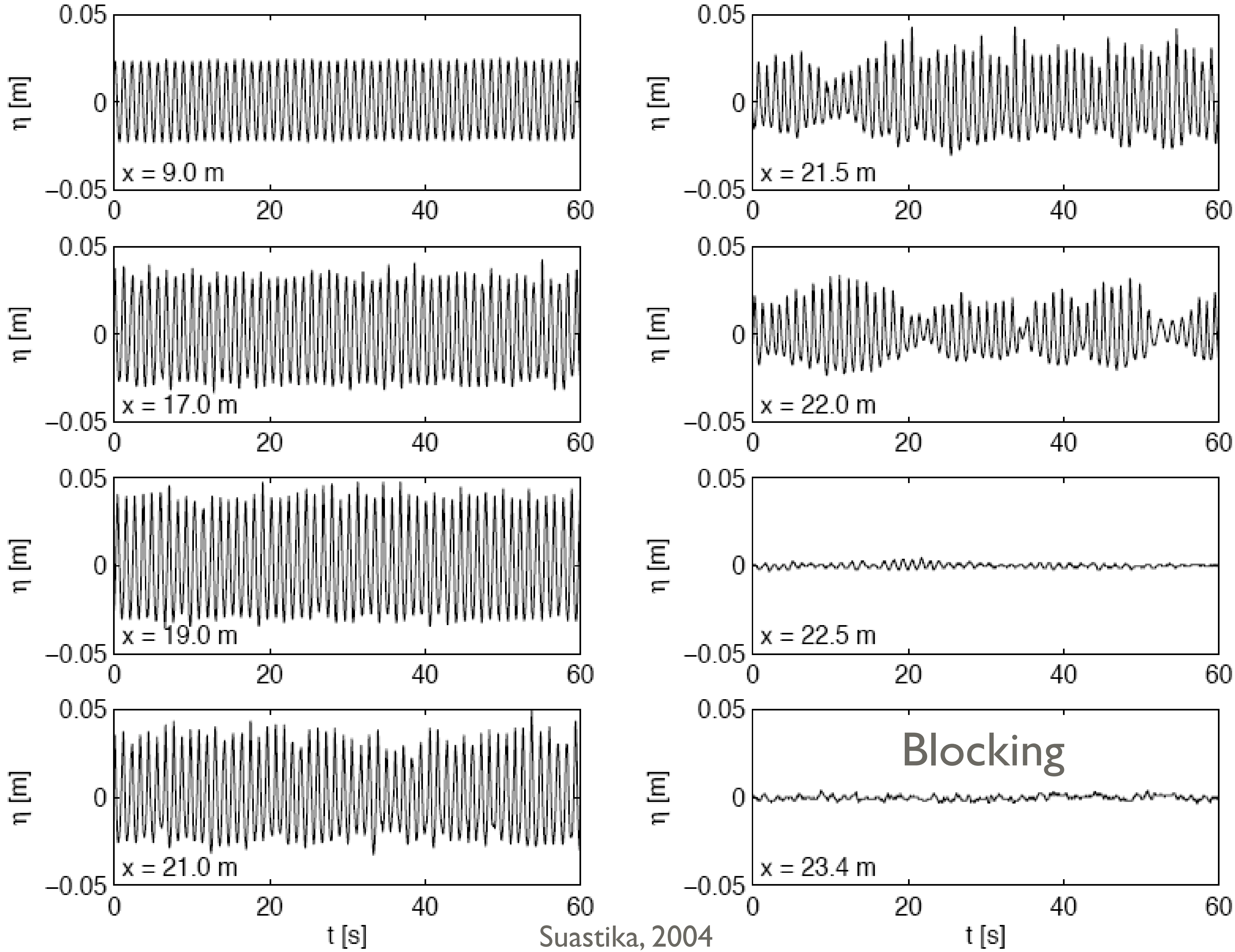


$x = 13.0$ m

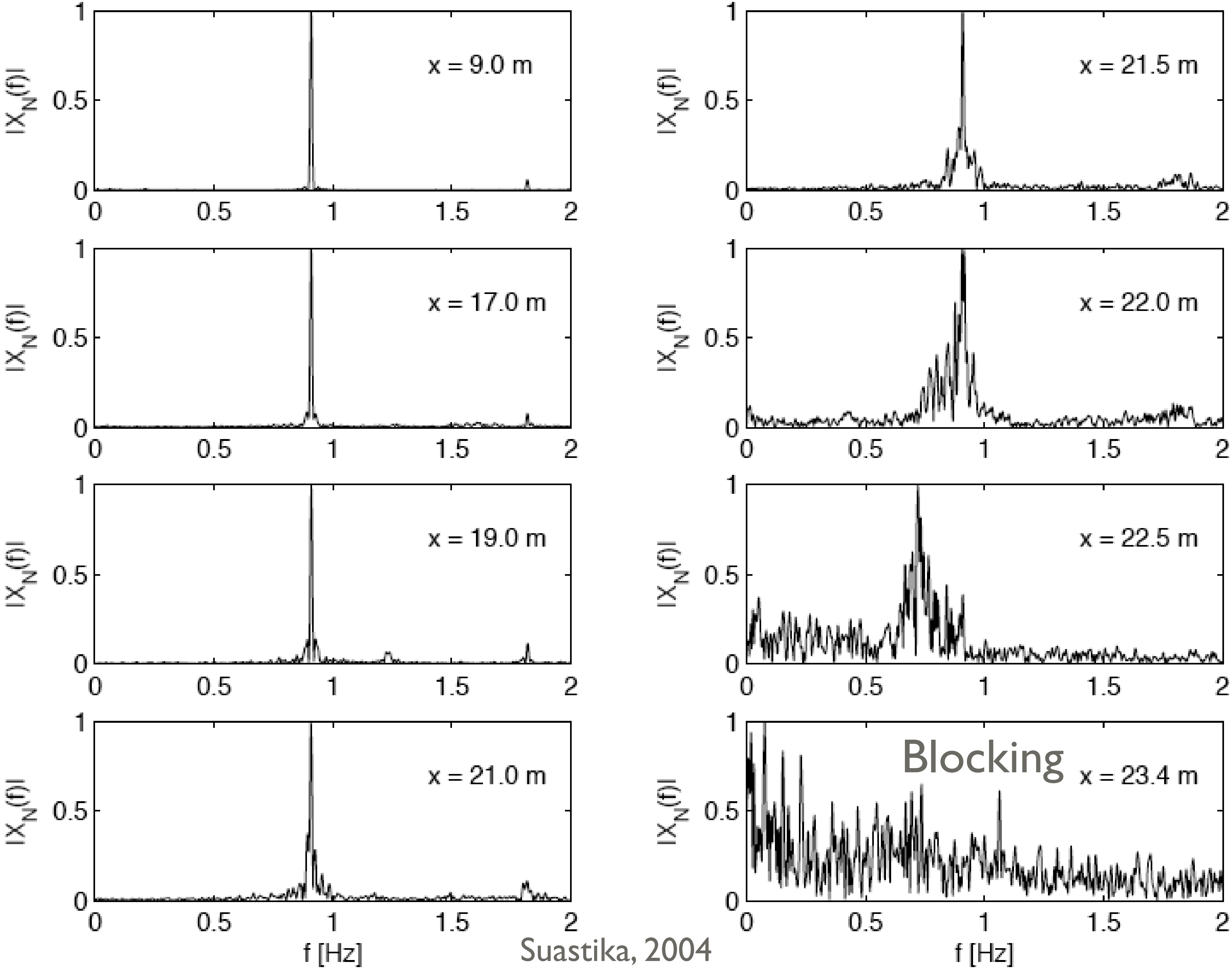


$x = 15.0$ m

$x = 17.0$ m

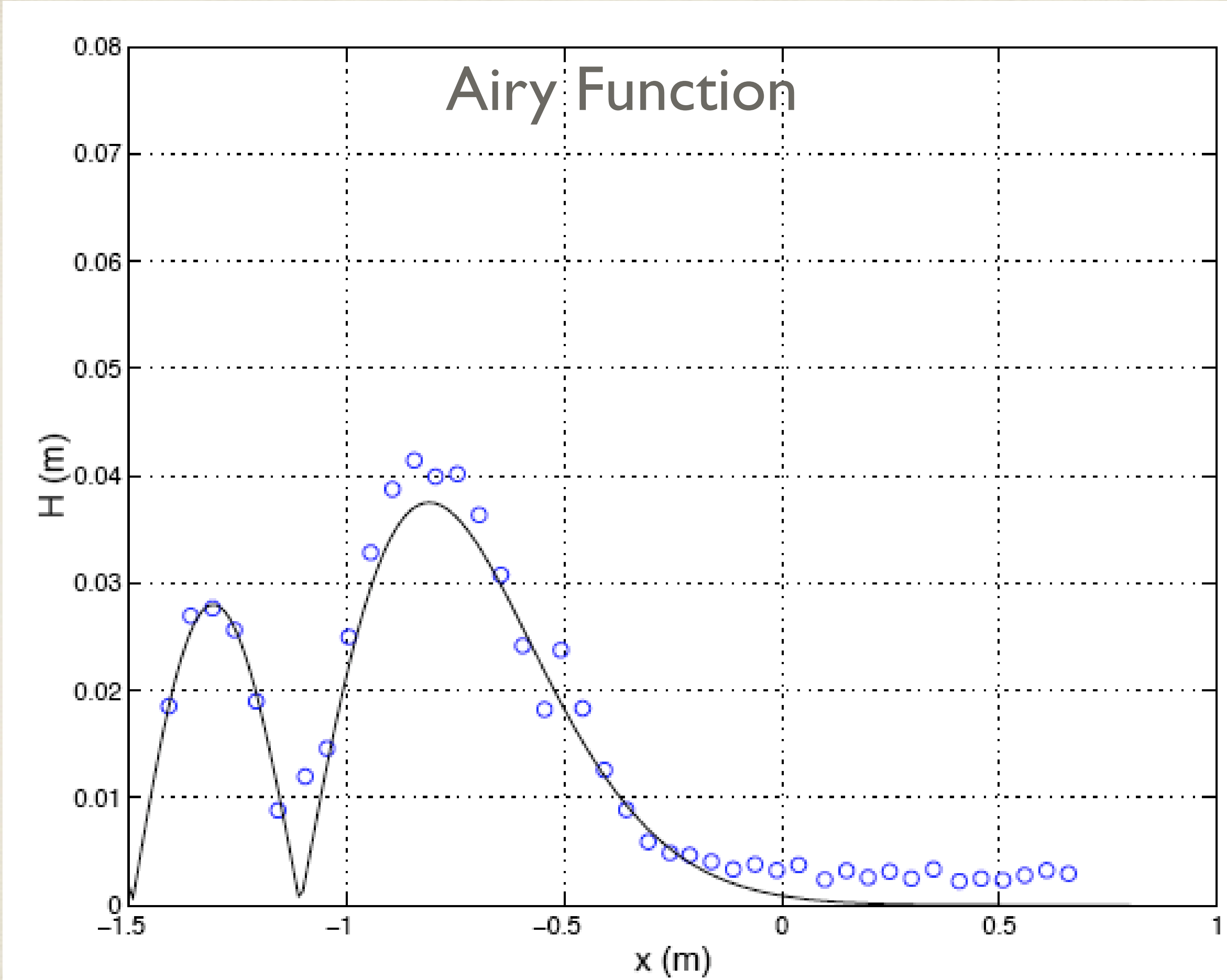


Suastika, 2004



Suastika, 2004

Wave Envelope near the Blocking Line

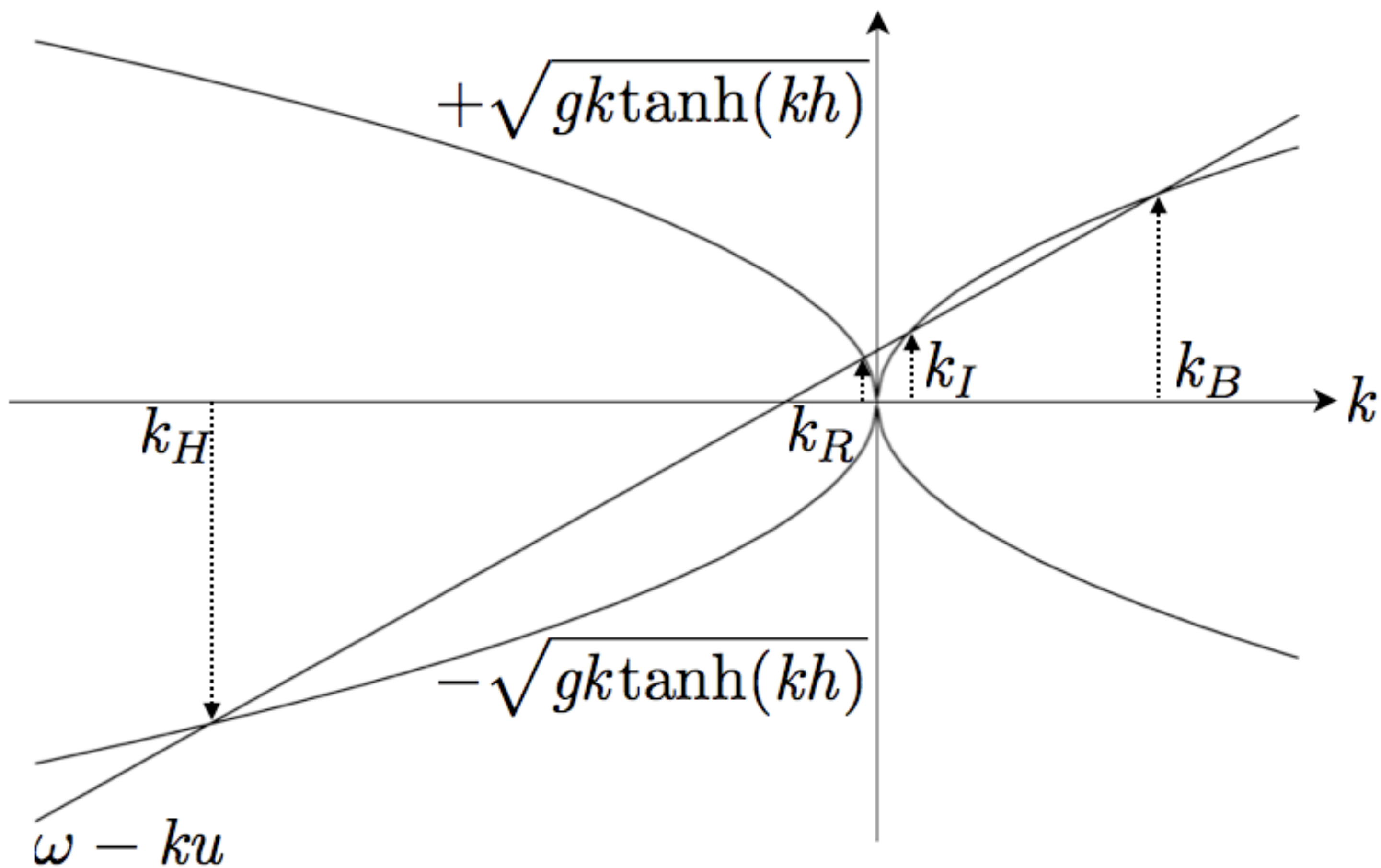


Chawla & Kirby, 2002

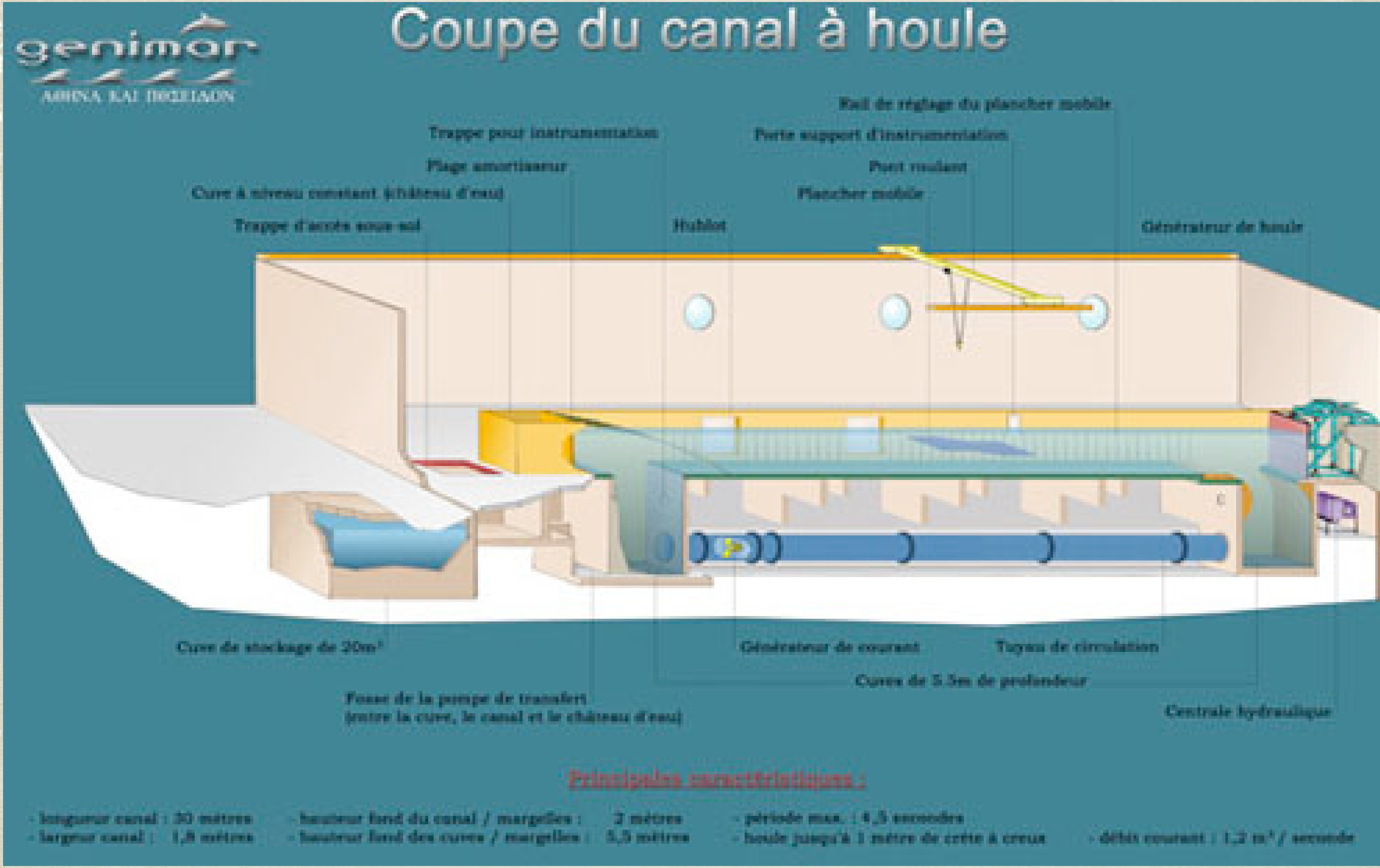
Nice Experiments in Nice

The Full Dispersion Relation and its solutions (4 or 2)

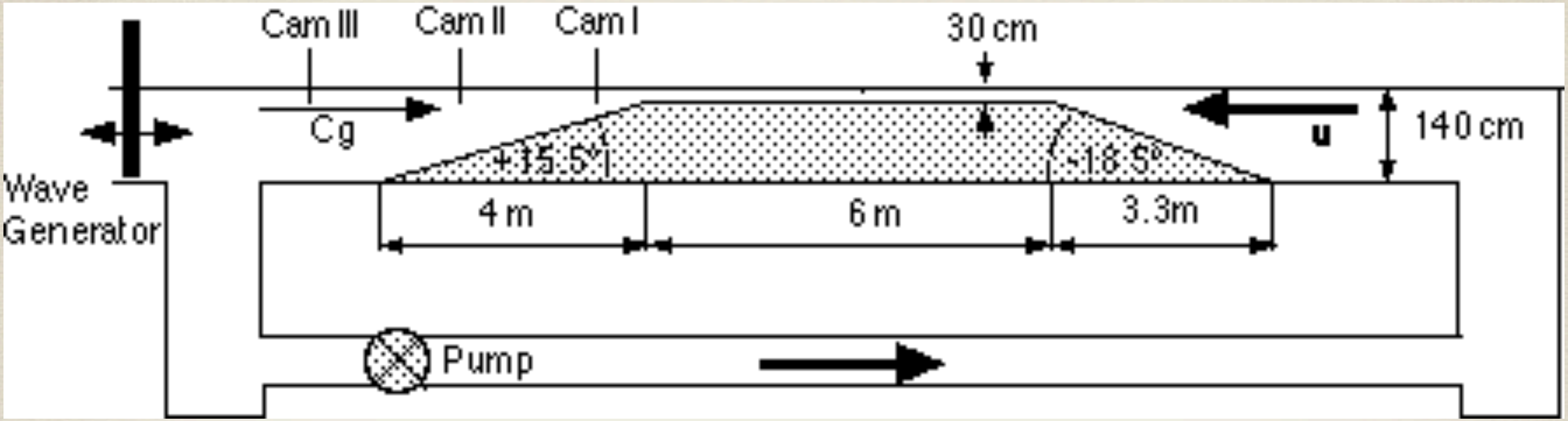
$$(\omega - kU)^2 = gk\tanh(kh)$$

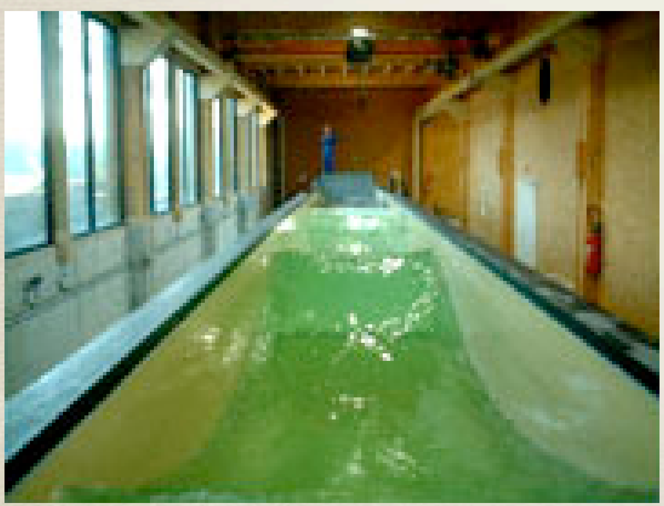


ACRI : GENIMAR Laboratory



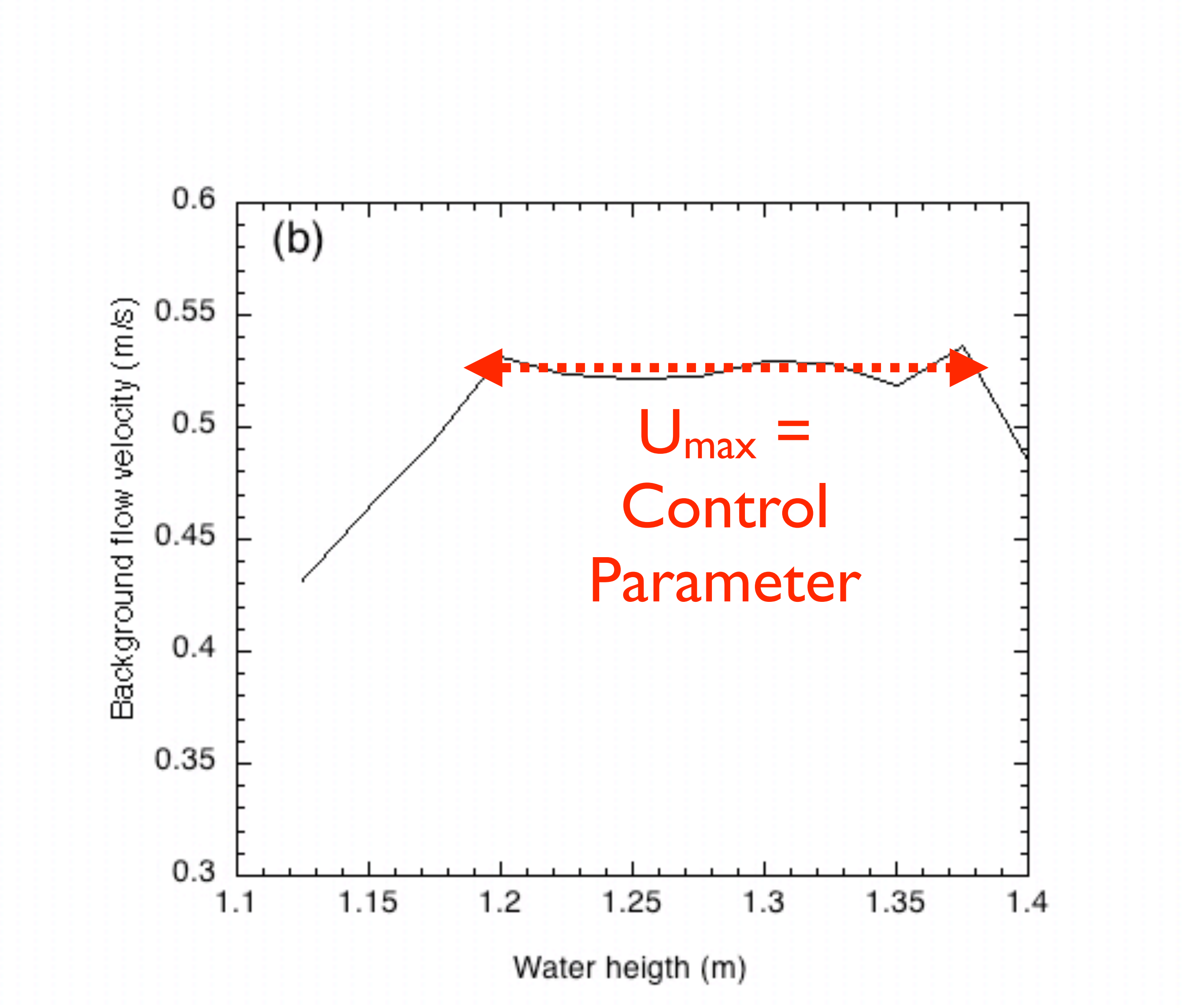
The Ramp


Cam III
Cam II
Cam I
30 cm
Cg
u
140 cm
+15.5°
-18.5°
Wave Generator
4 m
6 m
3.3m
Pump



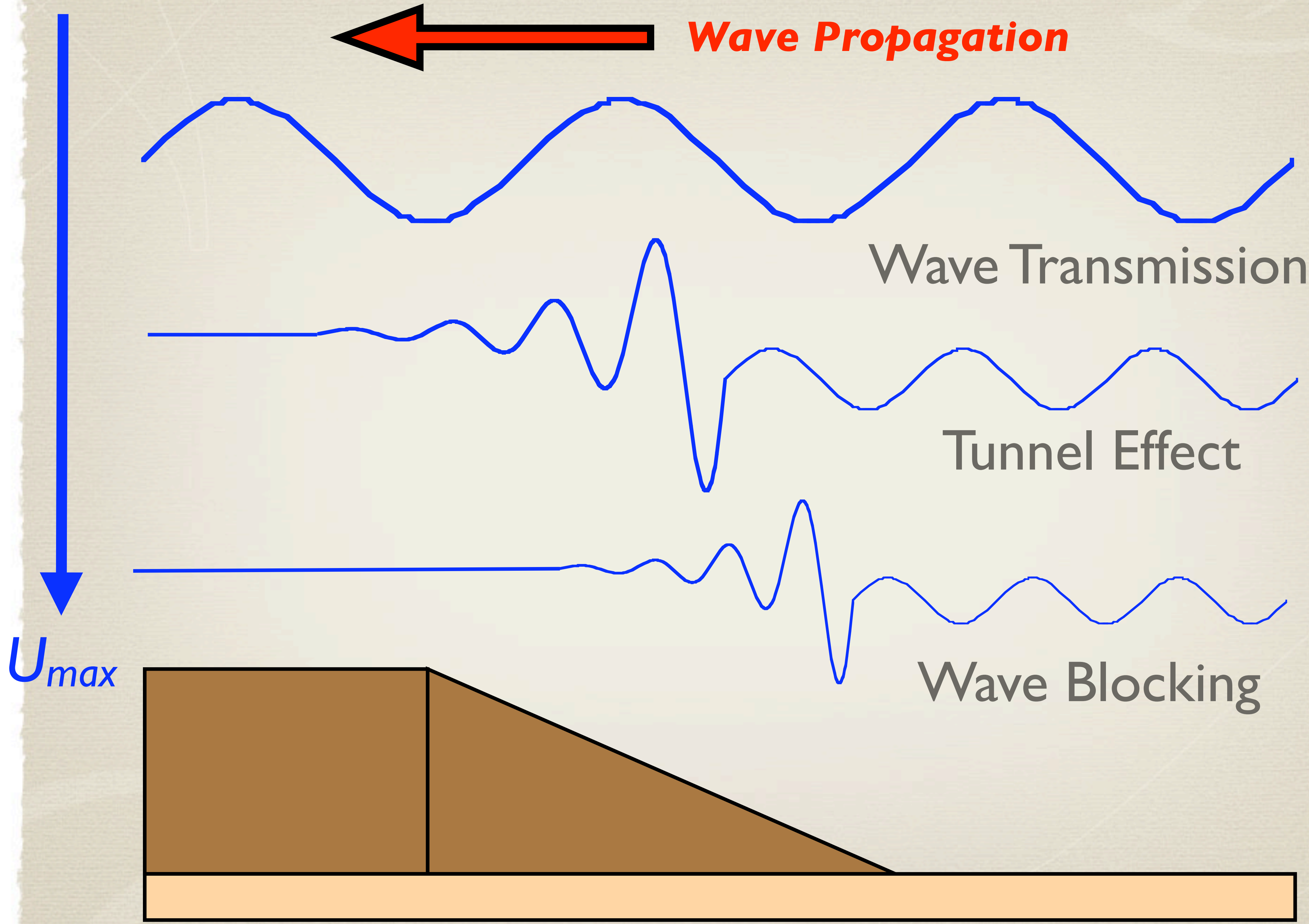
Experimental Setup

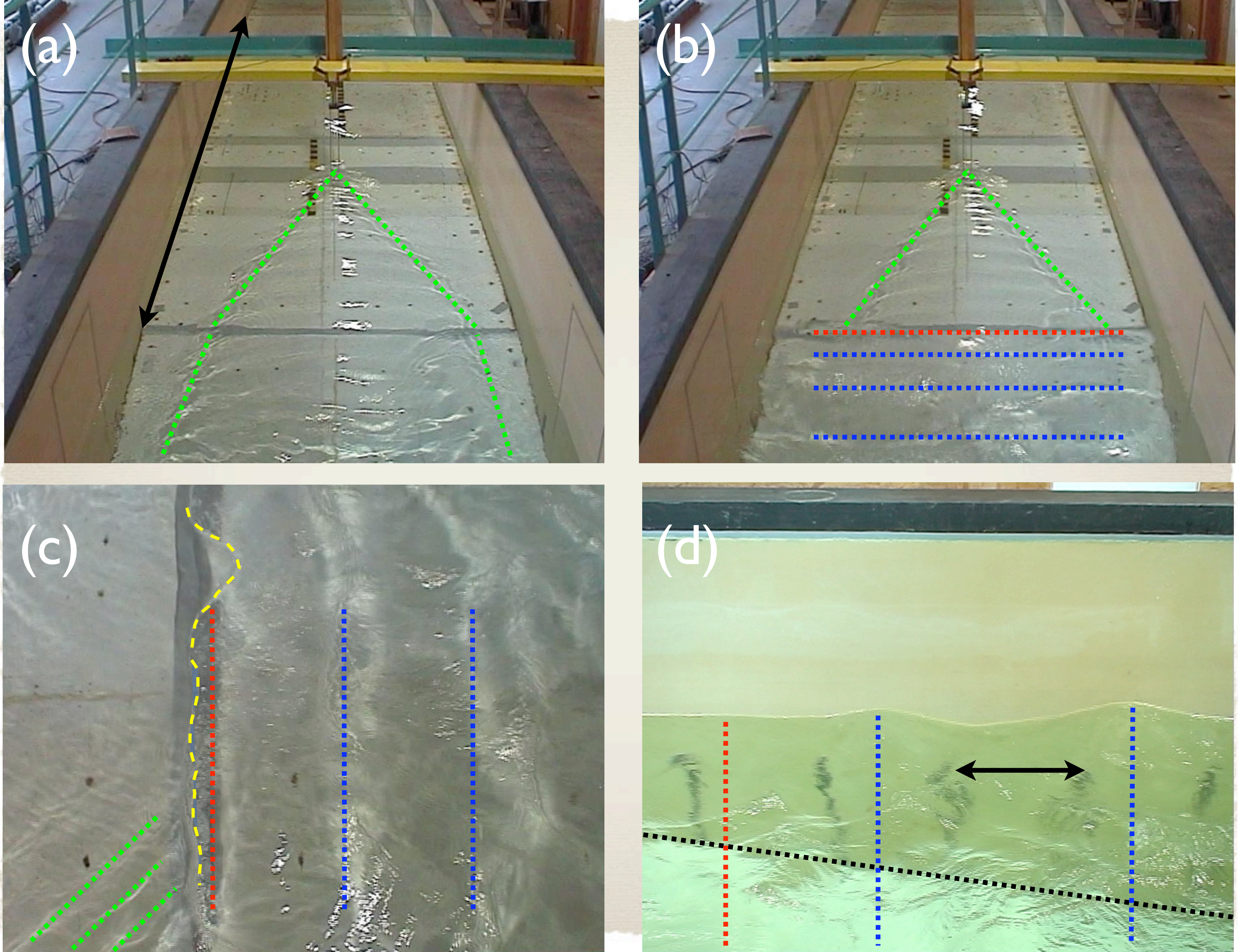
Plug Flow



Flat Part without Fluctuations

Three Experimental Regimes

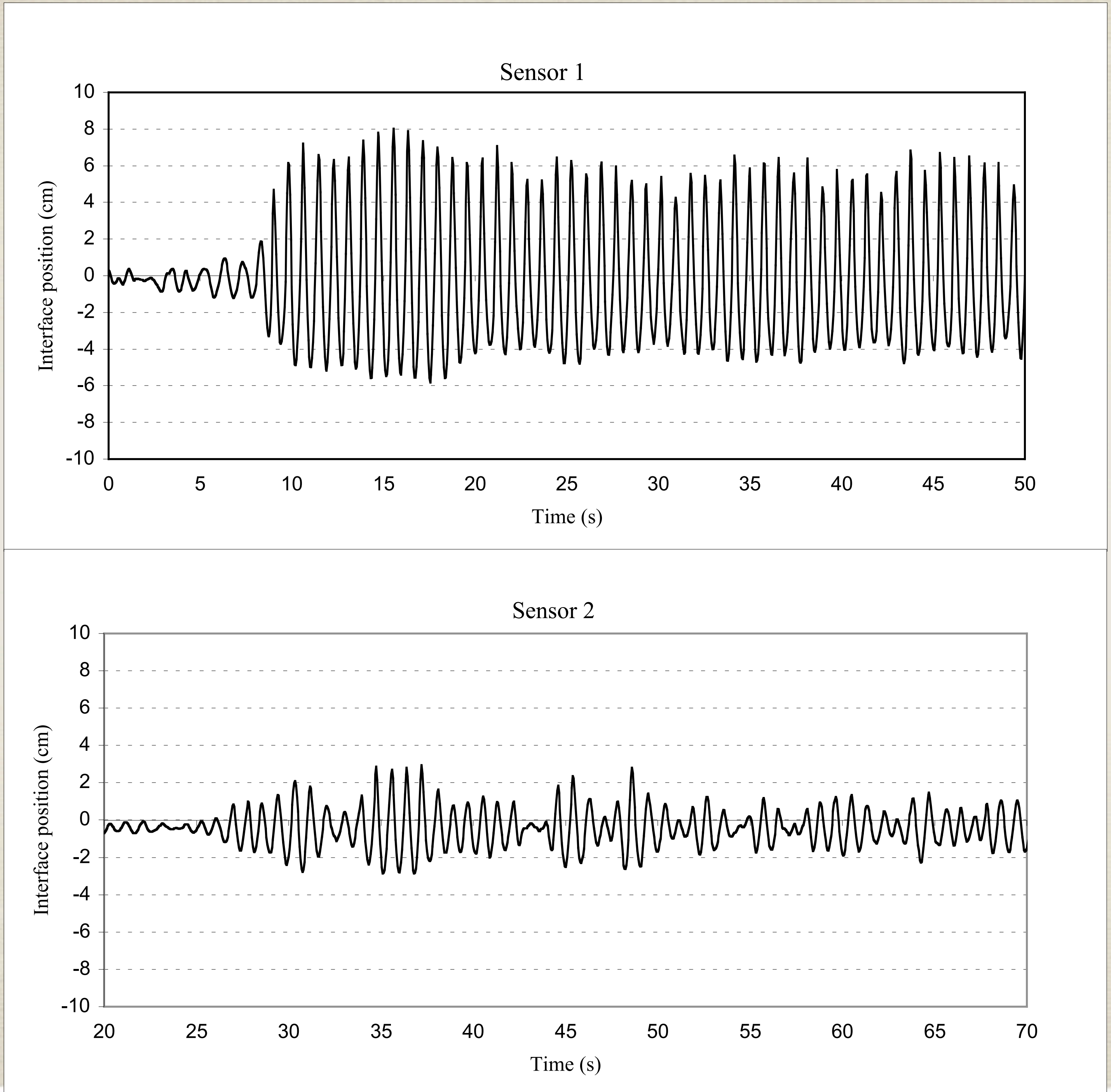



(a)
(b)
(c)
(d)

Transmission and Reduction

Ascending Slope for the Waves

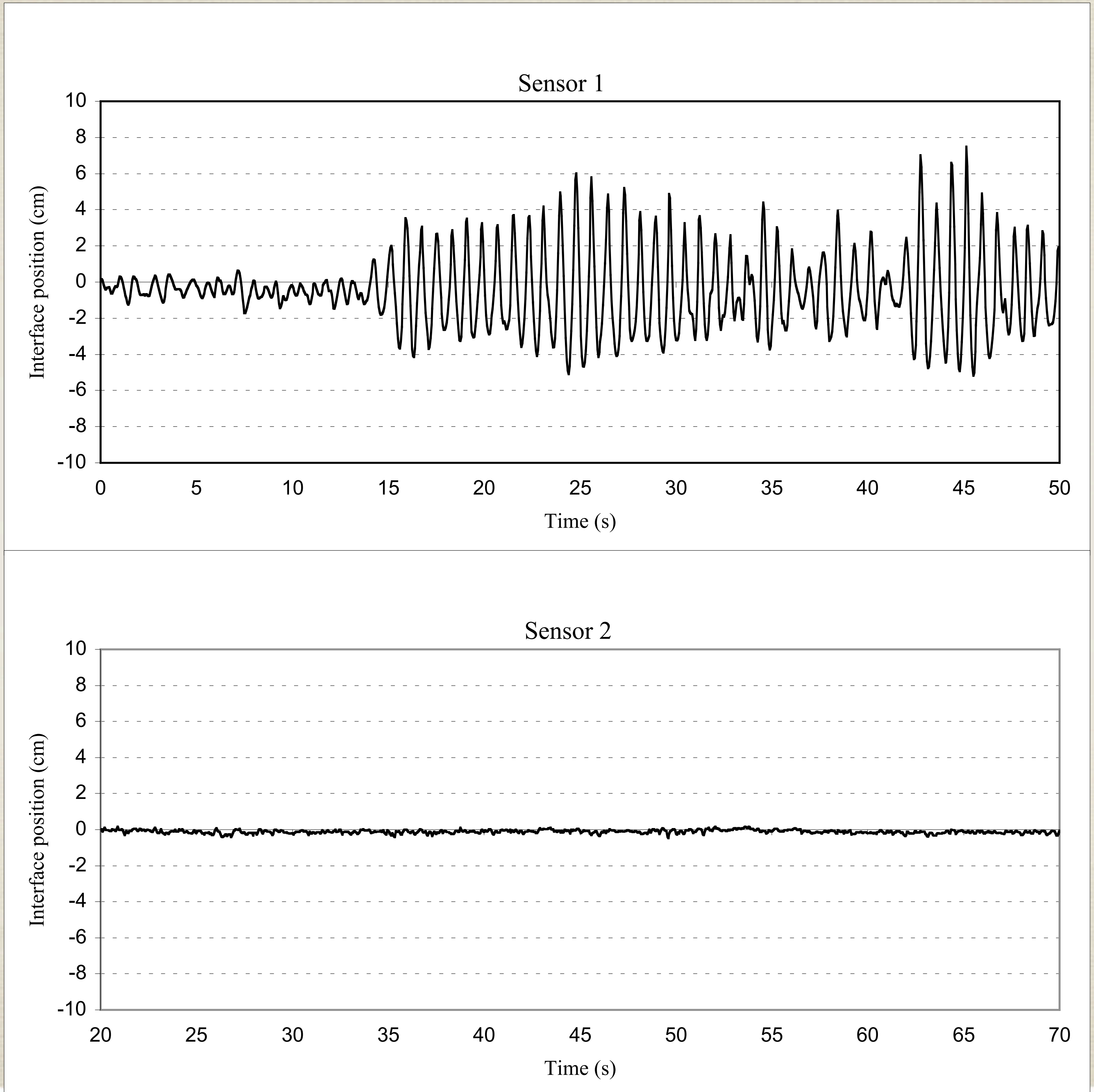
Flat Part



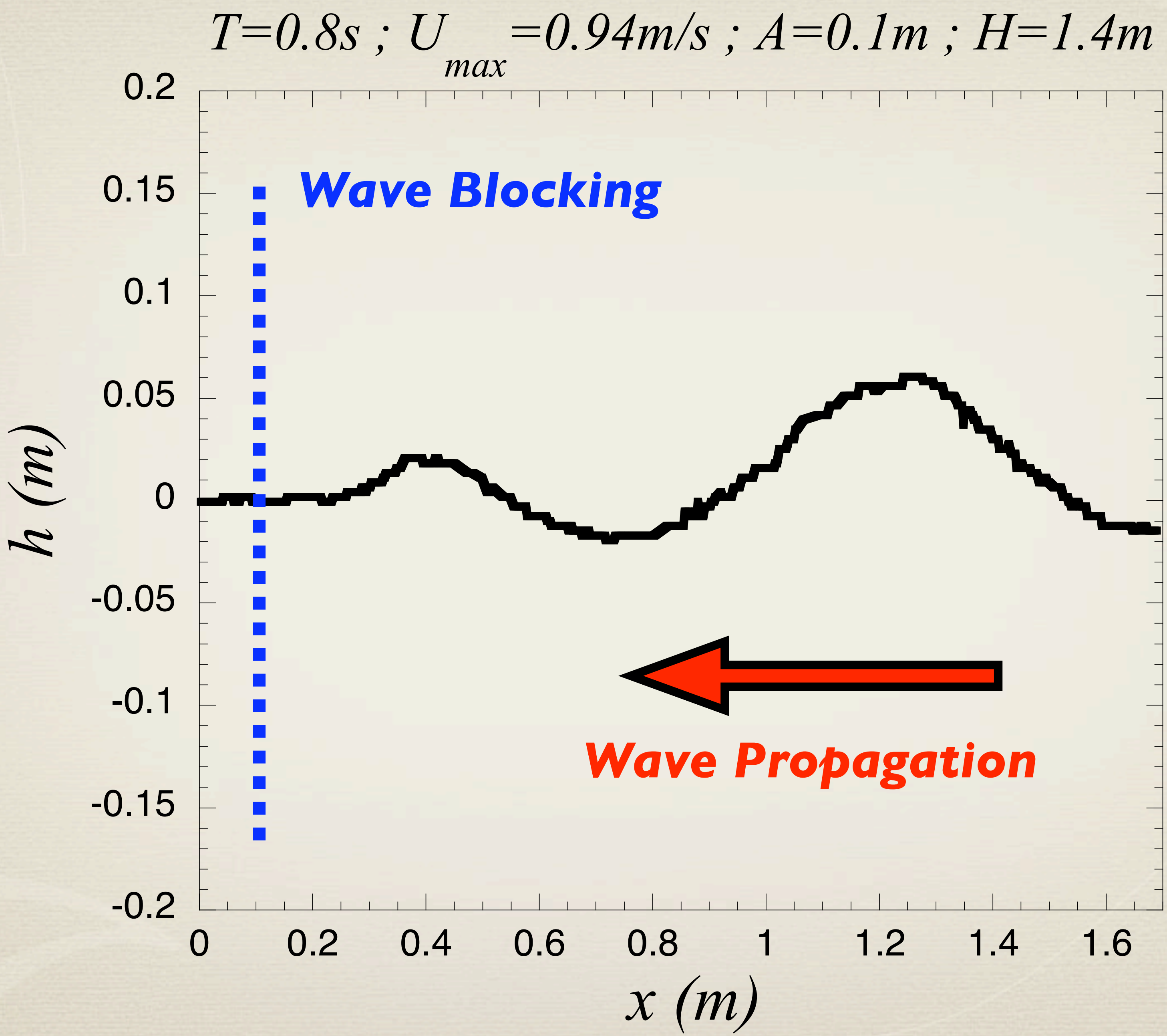
Wave Blocking and Beating

Ascending Slope for the Waves

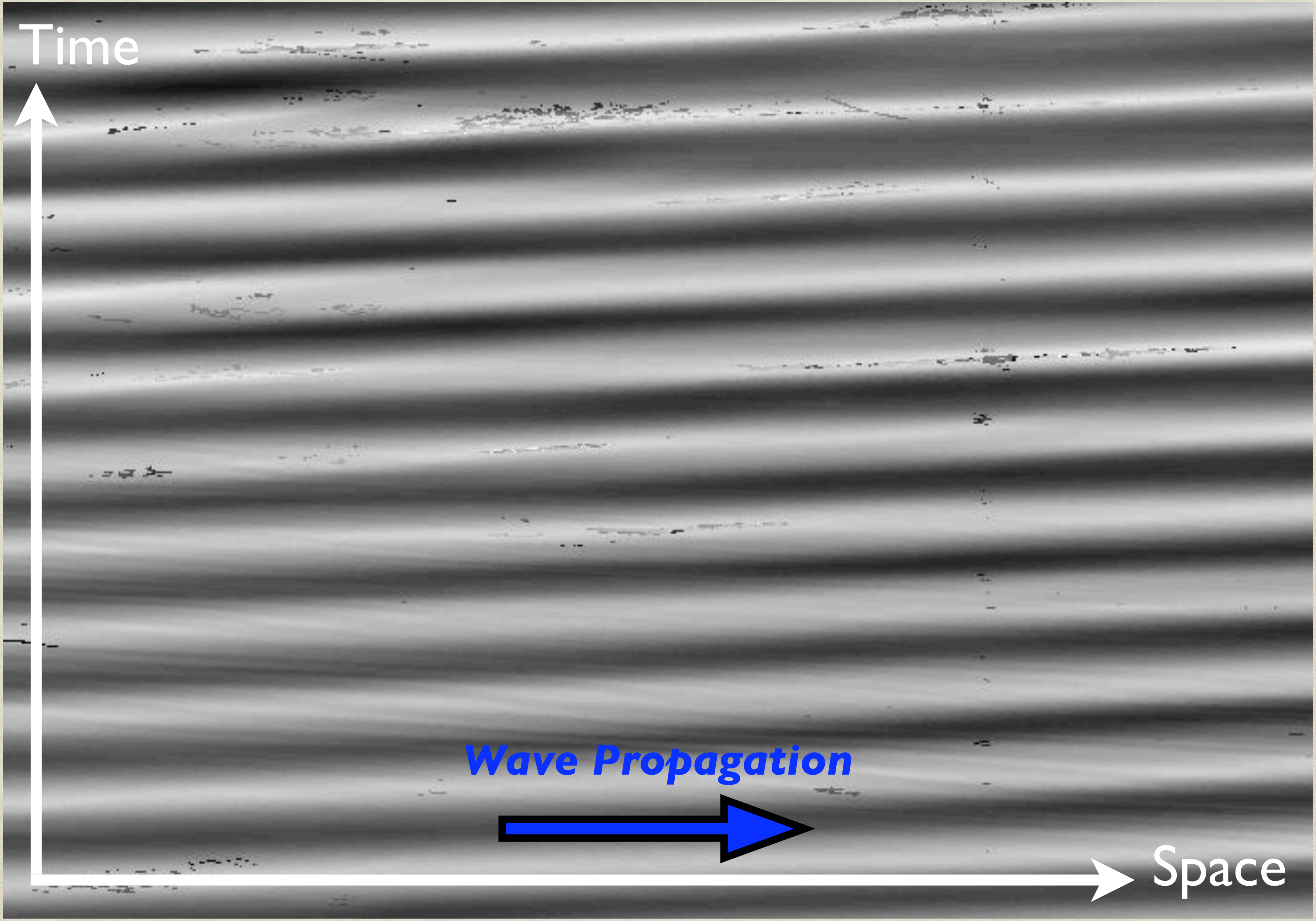
Flat Part



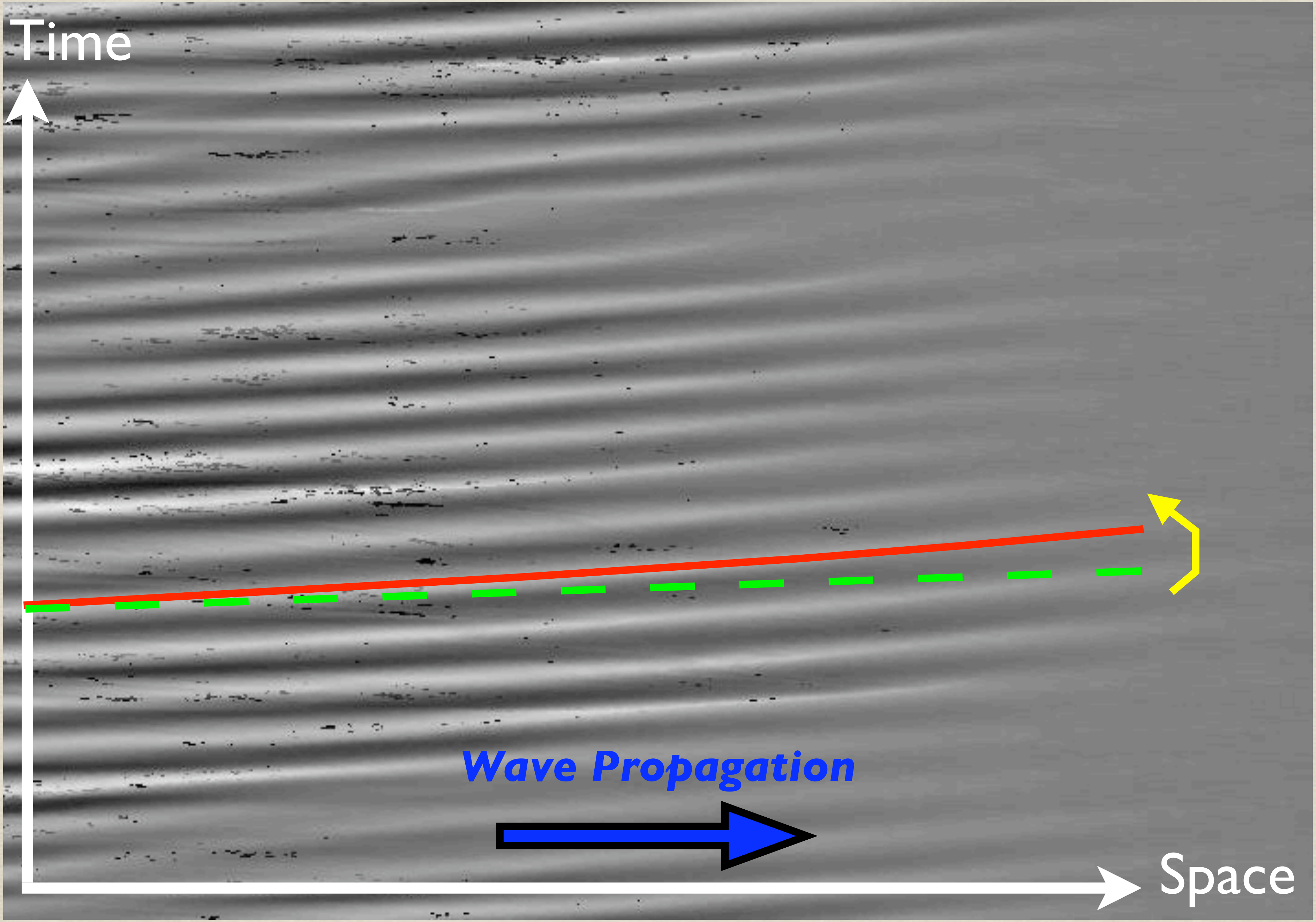
Experimental White Horizon



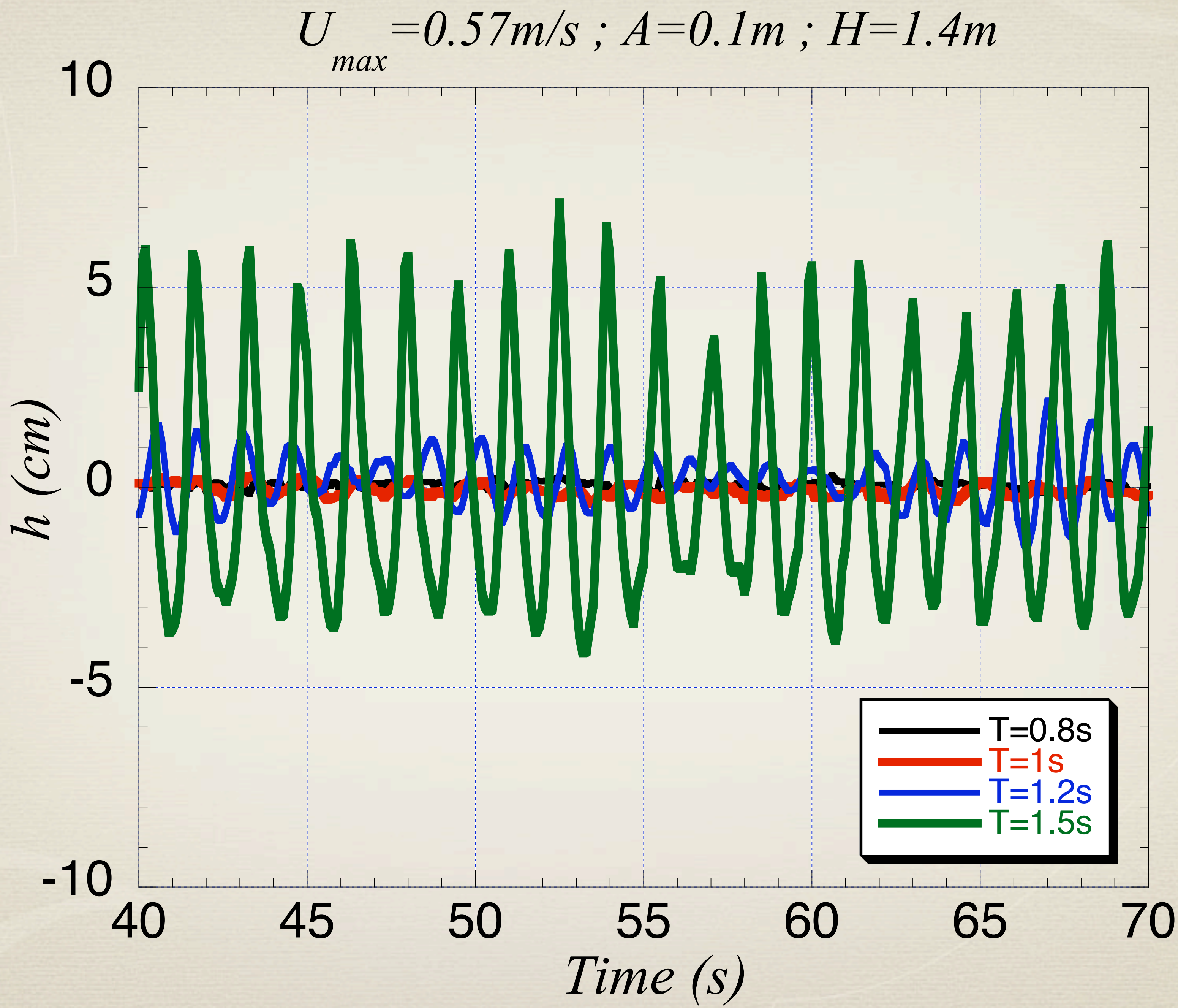
Wave Steepening (no blocking line)

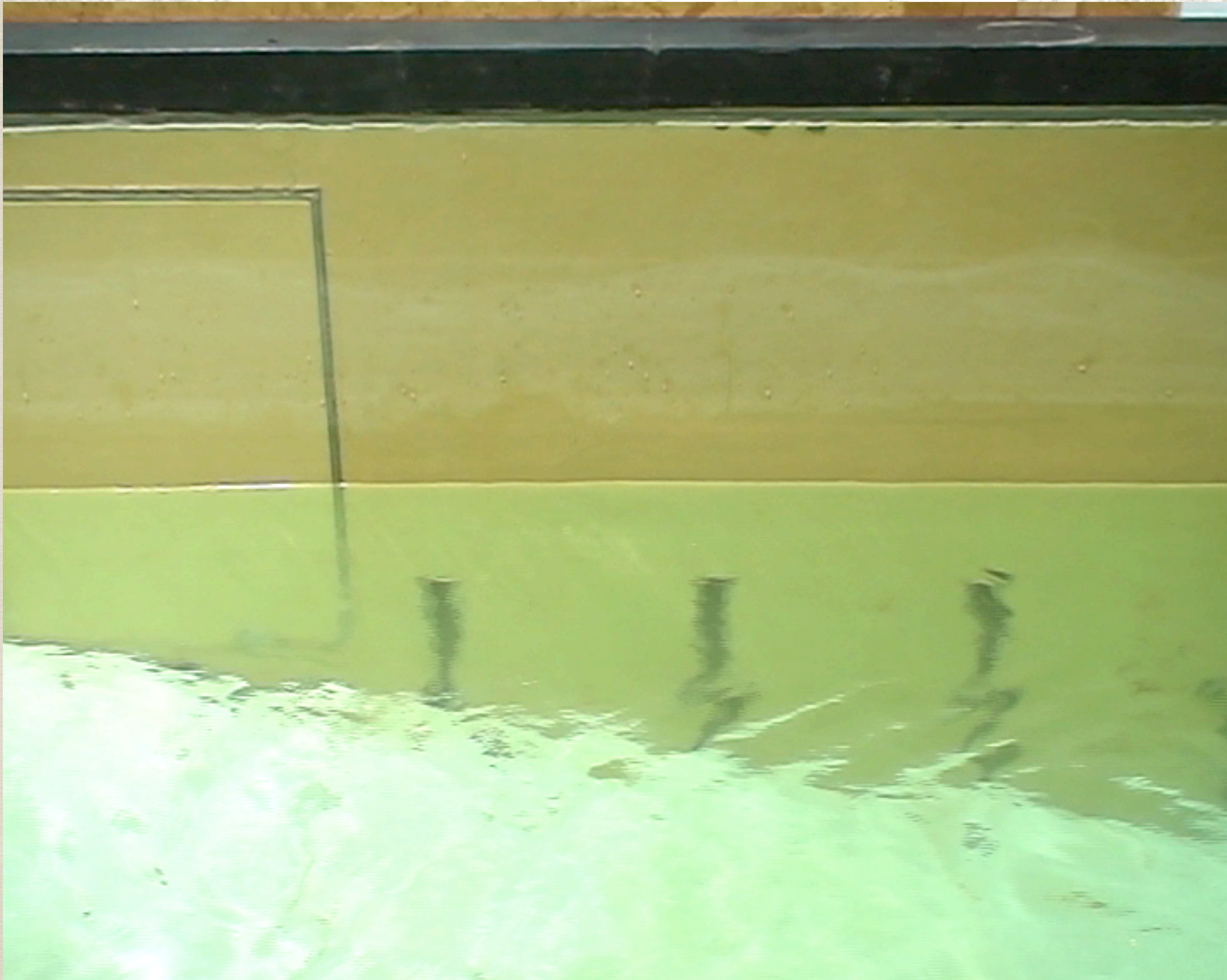


Phase Velocity at the Horizon

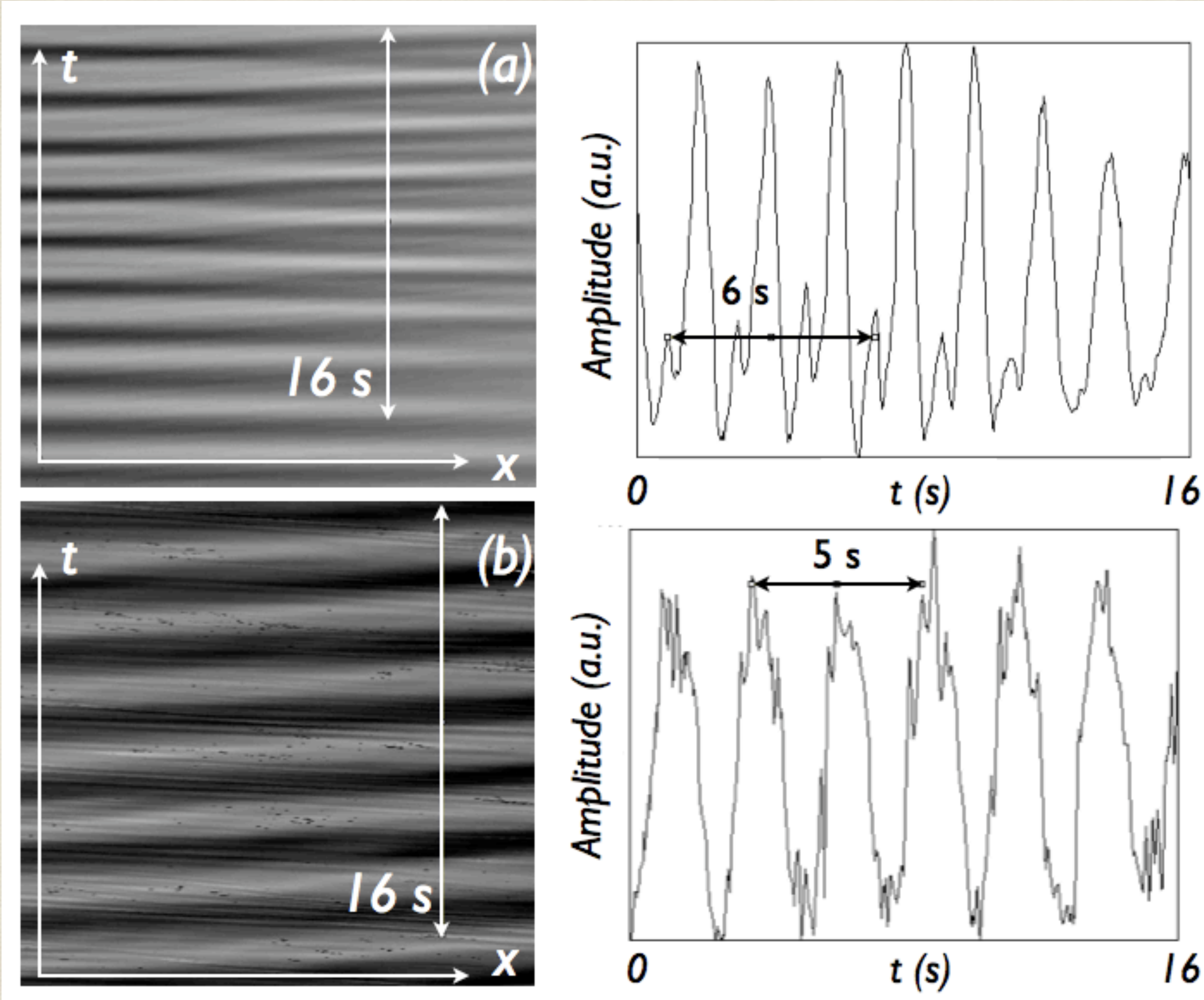


Influence of Dispersion





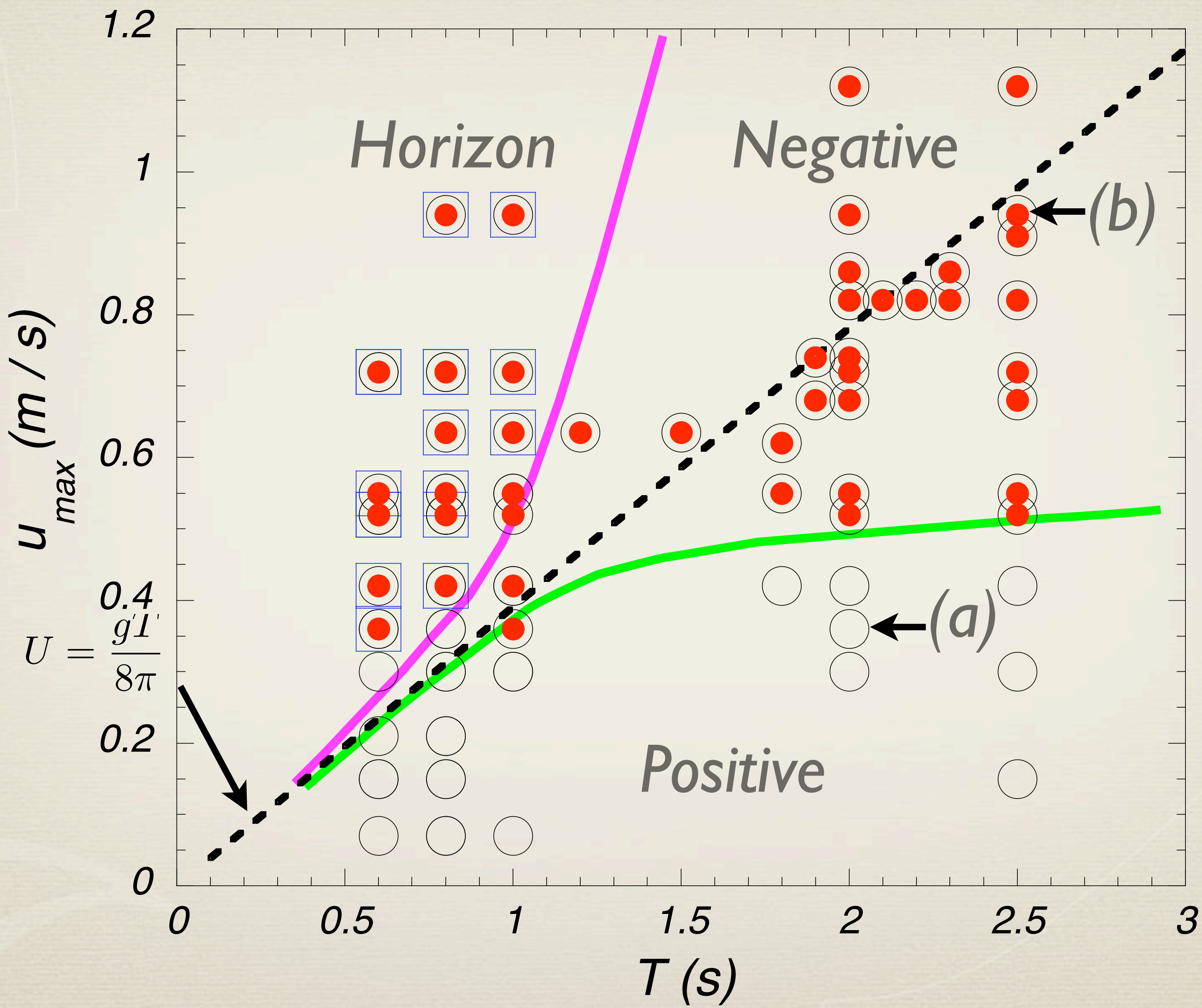
Experimental Mode Conversion



Ascending Slope for the Waves

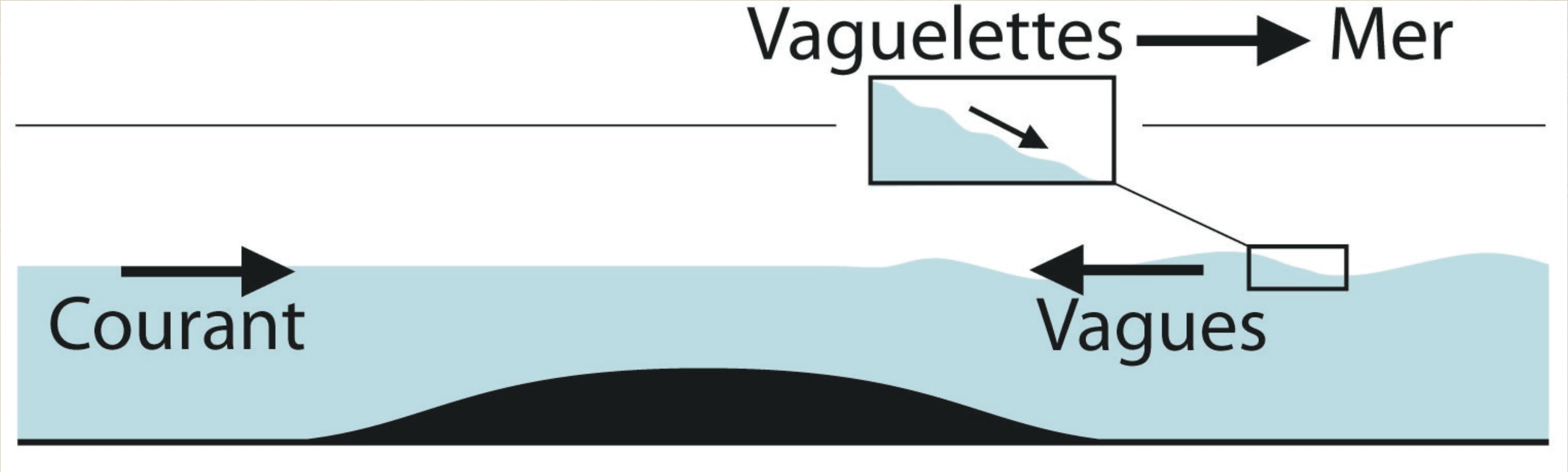
The Experimental Phase-Space

Rousseaux et al. 2008, New Journal of Physics

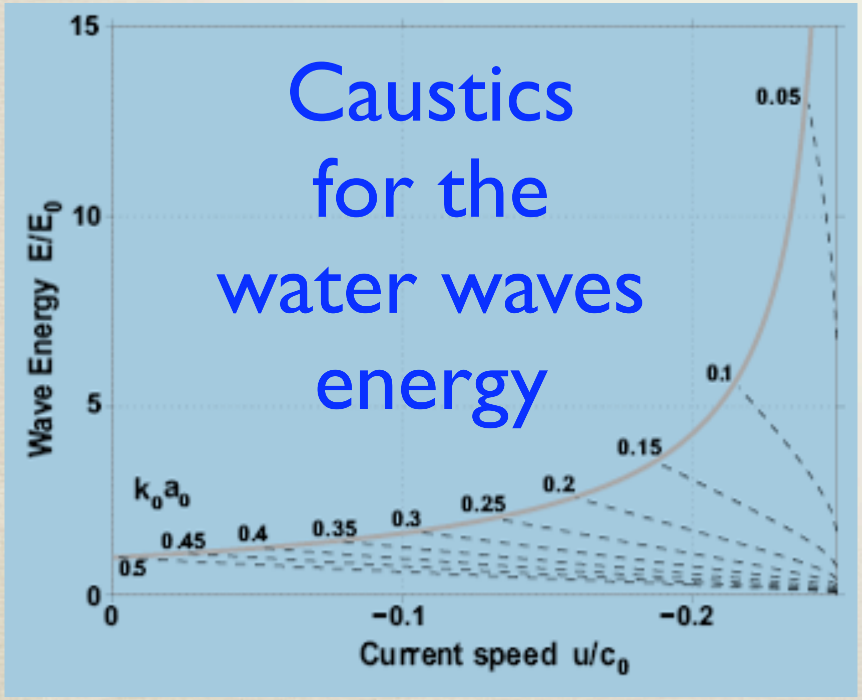


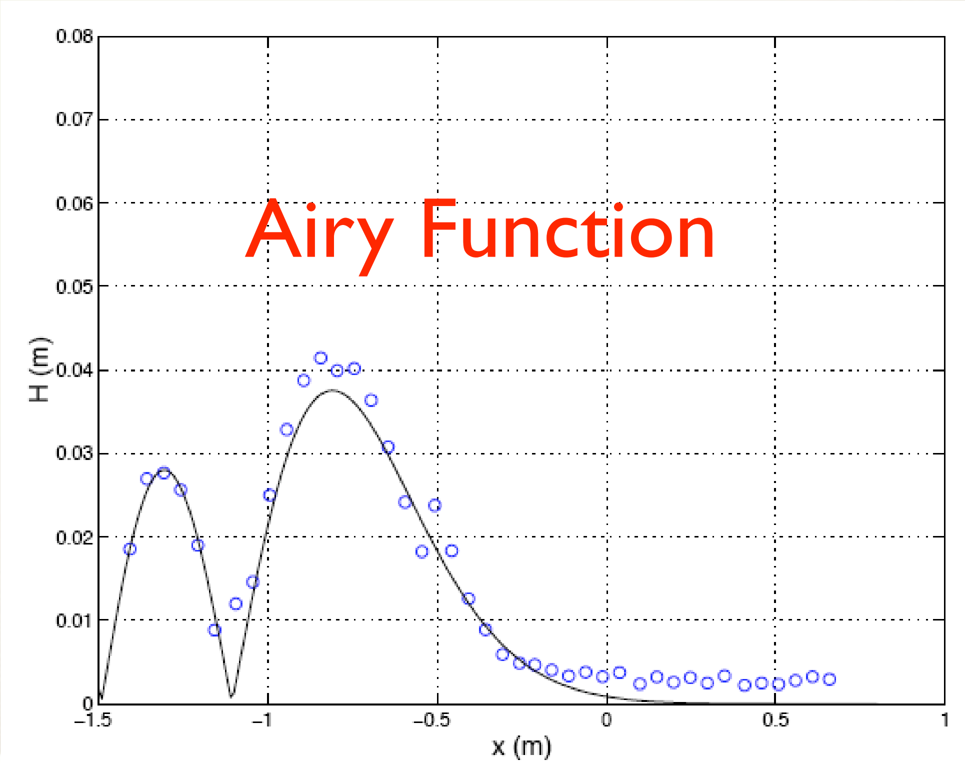
Fluid, Optical and Gravitational Horizons : a Catastrophic Story of Caustics

Wave-Current Interaction

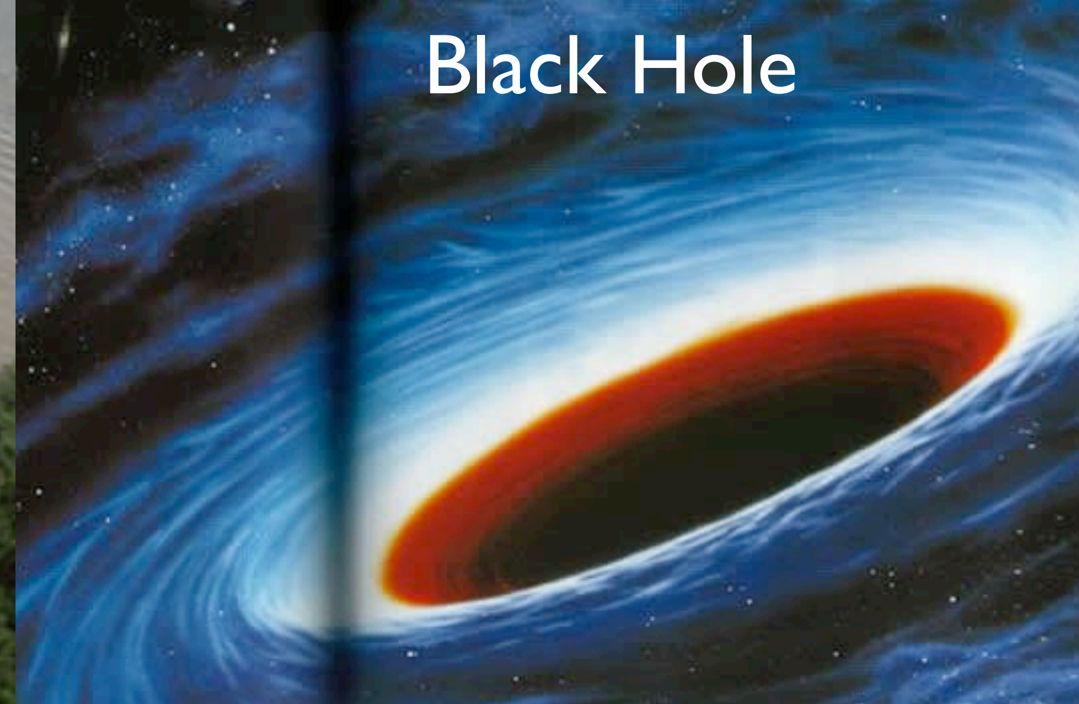


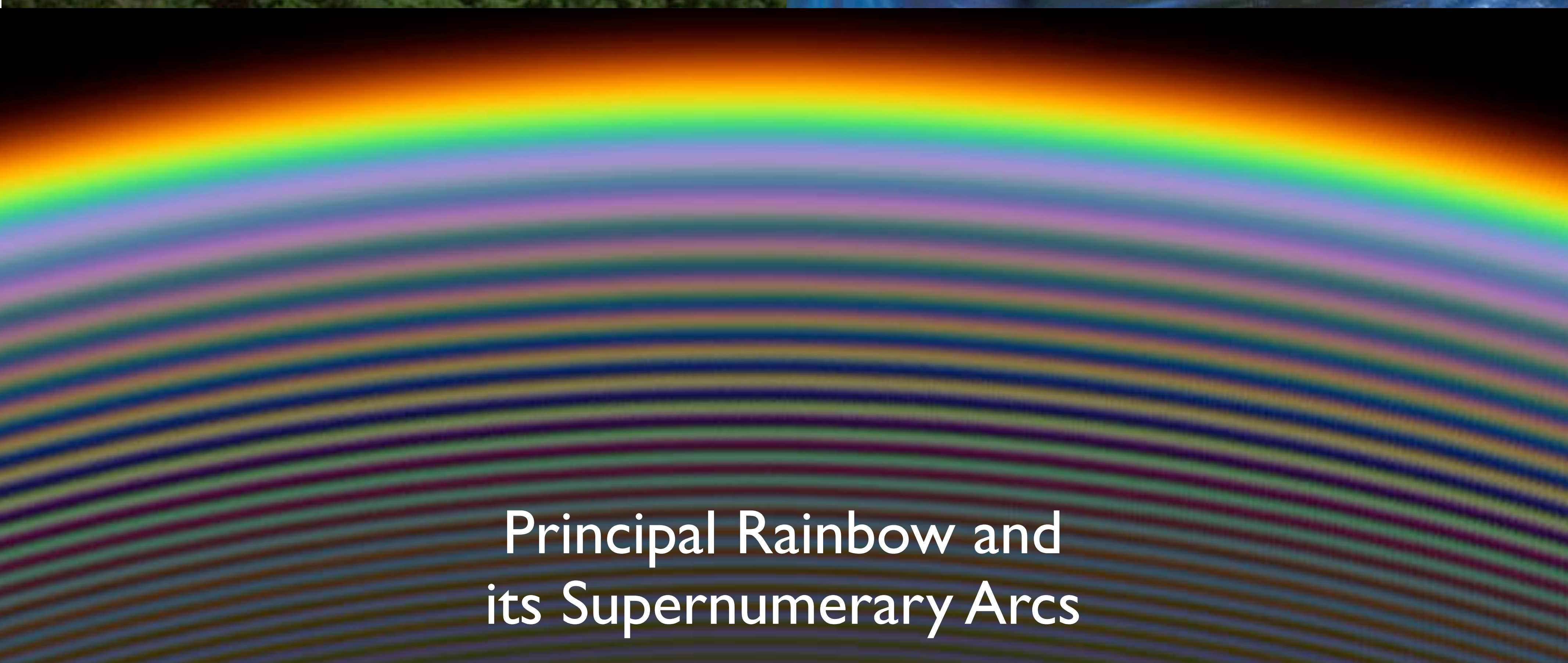
Envelope of the free surface



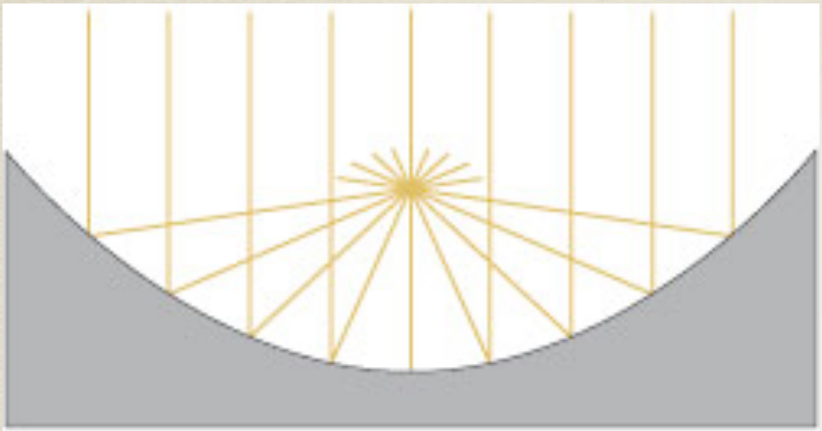



Wave-Current Interaction

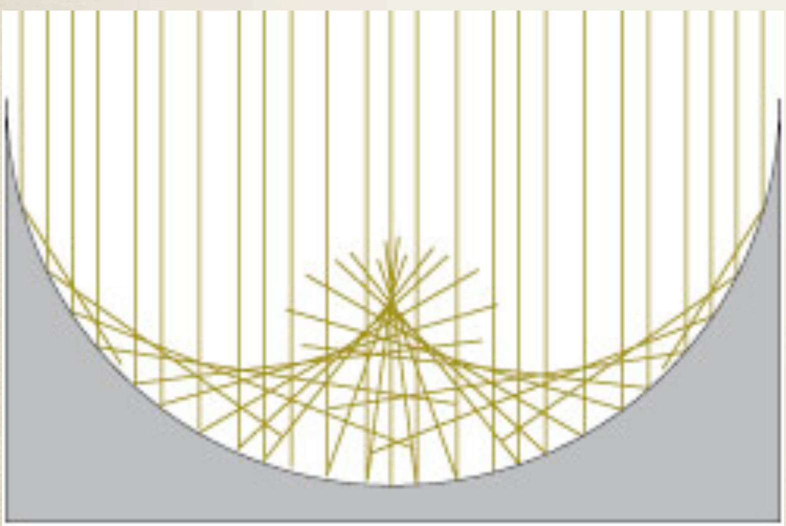

Black Hole


Principal Rainbow and
its Supernumerary Arcs

Optical Caustics



Parabolic Mirror

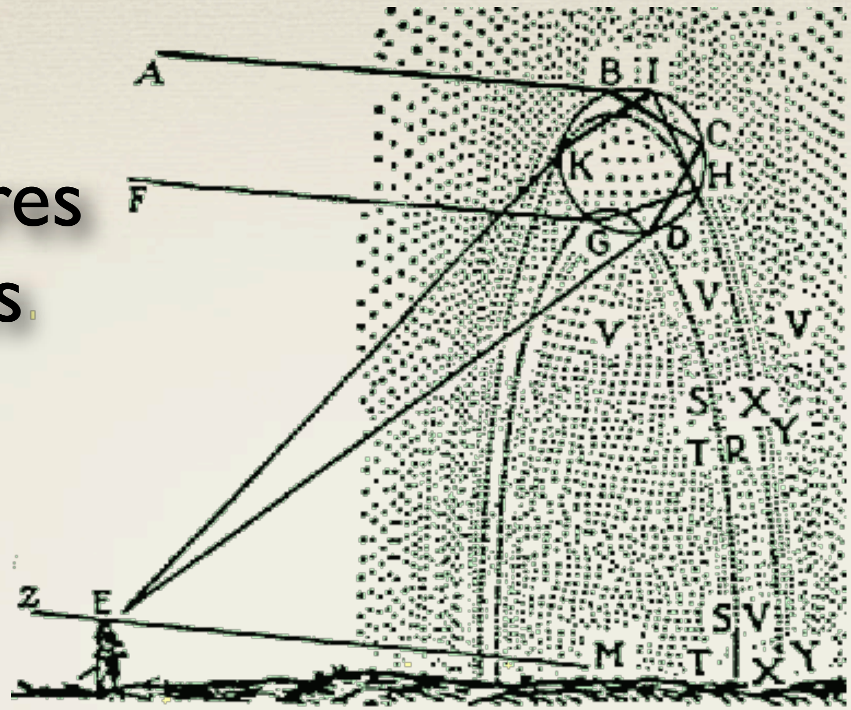


Cylindrical Mirror

kausticos : to burn in ancient greek

Rainbow

Les Météores
Descartes
1637



Drop = Prism :
Colours of the
Rainbow

Principal
Arc

Secondary
Arc

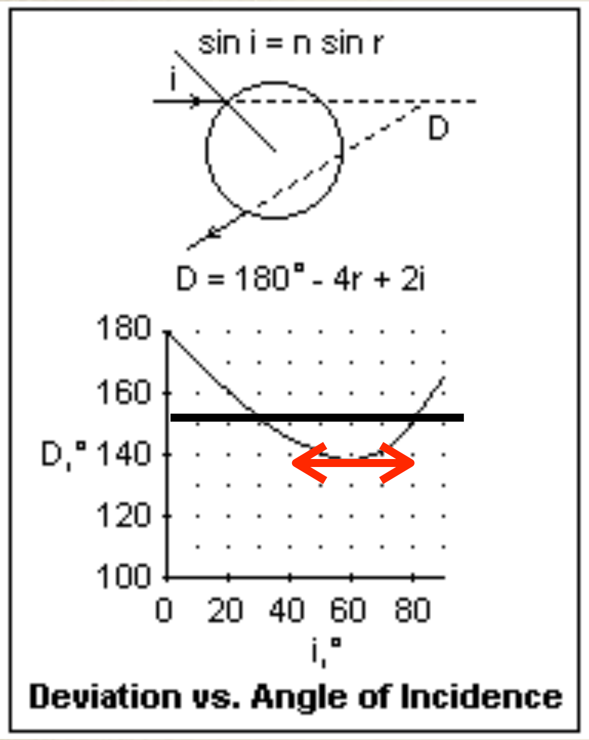
Surnumerary
Arcs

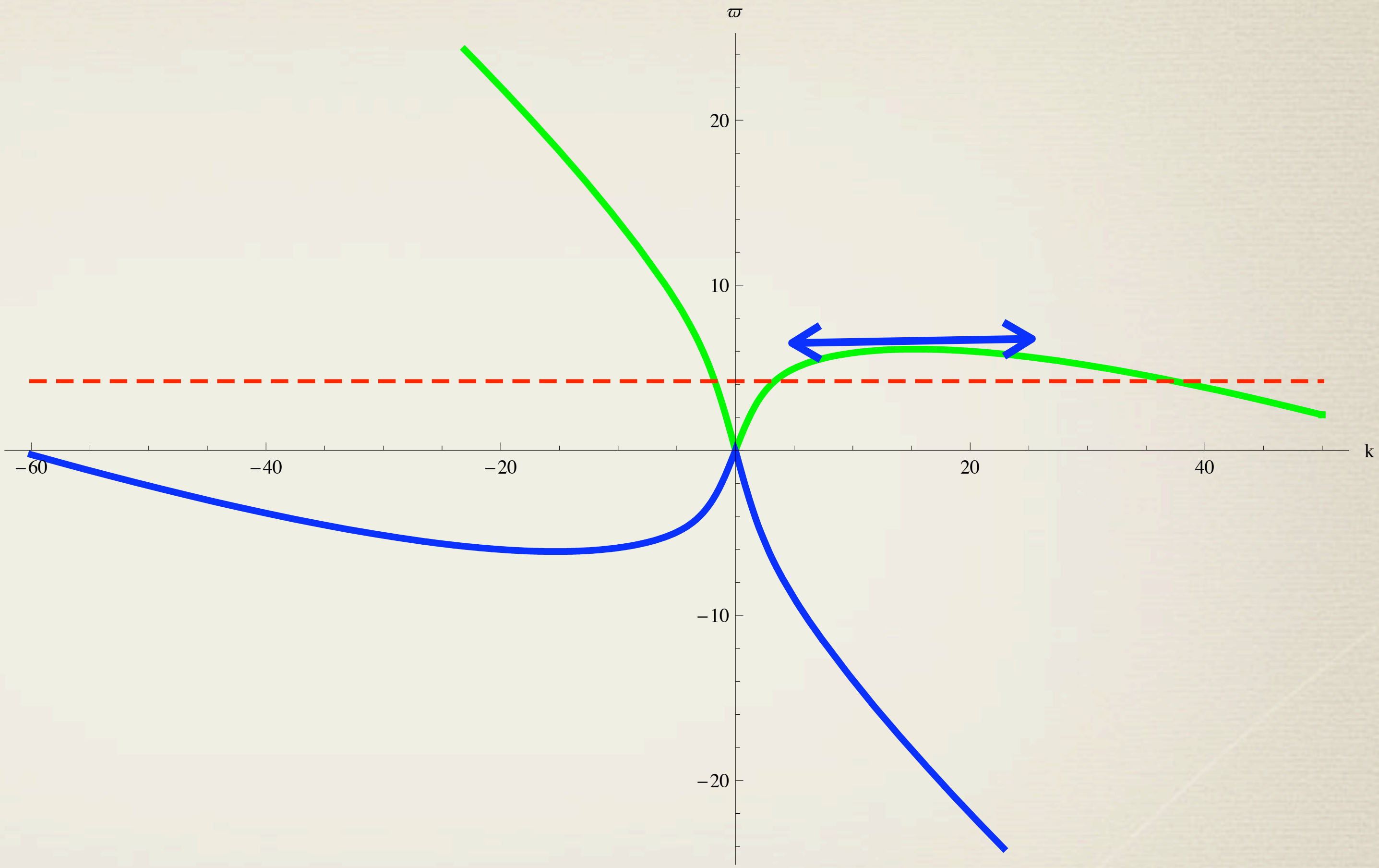
Rays Theory

Optics

Hydrodynamics

$$\omega = \Omega(k) = Uk \pm (gkTanh[kh])^{1/2}$$





$$D(i) \simeq D_{min} + \frac{1}{2}\frac{\partial^2 D}{\partial i^2}(i - i_{min})^2 \quad \Omega(k) = \Omega(k^*) + \frac{1}{2}\frac{\partial^2 \Omega}{\partial^2 k}(k - k^*)^2$$

Wave-current interaction as a spatial bifurcation

Dispersion relation in deep water :

$$\omega^2 - 2\omega U k + U^2 k^2 \approx g k$$

Reduced velocity :

$$\mu = -4\left(\frac{U - U^*}{U^*}\right), U^* = -\frac{g}{4\omega}$$

Reduced wavelength :

$$\kappa = \frac{k - k^*}{k^*}, k^* = \frac{4\omega^2}{g}$$

Normal form :

$$\mu - \kappa^2 = 0$$

Wave-blocking is a spatial bifurcation of the saddle-node type (fold caustic) due to the resonance of the incoming and the blue-shifted waves

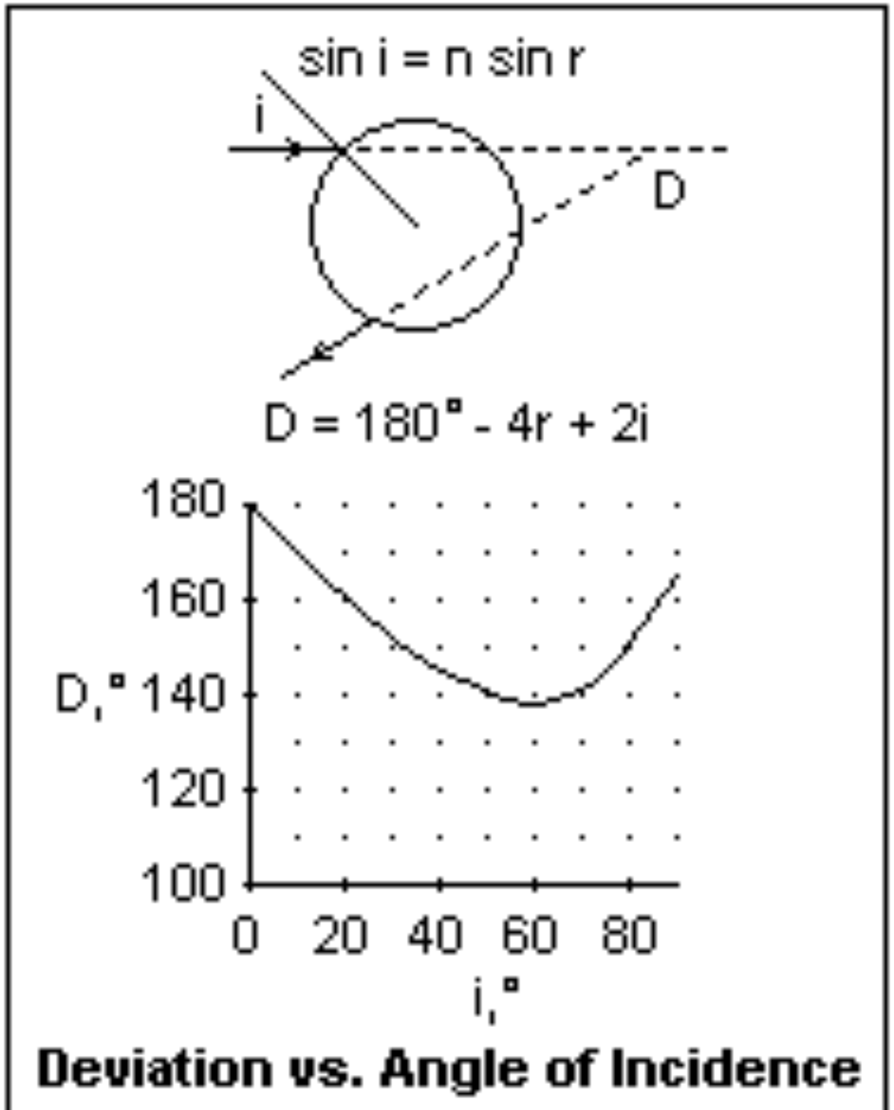
Rousseaux et al. 2009, submitted

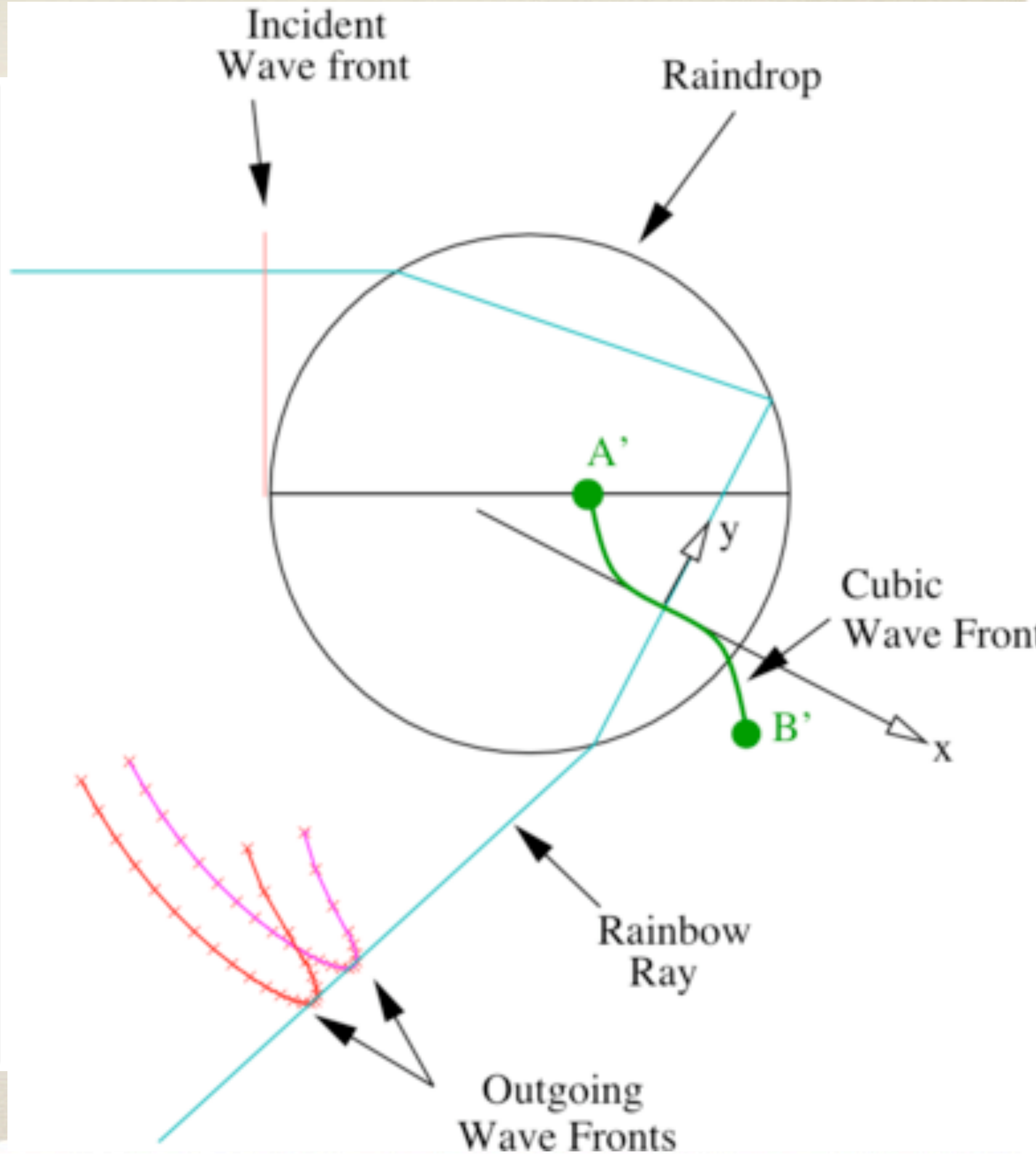
Geometrical Optics :

Minimum of deviation

Wave Optics :

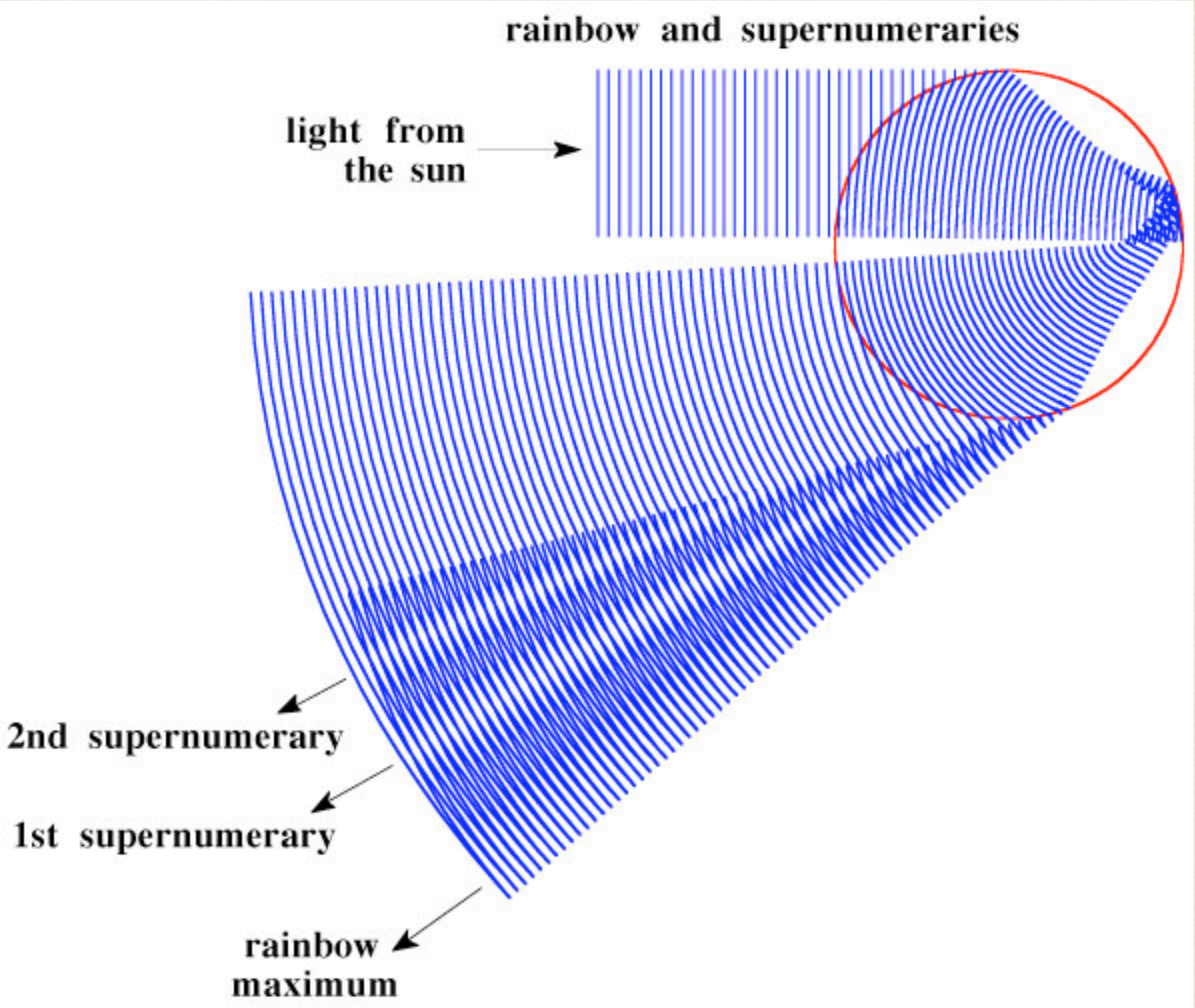
Self-wrapping of the wave front





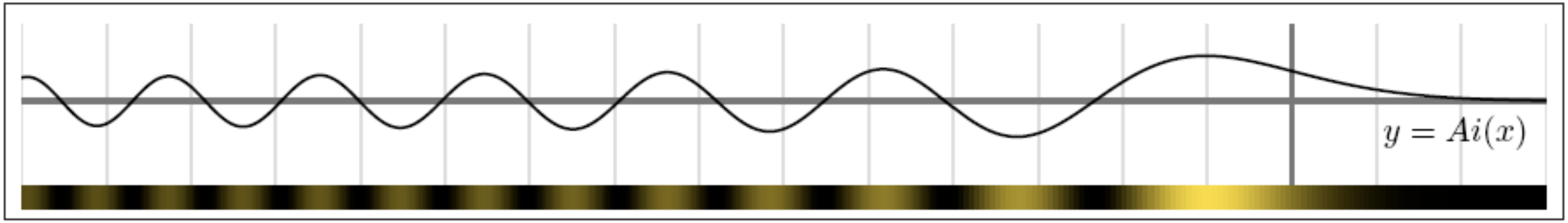
$$I(D) \approx \frac{\Theta(D - D_{min})}{\sqrt{D - D_{min}}}$$

Light Interferences: Classical Regularisation of the Caustic



Airy Function

Light Intensity



A Fold Caustic

Control parameter of the spatial bifurcation :

$$\mu = -4\left(\frac{U - U^*}{U^*}\right), U^* = -\frac{g}{4\omega}$$

Phase velocity in presence of a counter-current U* :

$$c_\varphi = -4U^*\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{U}{U^*}}\right)$$

Conservation of wave action :

$$\frac{E}{E_{far}} = \left(\frac{a}{a_{far}}\right)^2 = \frac{16U^{*2}}{(2U + c_\varphi)c_\varphi}$$

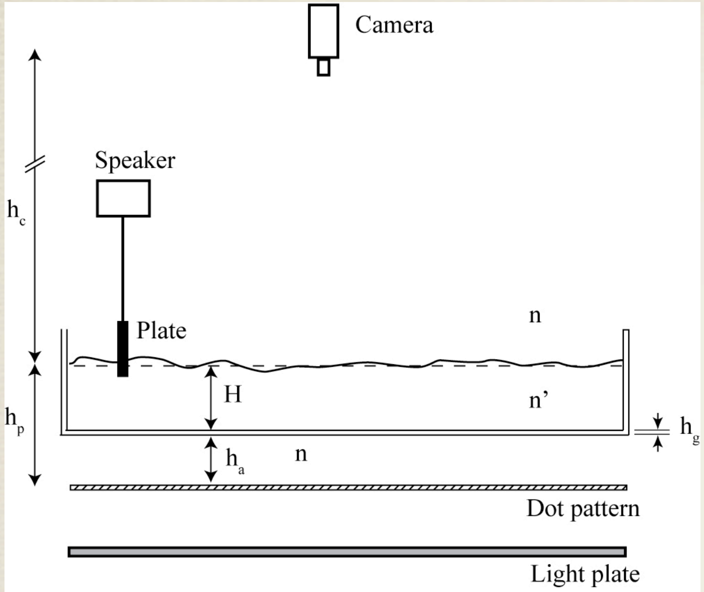
Divergence of water waves energy :

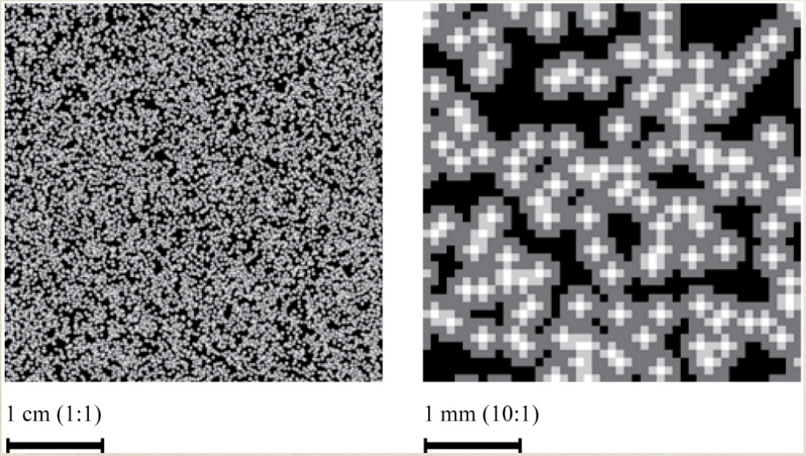
$$E(\mu) \approx \Theta(\mu)\mu^{-1/2}$$

Rousseaux et al. 2009, submitted

Perspectives

Experimental setup





Courtesy Moisy & Rabaud

1. Acquisition:

- Caméra 2048^2

(+time-resolved exp. 1024^2 @100Hz)

- optimized random dot pattern

2. Digital Image Correlation:

- PIV algorithm by DaVis (LaVision GmbH)
- 16x16 interrogation windows, 50% ovlp
- $\delta\mathbf{r}$ defined on a 256^2 grid (128^2)

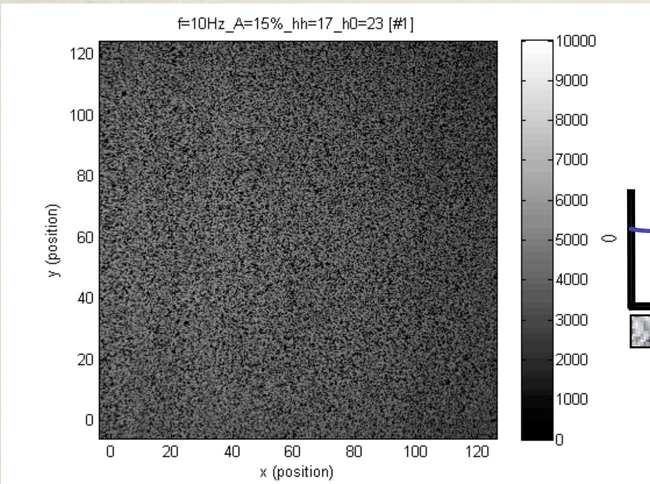
3. Surface reconstruction:

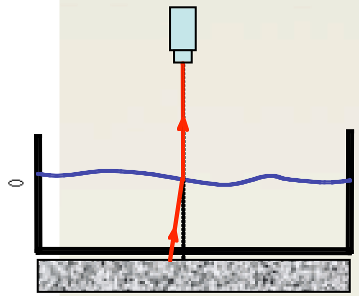
Least-square gradient inversion of $\delta\mathbf{r}$ using Matlab*

* `int2grad` by J. D'Errico, Matlab Central Server

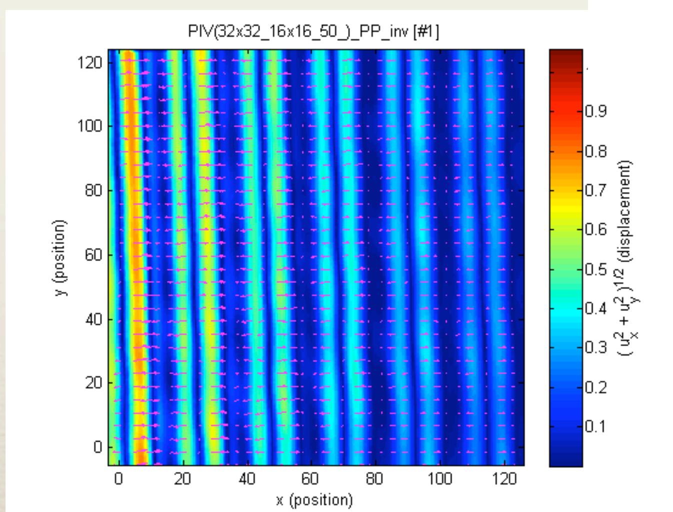
Principle of the method

Step 1: Image acquisition of a refracted random dot pattern

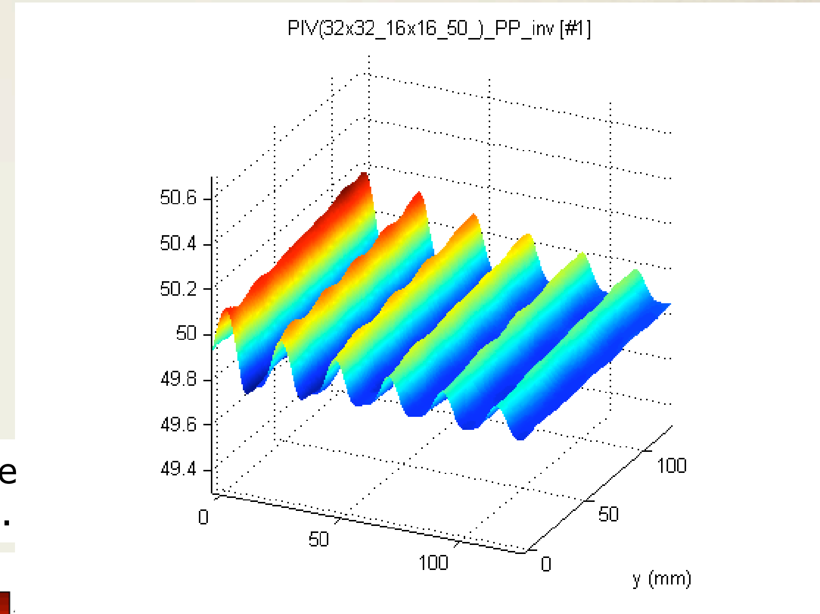




Step 2: Computation of the displacement field by D.I.C.

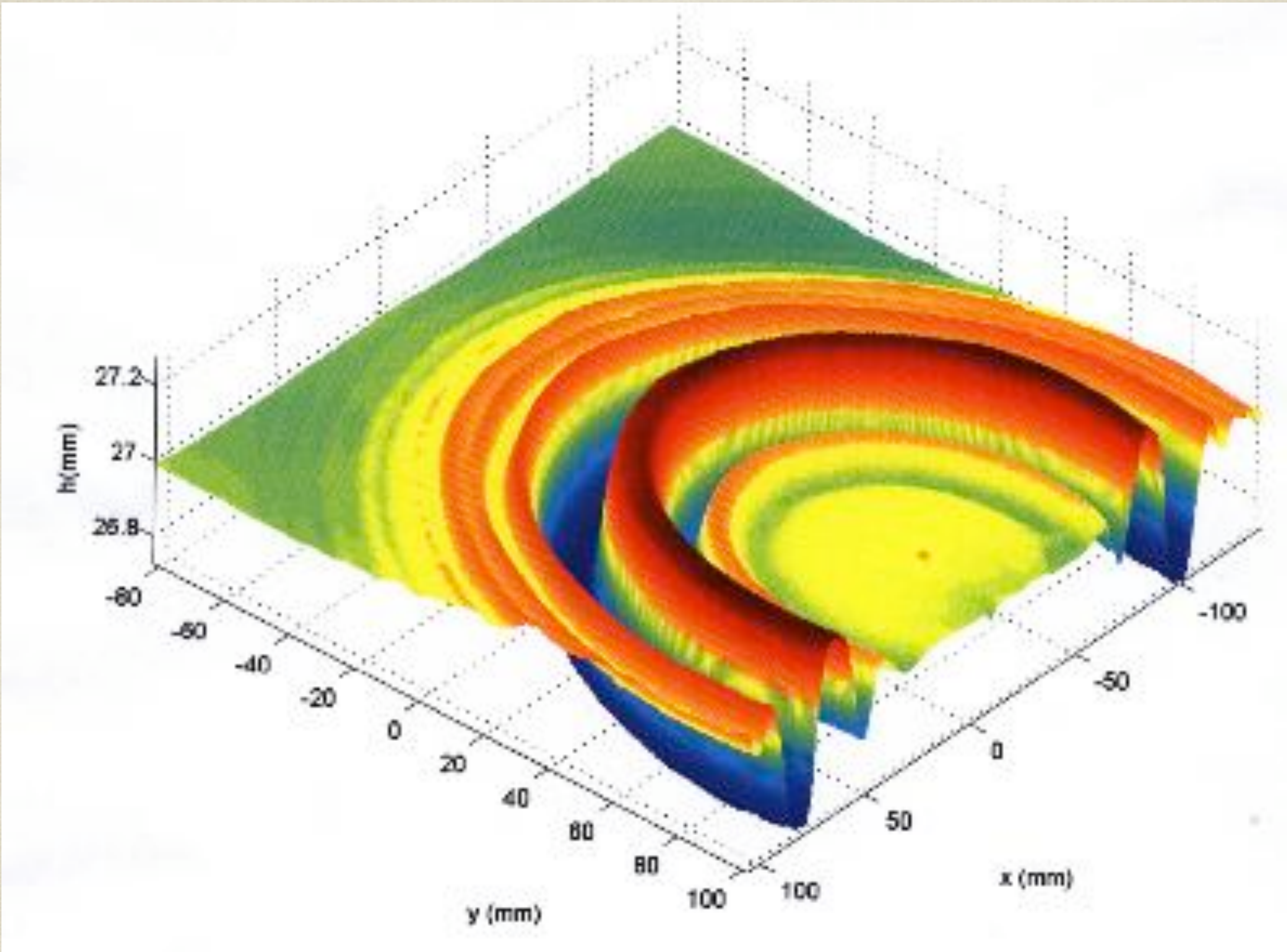


Step 3: Free surface reconstruction by a least-square gradient inversion



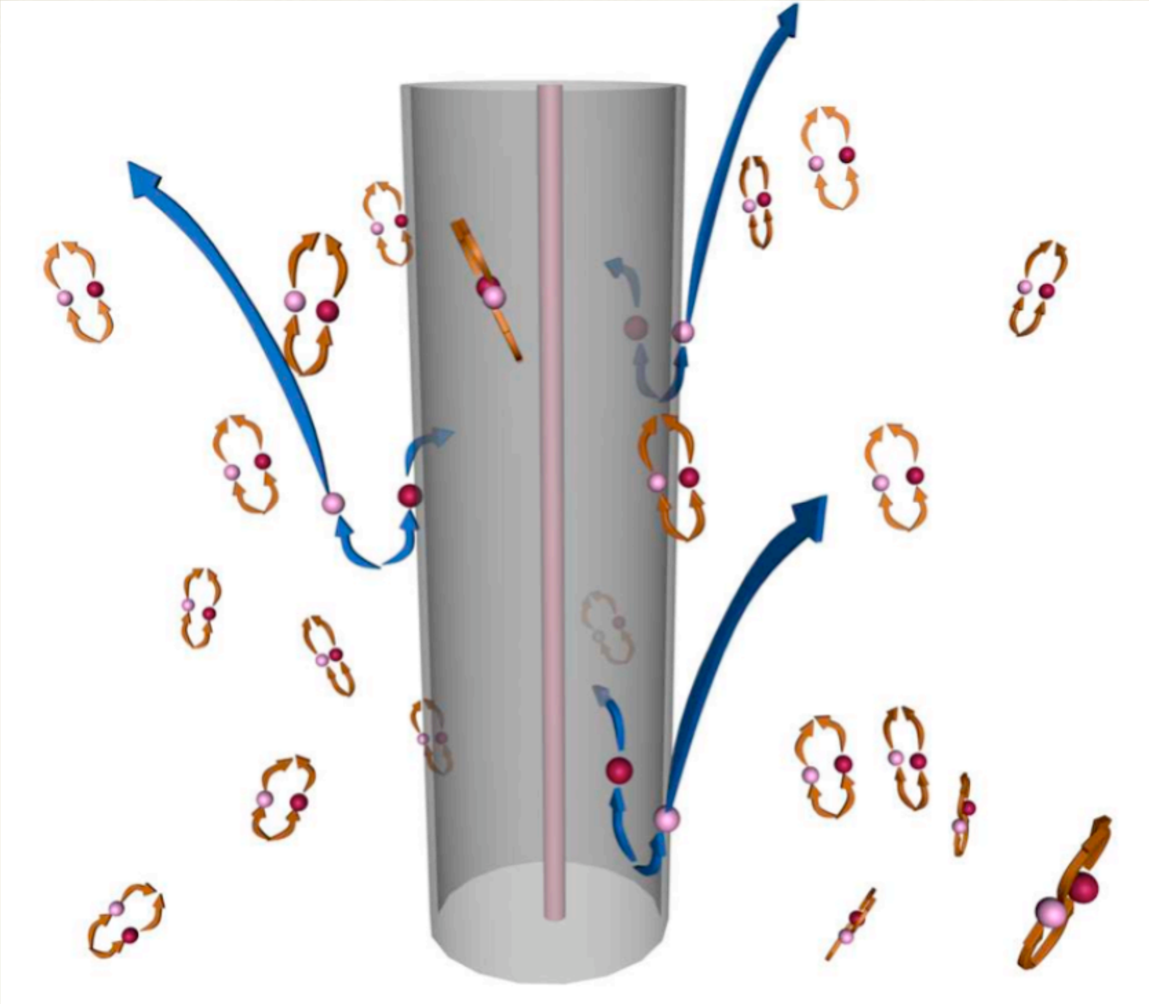
Courtesy Moisy & Rabaud

Water drop impacting a free surface



QuickTime™ et un
décompresseur
sont requis pour visionner cette image.

Hawking Radiation (1974)



Black Holes are NOT black !!!

- Spontaneous Emission : Creation and Annihilation
- Planckian Distribution : Black Body !!!

Hawking/Unruh Temperature

$$E = h\nu = \frac{hc}{\lambda} \qquad \lambda \simeq R_S = \frac{2GM}{c^2}$$

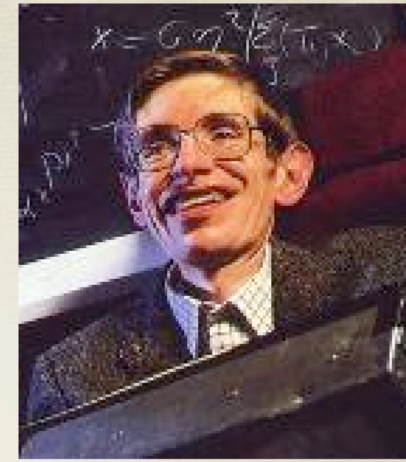
$$E \simeq \frac{hc^3}{2GM} \qquad E \simeq k_B T$$

$$T \simeq \frac{hc^3}{2k_B GM} \qquad T_H = \frac{hc^3}{16\pi^2 k_B GM}$$

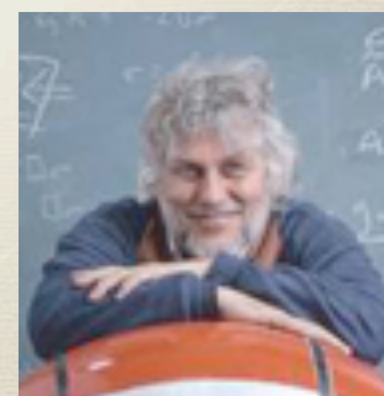
$$g_S = \frac{v_L(R_S)^2}{R_S} = \frac{2GM}{R_S^2} = \frac{c^4}{2GM}$$

$$T_H = \frac{\hbar g_S}{2\pi k_B c} \qquad T_U = \frac{\hbar a}{2\pi k_B c}$$

"God not only plays dice, he also sometimes throws the dice where they cannot be seen."
Stephen Hawking



Quantum Physics
+
Relativity
+
Thermodynamics



Uniform Accelerated Motion

$$\frac{dv}{dt} = a\left(1 - \frac{v^2}{c^2}\right)^{3/2}$$

$$v(t) = \frac{at}{\sqrt{1 + a^2t^2/c^2}}$$

$$t(\tau) = \frac{c}{a} sinh(\frac{a\tau}{c})$$

$$dt = \frac{d\tau}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}$$

$$v(\tau) = \frac{c}{a} tanh(\frac{a\tau}{c})$$

$$x(\tau) = \frac{c^2}{a} cosh(\frac{a\tau}{c})$$

$$\omega'(\tau) = \omega(\tau)\frac{1 \mp v/c}{\sqrt{1 - v^2/c^2}}$$

$$\omega'(\tau) = \omega(\tau)e^{\mp a\tau/c} \simeq \omega(\tau)(1 \mp a\tau/c)$$

An accelerated observer sees waves with a time-dependent phase

$$\Delta\phi(\tau) = \int^{\tau} \omega'(\tau')d\tau' = \frac{\omega c}{a} e^{\mp a\tau/c}$$

=> Exponential Red-Shift or Blue-Shift

Suastika, 2004