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# Heterogeneous Network Games: Conflicting Preferences

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**Abstract**

We propose a model of network games with heterogeneity introduced by endowing players with types that generate preferences among their choices. We study two classes of games: strategic complements or substitutes in payoffs. The payoff function depends on the network structure, and we ask how does heterogeneity shape players' decision making, what is its effect on equilibria, conditions of stability, and welfare. Heterogeneity in players' type establishes the existence of thresholds which determine the Nash equilibrium conditions in a network game. Network configurations in equilibrium can be *satisfactory* if each player chooses the action corresponding to her type or *frustrated* when at least one player is not. Also, equilibria can be *specialized* if all players are choosing the same action (only in strategic complements), or *hybrid* when both actions coexist. A refinement of the Nash equilibria through stochastic mutations of pairs of neighbors limits multiplicity to a subset of Stable Equilibrium Configurations. We find that the Nash networks are absorbing states from where it is possible to leave only through mutations and that such mutations in most cases will lead to a frustrated hybrid configuration which, for most networks, is the risk dominant equilibrium.

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# 1 Introduction

In a population or society, agents can be involved in different social and economic interactions, aiming to coordinate (anti-coordinate) choices with their counterparts. Interactions thus define a social network, the neighbors of a specific agent being those with whom she interacts. As a consequence, agents' wellbeing depends on the behavior adopted by themselves and their neighbors. Examples such as acquiring a specific technology (input) between companies, getting involved in a riot, or job search, show the relevance of the network for the decisions made by the agents. Within this approach, past and current literature have considered players as homogeneous, in the sense that their only difference arises from the specific number of contacts every agent has. In such framework, it is generally the case that an agent chooses an action if the number of her neighbors making that same choice is higher than a given threshold. This paper proposes a model of network games with individual-level heterogeneity introduced by endowing players with types that generate preferences over an action among their choices. We advance the field in the direction of a more realistic modeling by going beyond contextually different agents and considering intrinsically different agents. Thus, this paper is motivated by the consideration that when endowing players with preferences over their choice set, the game played might be different depending on a player's and her opponents' type, even if they have the same degree. We ask how does heterogeneity in types and degree shape players' decision making and payoffs, what is its effect upon equilibrium when local information is available, the conditions of stability for equilibrium, and how does this affect welfare.

As a general framework for the strategic interactions that take place in such setting, we study two classes of network games: strategic complements (SC) or strategic substitutes (SS). For both cases we find how heterogeneity affects the structure and conditions of the game depending on the type of players interacting. The games can be played between two players of different or same type, generating multiple cases. This will lead to conflicting preferences when two players of different (the same) type interact in games with strategic complements (substitutes).

A game with strategic complements in payoffs can be considered as a coordination game, where each player faces a binary choice set. When players are endowed with types, they will prefer one action rather than the other, so that even though they wish to coordinate, the payoff differs if the coordination occurs in the preferred choice or in the one a player dislikes. However, payoffs are higher by coordinating in the disliked option than in the case of anti-coordination, when the agent is left alone in choosing her favorite action. There are many examples of strategic complements in the literature. One simple case is that of coauthors choosing an operating system or a specific technology to work with. Consider two brands $A$ and $B$ among which they can choose. Type $A$ players will receive higher payoff when coordinating in $(A, A)$, whereas type B players will receive higher payoff when coordinating on $(B, B)$. However, it is clear that, due to their interest in working together, players of both types prefer to coordinate in the action they dislike rather than sticking to their preferred action and being left with an incompatible operating system or technology.

The situation is the opposite in games with strategic substitutes, which can be seen as anti-coordination games. Players are better off when anti-coordinating. If each one chooses the action they prefer in view of their type, both maximize their payoffs. As before, even if each one chooses the disliked option, but still anti-coordinate, they will receive a higher payoff than in the case of coordination, while coordinating in the disliked choices gives the lowest payoff. This type of games are very common in examples such as differentiation of a product between two companies. Say for example, they can either produce in low quantities at high prices (high quality) focusing on a segment of the population with high income, or choose for high levels of production at low prices (low quality) targeting a wider range of population, that of a lower income. When each firm has a level of capital (type) that makes it prefer a particular choice between high and low quality, and their preferences are opposite, the best possible outcome arises when each one of them produces what they like. On the contrary, for each one of them the worst situation arises when competing for the same segment of customers in a quality setting that is not the firm's preferred one.

To go from the above described 2×2 setting to a game in a social network, we assume that there is a fixed social (or geographic, or financial) structure where each player wants to adopt an action determined by her type, and interacts strategically with her neighbors. In this scenario, our main results can be summarized as follows: To begin with, we obtain two specific thresholds, one for each type of player and depending on her degree, such that her action changes when the number of her neighbors adopting the same action as hers goes above or below the threshold. In other words, we show that a player has incentives to adopt the behavior she likes if the number of her neighbors choosing the same (opposite) action exceeds a given type-dependent threshold for games with SC (SS). In this context, a relevant feature of our model is that heterogeneity in types generates heterogeneity in thresholds. A player who likes a specific action has a lower threshold of acceptance for such choice than a player who dislikes it, even with the same degree. Heterogeneity in players' degree also provides a wider range of thresholds, because a specific player's threshold depends on her type and degree. We subsequently study equilibria, finding specific network configurations that depend on the class of game being played. Thus, we obtain networks that we denote as *satisfactory*, where all players choose the action they like, which is the action corresponding to their type. We also find *frustrated* networks, situations when at least one player chooses the action she dislikes. These configurations, when considering the action profiles, are in turn subdivided into *specialized*, where every player chooses the same action, which is only possible in games with SC, and *hybrid* where both actions coexist.

We then refine this set of six (two) network game configurations in equilibrium in SC (SS) through a process of stochastic mutations. We follow the pioneering work of Foster and Young (1990), who were the first to argue that in games with multiple strict Nash equilibria, some equilibria are more likely to emerge than others in the presence of continual small stochastic shocks. To this end, we use myopic best response as our dynamical rule and consider a class of mutations that affects two neighbors simultaneously. In this way, we obtain a proper subset of equilibria configurations which are stable, and show that most of them are in the class of frustrated hybrid networks, implying that full coordination is very problematic in SC, and that there will generally be agents that choose the action they do not like. As a last step in the characterization of equilibria, we carry out a welfare analysis and show that the stable equilibria are risk-dominant in the sense of Harsanyi and Selten (1989), and in general they have the smallest overall welfare, an interesting result that connects with the classical one by Kandori *et al.* (1993).

There is an increasingly growing literature in the last decade on the different effects of considering a network as the substrate where strategic interactions or diffusion of information take place; for a review, see Vega-Redondo (2007), Goyal (2007), and Jackson (2008)[1]. Among this research, the paper by Galeotti *et al.* (2010) is specially relevant as it models network games with incomplete information, considering also both SC and SS, where the type of a player is her degree. Bramoullé and Rogers (2010) uses the same consideration of a player's type to model homophily, and Bramoullé and Kranton (2007) models games with SS. Analysis of equilibrium in network games are found in Bloch and Jackson (2006) and Jackson and Yariv (2007). Our contribution to the aforementioned literature is the modeling of intrinsic heterogeneity in SC and SS games played on networks by players with types who have preferences between actions. We assume local information, so that each player knows her degree, her type and the actions chosen by her neighbors. It is also worth mentioning the literature on behavioral models with thresholds for changing actions. Indeed, Granovetter (1978) used an example of collective behavior where each person decides to get involved in a riot or not conditioned to a given proportion of people they see doing it first. The aforementioned paper by Galeotti *et al.* (2009) use thresholds to characterize the Bayesian Nash equilibrium for network games showing that they have properties of monotonicity in the degree of the players. Finally, Chiang (2007) is related to our findings in so far they work with a model of threshold heterogeneity in networks.

The document is structured in four sections. In Sec. 2, we introduce the model considering the $2 \times 2$ games with types of players and the heterogeneous network games, where the relation between strategies and thresholds is exposed. We obtain and classify here the Nash equilibria networks of the model. Subsequently, in Sec. 3 we present the dynamic model, define stability for networks in equilibrium in terms of simultaneous mutations of two agents, and find the networks that are stable under this process. Section 4 presents a welfare analysis of the equilibrium networks, and finally Sec. 5 collects the discussion of our main results as well as additional concluding remarks, and closes the paper.

# 2 The Model

## 2.1 The 2x2 Games

**Strategic Complements (Coordination):** *Let SC be a 2-person game where every player has two types* $\Theta_i = \{0, 1\}$ *and the finite set of actions* $A_i = \{0, 1\}$. *The payoff matrix depends on each player's choices and type as follows:*

We consider[2] $a > b > c > d$.Each $2 \times 2$ coordination game, fixed the types of players, has two

[1] The interest on this kind of problems has gone well beyond the economics tradition, see e.g. Szabó and Fáth (2007).

[2] Generally, the relation of payoffs presented in coordination games is that $a > d$ and $b > c$, but for the

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **0** | | | | **1** | | | | **0** | | |
| | | 1 | 0 | | | 1 | 0 | | | 1 | 0 | |
| **1** | 1 | a,b | c,c | **1** | 1 | a,a | c,d | **0** | 1 | b,b | d,c | |
| | 0 | d,d | b,a | | 0 | d,c | b,b | | 0 | c,d | a,a | |

$\theta_1 = 1; \theta_2 = 0$ $\qquad$ $\theta_1 = 1; \theta_2 = 1$ $\qquad$ $\theta_1 = 0; \theta_2 = 0$

Table 1: Payoff matrices for strategic complements games.

Nash equilibria in pure strategies $(1, 1), (0, 0)$ and one in mixed strategies. The game can be played between two players of equal or opposite types. The case of a game of 2 players with opposite types, where each one likes a different action and both prefer to coordinate, presents conflicting preferences and it is not possible to Pareto rank equilibria. In games between equally-typed players there is no conflict in preferences because each one likes the same action, and the equilibrium when both choose the action corresponding to their type is Pareto dominant in payoffs: $(1, 1)$ Pareto dominates $(0, 0)$ if two players of type 1 are playing, and the opposite for two players of type 0.

**Strategic Substitutes (Anti-Coordination):** *Let SS be a 2-person game where every player has two types* $\Theta_i = \{0, 1\}$ *and the finite set of actions* $A_i = \{0, 1\}$. *The payoff matrix depends on each player's choices and type as follows:*

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **0** | | | | **1** | | | | **0** | | |
| | | 1 | 0 | | | 1 | 0 | | | 1 | 0 | |
| **1** | 1 | c,d | a,a | **1** | 1 | c,c | a,b | **0** | 1 | d,d | b,a | |
| | 0 | b,b | d,c | | 0 | b,a | d,d | | 0 | a,b | c,c | |

$\theta_1 = 1; \theta_2 = 0$ $\qquad$ $\theta_1 = 1; \theta_2 = 1$ $\qquad$ $\theta_1 = 0; \theta_2 = 0$

Table 2: Payoff matrices for strategic substitutes games.

As in SC, payoffs follow the condition $a > b > c > d$. The pure Nash equilibria of the game are in the anti-coordination combinations $(1, 0), (0, 1)$. When two players of opposite type interact there are no conflicting preferences because both players are better-off when choosing the action corresponding to each of their types, which is the Pareto dominant Nash equilibrium of the game. For our example: $(1, 0)$ Pareto dominates $(0, 1)$. Conflicting preferences arise when two players of the same type interact, because both of them like the same action and would want to anti-coordinate.

case of players with types it is relevant to specify the combinations of actions that are not an equilibrium. In the case of anti-coordination in SC a player receives a lower payoff when being alone in the action she dislikes than being alone in the favorite action. If payoffs in anti-coordination where equal, $c = d$, a player would be indifferent between being alone in either action, and the specification of types would allow for some ambiguity.

## 2.2 The Network Game

Our next step is to proceed from the $2 \times 2$ games to the network game. To that end, we need to define a network structure to model the manner in which agents interact. This social network is denoted as $\Gamma$ and represented by $(\mathbf{N}, \mathbf{g})$, where $\mathbf{N} = \{1, \ldots, N\}$ is a finite set of players, and $\mathbf{g}$ is the set of undirected links in the network, given by the adjacency matrix of the corresponding graph. The relationship between two players $i$ and $j$ in the network $(\mathbf{N}, \mathbf{g})$ is expressed by $g_{ij} \in \{0, 1\}$. When there is a link between them $g_{ij} = 1$, and we say $(i, j)$ are neighbors. In case they are not connected $g_{ij} = 0$. The set of $i$'s neighbors is $N_i(g) = \{j \in \{1, \ldots, N\} | g_{ij} = 1\}$. A player's degree is $k_i(g) = |N_i(g)| = n_i$ the cardinality of the set $N_i(g)$.

On top of this social network, we introduce the network game in the following way: Players can choose actions in a binary set $A_i = \{0, 1\}$ and have a type that belongs to a set of types $\Theta_i = \{0, 1\}$. A player $i$ of type $\theta_i = 1$ likes action $a_i = 1$ and dislikes $a_i = 0$, which symmetrically holds for a player of type $\theta_i = 0$. We use linear payoff functions dependent on a player's type and choice, where each receives benefit from own and neighbor's actions. We denote $a_{N_i}(\Gamma)$ as the vector of actions taken by $i$'s neighbors. The class of game played is either SC or SS, and correspondingly the payoff function of player $i$ is:

$$v_i(\theta_i, a_i, a_{N_i}(\Gamma)) = \lambda_{a_i}^{\theta_i}[1 + \delta_m \sum_{j=1}^{n_i} I_{\{a_j = a_i\}} + (1 - \delta_m) \sum_{j=1}^{n_i} I_{\{a_j \neq a_i\}}], \tag{1}$$

where $I_{\{a_j = a_i\}}$ is the indicator function of those neighbors choosing the same action as player $i$, and $I_{\{a_j \neq a_i\}}$ indicates neighbors choosing the opposite; the parameter $\lambda_{a_i}^{\theta_i} = \alpha$ if $a_i = \theta_i$, and $\lambda_{a_i}^{\theta_i} = \beta$ if $a_i \neq \theta_i$. The type of game played is specified through the multiplier $\delta_m$, that takes value 1 in SC games and 0 in SS games. In all cases, we will assume that payoffs verify the condition $0 < \beta < \alpha < 2\beta$, which as we will see below, has different implications for the actions of the players in the two games. Thus, the network game is represented by

$$\Gamma = \langle \{1, \ldots, N\}, \{g_{ij}\}_{i,j \in \{1,\ldots,N\}}, v_i, \Theta_i, A_i \rangle. \tag{2}$$

In what follows, we will assume *local partial information.* By this, we mean that a player in the network knows her degree $k_i$, her type $\theta_i$, and the set of actions associated to her neighbors $a_{N_i}$, but not their types. In this informational context, player $i$'s strategy can then be described as the following map:

$$\sigma_i : \theta_i \to A_i, \quad i \in \{1, \ldots, N\}. \tag{3}$$

As can be seen from Eq. (3) the payoff functions depend on the player's degree $k_i$ and identity, that is, on her type $\theta_i$. Hence, it is not sufficient for two players of the same degree $k_i = k_j$ to make the same choice $a_i = a_j$ in order to have the same payoff function nor to receive equal payoffs. There is a different payoff function for a player when choosing the action she likes than when not doing so.

Given the local [3] information context we are considering, the equilibrium concept to focus on in the next step is Nash equilibrium.

**Definition 1** ***Nash Equilibrium:*** *An action profile* $(\sigma_1^*, \ldots, \sigma_N^*)$ *is a Nash equilibrium in the network game* $\Gamma$*, if and only if*

$$v_i(\theta_i, \sigma_1^*, \ldots, \sigma_N^*) \geq v_i(\theta_i, \sigma_1^*, \ldots, \hat{\sigma}_i, \ldots, \sigma_N^*), \quad \forall\ \sigma_i^* \neq \hat{\sigma}_i, \quad i \in \mathbf{N}, \quad \text{and} \quad \theta_i \in \Theta. \tag{4}$$

We now study separately the equilibria for the two classes of games we are considering.

#### 2.2.1 Strategic Complements

For games with strategic complements, the four payoff functions depending on the player's type and choice can be written as

$$\begin{aligned} v_i(1,1,(a_{j_1},\ldots,a_{j_{n_i}})) &= \alpha(1+\chi_i), \\ v_i(1,0,(a_{j_1},\ldots,a_{j_{n_i}})) &= \beta(1+n_i-\chi_i), \\ v_i(0,0,(a_{j_1},\ldots,a_{j_{n_i}})) &= \alpha(1+n_i-\chi_i), \\ v_i(0,1,(a_{j_1},\ldots,a_{j_{n_i}})) &= \beta(1+\chi_i), \end{aligned} \tag{5}$$

where $\chi_i = \sum_{j=1}^{n_i} I_{\{a_j=1\}}$ is the number of $i$'s neighbors choosing 1, and $(n_i - \chi_i) = \sum_{j=1}^{n_i} I_{\{a_j=0\}}$ those choosing action 0.

To find the Nash equilibria of the network game in SC, notice that a player is better off choosing the action she likes than not doing so with the same number of neighbors choosing her same and opposite action. In particular, the condition on the values of $\alpha$ and $\beta$ introduced above implies that being alone in the choice one likes gives lower payoffs than choosing the disliked action and having neighbors making the same choice, i.e., $\beta(1+n_i) > \alpha$.

In order to formalize the Nash equilibria, let us define two thresholds, $\underline{\tau}(n_i)$ and $\overline{\tau}(n_i)$, that will be functions of player $i$'s degree for each type of player in a network game. We define these thresholds independently of the class of game being played as they will be useful in both cases. The thresholds are

$$\underline{\tau}(n_i) = \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} \rceil \tag{6}$$

$$\overline{\tau}(n_i) = \lfloor \frac{\alpha}{\alpha+\beta} n_i + \frac{\alpha-\beta}{\alpha+\beta}, \rfloor \tag{7}$$

where $\lceil \ldots \rceil$ and $\lfloor \ldots \rfloor$ denote respectively the maximum lower integer or the minimum higher integer of the real number considered. It can be shown from the payoff functions that $\overline{\tau}(n_i) > \underline{\tau}(n_i)$; in fact,

$$\overline{\tau}(n_i) - \underline{\tau}(n_i) = \frac{\alpha-\beta}{\alpha+\beta} n_i + 2(\frac{\alpha-\beta}{\alpha+\beta}) - 2 > 0. \tag{8}$$

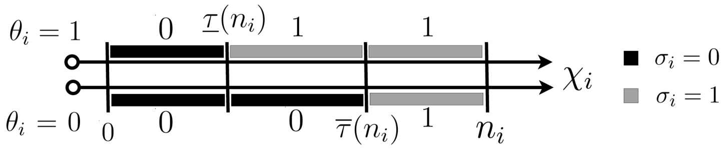


Figure 1: SC Thresholds

Figure 1 illustrates the thresholds for the two types of players $\theta_i = 1$ and $\theta_i = 0$, and the choices each makes in a Nash equilibrium depending on the size of $\chi_i$.

With the thresholds we have introduced, we can now go to the Nash equilibria:

**Proposition 1** *An action profile $(\sigma_i^*, \dots, \sigma_N^*)$ is a Nash equilibrium in $\Gamma$ for network games with strategic complements if:*

$$\begin{cases} \sigma_i^* = 1, & \text{if } \theta_i = 1 \text{ and } \chi_i \geq \underline{\tau}(n_i), \\ \sigma_i^* = 0, & \text{if } \theta_i = 1 \text{ and } \chi_i < \underline{\tau}(n_i), \\ \sigma_i^* = 0, & \text{if } \theta_i = 0 \text{ and } \chi_i \leq \overline{\tau}(n_i), \\ \sigma_i^* = 1, & \text{if } \theta_i = 0 \text{ and } \chi_i > \overline{\tau}(n_i). \end{cases} \tag{9}$$

*Proof:* A player $i \in \mathbf{N}$ of type $\theta_i = 1$ chooses her favorite action $a_i = 1$ instead of the action she dislikes $a_i = 0$ if $v_i(1, 1, ((\sigma_1^*, \dots, \sigma_{i-1}^*, \sigma_{i+1}^*, \dots, \sigma_N^*)) \geq v_i(1, 0, (\sigma_1^*, \dots, \sigma_{i-1}^*, \sigma_{i+1}^*, \dots, \sigma_N^*))$. Hence,

$$v_i(1, 1, (\sigma_1^*, \dots, \sigma_{i-1}^*, \sigma_{i+1}^*, \dots, \sigma_N^*)) = \alpha(1+\chi_i) \geq \beta(1+n_i-\chi_i) = v_i(1, 0, (\sigma_1^*, \dots, \sigma_{i-1}^*, \sigma_{i+1}^*, \dots, \sigma_N^*)), \tag{10}$$

from which we deduce that

$$\chi_i \geq \frac{\beta}{\alpha+\beta}n_i - \frac{\alpha-\beta}{\alpha+\beta} \Rightarrow \chi_i \geq \lceil \frac{\beta}{\alpha+\beta}n_i - \frac{\alpha-\beta}{\alpha+\beta} \rceil = \underline{\tau}(n_i). \tag{11}$$

On the other hand, a player $i \in \mathbf{N}$ of type $\theta_i = 0$ chooses her favorite action $a_i = 0$ instead of the action she dislikes $a_i = 1$ if $v_i(0, 0, (\sigma_1^*, \dots, \sigma_{i-1}^*, \sigma_{i+1}^*, \dots, \sigma_N^*)) \geq v_i(0, 1, (\sigma_1^*, \dots, \sigma_{i-1}^*, \sigma_{i+1}^*, \dots, \sigma_N^*))$, which leads to

$$v_i(0, 0, (\sigma_1^*, \dots, \sigma_{i-1}^*, \sigma_{i+1}^*, \dots, \sigma_N^*)) = \alpha(1+n_i-\chi_i) \geq \beta(1+\chi_i) = v_i(0, 1, (\sigma_1^*, \dots, \sigma_{i-1}^*, \sigma_{i+1}^*, \dots, \sigma_N^*)) \tag{12}$$

and

$$\chi_i \leq \frac{\alpha}{\alpha+\beta}n_i + \frac{\alpha-\beta}{\alpha+\beta} \Rightarrow \chi_i \leq \lfloor \frac{\alpha}{\alpha+\beta}n_i + \frac{\alpha-\beta}{\alpha+\beta} \rfloor = \overline{\tau}(n_i). \tag{13}$$

□

**Remark 1** *For a player of type $\theta_i = 1$ to play her preferred action, 1, she needs to be connected to at least $\chi_i$ neighbors choosing action 1, that is, $\chi_i \geq \underline{\tau}(n_i)$. In case $\chi_i < \underline{\tau}(n_i)$*

[3]We note that local and global information give the same Nash equilibrium strategies in each class of games. There exists global information when all players know the whole configuration of types.

*player $i$ adopts her disliked behavior. On the contrary, a player $i$ of type $\theta_i = 0$ needs to be connected to at most $\chi_i$ neighbors choosing action 1, that is, $\chi_i \leq \overline{\tau}(n_i)$. In case $\chi_i > \overline{\tau}(n_i)$ player $i$ adopts her disliked behavior. If the number of neighbors choosing 1 a given player has is $\chi_i < \underline{\tau}(n_i)$, independently of her type, such player chooses $\sigma_i = 0$, and when $\chi_i > \overline{\tau}(n_i)$ she chooses $\sigma_i = 1$. The case in between, where $\underline{\tau}(n_i) \leq \chi_i \leq \overline{\tau}(n_i)$ grants any player to choose the action corresponding to her type.*

### 2.2.2 Strategic Substitutes

Let us now turn to games with SS, in which as before there are four payoff functions depending on a player's type and choice:

$$\begin{aligned} v_i(1,1,(a_1,\ldots,a_{j_{n_i}})) &= \alpha(1+n_i-\chi_i) \\ v_i(1,0,(a_1,\ldots,a_{j_{n_i}})) &= \beta(1+\chi_i) \\ v_i(0,0,(a_1,\ldots,a_{j_{n_i}})) &= \alpha(1+\chi_i) \\ v_i(0,1,(a_1,\ldots,a_{j_{n_i}})) &= \beta(1+n_i-\chi_i). \end{aligned} \tag{14}$$

Reasoning along the same lines as in the previous case, we see that in the case of games with SS, a player is better off choosing the action she likes than not doing so with the same number of neighbors choosing her same and opposite action. The extreme case of being alone in the choice one likes having all the neighbors coordinating in the same action gives lower payoffs than making the disliked choice and having neighbors making the opposite decision, $\beta(1+n_i) > \alpha$.

The threshold functions for both classes of games, strategic substitutes and strategic complements are the same as in SC, but the relation between thresholds and players' strategies are not. Figure 2 illustrates the thresholds for the two types of players $\theta_i = 1$ and $\theta_i = 0$, and the choices each makes depending on the size of $\chi_i$.

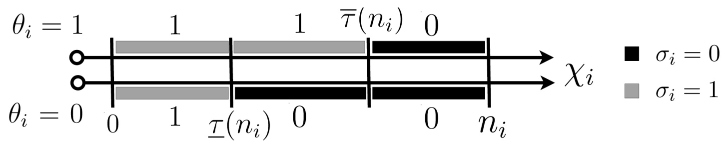


Figure 2: SS Thresholds

**Proposition 2** *An action profile $(\sigma_i^*, \ldots, \sigma_N^*)$ is a Nash equilibrium in $\Gamma$ for network games with strategic substitutes if:*

$$\begin{cases} \sigma_i^* = 1, & \text{if } \theta_i = 1 \text{ and } \chi_i \leq \overline{\tau}(n_i), \\ \sigma_i^* = 0, & \text{if } \theta_i = 1 \text{ and } \chi_i > \overline{\tau}(n_i), \\ \sigma_i^* = 0, & \text{if } \theta_i = 0 \text{ and } \chi_i \geq \underline{\tau}(n_i), \\ \sigma_i^* = 1, & \text{if } \theta_i = 0 \text{ and } \chi_i < \underline{\tau}(n_i). \end{cases} \tag{15}$$

*Proof:* As in the previous case: Player $i \in \mathbf{N}$ of type $\theta_i = 1$ chooses her favorite action $a_i = 1$ instead of the action she dislikes $a_i = 0$ if $v_i(1, 1, (\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_{i+1}^*, \ldots, \sigma_N^*)) \geq v_i(1, 0, (\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_{i+1}^*, \ldots, \sigma_N^*))$, and therefore

$$v_i(1, 1, (\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_{i+1}^*, \ldots, \sigma_N^*)) = \alpha(1+n_i-\chi_i) \geq \beta(1+\chi_i) = v_i(1, 0, (\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_{i+1}^*, \ldots, \sigma_N^*)), \tag{16}$$

leading to

$$\chi_i \leq \frac{\alpha}{\alpha+\beta} n_i + \frac{\alpha-\beta}{\alpha+\beta} \Rightarrow \chi_i \leq \lfloor \frac{\alpha}{\alpha+\beta} n_i + \frac{\alpha-\beta}{\alpha+\beta} \rfloor = \overline{\tau}(n_i). \tag{17}$$

In the opposite case, player $i \in \mathbf{N}$ of type $\theta_i = 0$ chooses her favorite action $a_i = 0$ instead of the action she dislikes $a_i = 1$ if $v_i(0, 0, (\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_{i+1}^*, \ldots, \sigma_N^*)) \geq v_i(0, 1, (\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_{i+1}^*, \ldots, \sigma_N^*))$, and we have

$$v_i(0, 0, (\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_{i+1}^*, \ldots, \sigma_N^*)) = \alpha(1+\chi_i) \geq \beta(1+n_i-\chi_i) = v_i(0, 1, (\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_{i+1}^*, \ldots, \sigma_N^*)), \tag{18}$$

and hence

$$\chi_i \geq \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} \Rightarrow \chi_i \geq \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} \rceil = \underline{\tau}(n_i). \tag{19}$$

□

**Remark 2** *For a player of type $\theta_i = 1$ to play her preferred action 1, she needs to be connected to at most $\chi_i$ neighbors choosing action 1, that is, $\chi_i \leq \overline{\tau}(n_i)$. In case $\chi_i > \overline{\tau}(n_i)$ player i adopts her disliked behavior. A player i of type $\theta_i = 0$ needs to be connected to at least $\chi_i$ neighbors choosing action 1, that is, $\chi_i \geq \underline{\tau}(n_i)$. In case $\chi_i < \underline{\tau}(n_i)$ player i adopts her disliked behavior. If the number of neighbors a given player has, choosing* 1 *is $\chi_i < \underline{\tau}(n_i)$, independently of her type, a player chooses $\sigma_i = 1$, and when $\chi_i > \overline{\tau}(n_i)$ she chooses $\sigma_i = 0$. The case in between, where $\underline{\tau}(n_i) \leq \chi_i \leq \overline{\tau}(n_i)$ grants any player to choose the action corresponding to her type.*

**Remark 3** *For the network games discussed in Galeotti* et al. *(2010), under incomplete information the authors show that the Bayesian Nash equilibrium payoff is non-decreasing (non-increasing) in the degree of the players for the case of SC (SS). While, due to the existence of types in the population, a similar result cannot be proven here, we find it noticeable that the Nash equilibria we obtain are reminiscent of that structure in the sense that the payoff is non-increasing or non-decreasing on the number of neighbors playing 1, $\chi_i$. In fact, the result of Galeotti* et al. *is recovered when there is only one type of players in the network.*

## 2.3 Equilibrium Configurations

In the previous subsection we have seen what are the conditions for players of each type to choose an action. These conditions, when applied to the network game as a whole, lead to a variety of equilibrium configurations associated to the choices each player makes and to the distribution of types. In order to classify all these equilibrium networks, in what follows we introduce some notation. Beginning with the actions, a network can be *specialized*

($S$) or *hybrid* ($H$). A specialized action profile is one in which all players make the same choice. Therefore, there are configurations specialized in action one, $S1$, or in action zero, $S0$. In a hybrid action profile, both actions are present. Second, regarding players' types, a network can be *frustrated* ($F$) or *satisfactory* ($S$). In a satisfactory configuration, each player chooses her favorite action corresponding to her type, and in the frustrated one *at least one* player adopts the disliked choice. As a result, we may observe six (two) different network configurations in SC (SS) depending on the distribution of types and the action profile: $S_{S1}, S_{S0}, F_{S1}, F_{S0}, S_H, F_H$ ($S_H, F_H$) (subindices refer to the category in terms of actions). Note that we are claiming that specialized networks only exist in SC games. This is due to the fact that the anti-coordination condition on the payoffs for SS does not allow an equilibrium where all players make the same choice. Finally, satisfactory specialized configurations are only possible under a very strong restriction to the type of the players: Indeed, $S_{S1}$ and $F_{S1}$ require that the set $\{\theta_i = 0\} = \emptyset$, whereas $S_{S0}$ and $F_{S0}$ require $\{\theta_i = 1\} = \emptyset$.

We now discuss the condition under which the different equilibrium configurations arise. As indicated above, the first proposition below refers only to SC games:

**Proposition 3** *The configuration of a network* $\Gamma(\mathbf{N}, \mathbf{g})$ *in equilibrium is **frustrated specialized** when all players choose one same action, so that* $a_i = a_j$, $\forall\ i, j \in \{1, \ldots, N\}$*. A network is specialized in action* $1(0)$ *if and only if the following three conditions are jointly satisfied:*

1. *Players of type* $\theta_i = 1$ *have* $\chi_i \geq \underline{\tau}(n_i)(\chi_i < \underline{\tau}(n_i))$ *neighbors*
2. *Players of type* $\theta_i = 0$ *have* $\chi_i > \overline{\tau}(n_i)(\chi_i \leq \overline{\tau}(n_i))$ *neighbors*
3. *All players of degree* $k_i(g) = 0$ *are* $\theta_i = 1(\theta_i = 0)$

**Example 1** *If a situation in which all players are of degree* $k_i(g) = 2$*, like in a circle (see Fig. 3, middle graph), with just one neighbor of the same type, all players can sustain the action they like. To specialize such a network, it is necessary for a player who dislikes the specialized choice to have both of her neighbors of the opposite type. The same idea and opposite symmetric conditions hold for the case of a network specialized in action* $0$*. Figure 3 illustrates some cases.*

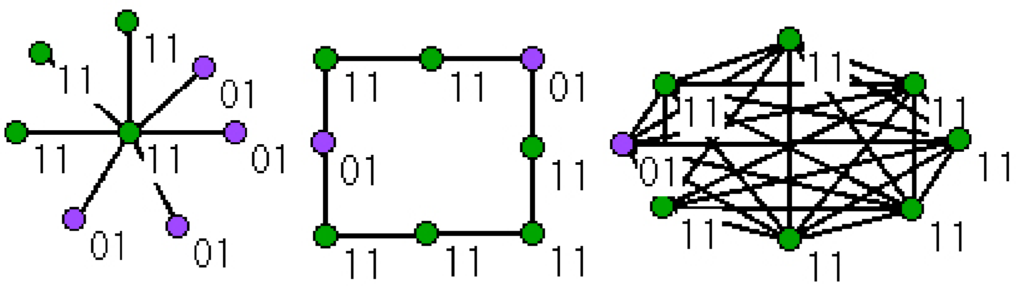


Figure 3: Frustrated Specialized Configurations. The first digit refers to the type, the second digit refers to the action.

**Proposition 4** *The configuration of a network* $\Gamma(\mathbf{N}, \mathbf{g})$ *in equilibrium is* ***satisfactory hybrid*** *in SC (SS) when each player chooses the action corresponding to her type, so that* $a_i = \theta_i$, $\forall\ i \in \{1, \ldots, N\}$. *A network is satisfactory hybrid if and only if the following four conditions are jointly satisfied:*

1. *Players of type* $\theta_i = 1$ *have* $\chi_i \geq \underline{\tau}(n_i)(\chi_i < \overline{\tau}(n_i))$ *neighbors*
2. *Players of type* $\theta_i = 0$ *have* $\chi_i \leq \overline{\tau}(n_i)(\chi_i > \underline{\tau}(n_i))$ *neighbors*
3. *Two players* $i$ *and* $j$ *both of degree* $k(g) = 1$, *such that* $g_{ij} = 1$, *are of the same type:* $\theta_i = \theta_j$
4. *The set of types* $\{\theta = 1\}$ *and* $\{\theta = 0\}$ *are non-empty.*

$S_H$ networks can result in games with SC or SS. This implies that there is multiplicity of equilibria.

**Example 2** *The star network cannot sustain satisfactory profiles in SC unless all players are of the same type, but it does in SS. In a circle where all players are of degree* $k_i(g) = 2$, *specialized configurations are possible if not all players have both of their neighbors of the same type. For higher levels of connectivity, the threshold relation is the only relevant feature. Figure 4 illustrates some cases. The first three graphs of the example correspond to SC and the other three to SS games, although the complete graph configuration (upper row, right) is an equilibrium configuration for both classes of games.*

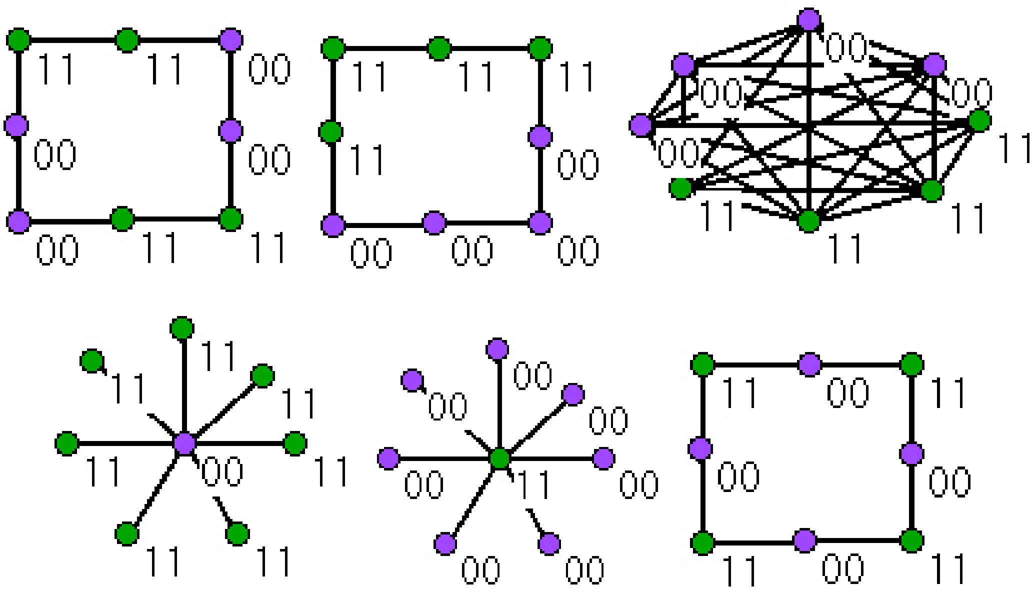

Figure 4: Satisfactory Hybrid Configurations. The first digit refers to the type, the second digit refers to the action.

**Proposition 5** *The configuration of a network* $\Gamma(\mathbf{N}, \mathbf{g})$ *in equilibrium is* ***frustrated hybrid*** *when both actions coexist, and there is at least one player who chooses the action not* *corresponding to her type, so that* $\exists\ i : a_i \neq \theta_i$.

**Example 3** *Again, in SC the star cannot support a frustrated hybrid equilibrium, but in SS it occurs when a peripheral and the central node have the same type. Complete networks can give rise to frustrated hybrid configurations in any class of game. A circle network is* $F_H$ *in SS when a player has two neighbors of her same type, but not all the circle has the same condition. Figure 5 illustrates some cases. The first row of graphs are examples of SC and the second shows graphs for SS.*

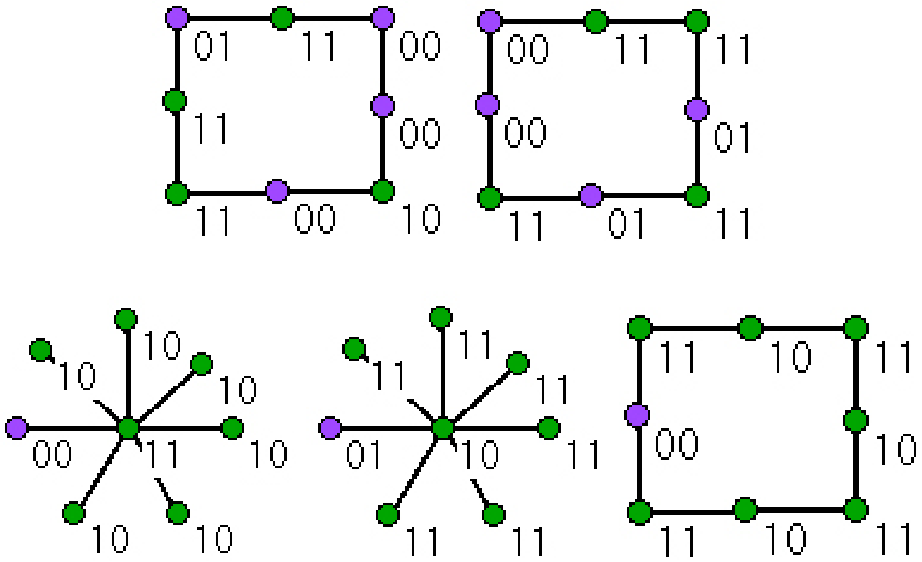


Figure 5: Frustrated Hybrid Configurations

As a final illustration of the conditions for the different types of equilibria to arise, Figure 6 illustrates the distribution of network configurations *in the special case in which all players of a given type have the same range of* $\chi_i$ *neighbors*, either below, between or above the thresholds, and face a player of the opposite type. The graph on the left represents games with SC and the one on the right games with SS. Note that in this situation satisfactory specialized configurations, being a very specific kind of equilibrium, do not appear.

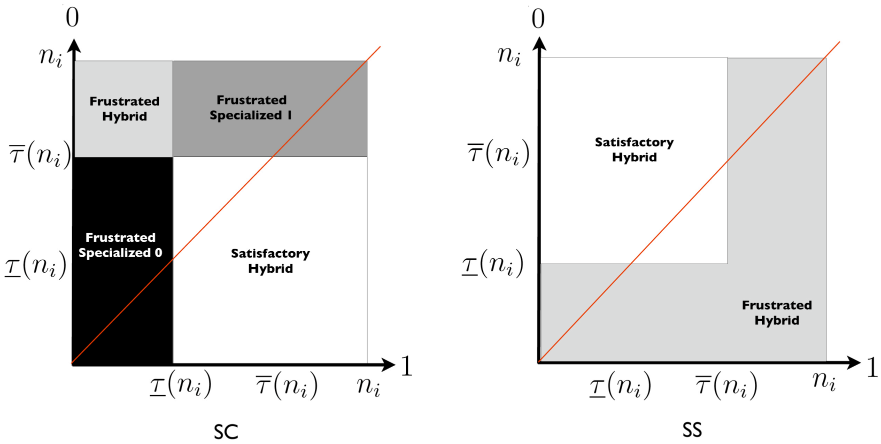


Figure 6: Network Equilibrium Configurations when all players of the same type have the same range of $\chi_i$.

# 3 The Dynamic Model

Up to this point, we have analyzed network games both in SC and SS, finding and classifying all possible pure equilibria. As we may see, there are very many different possibilities for equilibria, and therefore we need some criteria to refine this set and reduce the number of possible equilibrium configurations. In this section we address this issue by introducing a dynamical framework that allows us to define stability. To this end, we begin by considering a discrete set of time $\{0, 1, 2, \ldots\}$. At period $t = 0$ the initial state is a network $\Gamma(\mathbf{N}, \mathbf{g})$ with a distribution of types from the set $\Theta_i = \{0, 1\}$ and an action profile $(\sigma_1^*, \ldots, \sigma_N^*)$ that is a Nash equilibrium, which belongs to the class of either satisfactory or frustrated (hybrid or specialized) configurations. The dynamics is defined as follows: At each period $t \geq 1$ there are two steps:

*Step 1:* With probability $p \in (0, 1)$, two connected players $i, j \in \mathbf{N}$ such that $g_{ij} = 1$, independently receive an opportunity to experiment and change their action. The new actions are denoted by $\tilde{a}_i \neq a_i(t)$ and $\tilde{a}_j \neq a_j(t)$.

*Step 2:* After the mutation has taken place, $i$ and $j$ receive a revision opportunity and each player uses a myopic best response to the mutated action profile. In a myopic best response, as introduced by Ellison (1993), the player that is updating her strategy assumes that the actions of her neighbors in the next time step will be the same they have presently done, and computes what is the best response to those actions. In our specific framework, player $i$ assumes that the mutated action of $j$ and the actions of the rest of her neighbors are fixed and then chooses $a_i(t + 1) = a_i(t)$, returning to the initial choice, or maintains the mutated choice so that $a_i(t + 1) \neq a_i(t)$ (such conditions are symmetric for player $j$), depending on which one is the best response. Both players revise simultaneously. In case any of the mutated actions is retained, the resulting network configuration is let to evolve under myopic best response, choosing a new player at random to update her strategy.

**Remark 4** *In the case when either $i$ or $j$ have degree $k = 0$ the deviation always holds if*

*$x_i(t+1) = \theta_i$. Therefore, the case of an isolated component can be treated as a network on its own, and a network of only one node lacks of any interest. If $g_{ij} = 0$, so that i and j are not neighbors, the mutation corresponds to an equivalent case of a unilateral deviation, which is never a Nash equilibrium; myopic best response dynamics would immediately revert the action to the original choice. The same argument holds if only one player mutates, either i or j, even if they are connected. It is important to note that the resulting network after the dynamics does not need to be a Nash equilibrium configuration, although it can subsequently converge to one.*

**Stable Equilibrium:** *A Nash-network $E(\Gamma)$ is stable if in time $t+1$, for any pair of mutated neighbors $i, j \in \mathbf{N}$ their myopic best response coincides with the previous action: $a_i(t+1) = a_i(t)$ and $a_j(t+1) = a_j(t)$, i.e., the network configuration returns to its initial state. We will denote by $SSC(\Gamma)$ the set of stable networks in games with strategic complements, and $SSS(\Gamma)$ the set of stable networks in games with strategic substitutes.*

## 3.1 Stable Equilibria in SC

As before, we discuss separately the two cases of SS and SC games, beginning with the former.

**Proposition 6** *The set of Stable Equilibria in SC is a proper non-empty set of the Nash equilibrium network: $SSC(\Gamma) \subset E(\Gamma)$*

*Proof*: In order to prove that there exist equilibrium networks that are not stable in SC, we proceed to compute the stability conditions for each possible combination of types and actions. We obtain conditions that depend on $\chi_i$, which is the number of neighbors of player $i$ taking action 1. However, to those conditions one needs to add the corresponding ones arising from the fact that the starting configuration is a Nash equilibrium. These two sets of constraints, taken together, lead to a more restrictive condition than the Nash equilibrium one alone, and therefore, the set of stable networks is included in the set of Nash equilibrium.

The details of the proof go as follows: Let us fix two mutated and connected nodes $i, j \in \mathbf{N}$. First, take the same type $\theta_i = \theta_j$ and the same associated actions $a_i = a_j$ for 0 or 1. Second, one has to consider the case of type inequality, $\theta_i \neq \theta_j$, and explore two cases: When each agent chooses her preferred action, $\theta_i = a_i, \theta_j = a_j$, and the opposite case, $\theta_i \neq a_i$ or $\theta_j \neq a_j$. This study of six cases covers the most relevant instances and the remaining possibilities can be checked in the same way. We now go into the details of these cases, whereas the complete set of conditions is provided in table 3.

**I-** *Let $i, j \in \mathbf{N}$ be two connected nodes, $g_{ij} = 1$, such that $i \neq j$ and $\theta_i = \theta_j = 1$.* Suppose that the choices made by the pair of players at time $t$ are $a_i = a_j = 1$. The dynamics act as follows:

*Step 1:* A mutation affects $(i, j) \Rightarrow \tilde{a}_i = \tilde{a}_j = 0$.

*Step 2:* Myopic Best Response (MBR). The new actions are: Player at node $i$ plays $a_i(t+1) = 1$ (same condition holds for player $j$), iff:

$$\begin{aligned} \chi_i - 1 &\geq \underline{\tau}(n_i) = \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} \rceil \Rightarrow \\ \chi_i &\geq \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} \rceil + 1 = \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} + 1 \rceil = \lceil \frac{\beta(n_i+2)}{\alpha+\beta} \rceil. \end{aligned} \tag{20}$$

**II-** *Let $i, j \in \mathbf{N}$ be two connected nodes, $g_{ij} = 1$, such that $i \neq j$ and $\theta_i = \theta_j = 0$.* Suppose that the choices made by the pair of players at time $t$ are $a_i = a_j = 0$. The dynamics act as follows:

*Step 1:* A mutation affects $(i, j) \Rightarrow \tilde{a}_i = \tilde{a}_j = 1$.

*Step 2:* MBR: Player at node $i$ plays $a_i(t+1) = 0$ (same condition holds for player $j$), iff:

$$\begin{aligned} \chi_i + 1 &\leq \overline{\tau}(n_i) = \lfloor \frac{\alpha}{\alpha+\beta} n_i + \frac{\alpha-\beta}{\alpha+\beta} \rfloor \Rightarrow \\ \chi_i &\leq \lfloor \frac{\alpha}{\alpha+\beta} n_i + \frac{\alpha-\beta}{\alpha+\beta} \rfloor - 1 = \lfloor \frac{\alpha}{\alpha+\beta} n_i + \frac{\alpha-\beta}{\alpha+\beta} - 1 \rfloor = \lfloor \frac{\alpha n_i - 2\beta}{\alpha+\beta} \rfloor. \end{aligned} \tag{21}$$

**III-** *Let $i, j \in \mathbf{N}$ be two connected nodes, $g_{ij} = 1$, such that $i \neq j$ and $\theta_i = \theta_j = 1$.* Suppose that the choices made by the pair of players at time $t$ are $a_i = 1 = \theta_i, a_j = 0 \neq \theta_j$. The dynamics act as follows:

*Step 1:* A mutation affects $(i, j) \Rightarrow \tilde{a}_i = 0, \tilde{a}_j = 1$.

*Step 2:* MBR: Player at node $i$ plays $a_i(t+1) = 1$ iff:

$$\begin{aligned} \chi_i + 1 &\geq \underline{\tau}(n_i) = \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} \rceil \Rightarrow \\ \chi_i &\geq \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} \rceil - 1 = \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} - 1 \rceil = \lceil \frac{\beta n_i - 2\alpha}{\alpha+\beta} \rceil. \end{aligned} \tag{22}$$

Player at node $j$ plays $a_j(t+1) = 0$ iff:

$$\begin{aligned} \chi_j - 1 &< \underline{\tau}(n_j) = \lceil \frac{\beta}{\alpha+\beta} n_j - \frac{\alpha-\beta}{\alpha+\beta} \rceil \Rightarrow \\ \chi_j &< \lceil \frac{\beta}{\alpha+\beta} n_j - \frac{\alpha-\beta}{\alpha+\beta} \rceil + 1 = \lceil \frac{\beta}{\alpha+\beta} n_j - \frac{\alpha-\beta}{\alpha+\beta} + 1 \rceil = \lceil \frac{\beta(n_j+2)}{\alpha+\beta} \rceil. \end{aligned} \tag{23}$$

**IV-** *Let $i, j \in \mathbf{N}$ be two connected nodes, $g_{ij} = 1$, such that $i \neq j$ and $\theta_i = \theta_j = 0$.* Suppose that the choices made by the pair of players at time $t$ are $a_i = a_j = 1 \neq \theta_i = \theta_j$. The dynamics act as follows:

*Step 1:* A mutation affects $(i, j) \Rightarrow \tilde{a}_i = \tilde{a}_j = 0$.

*Step 2:* MBR : Player at node $i$ plays $a_i(t+1) = 1$ (same condition holds for player $j$), iff:

$$\begin{aligned}
\chi_i - 1 &> \overline{\tau}(n_i) = \lfloor \frac{\alpha}{\alpha+\beta} n_i + \frac{\alpha-\beta}{\alpha+\beta} \rfloor \Rightarrow \\
\chi_i &> \lfloor \frac{\alpha}{\alpha+\beta} n_i + \frac{\alpha-\beta}{\alpha+\beta} \rfloor + 1 = \lfloor \frac{\alpha}{\alpha+\beta} n_i + \frac{\alpha-\beta}{\alpha+\beta} + 1 \rfloor = \lfloor \frac{\alpha n_i - 2\beta}{\alpha+\beta} \rfloor. \qquad (24)
\end{aligned}$$

**V-** *Let $i, j \in \mathbf{N}$ be two connected nodes, $g_{ij} = 1$, such that $i \neq j$ and $\theta_i = 1, \theta_j = 0$.* Suppose that the choices made by the pair of players at time $t$ are $a_i = 1, a_j = 0$. The dynamics act as follows:

*Step 1:* A mutation affects $(i, j) \Rightarrow \tilde{a}_i = 0, \tilde{a}_j = 1$.

*Step 2:* MBR: Player at node $i$ plays $a_i(t+1) = 1$ iff:

$$\begin{aligned}
\chi_i + 1 &\geq \underline{\tau}(n_i) = \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} \rceil \Rightarrow \\
\chi_i &\geq \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} \rceil - 1 = \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} - 1 \rceil = \lceil \frac{\beta n_i - 2\alpha}{\alpha+\beta} \rceil. \qquad (25)
\end{aligned}$$

Player at node $j$ plays $a_j(t+1) = 0$ iff:

$$\begin{aligned}
\chi_j - 1 &\leq \overline{\tau}(n_j) = \lfloor \frac{\alpha}{\alpha+\beta} n_j + \frac{\alpha-\beta}{\alpha+\beta} \rfloor \Rightarrow \\
\chi_j &\leq \lfloor \frac{\alpha}{\alpha+\beta} n_j + \frac{\alpha-\beta}{\alpha+\beta} \rfloor + 1 = \lfloor \frac{\alpha}{\alpha+\beta} n_j + \frac{\alpha-\beta}{\alpha+\beta} + 1 \rfloor = \lfloor \frac{\alpha(n_j+2)}{\alpha+\beta} \rfloor. \qquad (26)
\end{aligned}$$

**VI-** *Let $i, j \in \mathbf{N}$ be two connected nodes, $g_{ij} = 1$, such that $i \neq j$ and $\theta_i = 1, \theta_j = 0$.* Suppose that the choices made by the pair of players at time $t$ are $a_i = 0 \neq \theta_i, a_j = 1 \neq \theta_j$. The dynamics act as follows:

*Step 1:* A mutation affects $(i, j) \Rightarrow \tilde{a}_i = 1, \tilde{a}_j = 0$

*Step 2:* MBR: Player at node $i$ plays $a_i(t+1) = 0$ iff:

$$\begin{aligned}
\chi_i - 1 &< \underline{\tau}(n_i) = \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} \rceil \Rightarrow \\
\chi_i &< \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} \rceil + 1 = \lceil \frac{\beta}{\alpha+\beta} n_i - \frac{\alpha-\beta}{\alpha+\beta} + 1 \rceil = \lceil \frac{\beta(n_i+2)}{\alpha+\beta} \rceil. \qquad (27)
\end{aligned}$$

Player at node $j$ plays $a_j(t+1) = 1$ iff:

$$\begin{aligned}
\chi_j + 1 &> \overline{\tau}(n_j) = \lfloor \frac{\alpha}{\alpha+\beta} n_j + \frac{\alpha-\beta}{\alpha+\beta} \rfloor \Rightarrow \\
\chi_j &> \lfloor \frac{\alpha}{\alpha+\beta} n_j + \frac{\alpha-\beta}{\alpha+\beta} \rfloor - 1 = \lfloor \frac{\alpha}{\alpha+\beta} n_j + \frac{\alpha-\beta}{\alpha+\beta} - 1 \rfloor = \lfloor \frac{\alpha n_j - 2\beta}{\alpha+\beta} \rfloor. \qquad (28)
\end{aligned}$$

| | $\theta_i = \theta_j = 1$ | $\theta_i = \theta_j = 0$ | $\theta_i = 1, \theta_j = 0$ |
|---|---|---|---|
| $a_i = a_j = 1$ | $\chi_i \geq \underline{\tau}(n_i) + 1$<br>$\chi_j \geq \underline{\tau}(n_j) + 1$ | $\chi_i > \overline{\tau}(n_i) + 1$<br>$\chi_j > \overline{\tau}(n_j) + 1$ | $\chi_i \geq \underline{\tau}(n_i) + 1$<br>$\chi_j > \overline{\tau}(n_j) + 1$ |
| $a_i = a_j = 0$ | $\chi_i < \underline{\tau}(n_i) - 1$<br>$\chi_j < \underline{\tau}(n_j) - 1$ | $\chi_i \leq \overline{\tau}(n_i) - 1$<br>$\chi_j \leq \overline{\tau}(n_j) - 1$ | $\chi_i < \underline{\tau}(n_i) - 1$<br>$\chi_j \leq \overline{\tau}(n_j) - 1$ |
| $a_i = 1, a_j = 0$ | * $\chi_i \geq \underline{\tau}(n_i) - 1$<br>* $\chi_j < \underline{\tau}(n_j) + 1$ | * $\chi_i > \overline{\tau}(n_i) - 1$<br>* $\chi_j \leq \overline{\tau}(n_j) + 1$ | * $\chi_i \geq \underline{\tau}(n_i) - 1$<br>* $\chi_j \leq \overline{\tau}(n_j) + 1$ |
| $a_i = 0, a_j = 1$ | * $\chi_i < \underline{\tau}(n_i) + 1$<br>* $\chi_j \geq \underline{\tau}(n_j) - 1$ | * $\chi_i \geq \overline{\tau}(n_i) + 1$<br>* $\chi_j > \overline{\tau}(n_j) - 1$ | * $\chi_i < \underline{\tau}(n_i) + 1$<br>* $\chi_j > \overline{\tau}(n_j) - 1$ |

Table 3: Conditions for a network configuration to be stable under every specific type of mutation. SC case.

Table 3 below summarizes the stability conditions for the six cases we have presented as examples and for the remaining ones.

Finally, to the above stability conditions one has to add the constraints arising from the fact that the original configuration is a Nash equilibrium, and those constraints actually forbid some of the possibilities included in Table 3 to take place. Recalling the conditions of Nash equilibria discussed in the preceding section, one finds the results collected in Table 4, that presents the necessary and sufficient conditions for the mutations to be accepted and the initial configurations in which each case can occur. This is a more restrictive set of conditions on the configurations, compatible of course with those for the configuration to be a Nash equilibrium, and therefore this is a proper subset of the equilibrium configurations, as we wanted to prove. □

**Remark 5** *For the cases marked with a star "*" in Table 3, one can check that when two players are making opposite choices, independently of their type, none has incentives to hold to the mutated choice in a Myopic Best Response, and then, on the contrary, they return to to their initial state. That is, a mutation of a neighbor supports a player's initial choice, independently if that choice is an action she likes or not, when they are initially anti-coordinating.*

| | $\theta_i = \theta_j = 1$ | $\theta_i = \theta_j = 0$ | $\theta_i = 1, \theta_j = 0$ |
|---|---|---|---|
| $a_i = a_j = 1$ | $\chi_i = \underline{\tau}(n_i)$<br>$\chi_j = \underline{\tau}(n_j)$ | $\chi_i = \overline{\tau}(n_i) + 1$<br>$\chi_j = \overline{\tau}(n_j) + 1$ | $\chi_i = \underline{\tau}(n_i)$<br>$\chi_j = \overline{\tau}(n_j) + 1$ |
| *Configurations* | $S_{S1}, F_{S1}, F_H, S_H$ | $F_{S1}, F_H$ | $F_{S1}, F_H$ |
| $a_i = a_j = 0$ | $\chi_i = \underline{\tau}(n_i) - 1$<br>$\chi_j = \underline{\tau}(n_j) - 1$ | $\chi_i = \overline{\tau}(n_i)$<br>$\chi_j = \overline{\tau}(n_j)$ | $\chi_i = \underline{\tau}(n_i) - 1$<br>$\chi_j = \overline{\tau}(n_j)$ |
| *Configurations* | $F_{S0}, H_F$ | $S_{S0}, F_{S0}, F_H, S_H$ | $F_{S0}, S_H$ |

Table 4: Conditions for mutations on a Nash equilibrium network configuration to be accepted. SC case.

We have just studied the possible mutations that can occur when two randomly chosen neighbors are allowed to experiment. Notice that the initial configurations depend on the distribution of types and choices made by the pair of randomly chosen neighbors. Hence, for example, no case in the first row in Table 4 can arise from a $F_{S0}$ as the initial condition, because at least two players $(i, j)$ must be choosing action *one*.

Following these analysis, the question arises as to what is the nature of the stable configurations. To gain insight into this issue, let us consider the resulting states and conditions for each particular case of mutation in SC, and illustrate the types of networks resulting after at least one of the two mutations holds. As we have found above and collected in Table 4, there are six possible mutations that can occur in games with SC. Taking into account the conditions necessary for them to be viable, the possible mutations are as follows:

$$\begin{cases} M1: \theta_i = \theta_j = 1 \quad \text{and} \quad a_i = a_j = 1 \Rightarrow \tilde{a}_i = \tilde{a}_j = 0, \\ M2: \theta_i = \theta_j = 0 \quad \text{and} \quad a_i = a_j = 1 \Rightarrow \tilde{a}_i = \tilde{a}_j = 0, \\ M3: \theta_i = 1, \theta_j = 0 \text{ and} \quad a_i = a_j = 1 \Rightarrow \tilde{a}_i = \tilde{a}_j = 0, \\ M4: \theta_i = \theta_j = 1 \quad \text{and} \quad a_i = a_j = 0 \Rightarrow \tilde{a}_i = \tilde{a}_j = 1, \\ M5: \theta_i = \theta_j = 0 \quad \text{and} \quad a_i = a_j = 0 \Rightarrow \tilde{a}_i = \tilde{a}_j = 1, \\ M6: \theta_i = 1, \theta_j = 0 \text{ and} \quad a_i = a_j = 0 \Rightarrow \tilde{a}_i = \tilde{a}_j = 1. \end{cases} \tag{29}$$

Consider now the possible resulting states after one of the six aforementioned mutations has occurred, and let us focus on the frequency in which each specific configuration (which, once again, needs not be a Nash equilibrium) can result after a mutation takes place. Figure 7 collects the possible transitions that can take place between types of SC upon acceptance of mutations.

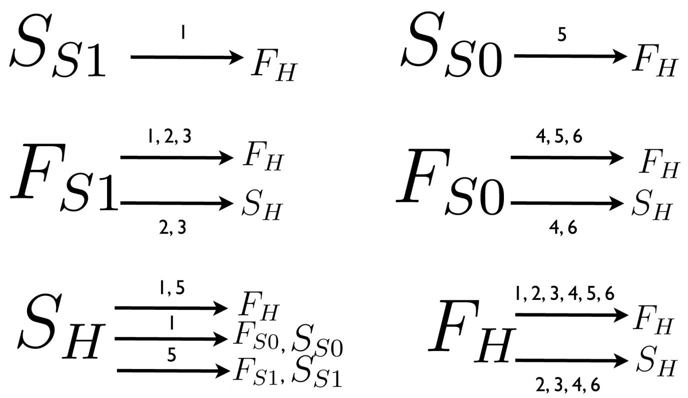


Figure 7: Resulting configurations in SC games after mutations are accepted.

The question then arises as to what is the relative frequency of the transformation processes of one type of network to a different one. To answer this question, let us consider the following set of conditions:

$$\begin{cases} C1: \nexists k \neq i \in \mathbf{N} | \theta_k = 1, \\ C2: \nexists k \neq j \in \mathbf{N} | \theta_k = 1, \\ C3: \nexists k \neq i \in \mathbf{N} | \theta_k = 0, \\ C4: \nexists k \neq j \in \mathbf{N} | \theta_k = 0, \\ C5: a_i(t+1) = 1, \\ C6: a_j(t+1) = 1, \\ C7: a_i(t+1) = 0, \\ C8: a_j(t+1) = 0. \end{cases} \tag{30}$$

With this notation, it is straightforward to see that the following transitions are possible:

- $\mathbf{F_{S1}} \to S_H$ iff conditions $c3, c4, c7, c8$ hold for mutation 2 (M2) in Eq. (29), or iff $c4, c8$ in M1.
- $\mathbf{F_{S1}} \to F_H$ iff $c7$ or $c7, c8$ hold in M3.
- $\mathbf{S_0} \to S_H$ iff $c1, c2, c7, c8$ hold in M4, or iff $c1, c5, c8$ hold in M6.
- $\mathbf{S_H} \to F_{S0}$ iff $c1, c2, c5, c6$ hold in M1.
- $\mathbf{S_H} \to F_{S1}$ iff $c3, c4, c5, c6$ hold in M5.
- $\mathbf{S_H} \to S_H$ iff $c3, c4, c7, c8$ hold in M2, or iff $c1, c2, c7, c8$ hold in M4, or iff $c1, c5, c8$ in M6.
- $\mathbf{S_H} \to F_{S1}$ iff $c4, c5, c6$ hold in M6.

In the remaining cases if one or both mutations hold, the resulting configuration will necessarily be frustrated hybrid. From this enumeration and the list of possible processes, it becomes clear that, generally speaking, the result of the acceptance of a mutation will very often be frustrated hybrid, and, on the other hand, specialized configurations are by far the most unstable ones.

In view of the large number of conditions and possibilities we have summarized above, we find it illuminating to discuss a few examples of the mutation process.

**Example 4** *Let $\Gamma$ be a complete frustrated specialized network $F_{S1}$ (see Fig. 8). There is a pair of neighbor players $\theta_i = 1$ and $\theta_j = 0$ whose actions mutate. Both players then revise simultaneously and follow their myopic best response. For definiteness, let us choose $2\alpha \geq 3\beta$, so that player $i$ has incentives to return to the same action she had in time $t$ in which $a_i(t) = a_i(t+1) = \theta_i$, however, player $j$, responding myopically to the action she has observed in her present time, chooses $a_j(t+1) = \theta_j$. The resulting state is a frustrated hybrid configuration that is not a Nash equilibrium.*

**Example 5** *Let $\Gamma$ be a complete satisfactory hybrid network $S_H$ (see Fig. 9). There is a pair of neighbor players $\theta_i = 1$ and $\theta_j = 0$ whose actions mutate. Both players then revise simultaneously and follow their myopic best response. Consider again that $2\alpha \geq 3\beta$,*

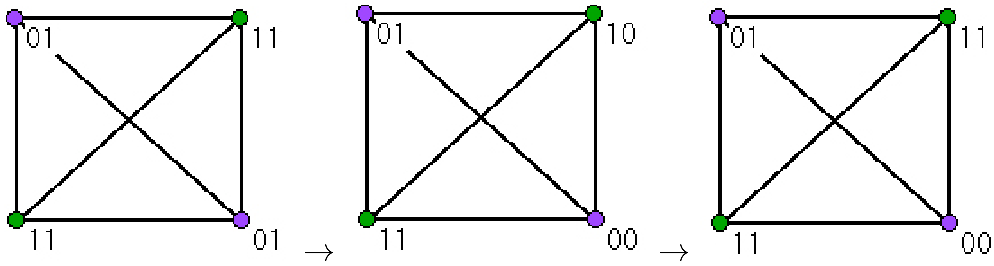


Figure 8: Discussion of an example mutation in a network that is frustrated specialized in action 1.

*so that player $i$ has incentives to return to the same action she had in time $t$ in which $a_i(t) = a_i(t+1) = \theta_i$. In a different example, if we had $3\alpha > 2\beta$, player $j$, responding myopically, returns as well to the initial choice $a_j(t) = a_j(t+1) = \theta_j$. The resulting state is the same initial satisfactory configuration.*

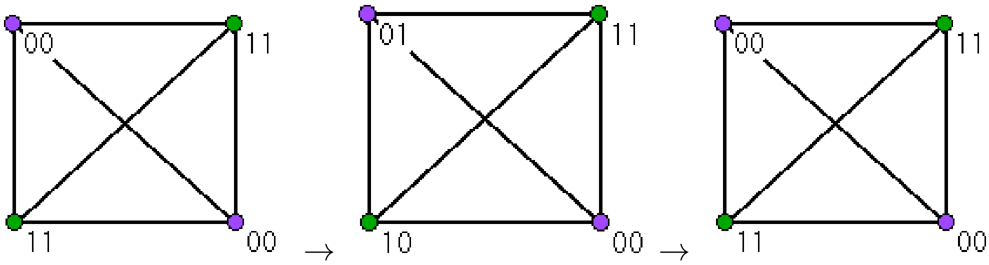


Figure 9: Discussion of an example mutation in a network that is satisfactory hybrid.

**Example 6** *Let $\Gamma$ be a complete satisfactory hybrid network $S_H$ (see Fig. 10). There is a pair of neighbor players $\theta_i = \theta_j = 1$ whose actions mutate. Both players then revise simultaneously and follow a myopic best response in the procedure. By the initial assumptions of the model $\alpha < 4\beta$ so that no player has incentives to return to the same action she had in time $t$ and the network configuration results in a frustrated specialized configuration in action zero, $S_0$ which is a Nash equilibrium.*

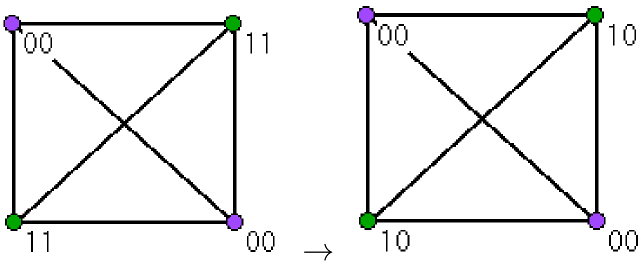


Figure 10: Discussion of an example mutation in a network that is satisfactory hybrid. In this example the mutation is accepted, opposite to what we had in the previous one.

**Example 7** *Let $\Gamma$ be a frustrated specialized network $F_{S1}$ (see Fig. 11). There is a pair of neighbor players $\theta_i = 1$ and $\theta_j = 0$ whose actions mutate. Both players then revise* *simultaneously and follow a myopic best response in the procedure. Consider that $2\alpha < 4\beta$, so that player $i$ has incentives to return to the same action she had in time $t$ in which $a_i(t) \neq \theta_i$, however, player $j$, responding myopically chooses $a_j(t+1) = \theta_j$, and the resulting state is a $F_H$ configuration that is not a Nash equilibrium.*

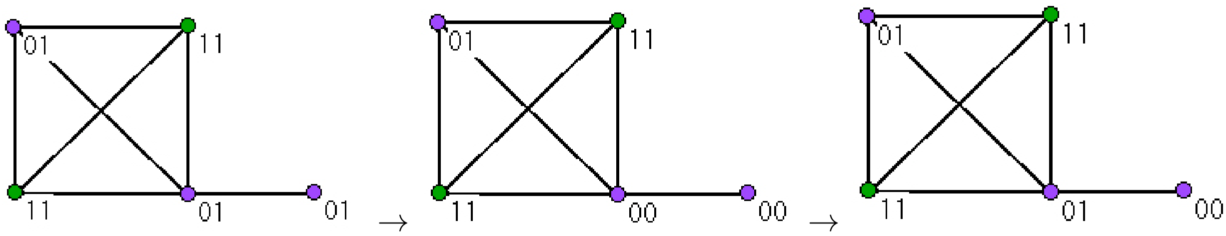


Figure 11: Discussion of an example mutation in a network that is frustrated specialized in the action 1.

## 3.2 Stable Equilibria in SS

**Proposition 7** *The set of Stable Equilibria in SS is a proper non-empty set of the Nash equilibrium network: $SSS(\Gamma) \subset E(\Gamma)$.*

The calculations needed to prove this result are essentially the same as we described in detail for the case of SC, and therefore we omit them, quoting directly the stability conditions and the table of mutations that can be accepted when the network configuration is a Nash equilibrium:

| | $\theta_i = \theta_j = 1$ | $\theta_i = \theta_j = 0$ | $\theta_i = 1, \theta_j = 0$ |
|---|---|---|---|
| $\mathbf{a_i = a_j = 1}$ | * $\chi_i \leq \overline{\tau}(n_i) + 1$<br>* $\chi_j \leq \overline{\tau}(n_j) + 1$ | * $\chi_i < \underline{\tau}(n_i) + 1$<br>* $\chi_j < \underline{\tau}(n_j) + 1$ | * $\chi_i \leq \overline{\tau}(n_i) - 1$<br>* $\chi_j < \underline{\tau}(n_j) + 1$ |
| $\mathbf{a_i = a_j = 0}$ | * $\chi_i > \overline{\tau}(n_i) - 1$<br>* $\chi_j > \overline{\tau}(n_j) - 1$ | * $\chi_i \geq \underline{\tau}(n_i) - 1$<br>* $\chi_j \geq \underline{\tau}(n_j) - 1$ | * $\chi_i > \overline{\tau}(n_i) + 1$<br>* $\chi_j \geq \underline{\tau}(n_j) - 1$ |
| $\mathbf{a_i = 1, a_j = 0}$ | $\chi_i \leq \overline{\tau}(n_i) - 1$<br>$\chi_j > \overline{\tau}(n_j) + 1$ | $\chi_i < \underline{\tau}(n_i) - 1$<br>$\chi_j \geq \underline{\tau}(n_j) + 1$ | $\chi_i \leq \overline{\tau}(n_i) - 1$<br>$\chi_j \geq \underline{\tau}(n_j) + 1$ |
| $\mathbf{a_i = 0, a_j = 1}$ | $\chi_i > \overline{\tau}(n_i) + 1$<br>$\chi_j \leq \overline{\tau}(n_j) + 1$ | $\chi_i \geq \underline{\tau}(n_i) + 1$<br>$\chi_j < \underline{\tau}(n_j) - 1$ | $\chi_i > \overline{\tau}(n_i) + 1$<br>$\chi_j < \overline{\tau}(n_j) - 1$ |

Table 5: Conditions for a network configuration to be stable under every specific type of mutation. SS case.

**Remark 6** *In the cases marked with "$*$" one can observe that when the two players are making the same choice, none has incentives to maintain the mutated action in a MBR. Table 4 presents the conditions necessary for the mutations to be accepted and the initial configurations in which each case can occur.*

As in the SC case, we now consider the resulting states and conditions for each particular case of mutation in SS, and illustrate the resulting types of networks after at least one of

| | $\boldsymbol{\theta_i = \theta_j = 1}$ | $\boldsymbol{\theta_i = \theta_j = 0}$ | $\boldsymbol{\theta_i = 1, \theta_j = 0}$ |
|---|---|---|---|
| $\mathbf{a_i = 1, a_j = 0}$ | $\chi_i = \overline{\tau}(n_i)$<br>$\chi_j = \overline{\tau}(n_j) + 1$ | $\chi_i = \underline{\tau}(n_i) - 1$<br>$\chi_j = \underline{\tau}(n_j)$ | $\chi_i = \overline{\tau}(n_i)$<br>$\chi_j = \underline{\tau}(n_j) + 1$ |
| *Configurations* | $F_H$ | $F_H$ | $F_H, S_H$ |
| $\mathbf{a_i = 0, a_j = 1}$ | $\chi_i = \overline{\tau}(n_i) + 1$<br>$\chi_j = \overline{\tau}(n_j)$ | $\chi_i = \underline{\tau}(n_i)$<br>$\chi_j = \underline{\tau}(n_j) - 1$ | $\chi_i = \overline{\tau}(n_i) + 1$<br>$\chi_j = \underline{\tau}(n_j) - 1$ |
| *Configurations* | $F_H$ | $F_H$ | $F_H$ |

Table 6: Conditions for mutations on a Nash equilibrium network configuration to be accepted. SS case.

the two mutations holds. We enumerate the six possible mutations as follows:

$$\begin{cases} M1 : \theta_i = \theta_j = 1 & \text{and } a_i = 1, a_j = 0 \Rightarrow \tilde{a}_i = 0, \tilde{a}_j = 1, \\ M2 : \theta_i = \theta_j = 0 & \text{and } a_i = 1, a_j = 0 \Rightarrow \tilde{a}_i = 0, \tilde{a}_j = 1, \\ M3 : \theta_i = 1, \theta_j = 0 & \text{and } a_i = 1, a_j = 0 \Rightarrow \tilde{a}_i = 0, \tilde{a}_j = 1, \\ M4 : \theta_i = \theta_j = 1 & \text{and } a_i = 0, a_j = 1 \Rightarrow \tilde{a}_i = 1, \tilde{a}_j = 0, \\ M5 : \theta_i = \theta_j = 0 & \text{and } a_i = 0, a_j = 1 \Rightarrow \tilde{a}_i = 1, \tilde{a}_j = 0, \\ M6 : \theta_i = 1, \theta_j = 0 & \text{and } a_i = 0, a_j = 1 \Rightarrow \tilde{a}_i = 1, \tilde{a}_j = 0. \end{cases} \tag{31}$$

The resulting network configurations for the previously enumerated mutations do not need to be Nash equilibrium. As before, we can now analyze the frequency in which each specific configuration can result after a mutation takes place in SS. The possible transitions are summarized in Fig. 12.

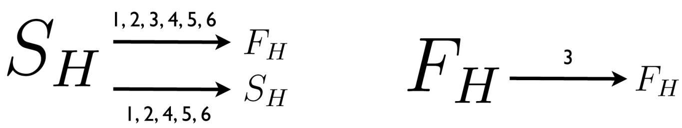


Figure 12: Resulting States (SS)

Consider the following set of conditions

$$\begin{cases} C1 : \nexists k \neq i \in \mathbf{N} | \theta_k = 1, \\ C2 : \nexists k \neq j \in \mathbf{N} | \theta_k = 1, \\ C3 : \nexists k \neq i \in \mathbf{N} | \theta_k = 0, \\ C4 : \nexists k \neq j \in \mathbf{N} | \theta_k = 0, \\ C5 : a_i(t+1) = 1, \\ C6 : a_j(t+1) = 1, \\ C7 : a_i(t+1) = 0, \\ C8 : a_j(t+1) = 0. \end{cases} \tag{32}$$

Therefore, we have

- $\mathbf{F_H} \to S_H$ iff conditions $c1, c2, c5, c6$ hold in M1, or iff $c3, c4, c7, c8$ hold in M2, or iff $c1, c2$ hold in M4, or iff $c1, c5$ hold in M5, or iff $c1, c2, c5$ hold in M6.

In the remaining cases if one or both mutations hold, the resulting configuration will be frustrated hybrid. This result will be clearer with the following examples:

**Example 8** *Let $\Gamma$ be a complete satisfactory hybrid network $S_H$ (see Fig. 13). There is a pair of neighbor players $\theta_i = 1$ and $\theta_j = 0$ whose actions mutate. Both players then revise simultaneously and follow a myopic best response in the procedure. Consider for the example that $2\alpha \geq 3\beta$, so that player $i$ has incentives to return to the same action she had in time $t$ in which $a_i(t) = a_i(t+1) = \theta_i$, and $j$ as well, $a_j(t) = a_j(t+1) = \theta_j$. The resulting state is equal to the initial satisfactory network, which is a Nash equilibrium.*

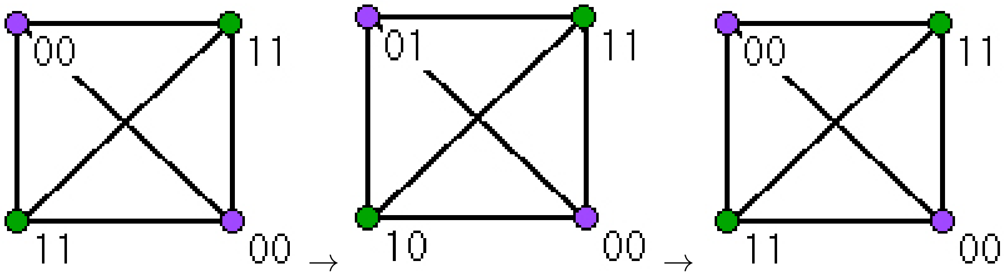


Figure 13: Discussion of an example mutation in a network that is satisfactory hybrid.

**Example 9** *Let $\Gamma$ be a complete satisfactory hybrid network $S_H$ (see Fig. 14). There is a pair of neighbor players $\theta_i = \theta_j = 0$ whose actions mutate. Both players then revise simultaneously and follow a myopic best response in the procedure. None have incentives to maintain the new action but rather return to the action in time $t$ in which $a_i(t) = a_i(t+1) = \theta_i$ and $a_j(t) = a_j(t+1) = \theta_j$. The resulting state is the same initial satisfactory configuration.*

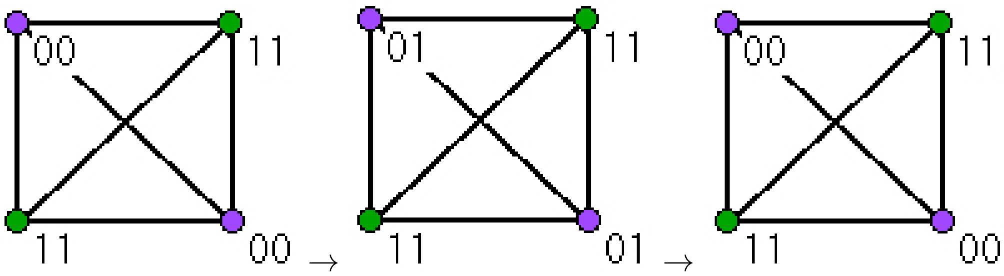


Figure 14: Discussion of an example mutation in a network that is satisfactory hybrid.

**Example 10** *Let $\Gamma$ be a frustrated hybrid network $F_H$ (see Fig. 15). There is a pair of neighbor players $\theta_i = \theta_j = 1$ whose actions mutate. Both players then revise simultaneously and follow a myopic best response in the procedure. Player $i$ has incentives to return to the same choice she had in time $t$ which corresponded to her type, $a_i(t) = a_i(t+1) = \theta_i$ while player $j$ prefers to hold to the new adopted behavior given her myopic best response, so that $a_i(t) \neq a_i(t+1) = \theta_i$. The resulting state is a satisfactory hybrid configuration that is not a Nash equilibrium.*

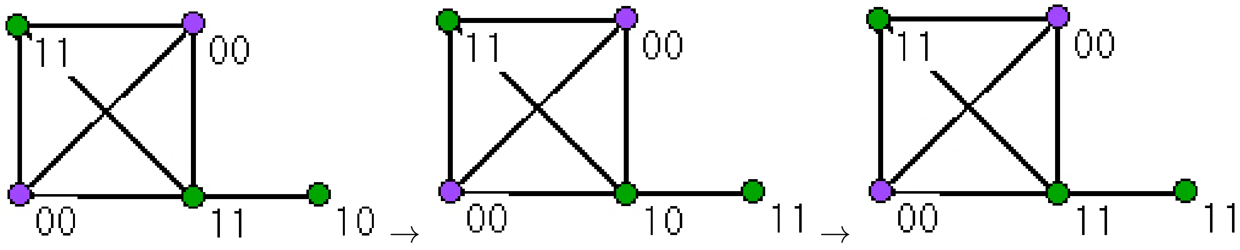


Figure 15: Discussion of an example mutation in a network that is frustrated hybrid. In this case the mutation is accepted.

# 4 Welfare Analysis

In the above section, we were concerned on individual choices and how they may alter the final configuration. Actually, we study refinements using a dynamic process. As we have already said, we obtain multiple equilibria configurations, therefore a natural question that arises is which classes of configurations give the highest utility for the whole network. The last step in the program of our research is to study the welfare provided by the Nash equilibrium configurations to the collective of players.

Welfare analysis is an important feature in economic studies and in particular in network setups. There is a vast literature where different criteria to measure welfare are employed. In this section, we analyze the set of Nash networks $E(\Gamma)$, through a utilitarian welfare function. We will consider welfare in society as the addition of individual payoffs: $W = \sum_{i=1}^{N} v_i$. In order to assess the welfare in each configuration we separate each player $i \in \mathbf{N} = \{1, \ldots, N\}$ in a network game $\Gamma$ into four classifications given their type and choice. These classifications are associated to the four payoff functions expressed both in games with SC or SS. Let $N_r^s$ be the set of agents with type $r$ and action $s$, for $r, s \in \{0, 1\}$. Namely, we have the four following sets:

$$\begin{cases} N_0^0 = \{i \in \mathbf{N} : \theta_i = a_i = 0\}, \\ N_1^1 = \{i \in \mathbf{N} : \theta_i = a_i = 1\}, \\ N_1^0 = \{i \in \mathbf{N} : 1 = \theta_i \neq a_i = 0\}, \\ N_0^1 = \{i \in \mathbf{N} : 0 = \theta_i \neq a_i = 1\}. \end{cases} \tag{33}$$

Notice that $N = N_0^0 \cup N_1^1 \cup N_1^0 \cup N_0^1$.

The welfare function for games with SC or SS can be expressed as:

$$W = \sum_{i=1}^{N} v_i = \sum_{i \in N_0^0} v_i + \sum_{i \in N_1^1} v_i + \sum_{i \in N_1^0} v_i + \sum_{i \in N_0^1} v_i, \tag{34}$$

## 4.1 Strategic Complements: Welfare Functions

As it has been presented above, games with SC portray six different configurations: satisfactory specialized in one ($S_{S1}$) or in zero ($S_{S0}$), frustrated specialized in one ($F_{S1}$) or in zero ($F_{S0}$), satisfactory hybrid ($S_H$), and frustrated hybrid ($F_H$). In games with strategic complements the welfare function is:

$$W_{sc} = \sum_{i \in N_0^0} \alpha(1 + n_i - \chi_i) + \sum_{i \in N_1^1} \alpha(1 + \chi_i) + \sum_{i \in N_1^0} \beta(1 + n_i - \chi_i) + \sum_{i \in N_0^1} \beta(1 + \chi_i)$$

The following are the welfare functions for each configuration of SC network games in equilibrium:

**I-** *Satisfactory specialized welfare function:* Let a network $\Gamma$ be $S_{S0}$ iff $\{N_0^0 \neq \emptyset, N_1^1 = \emptyset, N_1^0 = \emptyset, N_0^1 = \emptyset\}$ so that:

$$W(S_{S0}) = \sum_{i=1}^{N} v_i = \sum_{i \in N_0^0} \alpha(1 + n_i)$$

or let $\Gamma$ be $S_{S1}$ iff $\{N_1^1 \neq \emptyset, N_0^0 = \emptyset, N_1^0 = \emptyset, N_0^1 = \emptyset\}$ so that:

$$W(S_{S1}) = \sum_{i=1}^{N} v_i = \sum_{i \in N_1^1} \alpha(1 + n_i)$$

**II-** *Frustrated specialized welfare function:* Let a network $\Gamma$ be $F_{S0}$ iff $\{N_0^0 \neq \emptyset, N_1^1 = \emptyset, N_1^0 \neq \emptyset, N_0^1 = \emptyset\}$ so that:

$$W(F_{S0}) = \sum_{i \in N_0^0} \alpha(1 + n_i - \chi_i) + \sum_{i \in N_1^0} \beta(1 + n_i - \chi_i)$$

or let $\Gamma$ be $F_{S1}$ iff $\{N_1^1 \neq \emptyset, N_0^0 = \emptyset, N_1^0 = \emptyset, N_0^1 \neq \emptyset\}$ so that:

$$W(F_{S1}) = \sum_{i \in N_1^1} \alpha(1 + \chi_i) + \sum_{i \in N_0^1} \beta(1 + \chi_i)$$

**Remark 7** *Since $N_0^1 \neq \emptyset$ in **II** $\Rightarrow N = N_1^1 \cup N_0^1$ and an addition with a $\beta$ condition appears. Notice that as $\beta < \alpha$, when fixing $(\mathbf{N}, \mathbf{g})$, this gives a Nash network where $W(S_{S1}) > W(F_{S1})$. The conditions hold symmetrically for the case of $W(S_{S0}) > W(F_{S0})$.*

**III-** *Satisfactory hybrid welfare function:* Let a network $\Gamma$ be $S_H$ iff $\{N_1^1 \neq \emptyset, N_0^0 \neq \emptyset, N_1^0 = \emptyset, N_0^1 = \emptyset\}$ in SC so that:

$$W(S_H)_{sc} = \sum_{i \in N_0^0} \alpha(1 + n_i - \chi_i) + \sum_{i \in N_1^1} \alpha(1 + \chi_i)$$

**IV-** *Frustrated hybrid welfare function:* Let a network $\Gamma$ be $F_H$ iff $\{N_1^1 \neq \emptyset, N_0^0 \neq \emptyset, N_1^0 \neq \emptyset, N_0^1 \neq \emptyset\}$ in SC so that:

$$W(F_H)_{sc} = \sum_{i \in N_0^0} \alpha(1 + n_i - \chi_i) + \sum_{i \in N_1^1} \alpha(1 + \chi_i) + \sum_{i \in N_1^0} \beta(1 + n_i - \chi_i) + \sum_{i \in N_0^1} \beta(1 + \chi_i)$$

**Remark 8** *Since either $N_0^1 \neq \emptyset$ or $N_1^0 \neq \emptyset$ or both, in **IV** $\Rightarrow N = N_1^1 \cup N_0^0 \cup N_1^0 \cup N_0^1$ and an addition with a $\beta$ condition appears. Given that $\beta < \alpha$, when fixing $(\mathbf{N}, \mathbf{g})$, this gives a*

*Nash network where $W(U_H) > W(F_H)$. We can generate a relation between **I** and **III** where $W(U_S) > W(S_H)$ given that in the specialized network each player receives benefit from all of her links but in the hybrid each player receives only from those neighbors making her same action. Nevertheless, we cannot condition that $W(F_S) \lesseqgtr W(F_H)$ nor that $W(F_S) \lesseqgtr W(S_H)$. Each of this relations depends on the distribution of types even when fixing $(\mathbf{N}, \mathbf{g})$. This is supported by the consideration that a specialized network allows that every player receives benefit from each of her neighbors, and even if a player $i$ is choosing her disliked action, she can receive a higher payoff in a specialized configuration than in an satisfactory hybrid, where she makes the choice she likes but is connected to the minimum neighbors necessary to support it. Therefore we can generate the following sequence of welfare functions:*

*If $W(F_H) \geq W(F_S) \Rightarrow W(U_S) > W(S_H) > W(F_H) \geq W(F_S)$*

*If $W(S_H) > W(F_S) \geq W(F_H) \Rightarrow W(U_S) > W(S_H) > W(F_S) \geq W(F_H)$*

*If $W(F_S) \geq W(S_H) \Rightarrow W(U_S) > W(F_S) > W(S_H) > W(F_H)$*

### 4.2 Strategic Substitutes: Welfare Functions

Games with SS portray two different configurations: satisfactory hybrid ($S_H$), and frustrated hybrid ($F_H$). The welfare function in SS is:

$$W_{ss} = \sum_{i \in N_0^0} \alpha(1 + \chi_i) + \sum_{i \in N_1^1} \alpha(1 + n_i - \chi_i) + \sum_{i \in N_1^0} \beta(1 + \chi_i) + \sum_{i \in N_0^1} \beta(1 + n_i - \chi_i)$$

In each configuration the welfare function for a network game in equilibrium is:

**I-** *Satisfactory hybrid welfare function:* Let a network $\Gamma$ be $S_H$ iff $\{N_1^1 \neq \emptyset, N_0^0 \neq \emptyset, N_1^0 = \emptyset, N_0^1 = \emptyset\}$ in SS so that:

$$W(S_H)_{ss} = \sum_{i \in N_0^0} \alpha(1 + \chi_i) + \sum_{i \in N_1^1} \alpha(1 + n_i - \chi_i)$$

**II-** *Frustrated hybrid welfare function:* Let a network $\Gamma$ be $F_H$ iff $\{N_1^1 \neq \emptyset, N_0^0 \neq \emptyset, N_1^0 \neq \emptyset, N_0^1 \neq \emptyset\}$ in SS so that:

$$W(F_H)_{ss} = \sum_{i \in N_0^0} \alpha(1 + \chi_i) + \sum_{i \in N_1^1} \alpha(1 + n_i - \chi_i) + \sum_{i \in N_1^0} \beta(1 + \chi_i) + \sum_{i \in N_0^1} \beta(1 + n_i - \chi_i)$$

**Remark 9** *Since either $N_0^1 \neq \emptyset$ or $N_1^0 \neq \emptyset$ or both, in **II** $\Rightarrow N = N_1^1 \cup N_0^0 \cup N_1^0 \cup N_0^1$ and an addition with a $\beta$ condition appears. Given that $\beta < \alpha$, when fixing $(\mathbf{N}, \mathbf{g})$, this gives a Nash network where $W(U_H) > W(F_H)$.*

# Concluding Remarks

Networks of economic, technological or social interaction are nowadays recognized as a key structure to understand how agents behave and contribute to the general economic activity.

However, for all their ubiquity, they have not been considered in the body of economic literature until the beginning of this century. Work carried out so far on this subject has focused on modeling and understanding the effects of having a (possibly complex) network of interactions among identical actors, in a homogeneous setup, where the only source of difference is the pattern of connections a given agent has. The main novelty of this paper is the introduction of intrinsic diversity in this scenario by analyzing the case in which there are two types of agents. While, admittedly, this is still a very simplified model, our results show that allowing for heterogeneity in the economic interactions on the network leads to a wealth of interesting results even when sufficiently detailed local information is available.

The results we have obtained by studying a heterogeneous model are a noteworthy contribution to the research on games on networks. Thus, by means of a stability concept, we advance the field in the direction of equilibrium refinements in an informationally rich (albeit not complete) context. We have shown that the knowledge of the neighbors an agent has as well as their actions does not prevent us from classifying the possible equilibria and from later refining them to a proper subset dominated by frustrated hybrid configurations. For SC games, this implies that even if the desirable outcome is that every player contributes, it will not be possible to reach such a situation in general. For both SC and SS games, the consequence of this result is that most of the times there will be frustrated players playing the action they do not like. Looking in detail to the structure of the network, it can be seen that those frustrated individuals will be those with the smallest degrees, in particular the leaves of the network. It is also interesting to note that the equilibria we have found when there are two types of agents on the network is not unrelated to the homogeneous case considered by Galeotti *et al.* (2010), in the sense that we have been able to show a monotonicity property on the number of neighbors an agent has choosing one of the two actions. On the other hand, our welfare analysis shows that the most frequent equilibria correspond to low benefits for the society, a result reminiscent of that of Kandori *et al.* (1993) about the selection of the risk-dominant equilibrium when everybody interacts with everybody else (complete graph). This result deserves further study because of the implications it may have in situations such as technologies competing for different segments of the market. We note in addition that while we have been looking mainly at a local information setup, our study should be further extended to other informational contexts, in order to check what can be said about the equilibria under other hypotheses.

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