

Weighted Solidarity Values*

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Abstract

We present a noncooperative bargaining protocol among n players, applied to the setting of cooperative games in coalitional form with transferable utility. In this model, players are chosen randomly to make proposals until one is accepted unanimously, and after each proposal rejection, the probability that players leave the game increases. If after a rejection, some players withdraw the bargaining, the remaining players continue the process. We define a new family of values, called the *weighted solidarity values*, and we show that these values arise as the associated equilibrium payoffs of this bargaining protocol. In these values players have an altruistic behavior between them as the null player property is not satisfied.

Keywords: n -person bargaining; coalitional games; altruism; Solidarity value; Shapley value.

JEL Classification: C71

1 Introduction

In a cooperative setting, a value expresses a particular way in which players share the benefits of their cooperation. Following the *Nash program*, a particular value can be determined either by a set of properties that the value (and only it) satisfies, which is the axiomatic approach; or by a non cooperative game trying to reflect a plausible negotiation process where the cooperative agreement is obtained as the equilibrium payoffs of the game, which is the strategic approach. Both are considered as complementary and hopefully of mutual reinforcement.

We focus on the strategic approach in this paper. Our intention is to add some perspective regarding what rational players should expect to obtain in a multilateral negotiation process, where partial cooperation is likewise possible. This means that not only the coalition of all players can make profitable agreements, but also any strict subcoalition can do the same in case some players leave the negotiations. In this way, the likelihood of partial agreements has an influence on the agreement of the grand coalition.

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Most of the bargaining models proposed¹ can be better interpreted as suitable protocols that can implement in a decentralized way a particular cooperative solution which we have in mind, and then the cooperative solution becomes the target and the bargaining protocol is only a means to find it. We wish to proceed in the opposite direction: present a simple and natural bargaining model and determine what type of payoffs can be expected. We therefore need to specify how players find agreements and what role the coalitions play in the process.

Firstly, the agents involved can make feasible offers and counteroffers, up to the moment when a proposal is unanimously accepted. It is not specified who is the proposer each time, hence we model this fact assuming that the selection of the proposer is a random process. If the proposal is rejected by even one player, another proposer (who may be the same) is selected randomly until an offer is accepted by the remaining players. All players are initially involved in negotiations (they are "active" players).

Secondly, we assume that the time is costly in a very specific sense: The players' live time is not infinite. When players reject offers, they enlarge the time spent in the bargaining and this cannot be carried on indefinitely. As time goes by, the probability of the players leaving the negotiation increases. If after a rejection, some players leave the game, we assume that the remaining players continue the bargaining but restricted now only to the feasible payoffs that can be achieved by the remaining players. Again, it is not specified who are the candidates to leave the bargaining each time. Then the selection of the new active set is modeled as a random process which depend of the probabilities that each player has to remain still in the bargaining.

This is a noncooperative game which has stationary subgame perfect equilibria. In Section 2, we show in Theorem (1) that the equilibrium proposals are easily characterized: The proposing player offers every active player their continuation value and claims the rest of the pie. It turns out that being the proposer is always an advantage, but when the probability of all players continuing in the game, after a proposal rejection, converges to one, the difference between what a player obtains as a proposer and what as a respondent vanishes.

Two different sources of asymmetries among players are allowed in our model: Players can have a different probability to be selected as a proposer and players can have a different probability to leave the game after a proposal rejection. The first class of asymmetries are appropriate when players are representatives of teams, parties, cities, countries, etc. of different size. It can also be the case that some players have a better knowledge of the particular setting in which they are involved, or greater skill and experience in negotiations, that give they more chance to make offers and counteroffers. Whereas the second kind of asymmetry is more appropriate to capture differences in fitness characteristics of players, as expected time of life or vitality, or different economic opportunities outside the game. The main result of this paper is performed in Section 3. In Theorem (3), we are able to calculate these limit proposals for each specification of these probabilities, in what is referred to as the *weighted solidarity value*.

This value (in the symmetric case) was first given in Sprumont (1990) as an example of a population monotonic allocation scheme. Subsequently, and independently, Novak and Radzik (1995) provide an

¹The Literature in this topic is rather extensive now. The reader can find a good survey on the Nash program in Serrano (2005).

alternative, and equivalent, formulation, by offering an axiomatic characterization of it. Not a great deal of attention has been paid on this value so far.

It is usual to define altruism as actions taken by an agent at their own cost for the benefit of another. Altruism is incompatible with the null player property. A null player contributes with nothing to all coalitions he may join. A value satisfies the null player property if it yields nothing to null players. A well known value which satisfies this property is the Shapley value (Shapley 1953b). A simple example of value that does not satisfy the null player property is the egalitarian solution, which shares the worth of the coalition equally among all its members (whether or not they are null players). We can say that if a null player receives a positive reward in a monotonic game, the rest of players behave altruistically with respect to him. A very interesting class of games where studying the altruistic behavior of a value is the class of additive games. In an additive game, the worth of a coalition is the sum of the individual worth² of its members. In this context, there are no positive gains in cooperation and the marginal contribution of each player to every coalition is only his own individual worth. Hence, we are in a pure redistributive setting. If we find transfers between players, we can say that players that obtain less than their individual worth behave altruistically with respect to players that increase their individual worth.

Section 4 shows that the weighted solidarity values do not satisfy the null player axiom. We study the behavior of the weighted solidarity value in some detail in this class of additive games. In particular, for the two person case, we find:

- If players are symmetric (in the probabilities) then richer agents transfer money to poor agents.
- If players start initially with the same endowments, then powerless players (with higher probability of leaving the game and lower probability of being a proposer) transfer money to powerful players (with lower probability of leaving the game and high probability of being a proposer).

For the general n-person case:

- If players interact periodically by using the weighted solidarity values obtained each period as the initial endowments of the next period, this process end up with a unique limit point (independently of the initial endowments). Moreover, the players' payoffs in the limit are in proportion to their relative bargaining power (Theorem (3)).

Section 5 is devoted to some complementary remarks.

In the first subsection, we briefly review the literature related to the solidarity value and a comparison is performed with a family of values given by a convex combinations of the Shapley value and the egalitarian solution. This class was considered by Joosten (1996) and referred to as egalitarian Shapley values. We show that the solidarity value is not an element of this family.

In the second subsection, our multilateral bargaining is compared with the one proposed in Hart and Mas-Colell (1996). In that paper, the authors propose a bargaining procedure that, in the particular case of cooperative games with transferable utility, implements the Shapley value (Shapley, 1953). Both models have in common that every respondent have the same veto right to reject unsatisfactory offers. The only difference between both procedures lies in what happens after the rejection of a proposal: In the Hart and Mas-Colell procedure, only the proposer has a probability of defeat, while in our model every

²The worth that each player can obtain alone, i.e. when the remaining players are out of the game.

subset of players can leave after a rejection. The consequence of this breakdown difference is that the Hart and Mas-Colell model implements a value that satisfies the null player axiom (the Shapley value), whereas our model implements a value that violates the null player axiom (the weighted solidarity value). Hence, it should be clear that the source of the altruistic outcome is not given by the requirement of the unanimity in the agreement (equivalently the veto right). On the contrary, it is given by the differences in the opportunity to put the remaining players in front of an ultimatum situation: If an offer is rejected, “I can leave the game and then you will lose my marginal contribution”. In the case of the Shapley value leaves only the proposer, whereas in the weighted solidarity value every subset of players can leave. Therefore, the power of this ultimatum position is spread from the proposer (Shapley value) to all players, proposer and respondents (weighted solidarity value).

2 Bargaining

We first start with some definitions. Let $U = \{1, 2, \dots\}$ be the (infinite) set of potential players. A *cooperative game* with transferable utility (TU-game) is a pair (N, v) where $N \subset U$ is a non empty and finite set and $v : 2^N \rightarrow \mathbb{R}$ is a *characteristic function*, defined on the power set of N , satisfying $v(\emptyset) = 0$. An element i of N is called a *player* and every non empty subset S of N a *coalition*. The real number $v(S)$ is called the *worth* of coalition S , and it is interpreted as the total payoff that the coalition S , if it forms, can obtain for its members. Let G^N denote the set of all cooperative TU-games with player set N . Risk neutral players who use a totally divisible good to make the coalitional payoffs is an example of this type of games. Player $i \in N$ is a *null player* in (N, v) if $v(S \cup i) = v(S)$ for each $S \subseteq N \setminus i$.

For each $S \subseteq N$, we denote the restriction of (N, v) to S as (S, v) . For simplicity, we write $S \cup i$ instead of $S \cup \{i\}$, $N \setminus i$ instead of $N \setminus \{i\}$, and $v(i)$ instead of $v(\{i\})$. For each vector $x \in \mathbb{R}^N$, let $x(S) := \sum_{i \in S} x_i$ for each $S \subseteq N$.

A TU-game is said *monotonic* if $v(T) \leq v(S)$ whenever $T \subseteq S$. In our setting we made the explicit assumption that the utilities are previously normalized in such a way that when any player leaves the game, the payoff that it obtains is *zero*. Monotonicity implies that, for any subcoalition S , the players have an incentive to cooperate as every player can better attain payoffs than it will obtain being alone, “out of” the game. Note also that the payoff $v(i)$ is what player i obtains if the remaining $N \setminus i$ players have left the game, so $v(i) = 0$ is not need. For example, consider a bankrupt situation where a good, of liquidation value of one, should be divided between two creditors $\{i, j\}$, each one claiming the totality of the value. The possible outcomes are either the good is owned by only one of the players, say i obtains his claim and j receives nothing, i.e. $v(i) = 1$ and j leaves the game receiving zero, or the good can be shared by both, i.e. $v(\{i, j\}) = 1$. This is a monotonic game in which players cannot guarantee their $v(i)$. If we want to consider examples, as market games, where players can guarantee their initial endowments without the help of the remaining players we only need to normalize the utilities such that $v(i) = 0$.

A *value* is a function γ which assigns to every TU-game (N, v) and every player $i \in N$, a real number $\gamma_i(N, v)$, which represents an assessment made by i of his gains from participating in the game. A *payoff configuration* is an element of $(\mathbb{R}^S)_{S \subseteq N}$.

The process that players follow to find a cooperative agreement is modeled by an *alternating random proposer* protocol. This is a sequential noncooperative game where the proposer is chosen at random at every step:

Consider a TU-game (N, v) . In each round there is a set $S \subseteq N$ of “active” players, and a “proposer” which is chosen randomly, from a probability distribution $[p_i^S \geq 0, \text{ for each } i \in S; \sum_{i \in S} p_i^S = 1]$ among them. In the first round, all players are active, i.e. $S = N$. The proposer i makes a feasible offer $a^{S,i} \in \mathbb{R}^S$, i.e. $\sum_{j \in S} a_j^{S,i} \leq v(S)$. If the rest of players accept it, then the process ends with this offer. If it is rejected by even one player, we move to the next round where, from a probability distribution³ $[P_T^S \geq 0, \text{ for each } T \subseteq S; \sum_{T \subseteq S} P_T^S = 1]$, a coalition $T \subseteq S$ is chosen randomly to be the new set of active players and all players out of T (i.e., $j \in S \setminus T$) leave the game receiving a payoff of zero⁴.

As happens in this type of games with three or more players, there is a broad range of associated subgame perfect equilibria. Hence, we follow the familiar route of considering only the *stationary* subgame perfect equilibria (in what follows SP equilibria).

Our first result characterizes the offers of an SP equilibrium.

Theorem 1 : *Let (N, v) be a monotonic TU-game. Then for each specification of the probability distributions to select a proposer and to be the new active set under rejection, i.e. $(p_i^S)_{i \in S}$ and $(P_T^S)_{T \subseteq S}$ for all $S \subseteq N$, there is an SP equilibrium. The proposals corresponding to an SP equilibrium are always accepted and they are characterized by:*

- (1) $a_i^{S,i} = v(S) - \sum_{j \in S \setminus i} a_j^{S,i}$ for each $i \in S \subseteq N$; and
 - (2) $a_j^{S,i} = \sum_{\substack{T \subseteq S \\ T \ni j}} P_T^S a_j^{T,i}$ for each $i, j \in S$ with $i \neq j$, and each $S \subseteq N$;
- where $a^S = \sum_{i \in S} p_i^S a^{S,i}$. Moreover, these proposals are unique and nonnegative.

In other words, (2) says that i proposes to j the expected payoff that j would get in the continuation of the game in case of rejection, as the probability of every coalition $T \subseteq S$ having to be a new active set is P_T^S , and (1) says that i gets for himself the remaining up to complete $v(S)$.

Proof. The proof is done by induction. The proposition holds for the 1-player case. And assume that it is true for less than n players. Let $a^{S,i}$, for $i \in S \subset N$, be the proposals of a given equilibrium, and denote by $c^S \in \mathbb{R}^S$ the expected payoff vector for the members of S in the subgame where S is the set of active players. By (1) and (2) it holds that $\sum_{j \in S} c_j^S = v(S)$. The induction hypothesis implies that $c^S = a^S$ for $S \neq N$.

Firstly, note that monotonicity of v implies that

$$\sum_{S \subseteq N} P_S^N v(S) \leq v(N).$$

³We assume stationarity in all probability distributions.

⁴Although the vectors a^S and $a^{S,i}$ are functions of the probability distributions $(p_i^S)_{i \in S}$ and $(P_T^S)_{T \subseteq S}$, we write a^S instead of $a^S((p_i^S)_{i \in S}, (P_T^S)_{T \subseteq S})$ to simplify notation.

Let $d^{N,i} \in \mathbb{R}^N$ be defined by

$$d_j^{N,i} := P_N^N c_j^N + \sum_{\substack{S \subsetneq N \\ S \ni j}} P_S^N a_j^S,$$

for each $j \neq i$ and

$$d_i^{N,i} := v(N) - \sum_{j \in N \setminus i} d_j^{N,i}.$$

The amount $d_j^{N,i}$ is the expected payoff of j following a rejection of i 's proposal, then $d^{N,i}$ is the best proposal for i among the proposals that will be accepted if i is the proposer. In addition, any proposal of i which is rejected yields to i at most

$$P_N^N c_i^N + \sum_{\substack{S \subsetneq N \\ S \ni i}} P_S^N a_i^S.$$

But

$$\begin{aligned} d_i^{N,i} - \left(P_N^N c_i^N + \sum_{\substack{S \subsetneq N \\ S \ni i}} P_S^N a_i^S \right) &= \\ v(N) - \sum_{j \in N \setminus i} \left(P_N^N c_j^N + \sum_{\substack{S \subsetneq N \\ S \ni j}} P_S^N a_j^S \right) - \left(P_N^N c_i^N + \sum_{\substack{S \subsetneq N \\ S \ni i}} P_S^N a_i^S \right) &= \\ v(N) - \sum_{S \subseteq N} P_S^N v(S) &\geq 0. \end{aligned}$$

Hence, player i will propose $a^{N,i} = d^{N,i}$ and the proposal will be accepted. Thus, it follows that $c^N = a^N$.

To show that the equilibrium proposals $a^{N,i}$ are nonnegative, note that the following strategy will guarantee to i a payoff of at least 0: Accept only if offered at least 0 and, when proposing, propose $a_j^{N,i} = 0$, for each $j \in N$. This implies that $a^{N,i} \geq 0$.

We now show that proposals $(a^{S,i})_{S \subseteq N, i \in S}$ satisfying (1) and (2) can be supported as stationary subgame perfect equilibria. Firstly, by construction they are feasible. Second, non negativity can be proven by induction. The 1-player case is immediate. Now assume $a^{T,i} \geq 0$ for each $i \in T$ and each $T \subsetneq S$. Let $S \subseteq N$. Condition (2) implies that $a_j^{S,i} = a_j^{S,k}$ for each $i, k, j \in S$ and $i, k \neq j$. Then, for each

$i \in S$,

$$\begin{aligned}
a_i^S &= \sum_{j \in S} p_j^S a_i^{S,j} = p_i^S a_i^{S,i} + \sum_{j \in S \setminus i} p_j^S a_i^{S,j} \\
&= p_i^S \left(v(S) - \sum_{j \in S \setminus i} a_j^{S,i} \right) + \sum_{j \in S \setminus i} p_j^S a_i^{S,j} \\
&= p_i^S \left(v(S) - \sum_{j \in S \setminus i} \sum_{\substack{T \subseteq S \\ T \ni j}} P_T^S a_j^T \right) + \left(\sum_{j \in S \setminus i} p_j^S \right) \sum_{\substack{T \subseteq S \\ T \ni i}} P_T^S a_i^T \\
&= p_i^S \left(v(S) - \sum_{j \in S} \sum_{\substack{T \subseteq S \\ T \ni j}} P_T^S a_j^T \right) + \sum_{\substack{T \subseteq S \\ T \ni i}} P_T^S a_i^T \\
&= p_i^S \left(v(S) - \sum_{T \subseteq S} P_T^S v(T) \right) + P_S^S a_i^S + \sum_{\substack{T \subsetneq S \\ T \ni i}} P_T^S a_i^T
\end{aligned}$$

and then

$$(1 - P_S^S) a_i^S = p_i^S \left(v(S) - \sum_{T \subseteq S} P_T^S v(T) \right) + \sum_{\substack{T \subsetneq S \\ T \ni i}} P_T^S a_i^T. \quad (1)$$

By induction, we know that $a_i^T \geq 0$ for all $i \in T \subsetneq S$, and by monotonicity,

$$v(S) \geq \sum_{T \subseteq S} P_T^S v(T),$$

therefore $a_i^S \geq 0$ for all $i \in S$. Then, for each $i, j \in S$ with $i \neq j$,

$$a_j^{S,i} = \sum_{\substack{T \subseteq S \\ T \ni j}} P_T^S a_j^T \geq 0,$$

and

$$\begin{aligned}
a_i^{S,i} - a_i^{S,j} &= v(S) - \sum_{j \in S \setminus i} \left(\sum_{\substack{T \subseteq S \\ T \ni j}} P_T^S a_j^T \right) - \sum_{\substack{T \subseteq S \\ T \ni i}} P_T^S a_i^T \\
&= v(S) - \sum_{j \in S} \left(\sum_{\substack{T \subseteq S \\ T \ni j}} P_T^S a_j^T \right) = v(S) - \sum_{T \subseteq S} P_T^S v(T) \geq 0.
\end{aligned} \quad (2)$$

Hence $a_i^{S,i} \geq a_i^{S,j} \geq 0$.

We can now check whether the strategies corresponding to these proposals do form an SP equilibrium. According to the induction hypothesis, this is so in any subgame with player set $S \neq N$. Fix a player i in N . Given the strategies of the other players, as a proposer, i cannot increase his payoff $a_i^{N,i}$ from proposals that are accepted, and making proposals that were systematically rejected would only lead to an expected payoff $a_i^{N,j}$, whereas the suggested strategy yields $a_i^{N,i}$ which is a better outcome. As a respondent, i can only deviate by rejecting offer $a_i^{N,j}$ made by another player j , but this amount is just

equal to his expectation in case of continuation. Therefore, the only conceivable gain can come from managing defeat. Yet this gives a payoff of 0, whereas the suggested strategy yields nonnegative payoffs.

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Remark 1 Note that (1) implies that $a^{\{i\},i} = v(i)$. So $a^{\{i\},i}$ is nonrandom, which iterating in (2) yields that $a^{S,i}$ is also nonrandom. Therefore, in an stationary subgame perfect equilibrium, mixed strategies only could appear when $a_i^{S,i} = a_i^{S,j}$. However, in this case, as proposer, player i can also claim an amount $b_i^{S,i} > a_i^{S,i}$ for himself. But this implies an amount $b_j^{S,i} < a_j^{S,i}$ for some $j \in S \setminus i$, and then this proposal $b^{S,i}$ will be rejected by j for sure. In this case, the expected payoff associated to this strategy is again $a_i^{S,i} = a_i^{S,j}$. Therefore, as a proposer, any mixed strategy between offering $a^{S,i}$ or $b^{S,i}$, always yield the same payoffs.

Remark 2 From Theorem (1) it follows that being the proposer is always an advantage, because $a_i^{S,i} \geq a_i^S \geq a_i^{S,j}$, for each $i, j \in S \subseteq N$, $i \neq j$.

This *proposer's advantage* effect follows from the existence of the defect probabilities. The extreme case appears when $P_T^S = 0$ for all $\emptyset \neq T \subseteq S$ and $P_\emptyset^S = 1$, which is just the ultimatum offers game. Here, the i 's proposal is $a_i^{S,i} = v(S)$, and $a_j^{S,i} = 0$ for each $j \neq i$. This advantage effect is in sharp contrast with the Hart and Mas-Colell (1996) result for the Shapley value, where being the proposer is not necessarily an advantage; it depends on the monotonicity degree of the TU-game⁵. Nevertheless, this effect vanishes when the probabilities of defeat are lower. This is the content of the next Proposition.

Proposition 1 The proposals corresponding to an SP equilibrium satisfy that $|a_i^{S,i} - a_i^{S,j}| \rightarrow 0$, when $P_S^S \rightarrow 1$, for each $i, j \in S$ and $S \subseteq N$.

Proof. It is straightforward taking into account (2) and that $P_S^S \rightarrow 1$ implies $P_T^S \rightarrow 0$ for all $T \subsetneq S$.

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Hence, when the probability of defeat is lower for all players, *all the proposals $a^{S,i}$ are small deviations of the average a^S* .

3 Solidarity values

When a cooperative solution is considered from an axiomatic point of view, asymmetric versions of the value appear when the property of symmetry⁶ is dropped from the set of axioms which characterizes the value. What type of reasons justifies each asymmetric value depends on the context at hand. It could be differences in the negotiation ability of players (whatever that means), or because they are representatives of groups of different size, etc. On the contrary, the noncooperative game that models the bargaining process must be completely specified in the strategic approach. Now, the source of asymmetric payoffs will correspond to a particular specification of certain parameters in the game.

⁵In the Hart and Mas-Colell model, the probabilities of continuing in the game depend of who was the proposer. In particular, if $i \in S$ was the proposer, then $P_S^{S,i} = \rho$, $P_{S \setminus i}^{S,i} = (1 - \rho)$ and $P_T^{S,i} = 0$ otherwise, for some $0 \leq \rho < 1$.

⁶Two players $i, j \in N$ are *symmetric* in (N, v) if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$. A value γ satisfies *symmetry* if $\gamma_i(N, v) = \gamma_j(N, v)$ whenever i and j are symmetric.

Consider our bargaining process with the probabilities being specified as follows: Given two fixed vectors $\omega, \alpha \in \mathbb{R}^N$ with $\omega_i > 0$ and $\alpha_i > 0$ for each $i \in N$, and a parameter $\rho \in \mathbb{R}$ with $0 \leq \rho < 1$, for all $S \subseteq N$ we have

$$\begin{aligned} p_i^S &= \frac{\alpha_i}{\alpha(S)}, \quad (i \in S), \\ P_T^S &= \prod_{i \in T} \rho^{\omega_i} \prod_{j \in S \setminus T} (1 - \rho^{\omega_j}), \quad (T \subseteq S). \end{aligned}$$

Denote by $a_i^{S,j}(\rho)$, $i, j \in S \subseteq N$, the proposals corresponding to an SP equilibrium.

Here, each player has their own (independent) probability $p_i^S = \frac{\alpha_i}{\alpha(S)}$ of being selected as a proposer and, after rejection, every player has their own (independent) probability ρ^{ω_i} of remaining as an active player. By increasing α_i , we increase the probability of being selected as a proposer, and by increasing ω_i we decrease the probability ρ^{ω_i} of continuing in the game after rejection. Given the independence assumption, the probability of coalition $T \subseteq S$ being a new active coalition is $P_T^S = \prod_{i \in T} \rho^{\omega_i} \prod_{j \in S \setminus T} (1 - \rho^{\omega_j})$, for each $T \subseteq S$.

An interpretation for these parameters is as follows.

The differences in α_i fits very well when players are representatives of groups of different size. For example, consider the problem of distributing profits among the teams/departments of a firm. Any amount of money given to a team can be freely distributed among its members. Assume that the team can only carry out its work when it is complete, as all of its members are equally necessary for its completeness, which means that they are symmetric players. Moreover, as individual members (when the team is not complete), each player alone does not contribute to the productivity of the other teams. So they must only jointly be taken into account with the rest of the team in the distribution of the profits. This provides the economic justification for replacing the original worker game by the team game. Suppose that all workers have the same probability of being selected as a proposer in an active coalition S of completed teams in the bargaining process, this implies that the probability of selecting a representative of a team i is $\frac{\alpha_i}{\alpha(S)}$, where α_i measures the amount of workers of the team i .

Other examples are: Simple voting games in which players are parties with different number of seats in a parliament; sharing a cost allocation of a public facility among cities, or communities, of different size; international agreements among countries of different population, and so on.

An interpretation of ρ^{ω_i} is based on the fact that negotiations take place during the time $\lambda \in [0, \infty)$. Each player i has his own probability $p_i(\lambda)$ of being in the game up to time λ . Suppose that the rate at which this probability changes is a proportion of the time, that is, $dp_i(\lambda) = -\omega_i \lambda$. Here, ω_i is a positive and constant coefficient of proportionality, fixed by some characteristics of the player. A possibility can be performed by some fitness characteristic, as expected time of life or vitality; another possibility could be an “outside options” type of economic interpretation, that is, players with better options outside the game have a greater chance of leaving the negotiation after successive rejections. The negative sign means that $p_i(\lambda)$ is decreasing in time. Taking the initial condition $p_i(0) = 1$, the solution of this ordinal differential equation is $p_i(\lambda) = e^{-\omega_i \lambda}$. Enumerate the sequence of rounds by $t = 0, 1, 2, \dots$, and denote by $\delta > 0$ the length of time that each round takes. We can now write $p_i(t) = e^{-\omega_i \delta t}$ as the probability

of player i being in the game at round t . Letting $\rho = e^{-\delta}$ we obtain $p_i(t) = \rho^{\omega_i t}$, under our stationary assumption this means that, after a rejection, the probability of being in the game at round t conditional on still being in the game at round $t - 1$ is ρ^{ω_i} . When the period of time of each round δ approaches zero, ρ converges to 1 and so $\rho^{\omega_i} \rightarrow 1$.

We now show that an explicit formula for the average proposals can be found when we take these limits.

Some definitions are first necessary. Let (N, v) be a TU-game. For each coalition $S \subseteq N$ and each player $i \in S$, define

$$\Delta^i(v, S) := v(S) - v(S \setminus i).$$

We call $\Delta^i(v, S)$ the *marginal contribution of player i to coalition S in the TU-game (N, v)* .

For each coalition $S \subseteq N$, define

$$\Delta_\omega^{av}(v, S) := \sum_{i \in S} \frac{\omega_i}{\omega(S)} \Delta^i(v, S).$$

We call $\Delta_\omega^{av}(v, S)$ the *weighted average of the marginal contributions of players within coalition S in the game (N, v)* .

We define the payoff configuration $Sl^w(v) = (Sl^w(S, v))_{S \subseteq N}$ inductively by⁷

$$Sl_i^w(S, v) = \frac{\alpha_i}{\alpha(S)} \Delta_\omega^{av}(v, S) + \sum_{j \in S \setminus i} \frac{\omega_j}{\omega(S)} Sl_i^w(S \setminus j, v), \quad (i \in S \subseteq N), \quad (3)$$

starting with

$$Sl_i^w(\{i\}, v) = v(i), \quad (i \in N).$$

We call Sl as the *weighted solidarity value*⁸.

Theorem 2 *Let (N, v) be a monotonic TU-game. Let $0 \leq \rho < 1$ and $\alpha_i > 0$ and $\omega_i > 0$ for each $i \in N$. Then, for every coalition S the average of the SP equilibrium payoff proposals $a_i^S(\rho) = \sum_{j \in S} \frac{\alpha_j}{\alpha(S)} a_i^{S,j}(\rho)$, converge as $\rho \rightarrow 1$ to $Sl_i^w(S, v)$ for each i in S .*

Proof. Let $S \subseteq N$. In order to show that, for each $i \in S$, $a_i^S(\rho)$ converges to $Sl_i^w(S, v)$ as $\rho \rightarrow 1$, the proof is done by induction. When $S = \{i\}$, it holds that

$$a_i^{\{i\}}(\rho) = v(i) = Sl_i^w(\{i\}, v).$$

Assume that $a_i^T(\rho) \rightarrow Sl_i^w(T, v)$ for each $T \subsetneq S$ and each $i \in T$. For each $i \in S$, following in formula (1), we have

$$\begin{aligned} (1 - \rho^{\omega(S)}) a_i^S(\rho) &= \frac{\alpha_i}{\alpha(S)} \left(v(S) - \sum_{T \subseteq S} \rho^{\omega(T)} \prod_{r \in S \setminus T} (1 - \rho^{\omega_r}) v(T) \right) + \sum_{\substack{T \subsetneq S \\ T \ni i}} \rho^{\omega(T)} \prod_{r \in S \setminus T} (1 - \rho^{\omega_r}) a_i^T(\rho) \\ &= \frac{\alpha_i}{\alpha(S)} \left(v(S) - \rho^{\omega(S)} v(S) - \sum_{T \subsetneq S} \rho^{\omega(T)} \prod_{r \in S \setminus T} (1 - \rho^{\omega_r}) v(T) \right) + \sum_{\substack{T \subsetneq S \\ T \ni i}} \rho^{\omega(T)} \prod_{r \in S \setminus T} (1 - \rho^{\omega_r}) a_i^T(\rho), \end{aligned}$$

⁷It is clear that these vectors Sl^w are functions of α and ω .

⁸In formula (3), the payoffs' homogeneity with respect to ω and α is clear. Therefore, payoffs are only sensitive to changes in the relative weights.

and then

$$a_i^S(\rho) = \frac{\alpha_i}{\alpha(S)} \left(v(S) - \sum_{T \subsetneq S} \frac{\rho^{\omega(T)} \prod_{r \in S \setminus T} (1 - \rho^{\omega_r})}{(1 - \rho^{\omega(S)})} v(T) \right) + \sum_{\substack{T \subsetneq S \\ T \ni i}} \frac{\rho^{\omega(T)} \prod_{r \in S \setminus T} (1 - \rho^{\omega_r})}{(1 - \rho^{\omega(S)})} a_i^T(\rho).$$

Applying the l'Hopital's rule, when $\rho \rightarrow 1$, we have

$$\lim_{\rho \rightarrow 1} \frac{\rho^{\omega(T)} \prod_{r \in S \setminus T} (1 - \rho^{\omega_r})}{(1 - \rho^{\omega(S)})} = \lim_{\rho \rightarrow 1} \frac{\omega(T) \rho^{\omega(T)-1} \prod_{r \in S \setminus T} (1 - \rho^{\omega_r}) - \rho^{\omega(T)} \sum_{r \in S \setminus T} \omega_r \rho^{\omega_r-1} \prod_{k \in (S \setminus T) \setminus r} (1 - \rho^{\omega_k})}{-\omega(S) \rho^{\omega(S)-1}}.$$

When $t = s - 1$ it holds that $S \setminus T = \{r\}$. Hence,

$$\begin{aligned} \lim_{\rho \rightarrow 1} \frac{\omega(T) \rho^{\omega(T)-1} \prod_{r \in S \setminus T} (1 - \rho^{\omega_r}) - \rho^{\omega(T)} \sum_{r \in S \setminus T} \omega_r \rho^{\omega_r-1} \prod_{k \in (S \setminus T) \setminus r} (1 - \rho^{\omega_k})}{-\omega(S) \rho^{\omega(S)-1}} &= \\ &= \lim_{\rho \rightarrow 1} \frac{\omega(T) \rho^{\omega(T)-1} (1 - \rho^{\omega_r}) - \rho^{\omega(T)} \omega_r \rho^{\omega_r-1}}{-\omega(S) \rho^{\omega(S)-1}} = \frac{\omega_r}{\omega(S)}, \end{aligned}$$

and for all $t < s - 1$ it holds that $|S \setminus T| \geq 1$, then

$$\lim_{\rho \rightarrow 1} \frac{\omega(T) \rho^{\omega(T)-1} \prod_{r \in S \setminus T} (1 - \rho^{\omega_r}) - \rho^{\omega(T)} \sum_{r \in S \setminus T} \omega_r \rho^{\omega_r-1} \prod_{k \in (S \setminus T) \setminus r} (1 - \rho^{\omega_k})}{-\omega(S) \rho^{\omega(S)-1}} = \frac{0}{-\omega(S)} = 0.$$

Hence, applying the induction hypothesis,

$$\begin{aligned} \lim_{\rho \rightarrow 1} a_i^S(\rho) &= \lim_{\rho \rightarrow 1} \left[\frac{\alpha_i}{\alpha(S)} \left(v(S) - \sum_{T \subsetneq S} \frac{\rho^{\omega(T)} \prod_{r \in S \setminus T} (1 - \rho^{\omega_r})}{(1 - \rho^{\omega(S)})} v(T) \right) + \sum_{\substack{T \subsetneq S \\ T \ni i}} \frac{\rho^{\omega(T)} \prod_{r \in S \setminus T} (1 - \rho^{\omega_r})}{(1 - \rho^{\omega(S)})} a_i^T(\rho) \right] \\ &= \frac{\alpha_i}{\alpha(S)} \left(v(S) - \sum_{r \in S} \frac{\omega_r}{\omega(S)} v(S \setminus r) \right) + \sum_{r \in S \setminus i} \frac{\omega_r}{\omega(S)} Sl_i^w(S \setminus r, v) \\ &= \frac{\alpha_i}{\alpha(S)} \left(\sum_{r \in S} \frac{\omega_r}{\omega(S)} (v(S) - v(S \setminus r)) \right) + \sum_{r \in S \setminus i} \frac{\omega_r}{\omega(S)} Sl_i^w(S \setminus r, v) = \\ &= \frac{\alpha_i}{\alpha(S)} \Delta_\omega^{av}(v, S) + \sum_{r \in S \setminus i} \frac{\omega_r}{\omega(S)} Sl_i^w(S \setminus r, v) = Sl_i^w(S, v). \end{aligned}$$

■

Remark 3 When $\rho \rightarrow 0$, $a_i^S(\rho) \rightarrow \frac{\alpha_i}{\alpha(S)} v(S)$, for each i in S . This is the weighted egalitarian payoff configuration $\left(\left(x_i^S = \frac{\alpha_i}{\alpha(S)} v(S) \right)_{i \in S} \right)_{S \subseteq N}$.

Remark 4 Note that for the particular case of the unanimity game in the grand coalition, (N, u_N) , it holds that $Sl_i^w(N, u_N) = \alpha_i / \alpha(N)$ for each i in N . This is because all subgames (S, u_N) , where $S \neq N$, are zero games and then all players obtain $Sl_i^w(S, u_N) = 0$, $i \in S \subsetneq N$. In the bargaining of the grand coalition, each player has the chance $\alpha_i / \alpha(N)$ of being the proposer, and if even one defeats after a rejection, all players obtain zero. After a rejection, the probability that all players continue the bargaining is then $\prod_{i \in N} \rho^{\omega_i}$. As far as all probabilities ρ^{ω_i} converge to one, $\prod_{i \in N} \rho^{\omega_i}$ converges to one too, independently of the values of weights ω_i . This is the reason why the values of ω do not influence the final payoffs in this pure bargaining game and they only depend on the relative weights of α .

Remark 5 Although the analysis has been performed for games with transferable utility, there are no conceptual difficulties for its extension to games without transferable utility (NTU-games). Under the standard assumptions of convexity and monotonicity in the feasible utility sets, the previous results can be reproduced step by step. In Calvo (2008), the definition of the symmetric solidarity value in NTU-games can be seen.

4 Additive games

A player $i \in N$ is a *dummy player* in a game (N, v) if, for each $S \subseteq N \setminus i$: $v(S \cup i) = v(S) + v(i)$. We say that a value γ satisfies the *dummy player property* if $\gamma_i(N, v) = v(i)$ when i is a dummy player in (N, v) . Many values considered in the cooperative game theory satisfy this property. We can cite the Shapley value (Shapley, 1953), and the Banzhaf value (Banzhaf, 1965), which are two elements of the family of *probabilistic values* (Weber, 1988). In any of these values, each player has his own probability distribution on the coalitions to which he belongs, i.e., for each player $i \in N$, there is a vector $(p_{S,i})_{S \subseteq N: S \ni i}$ such that $p_{S,i} \geq 0$ for each $S \subseteq N : S \ni i$ and $\sum_{S \subseteq N: S \ni i} p_{S,i} = 1$. Assuming that the payoff obtained in each coalition is their marginal contribution, then the probabilistic value δ is their marginal expected contribution to the game:

$$\delta_i(N, v) = \sum_{\substack{S \subseteq N \\ S \ni i}} p_{S,i} \Delta^i(v, S), \quad (i \in N),$$

The marginal contribution of a dummy player is a constant ($\Delta^i(v, S) = v(i)$), and it is then straightforward that the payoff for a dummy player i is $\delta_i(N, v) = v(i)$.

When all players in a game are dummy players, the game is said to be an *additive game*. In that case, starting from a vector $a \in \mathbb{R}^N$, we can build an additive game (N, a) by performing $a(S) = \sum_{i \in S} a_i$, for each $S \subseteq N$. If (N, a) is an additive game any value γ that satisfies the dummy player property must yield as the payoff's vector $\gamma(N, a) = a^9$. Note that if the additive game is also monotonic (i.e., $a \in \mathbb{R}_+^N$) the *core* of the game (Gillies, 1953) is just one point: $\mathcal{C}(N, a) = \{a\}$. At first glance, it seems reasonable in additive games that a player i should accept at least his constant contribution a_i .

Moreover, when the value γ is *efficient*, i.e., $\sum_{i \in N} \gamma_i(N, a) = a(N)$, if a player i receives $\gamma_i(N, a) > a_i$ another player j must necessarily receive $\gamma_j(N, a) < a_j$. In such a case, and when players voluntarily agree to follow this cooperative rule γ , we could say that a player j , such that $\gamma_j(N, a) < a_j$ behaves altruistically¹⁰. Obviously, it is impossible to find this behavior in additive games under any value which

⁹The property that $\gamma(N, a) = a$ whenever (N, a) is an additive game is also known as the *projection axiom*. It was considered for example in Aumann and Shapley (1974) and Dubey *et al.* (1981).

¹⁰The usual definition of altruism in economics is the individual behavior that benefits others at one's own expense and individualism is the self interest or unwillingness to benefit others, except perhaps at zero cost to oneself. This way of thinking is slightly confusing. It is not clear whether it is a characteristic of the agents that influence their behavior or a behavior product of the environment in which the agents are involved. We prefer the following ethological definition of altruism (Graham, 2008):

"Behavior by an individual that increases the fitness of another individual while decreasing the fitness of the actor"

satisfies the dummy player property.

The value Sl^w takes into account the weighted average of the marginal contributions Δ_ω^{av} instead of the own marginal contributions Δ^i , and it is therefore rather straightforward to check that Sl^w does not satisfies the dummy player property, not even in additive games. For this fact to be clear, consider a two-player case with $N = \{i, j\}$. Note that Sl^w is efficient by construction, and suppose that both players have the same weights, i.e. $\alpha_i = \alpha_j$ and $\omega_i = \omega_j$. In that case

$$Sl_i(\{i, j\}, a) - a_i = \frac{1}{4}(a_j - a_i),$$

then $Sl_i(\{i, j\}, a) < a_i$ if and only if $a_i > a_j$.

Suppose that we interpret the worth a_i as the initial rent that agent i has and the worth of a coalition as the sum of the rents that agents contribute to this coalition. In that case, the value Sl determines how *the total rent between the agents is to be redistributed*. In our case, if agent i is richer than j ($a_i > a_j$), i increases altruistically the rent of the "poor" agent j ($Sl_j(\{i, j\}, a) > a_j$) by decreasing his own rent ($Sl_i(\{i, j\}, a) < a_i$).

Alternatively, suppose that players start initially with the same endowments, $a_i = a_j = a$, then it holds that

$$Sl_i^w(\{i, j\}, a) = \left(\frac{\alpha_i}{\alpha_i + \alpha_j} + \frac{\omega_j}{\omega_i + \omega_j} \right) a,$$

then a greater α_i means a higher probability of being the proposer, $\alpha_i/\alpha(S)$, and then a bigger payoff Sl_i^w . A greater ω_i means a higher probability of defeat, $(1 - \rho^{\omega_i})$, and then a lower payoff Sl_i^w .

A very interesting question can now be considered: what happens if players interact periodically transferring rents among them? Can we find a steady state where no more transfers are given?

Consider that each player $i \in N$ starts initially with some endowments $a_i^0 = a_i$. They obtain $a_i^1 = Sl_i^w(N, a^0)$ in the first interaction and by efficiency the total sum is equal to $a(N)$ with some endowment transferences among players. In the following period, they interact again obtaining $a_i^2 = Sl_i^w(N, a^1)$, and so on so forth. We have a sequence $\{a^t\}_{t=0}^\infty$ of a redistributive process and some questions can be raised. Is there a limit point? If that is so, is it dependent on the initial starting point? Can something be said about the final payoffs' redistribution?

More formally, for each $a \in \mathbb{R}_+^N$, define $\Delta(a) := \{x \in \mathbb{R}_+^N : \sum_{i \in N} x_i = a(N)\}$ and $f : \Delta(a) \rightarrow \Delta(a)$ as

$$f(x) := Sl^w(N, x), \quad (x \in \Delta(a)).$$

The function f is well-defined, since, by definition, $Sl_i^w(N, x) \geq 0$ for each $i \in N$ and, by efficiency,

$$\sum_{i \in N} f_i(x) = \sum_{i \in N} Sl_i^w(N, x) = \sum_{i \in N} x_i = a(N), \quad (x \in \Delta(a)).$$

Moreover, f is linear. Let us define the following sequence $\{a^t\}_{t=0}^\infty$ by

$$\begin{aligned} a^0 &= a, \\ a^{t+1} &= f(a^t), \quad (t = 0, 1, 2, \dots). \end{aligned}$$

Theorem 3 For each $a \in \mathbb{R}_+^N$, the sequence $\{a^t\}_{t=0}^\infty$ converges, when $t \rightarrow \infty$, to a unique fixed point a^* of f . This fixed point is characterized by

$$(a) \quad \frac{\omega_i a_i^*}{\alpha_i} = \frac{\omega_j a_j^*}{\alpha_j} \quad (i, j \in N),$$

$$(b) \quad \sum_{i \in N} a_i^* = a(N).$$

Moreover, a^* is given by

$$a_i^* = \frac{\alpha_i}{\omega_i} \frac{1}{\sum_{j \in N} \frac{\alpha_j}{\omega_j}} a(N), \quad (i \in N). \quad (4)$$

Proof. Firstly, we shall prove that f satisfies¹¹

$$\|f(x) - f(y)\| < \|x - y\|, \quad (x, y \in \Delta(a), x \neq y). \quad (5)$$

By the definition, when x is considered as an additive game,

$$\Delta_\omega^{av}(x, S) = \sum_{j \in S} \frac{\omega_j x_j}{\omega(S)}, \quad (S \subseteq N, x \in \Delta(a)).$$

Then,

$$f_i(x) = Sl_i^w(N, x) = \frac{\alpha_i}{\alpha(N)\omega(N)} \sum_{j \in N} \omega_j x_j + \frac{1}{\omega(N)} \sum_{j \in N \setminus i} \omega_j Sl_i(N \setminus j, x), \text{ for each } i \in N \text{ and for each } x \in \Delta(a).$$

Let $x, y \in \Delta(a)$, $x \neq y$. Since $\sum_{i \in N} x_i = \sum_{i \in N} y_i = a(N)$, there must exist $k, l \in N$ such that $x_k > y_k$ and $x_l < y_l$, and then

$$\left| \frac{\omega_k(x_k - y_k) + \omega_l(x_l - y_l)}{(\omega_k + \omega_l)} \right| < \left| \frac{\omega_k(x_k - y_k)}{(\omega_k + \omega_l)} \right| + \left| \frac{\omega_l(x_l - y_l)}{(\omega_k + \omega_l)} \right|.$$

Denote $S^* = \{k, l\} \subseteq N$.

For each $i, j \in N$, we have

$$Sl_i^w(\{i, j\}, x) = \frac{\alpha_i}{\alpha_i + \alpha_j} \frac{\omega_i x_i + \omega_j x_j}{(\omega_i + \omega_j)} + \frac{\omega_j x_i}{\omega_i + \omega_j},$$

and

$$Sl_j^w(\{i, j\}, x) = \frac{\alpha_j}{\alpha_i + \alpha_j} \frac{\omega_i x_i + \omega_j x_j}{(\omega_i + \omega_j)} + \frac{\omega_i x_j}{\omega_i + \omega_j}.$$

Then, for each $i, j \in N$,

$$\begin{aligned} & |Sl_i^w(\{i, j\}, x) - Sl_i^w(\{i, j\}, y)| + |Sl_j^w(\{i, j\}, x) - Sl_j^w(\{i, j\}, y)| = |Sl_i^w(\{i, j\}, x - y)| + |Sl_j^w(\{i, j\}, x - y)| = \\ & \left| \frac{\alpha_i}{\alpha_i + \alpha_j} \frac{\omega_i(x_i - y_i) + \omega_j(x_j - y_j)}{(\omega_i + \omega_j)} + \frac{\omega_j(x_i - y_i)}{\omega_i + \omega_j} \right| + \left| \frac{\alpha_j}{\alpha_i + \alpha_j} \frac{\omega_i(x_i - y_i) + \omega_j(x_j - y_j)}{(\omega_i + \omega_j)} + \frac{\omega_i(x_j - y_j)}{\omega_i + \omega_j} \right| \leq \\ & \left| \frac{\omega_i(x_i - y_i) + \omega_j(x_j - y_j)}{(\omega_i + \omega_j)} \right| + \left| \frac{\omega_j(x_i - y_i)}{\omega_i + \omega_j} \right| + \left| \frac{\omega_i(x_j - y_j)}{\omega_i + \omega_j} \right| \leq \\ & \left| \frac{\omega_i(x_i - y_i)}{(\omega_i + \omega_j)} \right| + \left| \frac{\omega_j(x_j - y_j)}{(\omega_i + \omega_j)} \right| + \left| \frac{\omega_j(x_i - y_i)}{\omega_i + \omega_j} \right| + \left| \frac{\omega_i(x_j - y_j)}{\omega_i + \omega_j} \right| = |x_i - y_i| + |x_j - y_j| \end{aligned}$$

and, in particular, for $S^* = \{k, l\}$ it holds that

$$|Sl_k^w(S^*, x - y)| + |Sl_l^w(S^*, x - y)| < |x_k - y_k| + |x_l - y_l|.$$

¹¹Where $\|x\| := \sum_{i \in N} |x_i|$.

Suppose, by induction, that

$$\sum_{i \in S} |Sl_i^w(S, x - y)| \leq \sum_{i \in S} |x_i - y_i|, \quad (S \subsetneq N)$$

and, for each $S \subsetneq N$ such that $S^* \subseteq S$, it holds that

$$\sum_{i \in S} |Sl_i^w(S, x - y)| < \sum_{i \in S} |x_i - y_i|.$$

Then,

$$\begin{aligned} \|f(x) - f(y)\| &= \sum_{i \in N} |Sl_i^w(N, x) - Sl_i^w(N, y)| = \sum_{i \in N} |Sl_i^w(N, x - y)| = \\ &= \sum_{i \in N} \left| \frac{\alpha_i}{\alpha(N)} \Delta_\omega^{av}(x - y, N) + \sum_{j \in N \setminus i} \frac{\omega_j}{\omega(N)} Sl_i^w(N \setminus j, x - y) \right| \leq \\ &\leq |\Delta_\omega^{av}(x - y, N)| + \sum_{i \in N} \sum_{j \in N \setminus i} \frac{\omega_j}{\omega(N)} |Sl_i^w(N \setminus j, x - y)| \leq \\ &\leq \sum_{j \in N} \frac{\omega_j |x_j - y_j|}{\omega(N)} + \frac{1}{\omega(N)} \sum_{i \in N} \omega_i \sum_{j \in N \setminus i} |Sl_j^w(N \setminus i, x - y)|. \end{aligned}$$

Applying the induction hypothesis,

$$\begin{aligned} \|f(x) - f(y)\| &< \frac{1}{\omega(N)} \sum_{j \in N} \omega_j |x_j - y_j| + \frac{1}{\omega(N)} \sum_{i \in N} \omega_i \sum_{j \in N \setminus i} |x_j - y_j| = \\ &= \frac{1}{\omega(N)} \sum_{i \in N} \omega_i \left(|x_i - y_i| + \sum_{j \in N \setminus i} |x_j - y_j| \right) = \\ &= \frac{1}{\omega(N)} \sum_{i \in N} \omega_i \|x - y\| = \|x - y\|. \end{aligned}$$

Thus, expression (5) is already proved. But, as $\Delta(a)$ is a compact set, a nonnegative real number $C < 1$ exists, such that

$$\|f(x) - f(y)\| \leq C \|x - y\|, \quad (x, y \in \Delta(a)).$$

Therefore, f is a contraction mapping and, by the Banach fixed point theorem, f has a unique fixed point and the sequence $\{a^t\}_{t=0}^\infty$ converges, when $t \rightarrow \infty$, to the fixed point of f .

Now let a^* be the vector defined by (4). It can immediately be seen that a^* satisfies

$$\begin{aligned} (a) \quad & \frac{\omega_i a_i^*}{\alpha_i} = \frac{\omega_j a_j^*}{\alpha_j} \quad (i, j \in N), \\ (b) \quad & \sum_{i \in N} a_i^* = a(N). \end{aligned}$$

Hence this equation system ((a) and (b)) has a solution. Moreover, for any a^* which satisfies (a) and (b), for each $i \in S$ we have

$$\frac{\alpha_i}{\alpha(S)} \Delta_\omega^{av}(a^*, S) = \frac{\alpha_i}{\alpha(S)} \left[\sum_{j \in S} \frac{\omega_j}{\omega(S)} a_j^* \right] = \frac{\alpha_i}{\alpha(S)} \left[\sum_{j \in S} \frac{\alpha_j}{\alpha_i} \frac{\omega_i}{\omega(S)} a_i^* \right] = \frac{\omega_i a_i^*}{\omega(S)}, \quad (S \subseteq N).$$

Therefore,

$$\begin{aligned} Sl_i^w(\{i, j\}, a^*) &= \frac{\alpha_i}{\alpha_i + \alpha_j} \Delta_{\omega}^{av}(a^*, \{i, j\}) + \frac{\omega_j}{\omega_i + \omega_j} Sl_i^w(\{i\}, a^*) = \\ &= \frac{\omega_i}{\omega_i + \omega_j} a_i^* + \frac{\omega_j}{\omega_i + \omega_j} a_i^* = a_i^*, \quad (i, j \in N). \end{aligned}$$

Applying an induction argument,

$$\begin{aligned} Sl_i^w(N, a^*) &= \frac{\alpha_i}{\alpha(N)} \Delta_{\omega}^{av}(a^*, N) + \sum_{j \in N \setminus i} \frac{\omega_j}{\omega(N)} Sl_i^w(N \setminus j, a^*) = \\ &= \frac{\omega_i a_i^*}{\omega(N)} + \sum_{j \in S \setminus i} \frac{\omega_j}{\omega(N)} a_i^* = a_i^*, \quad (i \in N). \end{aligned}$$

Thus a^* must be the unique fixed point of f . ■

Then, the payoffs in the steady state are determined by the relative bargaining power of the players: A greater α_i means a higher probability of being the proposer, $\alpha_i/\alpha(S)$, and then a bigger payoff a_i^* . A greater ω_i means a higher probability of defeat, $(1 - \rho^{\omega_i})$, and then a lower payoff a_i^* .

5 Additional remarks

5.1 Related Literature

Some precedents exist in the literature for the family of values Sl^w introduced in this paper. To the best of our knowledge, the first author that introduced the value Sl defined recursively by

$$Sl_i(S, v) = \frac{1}{s} \Delta^{av}(v, S) + \sum_{j \in S \setminus i} \frac{1}{s} Sl_i(S \setminus j, v), \quad (i \in S \subseteq N), \quad (6)$$

where

$$\Delta^{av}(v, S) := \sum_{i \in S} \Delta^i(v, S),$$

starting with

$$Sl_i(\{i\}, v) = v(i), \quad (i \in N),$$

was Sprumont (1990; Section 5). He sought to show that, in the class of *increasing average marginal contributions (IAMC) games*, i.e. games such that $\Delta^{av}(v, S) \leq \Delta^{av}(v, T)$, whenever $S \subseteq T$, it is possible to find a *population monotonic allocation scheme (PMAS)*. A PMAS is a payoff configuration $(x^S)_{S \subseteq N} \in (\mathbb{R}^S)_{S \subseteq N}$ such that

- (i) For each $S \subseteq N$, $\sum_{i \in S} x_i^S = v(S)$,
- (ii) For each $S, T \subseteq N$ and each $i \in S$, $S \subseteq T \Rightarrow x_i^S \leq x_i^T$.

This is a population monotonicity property: No existing player is worst off by adding a new player in the game. Sprumont (1990, Proposition 4) shows that this payoff configuration is a PMAS in the class of IAMC games.

The next formula

$$Sl_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!(s-1)!}{n!} \Delta^{av}(v, S), \quad (i \in N), \quad (7)$$

was introduced by Nowak and Radzik (1994) in order to define what they called the *Solidarity value* of the game (N, v) . They notice that this value does not satisfy the null player axiom. Instead, they propose an average null player axiom. We say that a player $i \in N$ in the game (N, v) is an *average null player* if $\Delta^{av}(v, S) = 0$ for each coalition $S \subseteq N$ containing i . A value γ satisfies the *Average null player axiom* if $\gamma_i(N, v) = 0$ when i is an average null player in (N, v) . They offer the following axiomatic support of Sl , parallel to the characterization of the *Shapley value* (Shapley, 1953):

Theorem 4 (Nowak and Radzik, 1994) *A value γ on $\mathcal{G}^N$ satisfies efficiency, additivity, symmetry and Average null player axiom if, and only if, γ is the solidarity value.*

In Calvo (2008), definitions (6) and (7) are shown to be equivalent.

An alternative family of solutions which do not satisfy the null player axiom is the convex combination of the Shapley and the egalitarian solution (i.e. $E_i(N, v) = v(N)/n$, $i \in N$). This family was considered in Joosten (1996) as egalitarian Shapley values:

$$E^\lambda(N, v) = \lambda Sh(N, v) + (1 - \lambda) E(n, v), \quad (0 \leq \lambda \leq 1).$$

Note that for the two player case, it holds that $Sl = (1/2) Sh + (1/2) E$, but this relationship is not longer true for $|N| \geq 3$. Hence, the solidarity value is not an element of this family.

We can imagine alternative convex combinations of the egalitarian value with values that do not satisfy the null player axiom. Some of them can be found in van den Brink, Funaki and Yu (2011). All of them are parametrized by a parameter which indicates the closeness that the value has with the egalitarian solution, as a kind of "altruism" index. On the contrary, the solidarity value only emerges in our model as a consequence of the environment in which utility maximizer players interact seeking cooperative agreements. Hence, we do not need any kind of ethical predisposition of the agents to explain altruistic outcomes. Our approach is more in the spirit of Dawkins' selfish gene theory (Dawkins, 1976) which explains why individuals behave altruistically toward their close relatives (as they share many of their own genes and not because they are altruistic).

5.2 Related bargaining protocols

The first bargaining process which yields a strategic support to the Solidarity value appears in the work of Hart and Mas-Colell (1996). Their main objective is to propose, in the setting of games without transferable utility (NTU-games), an elegant and simple variation of the alternating offers protocol which supports the Consistent solution (Maschler and Owen, 1989 and 1992).

The Hart and Mas-Colell bargaining procedure goes as follows:

There is a fixed parameter ρ ($0 \leq \rho < 1$). In each round there is a set of "active" players, and a "proposer" which is chosen randomly (from uniform distribution) among them. In the first round, all players are active. The proposer makes a feasible offer. If the rest of the players accept it, then the process ends with this offer as the final payoff. If it is rejected by even one player, we move to the next round where, with probability ρ , the set of active players is the

same and, with probability $(1 - \rho)$, the *proposer drops out of the game* (receiving a payoff of zero), and the remaining players becomes the new active set.

This bargaining procedure has stationary subgame perfect equilibrium. Moreover, when the probability ρ goes to one, every limit of the noncooperative equilibria belongs to the set of consistent payoffs of the NTU-game. The consistent values in a TU-game are just the Shapley value, whereas the Nash bargaining solution (Nash, 1950) follows when it is a pure bargaining game.

Although the protocol proposed in Hart and Mas-Colell (1996) looks rather similar to the one proposed here, there is a substantial difference in what happens after the rejection of a proposal: In the Hart and Mas-Colell procedure, only the proposer has a probability of defeat, while in our model every player (proposer and respondents) has his own, and independent, probability of defeat. Both models have in common that every respondent has the same veto right to reject unsatisfactory offers. Nevertheless the consequences are very different: The Hart and Mas-Colell model implement a value that satisfies the null player axiom (the Shapley value), whereas our model implements a value that violates the null player axiom (the weighted solidarity value). Hence, it should be clear that *the source of the altruistic outcome is not given by the requirement of the unanimity in the agreement* (equivalently the veto right).

For a better understanding of what is happening in these random proposer models, it should be clear why the marginal contributions of the players to the coalitions arise in the computation of the limit of the equilibrium proposals. This comes from the fact that when a player makes an offer, he has the opportunity to put the remaining players in front of an ultimatum situation: If an offer is rejected, “I may leave the game and then you will lose my marginal contribution”. In the case of the Shapley value only the proposer i leaves, accordingly, the Shapley value makes an expectation of the marginal contribution to the coalitions that the proposer belongs to, i.e. an expectation of $\Delta^i(S, v) = v(S) - v(S \setminus i)$, $S \subseteq N$ s.t. $i \in S$. If the proposer is a null player, the average of his marginal contributions is zero.

Whereas in the weighted solidarity value every subset of players T , proposer and respondents, may leave after a rejection, and then, an expectation of the average of the marginal contributions $\Delta^T(S, v) = v(S) - v(S \setminus T)$, $T \subseteq S$, s.t. $i \in S$ must be taken into account. Hence, the expectation of the average of the marginal contributions can be positive for a null player.

Moreover, in the case of the weighted solidarity value, under the assumption that the probabilities ρ^{ω_i} are independent, when $\rho \rightarrow 1$ it happens that for all $T \subsetneq S \setminus j$, the probability $P_T^S = \prod_{k \in T} \rho^{\omega_k} \prod_{k \in S \setminus T} (1 - \rho^{\omega_k})$ converges faster to zero than $P_{S \setminus j}^S = (1 - \rho^{\omega_j}) \prod_{k \in S \setminus j} \rho^{\omega_k}$, that is, the terms corresponding to more than one player living the game become relatively negligible, and this is why, at the limit, the expectation of the weighted average of the marginal contributions, $\Delta_\omega^{av}(S, v)$ is computed.

The payoffs of the weighted solidarity value are obtained in expectation, as it happens in mechanisms where the proposer is selected randomly. Nevertheless, when the probability of leaving the game becomes small enough, the respondent takes into account that there is a high probability of the same situation being repeated, and he may be chosen as proposer. This means that the proposals depend very little on who is chosen to be the proposer. At the limit, the difference between being the proposer or being a respondent vanishes.

Additionally, Hart and Mas-Colell consider several variations of the random alternating offers protocol and characterize their associated payoffs. In one of these variations, after an offer rejection, all players (proposer and respondents) drop out with equal probability. The player that leaves the game receives a zero payoff, and the rest restart the bargaining process.

The authors mention that in the TU-case:

"The resulting solution is different from the previous ones (thus, it is neither the Shapley value nor the equal split solution¹²)."

In Calvo (2008), this variation (called the *random removal model*) is shown to yield the Solidarity value in TU-games as a limit payoffs. It is also proven that this random removal process yields a unique payoff vector in the NTU-games setting as a limit.

It is noteworthy that they also propose a further variation in which, after a rejection, the proposer drops out with probability $(1 - \rho)\lambda$, whereas each respondent drops out with equal probability $(1 - \rho)(1 - \lambda)/(1 - s)$. In this case, the game implements the corresponding egalitarian Shapley value E^λ .

At this point one also could think that there is a very little difference between the random removal model and ours. On the contrary, we think that there is justification to consider whether, in real-life negotiations, we observe that *only one agent* (either proposer or respondent) can withdraw from negotiations each time an offer is refused, as happens in the random removal protocol. This is a question related on how natural the negotiation models look. In our approach, on the one hand, players make proposals to be accepted. There is no specific order to follow. Each one can be the proposer every time. The process ends when an offer is accepted. On the other hand, the process cannot be enlarged indefinitely in time. As time goes by without an agreement being reached, the probability of players leaving the negotiation increases. If after a rejection, some players leave the game, the remaining players continue the process, but restricted to the payoffs that can be achieved by themselves. As far as these probabilities are of public knowledge (to be selected each one as a proposer, and to leave the game after each rejection) we can expect the weighted solidarity value to arise in a natural way as the associated equilibrium payoffs.

In all variations considered in Hart and Mas-Colell (1996), only one agent has the chance of withdrawing from the process each time: either the proposer for the Shapley value; or only one of the respondents for the egalitarian value; or only one of all agents (proposer and respondents) for the solidarity value; or with a probability of λ the proposer and with $(1 - \lambda)$ only one respondent for the λ -egalitarian Shapley values. We note that this *only one agent withdrawal restriction* is also common to the bidding mechanism proposed by Pérez-Castrillo and Wettstein (2001). Here, agents previously bid to choose the proposer. The winner agent has the right to make the proposal, which can be accepted or rejected. Now, in case of rejection, the proposer is sure to leave the game and the remaining agents continue the process. The way in which bids are performed guarantees that the payoffs associated to the subgame perfect equilibrium

¹²That is, the egalitarian solution.

yields exactly the Shapley value (and not only in expectation, as all the mechanisms that select the proposer randomly).

6 References

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