Norms of products of polynomials

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The factor problem for polynomials

General statement

If $(\mathcal{P}_k, \|\cdot\|_k)$ $k = a, b, c$ are three spaces of scalar polynomials with $P \cdot Q \in \mathcal{P}_c$ for all $P \in \mathcal{P}_a$, $Q \in \mathcal{P}_b$, we search for constants $\lambda, \mu > 0$ such that

$$\lambda \|P\|_a \|Q\|_b \leq \|P \cdot Q\|_c \leq \mu \|P\|_a \|Q\|_b$$

for all $P \in \mathcal{P}_a$ and $Q \in \mathcal{P}_b$.

The factor problem in Banach spaces

Conventional notations

For a Banach space E:

1. $\mathcal{P}_n(E)$: Space of bounded polynomials on E of degree $\leq n$.
2. $\mathcal{P}(^nE)$: Space of bounded n-homogeneous polynomials on E.
3. $\mathcal{P}_n(E)$ and $\mathcal{P}(^nE)$ are endowed with the usual sup norm over the closed unit ball B_E of E:

$$\|P\| = \sup\{|P(x)| : x \in \mathrm{B}_E\}.$$

Products of arbitrary polynomials

Definition

If E is a B. s. over $\mathbb{K}$, $M_{\mathbb{K}}(n,k;E)$ denotes the best constant in

$$\|P\| \cdot \|Q\| \leq M_{\mathbb{K}}(n,k;E)\|P \cdot Q\|$$

for all $P \in \mathcal{P}_n(E)$ and $Q \in \mathcal{P}_k(E)$.

Theorem (Benítez, Sarantopoulos and Tonge – 1998)

If E is any complex Banach space then

$$M_{\mathbb{C}}(n,k;E) \leq \frac{(n+k)^{n+k}}{n^n k^k}.$$

Equality is attained for $E = \ell_1(\mathbb{C})$ and

$$P((x_j)_{j=1}^{\infty}) = x_1 \cdots x_n,$$
$$Q((x_j)_{j=1}^{\infty}) = x_{n+1} \cdots x_{n+k}.$$

Products of arbitrary polynomials

Theorem (Araújo, Enflo, M., Rodríguez, Seoane (2021))

If E is any real B. s. then

$$M_{\mathbb{R}}(n,k;E) = \frac{1}{2} C_{n+k,n} C_{n+k,k},$$

where

$$C_{r,s} = 2^s \prod_{j=1}^{s} \left(1 + \cos \frac{(2j-1)\pi}{2r}\right) \quad \text{for } 1 \leq s \leq r.$$

Equality is attained when $E = \mathbb{R}$ and

P vanishes at the n roots of T_{n+k} closest to -1.

Q vanishes at the other k roots of T_{n+k}.

Products of homogeneous polynomials

Definition

If E is a B. s. over $\mathbb{K}$, $M_{\mathbb{K}}^{h}(n,k;E)$ denotes the best constant in

$$\|P\| \cdot \|Q\| \leq M_{\mathbb{K}}^{h}(n,k;E)\|P \cdot Q\|$$

for all $P \in \mathcal{P}(^{n}E)$ and $Q \in \mathcal{P}(^{k}E)$.

Theorem (Pinasco – 2012)

If H is a complex Hilbert space then

$$M_{\mathbb{C}}^{h}(n,k;H) = \sqrt{\frac{(n+k)^{n+k}}{n^n k^k}}.$$

Equality is attained for $H = \ell_2^2(\mathbb{C})$ and

$$P(z_1, z_2) = z_1^n \quad \text{and} \quad Q(z_1, z_2) = z_2^k.$$

Products of homogeneous polynomials

Theorem (Carando, Pinasco & Rodríguez (2013))

If $1 < p < 2$ then

$$M_{\mathbb{C}}^{h}(n,k;L_p(\mu)) = \sqrt[p]{\frac{(n+k)^{n+k}}{n^n k^k}}.$$

Equality is attained for $\ell_p^2(\mathbb{C})$ and

$$P(z_1,z_2) = z_1^n \quad \text{and} \quad Q(z_1,z_2) = z_2^k.$$

Products linear forms

Definition (Linear polarization constants)

If E is a B. s. over $\mathbb{K}$ then $c_{\mathbb{K}}(m;E)$ denotes the best constant in

$$\|L_1\|\cdots\|L_m\| \leq c_{\mathbb{K}}(m;E)\|L_1\cdots L_m\|$$

for all $L_1,\dots,L_m \in E^*$.

Theorem

If E is a B. s. over $\mathbb{K}$ with $\dim(E) \geq m$ then

$$c_{\mathbb{K}}(m;E) \leq m^m$$

with equality for $E = \ell_1(\mathbb{K})$ and $L_k((x_j)_{j=1}^{\infty}) = x_k$, $1 \leq k \leq m$.

- Benítez, Sarantopoulos and Tonge (1998) for $\mathbb{K} = \mathbb{C}$.
- Révész and Sarantopoulos (2004) for $\mathbb{K} = \mathbb{R}$.

Linear polarization constants

Theorem (Arias de Reyna – 1998)

If H is a complex Hilbert space with $\dim(H) \geq m$ then

$$c_{\mathbb{C}}(m; H) = m^{\frac{m}{2}}.$$

Equality is attained for

$$L_k(x) = \langle a_k, x \rangle \quad (1 \leq k \leq m)$$

where $\{a_1, \ldots, a_m\}$ is orthonormal.

Linear polarization constants

On the calculation of $c_{\mathbb{R}}(m, \ell_2)$

- It has been conjectured that $c_{\mathbb{R}}(m, \ell_2) = m^{\frac{m}{2}}$.
- Proved for $m \leq 5$: Benítez, Sarantopoulos and Tonge (1998).
- Proved for $m \leq 14$. Pinasco (2022).
- $\boxed{c_{\mathbb{R}}(m, \ell_2) \leq \left(\frac{3\sqrt{3}}{e} m\right)^{\frac{m}{2}}}$, where $\frac{3\sqrt{3}}{e} \approx 1.9115$: Frenkel (2008).
- $\boxed{c_{\mathbb{R}}(m, \ell_2) \leq m 2^{m/4} \cdot m^{\frac{m}{2}} = m(\sqrt{2} m)^{\frac{m}{2}}}$ for sufficiently large m: M. Sarantopoulos and Seoane (2010).

A few instances of interest

Common choices for the polynomial spaces

1. Polynomials on the real line of degree at most n and m and coefficients in $\mathbb{K}$.
2. Polynomials on the complex plane of degree at most n and m and coefficients in $\mathbb{K}$.
3. Polynomials in several variables of degree at most n and m and with coefficients in $\mathbb{K}$.
4. Even infinite series.

Common choices for the norms

1. Sup norm over $[-1, 1]$ or $\mathbb{D}$.
2. L_p like norms.
3. The ℓ_p norm of the coefficients.
4. Lacunary norms.

Polynomials in several variables

Definition

$P(x) = \sum_{|\alpha| \le n} a_\alpha x^\alpha$, $x \in \mathbb{C}^N$, $1 \le p < \infty$.

1. $|P|_p = \left(\sum_{|\alpha| \le n} |a_\alpha|^p\right)^{\frac{1}{p}}$.
2. $|P|_\infty = \max\{|a_\alpha| : |\alpha| \le n\}$.
3. $[P]_p = \left(\sum_{|\alpha| \le n} \left(\frac{n!}{\alpha!}\right)^{p-1} |a_\alpha|^p\right)^{\frac{1}{p}}$.
4. $\|P\|_p = \left(\int_0^{2\pi} \cdots \int_0^{2\pi} |P(e^{i\theta_1}, \ldots, e^{i\theta_N})|^p \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi}\right)^{\frac{1}{p}}$.
5. $\|P\|_\infty = \sup\{|P(e^{i\theta_1}, \ldots, e^{i\theta_N})| : \theta_1, \ldots, \theta_N \in \mathbb{R}\}$.

Problem

Estimate the best λ, μ in $\lambda\|P\| \cdot \|Q\| \le \|P \cdot Q\| \le \mu\|P\| \cdot \|Q\|$ for various combinations of the norms above.

Polynomials in several variables

Definition (Concentration)

$P \in \mathcal{P}_n(\mathbb{C}^N)$ has concentration $d \in (0,1]$ at degree k if

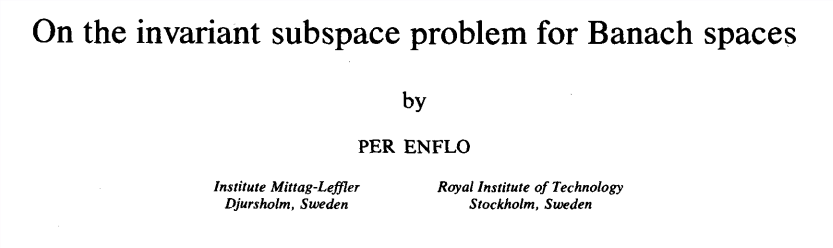
$$|P|_k|_1 = \sum_{|\alpha| \leq k} |a_\alpha| \geq d|P|_1.$$

Theorem (Enflo – 1987)

There is $\lambda(d_1, d_2, n', k') > 0$ such that for every $P \in \mathcal{P}_n(\mathbb{C}^N)$ with concentration d_1 at degree n' and every $Q \in \mathcal{P}_k(\mathbb{C}^N)$ with concentration d_2 at degree k' we have

$$|P \cdot Q|_1 \geq \lambda |P|_1 \cdot |Q|_1.$$

Acta Mathematica


On the invariant subspace problem for Banach spaces

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Homogeneous polynomials in several variables

Theorem (Enflo – 1987)

There is $\lambda(n,k) > 0$ such that for $P \in \mathcal{P}({}^n\mathbb{C}^N)$ and $Q \in \mathcal{P}({}^k\mathbb{C}^N)$,

$$|P \cdot Q|_1 \geq \lambda |P|_1 \cdot |Q|_1.$$

Theorem (Beauzamy, Bombieri, Enflo, Montgomery – 1990)

If P and Q are homogeneous of degree n and k and $1 \leq p \leq \infty$:

$$|P \cdot Q|_p \leq 2^{\frac{n+k}{p*}} |P|_p \cdot |Q|_p,$$

$$[P \cdot Q]_p \geq \sqrt{\frac{(n+k)!}{n!k!}} [P]_p \cdot [Q]_p.$$

Polynomials in one variable

Definition (Lacunary sets)

(a) The 0-lacunary sets are of the form $\{k\}$ with $k \in \mathbb{N}$.

(b) Given $k \in \mathbb{N}$ and $E \subset \mathbb{N}$, E is k-lacunary if for every positive integer m, $(m+E) \cap E$ is contained in a $(k-1)$-lacunary set.

The set of all k-lacunary subsets of $\mathbb{N}$ is denoted by Ω_k.

Definition (Polynomial lacunary norm)

$|h|_{k-lac} = \sup_{E \in \Omega_k} |h|_E|_1$ for $h = \sum_{j \geq 0} h_j x^j$, $h|_E(x) = \sum_{j \in E} h_j x^j$.

Proposition

$$|q|_1 = \lim_{k \to \infty} |q|_{k-\mathrm{lac}} \geq \cdots \geq |q|_{k-\mathrm{lac}} \geq \cdots \geq |q|_{0-\mathrm{lac}} = |q|_\infty .$$

Polynomials in one variable

Theorem (Araújo, Enflo, M., Rodríguez, Seoane – 2021)

Given n, C, K, i, and $Q > 1$, there is a $\beta = \beta(n, C, K, Q, i) > 0$ such that for all polynomials h and q satisfying

$$|h|_{i-\mathrm{lac}} \leq Q\,|h_0|\,,$$
$$|h|_1 \leq K\,|h|_{i-\mathrm{lac}}\,,$$
$$|q|_1 \leq C\,|q\,|_n|_1\,,$$

where $h_0 \neq 0$ is the independent term of h, we have

$$|hq|_{i-\mathrm{lac}} \geq \beta(n, C, K, Q, i)\,|h|_1\,|q|_1\,.$$

Thank you for your attention