Coll Positioning systems: a two-dimensional approach

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Abstract. The basic elements of Coll positioning systems (n clocks broadcasting electromagnetic signals in a n-dimensional space-time) are presented in the two-dimensional case. This simplified approach allows us to explain and to analyze the properties and interest of these relativistic positioning systems. The positioning system defined in flat metric by two geodesic clocks is analyzed. The interest of the Coll systems in gravimetry is pointed out.

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INTRODUCTION

The relativistic positioning systems were introduced by B. Coll a few years ago in the Spanish Relativity Meeting from Valladolid [1]. Remember that these systems are defined by four clocks broadcasting their proper time. Here we will name them for short 'Coll systems'. In a 'long contribution' to these proceedings B. Coll explains the interest, characteristics and good qualities of these relativistic positioning systems in the generic four-dimensional case. In this short communication we present a two-dimensional approach to the Coll systems.

The two-dimensional approach should help us understand better how these relativistic systems work and the richness of the elements of the Coll systems. Indeed, the simplicity of the 2-dimensional case allows us to use precise and explicit diagrams which improve the qualitative comprehension of the positioning systems. Moreover, two-dimensional examples admit simple and explicit analytic results.

Nevertheless, it is worth remarking that the two-dimensional case has particularities and results that cannot be generalized to the generic four-dimensional case. Consequently, the two-dimensional approach is suitable for learning basic concepts about positioning systems, but it does not allow us to study some specific positioning features that necessarily need a three- or a four-dimensional approach.

In the second section we introduce the basic elements of a Coll system: emission coordinates and its other essential physical components. In the third section we explain the analytic method to obtain the emission coordinates from an arbitrary null coordinate system and we use it to develop the positioning system defined in flat space-time by two geodesic clocks. We finish with some comments about other positioning subjects that we are considering at present.

BASIC CONCEPTS AND ESSENTIAL PHYSICAL COMPONENTS

In a two-dimensional space-time, let γ_1 and γ_2 be the world lines of two clocks measuring their proper times τ^1 and τ^2 respectively. Suppose they broadcast them by means of electromagnetic signals, and that these signals reach each other of the world lines. The future light cones (here reduced to pairs of 'light' lines) cut in the region between both emitters and they are tangent outside. Thus, these proper times do not distinguish different events on the emission null geodesics of the exterior region.

The internal region, bounded by the emitter world lines, defines a coordinate domain, the *emission coordinate domain* Ω . Indeed, every event on this domain can be distinguished by the times (τ^1, τ^2) received from the emitter clocks. In other words, the past light cone of every event on the emission domain cuts the emitter world lines at $\gamma_1(\tau^1)$ and $\gamma_2(\tau^2)$ respectively: then $\{\tau^1, \tau^2\}$ are the *emission coordinates* of this event. An important property of the emission coordinates we have defined is that they are null coordinates. The plane $\{\tau^1\} \times \{\tau^2\}$ in which the different data of the positioning system can be transcribed is called the *grid* of the positioning system.

An observer γ travelling throughout the emission coordinate domain and equipped with a receiver which allows to read the proper times (τ^1, τ^2) at each point of his trajectory is a user of this positioning system.

In defining the emission coordinates we have introduced the first essential physical components of a Coll system:

- The principal emitters γ_1 , γ_2 , which broadcast their proper time τ^1 , τ^2 .
- The users γ , travelling in the emitter coordinate domain Ω , receive the emitted times $\{\tau^1, \tau^2\}$ (their emitter coordinates).

These elements define a generic, free and immediate location system (it can be defined in a generic space-time; it can be defined without knowing the gravitational field; a user knows his coordinates without delay).

Any user receiving continuously the *user's positioning data* $\{\tau^1, \tau^2\}$ may extract his trajectory, $\tau^2 = F(\tau^1)$, in the grid. Nevertheless, whatever the user be, these data are insufficient to construct both of the two emitter trajectories.

In order to give to any user the capability of knowing the emitter trajectories in the grid, the positioning system must be endowed with a device allowing every emitter to also broadcast the proper time it is receiving from the other emitter:

- The emitters γ_1 , γ_2 are also transmitters: they receive the signals (such as a user) and broadcast them.
- The users γ also receive the transmitted times $\{\bar{\tau}^1, \bar{\tau}^2\}$.

In other words, the clocks must be allowed to broadcast *their emission coordinates* and then, any user receiving continuously the *emitter's positioning data* $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2\}$ may extract from them the equations $\bar{\tau}^2 = \varphi_1(\tau^1)$ and $\bar{\tau}^1 = \varphi_2(\tau^2)$ of the emitter trajectories. A positioning system so endowed will be called an *auto-located positioning system*.

Eventually, the positioning system can be endowed with complementary devices. For example, in obtaining the dynamic properties of the system:

- The emitters γ_1 , γ_2 can carry accelerometers and broadcast their acceleration.

- The users γ can also receive the emitter acceleration data $\{\alpha_1, \alpha_2\}$.

In some cases, it can be useful that the users generate their own data: they can carry a clock that measures their proper time τ and an accelerometer that measures their acceleration α .

Thus, a Coll positioning system can be performed in such a way that any user can obtain a subset of the user data: $\{\tau^1, \tau^2; \overline{\tau}^1, \overline{\tau}^2; \alpha_1, \alpha_2; \tau, \alpha\}$.

POSITIONING WITH GEODESIC EMITTERS IN FLAT METRIC

Let us assume the *proper time history of two emitters* to be known in a null coordinate system $\{u, v\}$:

$$\gamma_1 \equiv \begin{cases} u = u_1(\tau^1) \\ v = v_1(\tau^1) \end{cases} \qquad \gamma_2 \equiv \begin{cases} u = u_2(\tau^2) \\ v = v_2(\tau^2) \end{cases}$$
(1)

We can introduce the proper times as coordinates $\{\tau^1, \tau^2\}$ as follows:

$$u = u_1(\tau^1), v = v_2(\tau^2)$$
 (2)

This change defines *emission null coordinates* in the *emission coordinate domain* $\Omega \equiv \{(u,v) \mid F_2^{-1}(v) \le u, F_1(u) \le v\}$. In the region outside Ω this change also determines null coordinates which are an extension of the emission coordinates. But in this region the coordinates are not physical, i.e. are not the emitted proper times of the *principal emitters* γ_1 , γ_2 .

Now we use this procedure for the case of two *geodesic* emitters γ_1 , γ_2 in *flat spacetime*. In inertial null coordinates $\{u, v\}$ the proper time parametrization of the emitters are:

$$\gamma_{1} \equiv \begin{cases} u = \lambda_{1}\tau^{1} \\ v = \frac{1}{\lambda_{1}}\tau^{1} + v_{0} \end{cases} \qquad \gamma_{2} \equiv \begin{cases} u = \lambda_{2}\tau^{2} + u_{0} \\ v = \frac{1}{\lambda_{2}}\tau^{2} \end{cases}$$
(3)

Then, the emitter coordinates $\{\tau^1, \tau^2\}$ are defined by the change:

$$u = u_1(\tau^1) = \lambda_1 \tau^1, \qquad v = v_2(\tau^2) = \frac{1}{\lambda_2} \tau^2$$
 (4)

From here we can obtain the metric tensor in emitter coordinates $\{\tau^1, \tau^2\}$ and we obtain: $ds^2 = \lambda d\tau^1 d\tau^2$, $\lambda \equiv \frac{\lambda_1}{\lambda_2}$. On the other hand, in emission coordinates $\{\tau^1, \tau^2\}$, the equations of the emitter trajectories are:

$$\gamma_{1} \equiv \begin{cases} \tau^{1} = \tau^{1} \\ \tau^{2} = \varphi_{1}(\tau^{1}) \equiv \frac{1}{\lambda}\tau^{1} + \tau_{0}^{2} \end{cases} \qquad \gamma_{2} \equiv \begin{cases} \tau^{1} = \varphi_{2}(\tau^{2}) \equiv \frac{1}{\lambda}\tau^{2} + \tau_{0}^{1} \\ \tau^{2} = \tau^{2} \end{cases}$$
(5)

Let γ be a user of this positioning system. What information can this user obtain from the public data? Evidently (τ^1, τ^2) place the user on the user grid, and $(\bar{\tau}^1, \bar{\tau}^2)$, $\bar{\tau}^i = \varphi_j(\tau^j)$, place the emitters on the user grid. On the other hand, the metric component could be obtained from the emitter's positioning data $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2\}$ at two events. The space-time interval is:

$$ds^{2} = \sqrt{\frac{\Delta\tau^{1}\Delta\tau^{2}}{\Delta\bar{\tau}^{1}\Delta\bar{\tau}^{2}}} d\tau^{1}d\tau^{2}$$
(6)

DISCUSSION AND WORK IN PROGRESS

We finish this talk with some comments about other positioning subjects we are studying at present. Firstly, the interest of the Coll systems in gravimetry. If we suppose that the user has no previous information on the gravitational field, what metric information can a user obtain from the public and proper user data? Can a user do gravimetry by using our positioning system? We have shown that [2]:

- The public data $\{\tau^1, \tau^2; \overline{\tau}^1, \overline{\tau}^2; \alpha_1, \alpha_2\}$ determine the space-time metric interval and its gradient along the emitter trajectories.

- The public-user data $\{\tau^1, \tau^2; \tau, \alpha\}$ determine the space-time metric interval and its gradient along the user trajectory.

The development of a general method that offer a good estimation of the gravitational field from this information is still an open problem, but some preliminary results show its interest in determining the parameters in a given (parameterized) model [3].

On the other hand, some circumstances can lead to take another point of view: the user knows the space-time in which he is immersed (Minkowski, Schwarzschild,...) and we want to study the information that the data received by the user offer. We have undertaken this problem for the flat case an we have obtained interesting preliminary results. In particular, we have shown that [4]:

- If a user receives the emitter positioning data $\{\tau^1, \tau^2; \overline{\tau}^1, \overline{\tau}^2\}$ along his trajectory and the acceleration of one of the emitters during a sole echo interval (i.e., travel time of a two-way signal from an emitter to the other), then this user knows: his local unities of time and distance, the metric interval in emission coordinates everywhere, his own acceleration and the acceleration of the principal emitters, the change between emission and inertial coordinates, his trajectory and the emitter trajectories in inertial coordinates.

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