Emission and null coordinates: geometrical properties and physical construction

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Abstract. A Relativistic Positioning System is defined by four clocks (emitters) broadcasting their proper time. Then, every event reached by the signals is naturally labeled by these four times which are the emission coordinates of this event. The coordinate hypersurfaces of the emission coordinates are the future light cones based on the emitter trajectories. For this reason the emission coordinates have been also named null coordinates or light coordinates. Nevertheless, other coordinate systems used in different relativistic contexts have the own right to be named null or light coordinates. Here we analyze when one can say that a coordinate is a null coordinate and when one can say that a coordinate system is null. Moreover, we examine the physical construction and the geometrical properties of several "null coordinate systems": the emission and the reception coordinates, the radar coordinates, and the Bondi-Sachs coordinates, among others.

1. Introduction

The relativistic positioning systems were introduced by B. Coll ten years ago in the Spanish Relativity Meeting that took place in Valladolid [1]. In a relativistic positioning system we have four emitters. Then, the four families of future light cones based on the four emitters constitute coordinate hypersurfaces of a coordinate system: the *emission coordinates*. The emission coordinates of an event are the four proper time signals $\{\tau^A\}$ received by any observer at the event.

The basic properties of the emission coordinates have been analyzed by B. Coll and collaborators in several papers [2, 3, 4]. Let us remember that a coordinate system defines and is defined by their associated coordinate hypersurfaces. For the emission coordinates their coordinate hypersurfaces are future light cones and, consequently, the coordinate covectors are null and future-directed. This means that the contravariant metric has zeros in the diagonal $(g^{\alpha\alpha} = 0)$ and negative extra-diagonal terms $(g^{\alpha\beta} < 0, \alpha \neq \beta)$.

These coordinates associated with a relativistic positioning system have been considered by several authors [1, 3, 4, 5, 6, 7, 8, 9, 10] which have given them different names: null coordinates, emission coordinates, GPS coordinates, GNSS coordinates. The first name, null coordinates, has been recommended enough because this name is a reference to their geometrical properties. Nevertheless, we can find in different relativistic contexts other coordinate systems which have similar geometrical properties, a fact which justifies that all of them could be called null coordinates. For this reason we prefer to call *emission coordinates* the coordinates defined by a relativistic positioning system. Here, we analyze the physical and geometrical construction of several coordinate systems with "null properties". We begin with two conceptual topics: firstly we explain that the physical construction of a coordinate system is linked with the causal character of the coordinate lines, surfaces and hypersurfaces and, secondly, we show that the association of a causal character to a coordinate is not generically coherent. Consequently, it must be explained in what sense we can say that a coordinate is a null coordinate. Later we present a short report on several "null coordinate systems", and we comment on their physical feasibility and on the framework where they has been or could be used.

2. Coordinate systems: causal character and physical construction

In Relativity, lines, surfaces and hypersurfaces of the space-time may be spacelike, lightlike or timelike. These characters, of clear physical meaning, are usually referred to as causal characters. The causal character of lines, surfaces or hypersurfaces is strongly related to the different physical ingredients with which we can physically construct them in the physical space-time (dust, clocks, rods, strings, light beams, light flashes, etc).

By taking three, two or one coordinate as constants, coordinate systems locally define families of coordinate lines, coordinate surfaces or coordinates hypersurfaces, namely they define four three-parametric congruences of lines, six two-parametric families of surfaces and four oneparametric families of hypersurfaces. The causal character of these fourteen geometric elements are generically independent. The ordered set of these causal characters is called the *causal class* of the coordinate system. It is already a well known result that the number of different causal classes of coordinate systems in Relativity is 199 [11] while in Newtonian physics there exists only four causal classes [12].

3. Coordinate parameters and gradient coordinates

When the coordinate system is already known, say $\{x^{\alpha}\}$, the coordinate geometric elements are: the four one-parameter families of coordinate hypersurfaces $\{x^{\alpha} = \text{constant}\}$, the six two-parameter families of coordinate surfaces $\{x^{\alpha} = \text{constant}, x^{\beta} = \text{constant}\}$, and the four three-parameter families of coordinate lines $\{x^{\alpha} = \text{constant}, x^{\beta} = \text{constant}, x^{\gamma} = \text{constant}\}$ for superscripts α , β , γ all different.

In fact, in any space-time, every coordinate x^{α} plays two extreme roles: that of a (coordinate) hypersurface for every constant value, of gradient dx^{α} , and that of a (coordinate) line when the other coordinates remain constant, of tangent vector ∂_{α} . This simple fact shows that, in spite of our deep-seated custom of associating to a coordinate a causal character, saying that *it is* timelike, lightlike or spacelike, *this appellation is not generically coherent*. Causal characters are generically associated with directions or sets of directions of geometric objects, but not with space-time variables or parameters associated to them. In the case of a coordinate system, dx^{α} and ∂_{α} , have generically different causal characters. Only when both causal characters coincide, it is conceptually clear to extend to x^{α} itself the appellation of the common causal character of its two mentioned variations.

Consequently, we shall say of a coordinate x^{α} that it is a timelike, lightlike or spacelike gradient coordinate (respectively, coordinate parameter) when the causal characters of its variation dx^{α} (respectively, ∂_{α}) be timelike, lightlike or spacelike.

In addition, of a coordinate t which is a timelike coordinate parameter and a timelike (resp. spacelike) gradient coordinate, we shall say also that it defines a *spacelike* (resp. *timelike*) synchronization (the coordinate hypersurfaces t = constant being the synchronous event loci of the coordinate lines t = variable).

Every inertial time is a timelike coordinate parameter that defines a spacelike synchronization. However, a remarkable example where the notion of "timelike coordinate" becomes incoherent is the *local Solar time* (the time shown by a sundial). The local Solar time is the oldest timelike coordinate parameter known by humanity, but it is a spacelike gradient coordinate and, consequently, it not defines a spacelike but a timelike synchronization (see [12] for more details).

4. On the "null coordinate systems"

After the above considerations the appellation "null coordinate system" must be used carefully. One can use this appellation because one or several coordinate lines are null (null coordinate parameters), or because one or several coordinate hypersurfaces are null (null gradient coordinates). Or even because one or several coordinate surfaces are null. A quick glance to the table of the 199 causal classes of frames [11] (see also the recent paper [13]) shows that there are 175 classes with some of these null qualities.

It is worth remarking that no coordinate system exists with the four coordinate being both, null coordinate parameters and null gradient coordinates. At most two coordinates can keep this double null quality. This leads to a causal class which is a special case of the Bondi-Sachs coordinates. Nevertheless, there is a class with four null coordinate parameters, and another class with four null gradient coordinates. In what follows we comment on the interest and physical construction of these "null coordinate systems".

Null coordinate parameters $(g_{\alpha\alpha} = 0)$

When the four coordinates are null coordinate parameters, the four congruences of coordinate lines are null. Then, necessarily, they are spacelike gradient coordinates, that is, the four families of coordinate hypersurfaces are timelike. Moreover, the six families of coordinate surfaces are also timelike.

B. Coll was soon interested in this kind of coordinates and proposed their physical construction by means of light beams [14].

Null gradient coordinates $(g^{\alpha\alpha} = 0)$

When the four coordinates are null gradient coordinates, the four families of coordinate hypersurfaces are null. Then, necessarily, they are spacelike coordinate parameters, that is, the four congruences of coordinate lines are spacelike. Moreover, the six families of coordinate surfaces are also spacelike.

Emission-reception coordinates

Emission-reception coordinates are null gradient coordinates which can be physically built by emission and/or reception of times: an event receives the time broadcast by some emitters and the event broadcasts a signal that reaches some receivers. Then, the coordinate hypersurfaces are future and/or past light cones.

Radar coordinates

In two dimensions, and based in the Poincaré-Einstein protocol of synchronization, radar coordinates $\{t_e, t_r\}$ of an event are defined as follows: a signal is broadcast by an observer at the emission time t_e , and this signal is received by the same observer at the reception time t_r after been echoed by the event. Thus, radar coordinates are emission-reception coordinates defined by a sole emitter-receiver.

In a four-dimensional space-time, Ehlers-Pirani-Schild [15] radar coordinates are emissionreception coordinate defined by two emitter-receivers.

Emission coordinates

Emission coordinates of an event are defined by the four times that the clocks of four *emitters* (satellites) watched when they broadcast signals received at this event. The four families of coordinate hypersurfaces are future light cones with vertex on the emitter world lines.

In his first presentation B. Coll [1] named these coordinates "Coordinate Systems of GPS type", but after he and his collaborators opt for the use of "emission coordinates". The usefulness of these coordinates in building a fully relativistic theory for the Global Navigation Satellite Systems demands to improve their current geometrical and physical comprehension [2, 3, 4, 5, 6, 9, 10].

Reception coordinates

Reception coordinates of an event are defined by the four times that watch the clocks of four *receivers* (satellites) when they receive a signal broadcast from this event. Now, the four families of coordinate hypersurfaces are past light cones with vertex on the receiver world lines.

The reception coordinates will have an important role in a future theory of relativistic stereometry. B. Coll remarked this fact two years ago, in the GraviMAS-FEST [16].

Bondi-Sachs coordinates $(g^{00} = g_{11} = 0)$

The generalized Bondi-Sachs coordinates were introduced in order to study gravitational radiation [17]. Usually in their definition one also imposes the condition $g_{22}g_{33} - g_{23}^2 > 0$. Nevertheless, this condition is a consequence of $g^{00} = g_{11} = 0$. According with [11, 13], the generalized Bondi-Sachs coordinates belong to 13 different causal classes.

In these Bondi-Sachs causal classes there are a null gradient coordinate u and a null coordinate parameter r. The orthogonal lines to the coordinate hypersurfaces u = constant are the null coordinate lines r = variable, that is, the (null) gradient du and the vector ∂_r are collinear.

A way to provide Bondi-Sachs coordinates is to consider the future (past) light cones u = constant associated with an emitter (receiver) and to choose a parameter r on the generatrix of the cones.

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