

On the space-times admitting two shear-free geodesic null congruences

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Abstract We analyze the space-times admitting two shear-free geodesic null congruences. The integrability conditions are presented in a plain tensorial way as equations on the volume element U of the time-like 2-plane that these directions define. From these we easily deduce significant consequences. We obtain explicit expressions for the Ricci and Weyl tensors in terms of U and its first and second order covariant derivatives. We study the different compatible Petrov-Bel types and give the necessary and sufficient conditions that characterize every type in terms of U . The type D case is analyzed in detail and we show that every type D space-time admitting a $2 + 2$ conformal Killing tensor also admits a conformal Killing-Yano tensor.

Keywords Shear-free null geodesics · Weyl and Ricci tensors

1 Introduction

The family of space-times admitting two shear-free geodesic null congruences contains significant and well-known solutions of the Einstein equations. Hence, in the Schwarzschild and Kerr black holes these congruences define the outgoing and ingoing propagation of light in the radial direction. We find a

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similar situation in their charged counterparts, the Reissner-Nordström and Kerr-Newman solutions. Moreover, all the other vacuum Petrov-Bel type D solutions and their charged counterparts also belong to this family.

But this family contains other interesting space-times. Indeed, we have shown elsewhere [1] that in every type D metric with a vanishing Cotton tensor the two principal planes define an umbilical $2+2$ structure, and that this geometric property states, equivalently, that the two principal null directions determine shear-free geodesic congruences. It is worth remarking that the umbilical condition can be stated in terms of the *canonical 2-form* U , volume element of the time-like plane defined by the two congruences [1].

On the other hand, some first integrals of the geodesic equation are closely linked with this property. Thus, conformal and Killing tensors of type $2+2$ and non-null Killing-Yano tensors can only exist in space-times admitting two shear-free geodesic null congruences [2–5].

A shear-free geodesic null congruence determines a Debever principal null direction of the Weyl tensor, but is not, necessarily, a multiple Debever direction. Consequently, all the Petrov-Bel types, excepting type N, are compatible with an umbilical $2+2$ structure. What restrictions does this condition impose on the Weyl and Ricci tensors? In order to give an answer to this question one needs to study the integrability conditions of the umbilical equation.

These integrability conditions were presented in spinorial formalism by Dietz and Rüdiger [3] and they obtained some complementary restrictions on the type D space-times. Here, we develop this question in a plain tensorial way that allows us to analyze, not only all the compatible Petrov-Bel types, but also the restrictions on the Ricci tensor.

Our results can easily be summarized if one takes into account that when U defines an umbilical structure, its first derivatives can be collected in a complex vector χ [1,4]. Then the covariant derivative $\nabla\chi$ contains all the information on the second derivatives of U . We show that the Ricci tensor is determined, up to two functions, by U , χ and the symmetric part of $\nabla\chi$, and we give its explicit expression. On the other hand we give a general expression for the Weyl tensor in terms of the U , χ , $d\chi$ and the scalar curvature. Moreover we study the characterization of the different Petrov-Bel types and we show that the 2-forms U and $d\chi$ determine the canonical 2-forms associated with the Weyl tensor geometry.

From our study a result easily follows: if a non conformally flat space-time admits a non-null Maxwell field whose principal directions define shear-free geodesic null congruences, then the metric is type D and these directions are the double Debever directions of the Weyl tensor. This property has allowed elsewhere [4] to obtain the intrinsic characterization of the space-times admitting Killing-Yano or conformal Killing-Yano tensors, and to give an algorithm that determines these first integrals of the geodesic equation.

It is known that a type D vacuum solution admitting a Killing tensor also admits a Killing-Yano tensor [6,7]. On the other hand, all the type D vacuum solutions admit a conformal Killing tensor [8] and, from our results here and [4], they also admit a conformal Killing-Yano tensor. Can these results be

generalized to the non vacuum case? Here we show that *every type D solution with a $2 + 2$ conformal Killing tensor also admits a conformal Killing-Yano tensor*.

This paper is organized as follows. In Sect. 2 we introduce the notation and summarize some previous results needed in this work. In Sect. 3 we present the integrability conditions for the umbilical equation and we obtain the restrictions that these conditions impose on the Ricci and Weyl tensors. The compatible Petrov-Bel types are considered in Sect. 4 and we show that the Weyl canonical bivectors are determined by U and its differential concomitants. Finally, Sect. 5 is devoted to analyzing type D space-times in detail and we recover some known result and provide new ones.

2 Notation and previous results

Let (V_4, g) be an oriented space-time of signature $\{-, +, +, +\}$. The metric G on the space of 2-forms is $G = \frac{1}{2}g \wedge g$, \wedge denoting the double-forms exterior product, $(A \wedge B)_{\alpha\beta\mu\nu} = A_{\alpha\mu}B_{\beta\nu} + A_{\beta\nu}B_{\alpha\mu} - A_{\alpha\nu}B_{\beta\mu} - A_{\beta\mu}B_{\alpha\nu}$. If F and H are 2-forms, (F, H) denotes the product with G :

$$(F, H) \equiv G(F, H) = \frac{1}{4}G_{\alpha\beta\lambda\mu}F^{\alpha\beta}H^{\lambda\mu} = \frac{1}{2}F_{\alpha\beta}H^{\alpha\beta},$$

A self-dual 2-form is a complex 2-form \mathcal{F} such that $*\mathcal{F} = i\mathcal{F}$, where $*$ is the Hodge dual operator. We can associate biunivocally with every real 2-form F the self-dual 2-form $\mathcal{F} = \frac{1}{\sqrt{2}}(F - i * F)$. For short, we here refer to a self-dual 2-form as a *SD bivector*. The endowed metric on the 3-dimensional complex space of the SD bivectors is $\mathcal{G} = \frac{1}{2}(G - i \eta)$, η being the metric volume element of the space-time. The orthogonal complement of the SD bivectors space is the space of the anti-self-dual 2-forms, which are those satisfying $*F = -iF$.

If F is a 2-form and P and Q are double-2-forms, $P(F)$ and $P \circ Q$ denote, respectively, the 2-form and the double-2-form given by:

$$P(F)_{\alpha\beta} \equiv \frac{1}{2}P_{\alpha\beta}^{\mu\nu}F_{\mu\nu}, \quad (P \circ Q)_{\alpha\beta\rho\sigma} \equiv \frac{1}{2}P_{\alpha\beta}^{\mu\nu}Q_{\mu\nu\rho\sigma}$$

Every double-2-form, and in particular the Weyl tensor W , can be considered as an endomorphism on the space of the 2-forms. The restriction of the Weyl tensor on the SD bivectors space is the *self-dual Weyl tensor* and it is given by:

$$\mathcal{W} \equiv \mathcal{G} \circ W \circ \mathcal{G} = \frac{1}{2}(W - i * W)$$

For short we here refer to a p -dimensional distribution (set of vector fields generated by p independent vector fields) as a p -plane. The generalized second fundamental form Q_v of a non-null p -plane V is the $(2,1)$ -tensor:

$$Q_v(x, y) = h(\nabla_{v(x)}v(y)), \quad \forall x, y \quad (1)$$

where ν is the projector on V . Let us consider the invariant decomposition of Q_ν into its antisymmetric part A_ν and its symmetric part $S_\nu \equiv S_\nu^T + \frac{1}{p}\nu \otimes \text{tr } S_\nu$, where S_ν^T is a traceless tensor:

$$Q_\nu = A_\nu + \frac{1}{p}\nu \otimes \text{tr } S_\nu + S_\nu^T \quad (2)$$

The p -plane V is a *foliation* if, and only if, $A_\nu = 0$, and, similarly, V is said to be *minimal*, *umbilical* or *geodesic* if $\text{tr } S_\nu = 0$, $S_\nu^T = 0$ or $S_\nu = 0$, respectively.

A $p + q$ almost-product structure is defined by a p -plane V and its orthogonal complement q -plane H . The almost-product structures can be classified by taking into account the invariant decomposition of the covariant derivative of the so called structure tensor Π [9] or, equivalently, according to the foliation, minimal or umbilical character of each plane [1, 10]. We will say that a structure is integrable when both, V and H are a foliation. We will say that the structure is minimal (umbilical) if both, V and H are minimal (umbilical).

On the space-time, a 2+2 almost-product structure is defined by a time-like plane V and its space-like orthogonal complement H . Let ν and $h = g - \nu$ be the respective projectors and let $\Pi = \nu - h$ be the *structure tensor*. A 2+2 space-time structure is also determined by the *canonical* unitary 2-form U , volume element of the time-like plane V . Then, the respective projectors are $\nu = U^2$ and $h = -(* U)^2$, where $U^2 = U \cdot U$. Here, if A and B are two 2-tensors, we denote $A \cdot B$ the tensor with components $(A \cdot B)_{\alpha\beta} = A_\alpha^\mu B_{\mu\beta}$.

In working with 2+2 structures it is useful to introduce the canonical SD bivector $\mathcal{U} \equiv \frac{1}{\sqrt{2}}(U - i * U)$ associated with U , that satisfies $2\mathcal{U}^2 = g$ and, consequently, it is unitary, $(\mathcal{U}, \mathcal{U}) = -1$. If \perp denotes the projector on the space of the SD bivectors which are orthogonal to \mathcal{U} , and \mathcal{G}_\perp is the restriction of the metric on this space, for a 2-form F and a double-2-form P , we have:

$$\mathcal{G}_\perp = \mathcal{U} \otimes \mathcal{U} + \mathcal{G}, \quad F_\perp \equiv \mathcal{G}_\perp(F), \quad P_\perp \equiv \mathcal{G}_\perp \circ P \circ \mathcal{G}_\perp \quad (3)$$

If A and B are two 2-tensors, $[A, B]$ and $\{A, B\}$ denote the commutator and anti-commutator, respectively:

$$[A, B] = A \cdot B - B \cdot A, \quad \{A, B\} = A \cdot B + B \cdot A$$

With this notation a straightforward calculation leads to:

Lemma 1 *A real symmetric tensor E is determined, up to two scalars e_1 and e_2 , by the commutator $[\mathcal{U}, E]$, where \mathcal{U} is a unitary SD bivector. More precisely:*

$$E = \frac{1}{4}(e_1 g + e_2 \Pi) + \mathcal{U} \cdot \left([\mathcal{U}, E] + \{\mathcal{U}, \bar{\mathcal{U}} \cdot [\overline{\mathcal{U}}, E]\} \right) \quad (4)$$

where Π is the structure tensor associated with \mathcal{U} , $\Pi = 2\mathcal{U} \cdot \bar{\mathcal{U}}$, and $\bar{}$ stands for complex conjugate.

There are some first order differential concomitants of U that determine the geometric properties of the structure [1]. Moreover, the first order differential properties of a 2+2 structure admit a kinematical interpretation [11]. Now we summarize some of these results needed in the following sections. If i_ξ denotes the interior product with a vector field ξ , and δ the exterior codifferential, $\delta = *d*$, we have the following lemma [1]:

Lemma 2 *Let us consider the 2+2 structure defined by the canonical 2-form U . The three following conditions are equivalent:*

- (i) *The structure is umbilical*
- (ii) *The canonical SD bivector $\mathcal{U} = \frac{1}{\sqrt{2}}(U - i * U)$ satisfies:*

$$\Sigma[U] \equiv \nabla \mathcal{U} - i_\xi \mathcal{U} \otimes \mathcal{U} - i_\xi \mathcal{G} = 0, \quad \xi \equiv \delta U \quad (5)$$

- (iii) *The principal directions of U determine shear-free geodesic null congruences.*

In the following we refer to (5) as the *umbilical equation*.

On the other hand, let us define the 1-forms:

$$\begin{aligned} \Phi &\equiv \Phi[U] \equiv *U(\delta * U) - U(\delta U) \\ \Psi &\equiv \Psi[U] \equiv *U(\delta U) + U(\delta * U) \end{aligned} \quad (6)$$

where, for a 2-tensor A and a vector x , $A(x)_\mu = A_{\mu\nu}x^\nu$. Then, we have the following result [1]:

Lemma 3 *Let U be the canonical 2-form of a 2+2 almost product structure. Then it holds:*

- (i) *The structure is minimal if, and only if, U satisfies $\Phi[U] = 0$.*
- (ii) *The structure is integrable if, and only if, U satisfies $\Psi[U] = 0$.*

These characterizations of a minimal and an integrable structure, and the kinematic interpretation of these geometric properties given in [11] lead to call the Φ and Ψ given in (6) the *expansion vector* and the *rotation vector* of the structure, respectively.

Every non-null 2-form F can be written in the form $F = e^\phi [\cos \psi U + \sin \psi * U]$, where the *geometry* U is a unitary and simple 2-form that determines the 2+2 associated structure (principal planes), ϕ is the *energetic index* and ψ is the *Rainich index*. When F is solution of the source-free Maxwell equations, $\delta F = 0$, $\delta * F = 0$, one says that its intrinsic geometry U defines a *Maxwellian structure*. In terms of the intrinsic elements (U, ϕ, ψ) , Maxwell equations become [12, 13]:

$$d\phi = \Phi[U], \quad d\psi = \Psi[U] \quad (7)$$

Then, from (7) the Rainich theorem [12] follows:

Lemma 4 *A unitary 2-form U defines a Maxwellian structure if, and only if, the expansion and the rotation are closed 1-forms, namely U satisfies:*

$$d\Phi[U] = 0, \quad d\Psi[U] = 0 \quad (8)$$

When the Maxwell-Minkowski energy tensor T associated with a non-null 2-form is divergence-free, the underlying 2+2 structure is said to be *pre-Maxwellian* [14]. The conservation of T is equivalent to the first of the Maxwell-Rainich equations (7) [4]. Consequently: *a 2+2 structure is pre-Maxwellian if, and only if, the canonical 2-form satisfies the first equation in (8).*

We can collect the expansion vector Φ and the rotation vector Ψ in a complex vector with a simple expression in terms of the canonical SD bivector \mathcal{U} . Indeed, if we define $\chi \equiv \chi[U] = i_\xi \mathcal{U}$, $\xi \equiv \delta \mathcal{U}$, we have:

$$\chi = \frac{1}{2}(\Phi[U] + i\Psi[U]) \quad (9)$$

Then, conditions (8) that characterize a Maxwellian structure can be written as:

$$d\chi = 0 \quad (10)$$

It is worth remarking that in (10) there are just five independent complex equations (or in (8) ten real ones). This fact has been pointed out in [13] and is a consequence of the integrability condition $\delta\delta\mathcal{U} = 0$. Indeed this identity states, equivalently, that the complex 2-form $d\chi$ is orthogonal to \mathcal{U} :

$$(d\chi, \mathcal{U}) = 0 \quad (11)$$

condition that in real formalism becomes:

$$(d\Phi, U) + (d\Psi, *U) = 0, \quad (d\Phi, *U) - (d\Psi, U) = 0$$

3 Integrability conditions for umbilical 2+2-structures

As a consequence of Lemma 2 the Riemann tensor of the space-times admitting two shear-free geodesic null congruences must be submitted to the integrability conditions of the umbilical equation (5). We can obtain these conditions from the Ricci identities for the SD bivector \mathcal{U} :

$$\nabla_\alpha \nabla_\beta \mathcal{U}_{\mu\nu} - \nabla_\beta \nabla_\alpha \mathcal{U}_{\mu\nu} = \mathcal{U}_\mu^\lambda R_{\lambda\nu\beta\alpha} - \mathcal{U}_\nu^\lambda R_{\lambda\mu\beta\alpha} \quad (12)$$

The umbilical equation implies that the covariant derivative of the canonical 2-form \mathcal{U} is determined by the complex vector $\xi \equiv \delta\mathcal{U}$. Indeed, (5) may equivalently be written as

$$\nabla \mathcal{U} = i_\xi \mathcal{G}_\perp, \quad \xi \equiv \delta \mathcal{U}. \quad (13)$$

If one simplifies (12) by using (13), one obtains:

Proposition 1 *Under the umbilical condition (13) the Ricci identities (12) take the form:*

$$(Riem - K \wedge g) \circ \mathcal{G}_\perp = 0 \quad (14)$$

$$K \equiv T \cdot \mathcal{U}, \quad T \equiv \nabla \xi - \chi \otimes \xi \quad (15)$$

where $\xi = \delta \mathcal{U}$ and $\chi = i_\xi \mathcal{U}$.

Taking into account the usual decomposition of the Riemann tensor, we can write (14) as equations for the Weyl and Ricci tensors. Indeed, by considering the self-dual and the anti-self-dual parts in the first pair of indexes of this equation, we obtain:

Theorem 1 *Let \mathcal{W} , Ric and R be the self-dual Weyl tensor, the Ricci tensor and the scalar curvature of a space-time admitting an umbilical structure \mathcal{U} . The integrability conditions for the umbilical equation (5) may be written as:*

$$\mathcal{W}_\perp = \Omega \mathcal{G}_\perp, \quad \Omega \equiv - \left[\frac{R}{12} + (A, \mathcal{U}) \right] \quad (16)$$

$$\mathcal{W}(\mathcal{U})_\perp = -A_\perp \quad (17)$$

$$[\mathcal{U}, Ric] = 2\mathcal{U} \cdot S \cdot \mathcal{U} - S \quad (18)$$

where A and S are the antisymmetric and symmetric parts of the tensor T given in (15):

$$A = \frac{1}{2}(T - T^t), \quad S = \frac{1}{2}(T + T^t)$$

The integrability conditions (16) and (17) can be used to obtain an expression for the Weyl tensor whereas the third one (18) offers an expression for the Ricci tensor. Below we give explicit expressions of these two irreducible parts of the Riemann tensor, and we show that they are determined up to two scalars by \mathcal{U} and its derivatives. This fact is a consequence of the Codazzi relations, which give all the mixed components of the curvature tensor as a concomitant of the second fundamental forms of an arbitrary $p + q$ structure. Thus, only the total projections of the curvature tensor on the two planes are not determined by the second fundamental forms. In the case of a 2+2 structure these projections have one sole component. Thus, if $v(t)$ and $h(t)$ denote the total projection of a tensor t on the planes V and H , respectively, we have:

$$v(Riem) = -X \mathcal{U} \otimes \mathcal{U}, \quad h(Riem) = Y * \mathcal{U} \otimes * \mathcal{U} \quad (19)$$

Let us note that the two scalars X and Y are the Gauss curvatures of the respective 2-plane when the structure is a product one, that is, when the

space-time metric breaks down into two bi-dimensional metrics, $g = v_{ij}(x^k)dx^i dx^j + h_{AB}(x^C)dx^A dx^B$, $i, j, k = 0, 1$ and $A, B, C = 2, 3$.

Before obtaining the Ricci and Weyl tensors, we study some integrability restrictions on the structure U that are independent of the curvature tensor. The third integrability equation (18) can be written in the form:

$$[\mathcal{U}, Ric] = [\mathcal{U}, L], \quad L \equiv [S, \mathcal{U}] \quad (20)$$

Thus, the tensor L determines the commutator $[\mathcal{U}, Ric]$. But Lemma 1 implies that this commutator fixes the Ricci tensor up to two scalars. More precisely, considering that (18) implies $\tilde{\mathcal{U}} \cdot [\tilde{\mathcal{U}}, Ric] = \tilde{L}$, and taking $E = Ric$ in expression (4), we have:

$$\begin{aligned} Ric &= \frac{1}{4}(Rg + r\Pi) + \mathcal{U} \cdot [\mathcal{U}, L] + \mathcal{U} \cdot \{\mathcal{U}, \tilde{L}\} \\ &= \frac{1}{4}(Rg + r\Pi) + \frac{1}{2}(L + \tilde{L}) + \mathcal{U} \cdot (L - \tilde{L}) \cdot \mathcal{U} \end{aligned} \quad (21)$$

The Ricci tensor being real, the imaginary part of the last term in the above expression must vanish. Then, one obtains:

$$v(\text{Im}[L]) = 0, \quad h(\text{Im}[L]) = 0 \quad (22)$$

A straightforward calculation allows us to obtain its real and imaginary parts from the definition (20) of L :

$$4\text{Re}[L] = M - U \cdot M \cdot U + *U \cdot M \cdot *U - U \cdot N \cdot *U - *U \cdot N \cdot U \quad (23)$$

$$4\text{Im}[L] = N - U \cdot N \cdot U + *U \cdot N \cdot *U + U \cdot M \cdot *U + *U \cdot M \cdot U \quad (24)$$

M and N being the symmetric tensors:

$$M \equiv \mathcal{L}_\Phi g - \Phi \otimes \Phi + \Psi \otimes \Psi, \quad N \equiv \mathcal{L}_\Psi g - \Phi \tilde{\otimes} \Psi \quad (25)$$

where \mathcal{L}_s denotes the Lie derivative with respect to a vector field s and, for two arbitrary tensors, $A \tilde{\otimes} B = A \otimes B + B \otimes A$. We can make the integrability conditions (22) more explicit. Indeed, by using (24) we can easily write these equations as:

$$[v(N), U] = 0, \quad [h[N], *U] = 0$$

But, taking into account that the sole symmetric tensor in V (resp. H) which commutes with U (resp. $*U$) is proportional to v (resp. h), we have:

Proposition 2 *The canonical 2-form U of an umbilical structure satisfies the integrability conditions:*

$$v(\mathcal{L}_\Psi g - \Phi \tilde{\otimes} \Psi) = \lambda v, \quad h(\mathcal{L}_\Psi g - \Phi \tilde{\otimes} \Psi) = \mu h \quad (26)$$

where Φ and Ψ are given in (6), and λ and μ are two scalars that can be obtained by taking the trace.

3.1 The Ricci tensor

Taking its real part, the expression (21) of the Ricci tensor becomes:

$$\text{Ric} = \frac{1}{4}(Rg + r\Pi) + \text{Re}[L] - U \cdot \text{Im}[L] \cdot *U - *U \cdot \text{Im}[L] \cdot U \quad (27)$$

where the scalar curvature R and the trace r with the structure tensor Π depend on the scalars X and Y and the U derivatives as:

$$R = 2(X + Y) + \Phi^2 - \Psi^2 - 2\delta\Phi, \quad r = 2(X - Y) \quad (28)$$

Finally, if we substitute (23) and (24) in the expression (27), we obtain:

Theorem 2 *If a space-time admits an umbilical structure U , the Ricci tensor takes the expression:*

$$\text{Ric} = \frac{1}{4}(\hat{R}g + \hat{r}\Pi) + \frac{1}{2}(M - U \cdot N \cdot *U - *U \cdot N \cdot U) \quad (29)$$

M and N depending on the expansion and rotation vectors Φ and Ψ as (25), and

$$\begin{aligned} 2\hat{R} &\equiv 4(X + Y) + 3\Phi^2 - 3\Psi^2 - 2\delta\Phi \\ 2\hat{r} &\equiv 4(X - Y) + \Pi(\Phi, \Phi) - \Pi(\Psi, \Psi) + 2\delta\Pi\Phi \end{aligned}$$

where $\delta\Pi\Phi \equiv -\Pi^{\alpha\beta}\nabla_\alpha\Phi_\beta$.

This proposition shows that the Ricci tensor is determined, up to the scalars X and Y given in (19), by U , χ and $\mathcal{L}_\chi g$.

3.2 The Weyl tensor

As a consequence of theorem 1, the self-dual Weyl tensor \mathcal{W} is determined by the scalar Ω and the self-dual 2-form $\mathcal{Z} \equiv A_\perp$. Indeed, (16) gives the orthogonal components of \mathcal{W} , (17) gives the mixed components and the $\mathcal{U} \otimes \mathcal{U}$ component is determined by the traceless condition. On the other hand, from the expression (15) of T we can obtain:

$$2(A, \mathcal{U}) = \delta\chi - \chi^2 \quad (30)$$

$$\mathcal{Z} \equiv A_\perp = \frac{1}{2}(\text{d}\xi)_\perp = [\nabla\chi, \mathcal{U}]_\perp = \frac{1}{2}[\text{d}\chi, \mathcal{U}]_\perp = \frac{1}{2}[\text{d}\chi, \mathcal{U}] \quad (31)$$

and, consequently, we have:

Theorem 3 *If a space-time admits an umbilical structure \mathcal{U} , the self-dual Weyl tensor takes the expression:*

$$\mathcal{W} = 3\Omega \mathcal{U} \otimes \mathcal{U} + \Omega \mathcal{G} + \mathcal{U} \tilde{\otimes} \mathcal{Z} \quad (32)$$

where

$$\Omega = \frac{1}{2}(\chi^2 - \delta\chi) - \frac{R}{12}, \quad \mathcal{Z} = \frac{1}{2}[d\chi, \mathcal{U}] \quad (33)$$

Moreover, the Weyl tensor invariants are

$$a \equiv \text{tr } \mathcal{W}^2 = 2(3\Omega^2 - (\mathcal{Z}, \mathcal{Z})), \quad b \equiv \text{tr } \mathcal{W}^3 = 3\Omega((\mathcal{Z}, \mathcal{Z}) - 2\Omega^2)$$

and the Weyl tensor eigenvalues

$$\Omega, \quad \frac{1}{2} \left(-\Omega \pm \sqrt{9\Omega^2 - 4(\mathcal{Z}, \mathcal{Z})} \right) \quad (34)$$

This theorem shows that the Weyl tensor is determined by the scalar curvature R , and by U , χ and $d\chi$. This fact and Theorem 2, which gives the Ricci tensor, show that the Riemann tensor is determined, up to the scalars X and Y , by the first and second derivatives of U , in accordance with our comment before expression (19).

From expressions (32) and (33) we can obtain the real Weyl tensor. Indeed, if we substitute χ in terms of the expansion and rotation vectors Φ and Ψ , and we take the real part of (32), we obtain:

Proposition 3 *If a space-time admits an umbilical structure U the Weyl tensor takes the expression:*

$$\begin{aligned} W = & 3\Omega_1(U \otimes U - *U \otimes *U) + 3\Omega_2 U \tilde{\otimes} *U + \Omega_1 G + \Omega_2 \eta \\ & + U \tilde{\otimes} Z - *U \tilde{\otimes} *Z \end{aligned}$$

where

$$Z = \frac{1}{4}([d\Phi, U] + [d\Psi, *U]), \quad (35)$$

$$\Omega_1 \equiv \frac{1}{24}(\Phi^2 - \Psi^2 - 2\delta\Phi) - \frac{1}{6}(X + Y), \quad \Omega_2 \equiv \frac{1}{4}((\Phi, \Psi) - \delta\Psi) \quad (36)$$

4 The compatible Petrov-Bel types

The algebraic classification of the Weyl tensor W can be obtained by studying the traceless linear map defined by the self-dual Weyl tensor \mathcal{W} on the SD bivectors space. In terms of the invariants a and b the characteristic equation reads $x^3 - \frac{1}{2}ax - \frac{1}{3}b = 0$. Then, Petrov-Bel classification follows taking

into account both the eigenvalue multiplicity and the degree of the minimal polynomial. The algebraically regular case (type I) occurs when $6b^2 \neq a^3$ and so the characteristic equation admits three different roots. If $6b^2 = a^3 \neq 0$, there is a double root and a simple one and the minimal polynomial distinguishes between types D and II. Finally, if $a = b = 0$ all the roots are equal and so zero, and the Weyl tensor is of type O, N or III, depending on the degree of the minimal polynomial. A fully tensorial algorithm has been presented in [16] enabling us to determine the Petrov-Bel type.

On the other hand, let us remember that we have defined a Debever SD bivector as one whose principal directions are Debever principal null directions of the Weyl tensor [16]. As the principal directions of \mathcal{U} are null shear-free geodesics, they are Debever principal directions and, consequently, \mathcal{U} is always a unitary Debever SD bivector.

In the previous section we have obtained the Weyl tensor of a metric admitting a 2+2 umbilical structure as well as the invariants of the Weyl tensor. These invariants depend on Ω and $(\mathcal{Z}, \mathcal{Z})$ as given in theorem 3. Then, a straightforward calculation leads to:

$$a^3 - 6b^2 = -2(\mathcal{Z}, \mathcal{Z})(9\Omega^2 - 4(\mathcal{Z}, \mathcal{Z}))$$

Then, the algorithm given in [16] works and we can obtain the algebraic type of the Weyl tensor in terms of \mathcal{Z} and Ω :

Theorem 4 *If a space-time admits a 2 + 2 umbilical structure, then the canonical SD bivector \mathcal{U} is a unitary Debever SD bivector. Thus, the space-time cannot be of Petrov-Bel type N. Moreover:*

- i) *It is of type O if, and only if, $\Omega = 0, \quad \mathcal{Z} = 0$.*
- ii) *It is of type III if, and only if, $\Omega = 0, \quad \mathcal{Z} \neq 0, \quad (\mathcal{Z}, \mathcal{Z}) = 0$.*
- iii) *It is of type D if, and only if, $\Omega \neq 0, \quad \mathcal{Z} = 0$.*
- iv) *It is of type II if, and only if, $\Omega \neq 0 \neq \mathcal{Z}, \quad (\mathcal{Z}, \mathcal{Z})(9\Omega^2 - 4(\mathcal{Z}, \mathcal{Z})) = 0$.*
- v) *It is of type I if, and only if, $(\mathcal{Z}, \mathcal{Z})(9\Omega^2 - 4(\mathcal{Z}, \mathcal{Z})) \neq 0$.*

Now we analyze the relationship between the geometry of the Weyl tensor and the geometrical elements defined by \mathcal{U} and its derivatives. We show that both SD bivectors \mathcal{U} and \mathcal{Z} are closely related to the principal null directions or Debever directions.

For every Petrov-Bel type, we briefly summarize the geometrical elements that the Weyl tensor outlines. We use the notation and results used in the aforementioned paper [16], where an exhaustive approach to this topic can be found. We compare the canonical form of the Weyl tensor of every type with the expression of the Weyl tensor obtained in Theorem 3.

4.1 Petrov-Bel type III

The self-dual Weyl tensor of a type III space-time takes the canonical expression [16]:

$$\mathcal{W} = \mathcal{U} \tilde{\otimes} \mathcal{H}$$

where \mathcal{H} is the null eigen-bivector and \mathcal{U} the canonical unitary SD bivector. The two Debever directions are the principal directions of the canonical SD bivector \mathcal{U} , the triple one being the common null principal direction to \mathcal{H} and \mathcal{U} .

From Theorem 4, a type III space-time with an umbilical structure is characterized by $\Omega = 0$ and $(\mathcal{Z}, \mathcal{Z}) = 0$, that is, \mathcal{Z} is a null SD bivector. Then, the expression for the Weyl tensor in theorem 3 becomes

$$\mathcal{W} = \mathcal{U} \tilde{\otimes} \mathcal{Z}$$

Consequently, we find that \mathcal{U} coincides with the canonical unitary SD bivector and \mathcal{Z} with the null eigen-bivector. This way, we can conclude

Proposition 4 *If a type III space-time admits a $2 + 2$ umbilical structure \mathcal{U} , then the Debever SD bivectors are \mathcal{U} and \mathcal{Z} , that is, the Debever directions are the principal directions of \mathcal{U} , the triple one being the fundamental direction of the null SD bivector \mathcal{Z} .*

4.2 Petrov-Bel type D

Theorem 4 states that a type D space-time admits a $2+2$ umbilical structure \mathcal{U} if, and only if, $\Omega \neq 0$ and $\mathcal{Z} = 0$. Then, expression (32) of the Weyl tensor becomes:

$$\mathcal{W} = 3\Omega\mathcal{U} \otimes \mathcal{U} + \Omega\mathcal{G}$$

Thus, \mathcal{U} is the principal SD bivector of the Weyl tensor with associated eigenvalue Ω . Then the principal null directions of \mathcal{U} are the Debever directions of the Weyl tensor. Consequently, we conclude:

Proposition 5 *If a type D space-time admits a $2 + 2$ umbilical structure \mathcal{U} , then \mathcal{U} is the principal SD bivector of the Weyl tensor, that is, the two double Debever directions are the principal directions of \mathcal{U} .*

4.3 Petrov-Bel type II

A type II Weyl tensor has a unitary eigenbivector associated with the simple eigenvalue -2ρ and just a null one \mathcal{H} corresponding to the double eigenvalue ρ . The double Debever principal null direction l is the fundamental direction of \mathcal{H} and, moreover, there exist two simple Debever directions l_{\pm} . Thus, three unitary Debever SD bivectors $\{\mathcal{V}, \mathcal{V}_{\pm}\}$ can be considered, \mathcal{V} having l_{\pm} as principal directions, and \mathcal{V}_{\pm} having l and l_{\pm} , respectively, as principal directions. In terms

of these Debever SD bivectors the Weyl tensor takes these two alternative canonical expressions [16]:

$$\mathcal{W} = \rho \mathcal{G} + \frac{3}{2} \rho \mathcal{V}_+ \tilde{\otimes} \mathcal{V}_- = -2\rho \mathcal{G} + \mathcal{H} \tilde{\otimes} \mathcal{V} \quad (37)$$

As Theorem 4 states, type II can occur in two different ways, when \mathcal{Z} is a null SD bivector, $(\mathcal{Z}, \mathcal{Z}) = 0$, or when $9\Omega^2 - 4(\mathcal{Z}, \mathcal{Z}) = 0$, both Ω and \mathcal{Z} being non zero.

In the first case, expressions (34) give us the double eigenvalue $\rho = \Omega$. Moreover, from Theorem 3 we obtain that $\mathcal{W}(\mathcal{Z}) = \Omega \mathcal{Z}$ and so, \mathcal{Z} is the canonical null SD bivector. This way \mathcal{Z} determines the double Debever principal null direction. Moreover, in this case the expression (32) of the Weyl tensor becomes:

$$\mathcal{W} = \Omega \mathcal{G} + \frac{3}{2} \Omega \mathcal{U} \tilde{\otimes} \left[\mathcal{U} + \frac{2}{3\Omega} \mathcal{Z} \right]$$

By comparing this expression with the first in (37) we conclude that in this case $\mathcal{U} + \frac{2}{3\Omega} \mathcal{Z}$ and \mathcal{U} are the unitary Debever SD bivectors \mathcal{V}_\pm .

In the second case, \mathcal{Z} is a non null SD bivector, and expressions (34) imply that the double eigenvalue is $\rho = -\frac{1}{2}\Omega$. Then the SD bivector $\frac{3}{2}\Omega \mathcal{U} + \mathcal{Z}$ is null, and the expression (32) of the Weyl tensor can be written as:

$$\mathcal{W} - \Omega \mathcal{G} = \mathcal{U} \tilde{\otimes} \left[\frac{3}{2} \Omega \mathcal{U} + \mathcal{Z} \right]$$

Thus, we have $\frac{3}{2}\Omega \mathcal{U} + \mathcal{Z}$ coincides with the \mathcal{H} of (37) and, consequently, it is the canonical null SD bivector. These results are summarized as follows:

Proposition 6 *Let \mathcal{U} be the canonical SD bivector of an umbilical structure in a type II space-time. It holds:*

i) *If $(\mathcal{Z}, \mathcal{Z}) = 0$, then \mathcal{Z} is the canonical null SD bivector and thus the fundamental direction of \mathcal{Z} is the double Debever direction. This one is also the common principal directions of the Debever SD bivectors \mathcal{U} and $\mathcal{U} + \frac{2}{3\Omega} \mathcal{Z}$. The other two principal directions of these SD bivectors determine the two simple Debever directions.*

ii) *If $9\Omega^2 - 4(\mathcal{Z}, \mathcal{Z}) = 0$, then \mathcal{U} is the Debever SD bivector whose principal directions are the two simple Debever directions, and the double one is the fundamental direction of the null SD bivector $\frac{3}{2}\Omega \mathcal{U} + \mathcal{Z}$*

4.4 Petrov-Bel type I

In an algebraically general space-time the Weyl tensor has three different eigenvalues ρ_i , and an orthonormal frame $\{\mathcal{U}_i\}$ of eigenbivectors of the Weyl tensor can be built. On the other hand, four simple Debever principal null directions

exist that define six unitary Debever SD bivectors $\{\mathcal{V}_{i\pm}\}$. If we choose a value of the index i , say $i = 3$, the Weyl tensor can be written as [16, 17]:

$$\mathcal{W} = \rho_3 \mathcal{G} + \frac{\rho_2 - \rho_1}{2} \mathcal{V}_{3+} \tilde{\otimes} \mathcal{V}_{3-} \quad (38)$$

As a consequence of theorem 4, when the space-time admits an umbilical structure, the type I case can be characterized by the scalar condition $(\mathcal{Z}, \mathcal{Z})(9\Omega^2 - 4(\mathcal{Z}, \mathcal{Z})) \neq 0$. Then, the expression (32) of the Weyl tensor can be written as (38) by taking:

$$\rho_3 \equiv \Omega, \quad \rho_2 - \rho_1 \equiv 2\sqrt{9\Omega^2 - 4(\mathcal{Z}, \mathcal{Z})} \quad (39)$$

$$\mathcal{V}_{3+} \equiv \mathcal{U}, \quad \mathcal{V}_{3-} \equiv \frac{1}{\sqrt{9\Omega^2 - 4(\mathcal{Z}, \mathcal{Z})}}(3\Omega\mathcal{U} + 2\mathcal{Z}) \quad (40)$$

Thus, we can state:

Proposition 7 *Let \mathcal{U} be an umbilical structure in a type I space-time. Then the four Debever principal null directions are the principal directions of the Debever SD bivectors \mathcal{U} and $3\Omega\mathcal{U} + 2\mathcal{Z}$.*

The expression (38) is not adapted to the Weyl eigenbivectors; however, these can be obtained by means of the projectors given in [16]. Indeed, using (32) and the expression (34) of the eigenvalues, we obtain:

Proposition 8 *In a type I space-time with an umbilical structure \mathcal{U} , the orthonormal basis of eigenbivectors $\{\mathcal{U}_i\}$ associated with the eigenvalues (34) are, respectively:*

$$\mathcal{U}_3 \propto \mathcal{U} \cdot \mathcal{Z}, \quad \mathcal{U}_1 \propto (3\Omega - \alpha)\mathcal{U} + 2\mathcal{Z}, \quad \mathcal{U}_2 \propto (3\Omega + \alpha)\mathcal{U} + 2\mathcal{Z},$$

where $\alpha \equiv \sqrt{9\Omega^2 - 4(\mathcal{Z}, \mathcal{Z})}$.

5 Type D space-times admitting two shear-free geodesic null congruences

In Theorem 4 we have shown that the Petrov-Bel type D space-times admitting two geodesic shear-free null congruences, that is, an umbilical structure, are those non conformally flat space-times satisfying $\mathcal{Z} = 0$. Then, from the expression $2\mathcal{Z} = [d\chi, \mathcal{U}]$ (see (31)) and condition (10) which characterizes a Maxwellian structure, we obtain the following:

Proposition 9 *If a non conformally flat space-time admits an umbilical and Maxwellian structure, then the Weyl tensor is Petrov-Bel type D and the structure is aligned with the Weyl principal structure.*

This result has been used elsewhere [4] to characterize the type D space-times admitting a non-null Killing-Yano or conformal Killing-Yano tensor.

Nevertheless, type D space-times exist with an umbilical and non Maxwellian structure. In order to obtain complementary conditions that guarantee the Maxwellian character of the principal structure one should consider this lemma:

Lemma 5 *The complex 2-form $d\chi$ admits the following decomposition in its self-dual and anti-self-dual parts:*

$$d\chi = ([d\chi, \mathcal{U}] + \{d\chi, \mathcal{U}\}) \cdot \mathcal{U} \quad (41)$$

The decomposition (41) is valid for an arbitrary complex two form, the first term being a self-dual 2-form. The second term contains, generically, the \mathcal{U} -component but, for the 2-form $d\chi$, this component vanishes as a consequence of (11).

From lemma above, theorem 4 and expression (31) one finds: *a space-time with two shear-free geodesic null congruences is type D, if and only if, the 2-form $d\chi$ is anti-self dual*, as stated by Dietz and Rüdiger [3]. Then, if this anti-self-dual part vanishes, the umbilical principal structure of a type D space-time becomes Maxwellian. But the nullity of a self-dual or anti-self-dual 2-form is equivalent to the nullity of its real (or imaginary) part. Thus, we can state:

Proposition 10 *In a non conformally flat space-time, an umbilical structure \mathcal{U} is Maxwellian ($d\chi = 0$) if, and only if, two (and then all) of the following conditions hold:*

- (i) *The space-time is Petrov-Bel type D (and the structure is the principal one).*
- (ii) *The structure is pre-Maxwellian (i.e. the expansion 1-form is closed, $d\Phi = 0$).*
- (iii) *The rotation 1-form is closed, $d\Psi = 0$.*
- (iv) *$\{d\chi, \mathcal{U}\} = 0$ (i.e. $d\Phi = *d\Psi$).*

Now we analyze in detail the last condition (iv) in the proposition above. Under the umbilical condition, the anti-self-dual part of $d\chi$ is $\{S, \mathcal{U}\}$ as a consequence of the expression of the tensor S given in theorem 1 and the definition of χ . Then, from the integrability condition (18), we can state:

Lemma 6 *Let \mathcal{U} be the canonical SD bivector of an umbilical structure and S the tensor defined in theorem 1. The following statements are equivalent:*

- i) *$d\chi$ is a self-dual 2-form (i.e. $\{d\chi, \mathcal{U}\} = 0$)*
- ii) *$[Ric, \mathcal{U}] = 2S$.*

As a direct consequence of this lemma and proposition 10 we obtain:

Corollary 1 *In a non conformally flat space-time, let \mathcal{U} be the canonical SD bivector of an umbilical structure and S the tensor defined in theorem 1. The following statements are equivalent:*

- i) *\mathcal{U} defines a maxwellian structure.*
- ii) *The space-time is Petrov-Bel type D and $[Ric, \mathcal{U}] = 2S$.*

Despite this result, the Maxwellian character of an umbilical structure does not restrict, generically, the algebraic type of the Ricci tensor. Nevertheless, the Maxwellian condition can restrict the Ricci tensor if we impose complementary conditions. Thus, we analyze elsewhere [15] the important role played by tensor S in generalizing to other energy contents the commutative group of symmetries that the type D vacuum solutions admit, and we will show in [15] that in this case the Ricci tensor becomes algebraically special. In this study corollary 1 will play a significative role.

On the other hand, some conditions on the Ricci tensor can impose strong restrictions on the Weyl principal structure of a type D space-time. Indeed, we have shown elsewhere [1] that, for vacuum solutions or when the Cotton tensor vanishes, the principal structure of a type D space-time is umbilical and Maxwellian. Moreover, in this case, the Weyl eigenvalues are real if, and only if, the structure is integrable [1]. Generalizing this kind of results to other energy contents is underway and involves analyzing the Bianchi identities for a type D Weyl tensor and a Ricci restricted by the umbilical integrability conditions.

Nevertheless, we can obtain some simple results which do not depend on the Ricci tensor. Thus, from proposition 10 and the expression (36) of the imaginary part of the Weyl eigenvalue, we have:

Proposition 11 *If a Petrov-Bel type D space-time admits an umbilical and integrable structure, then the Weyl tensor has real eigenvalues and the structure is the principal one and Maxwellian.*

Moreover, a function ϕ exists such that $d\phi = \Phi$ and the metric tensor is conformal to a product one, the conformal factor being $e^{-2\phi}$.

The second statement of this proposition is a direct consequence of the changes that a conformal transformation produces on the differential properties of a structure. A summary of this subject can be found in [1].

It is worth remarking that all the degenerate static solutions as well as their charged counterparts satisfy the hypothesis of the Proposition 11. The canonical expression of the metric as conformal to a product one has allowed us to obtain an intrinsic and explicit labeling of both families of solutions [1, 18].

It has been known for years [2, 3] that the two null eigenvectors of a conformal or a Killing tensor of type 2+2 as well as the two principal directions of a non-null Killing-Yano or a conformal Killing-Yano 2-form are shear-free geodesic null congruences. More recently [4, 5] we have recovered this property and have studied the complementary conditions, as equations on the structure U defined by these null directions, that guarantee the existence of these first integrals of the geodesic equation. For conformal Killing tensors and conformal Killing-Yano tensors these complementary conditions are related with Maxwellian properties of the structure. More precisely, we have [4, 5]:

Lemma 7 *A 2 + 2 traceless symmetric tensor is a conformal Killing tensor if, and only if, it defines an umbilical and pre-Maxwellian structure.*

A non-null 2-form is a conformal Killing-Yano tensor if, and only if, it defines an umbilical and Maxwellian structure.

It is known that the type D vacuum solutions which admit a Killing tensor also admit a Killing-Yano tensor [6, 7], and all the type D vacuum solutions admit a conformal Killing tensor [8]. The last statement can be generalized to type D solutions with a vanishing Cotton tensor and extended to conformal Killing-Yano tensors as a consequence of the results in [1] and Lemma 7. Moreover, from this lemma and Proposition 10 we obtain an interesting result which does not depend on the energy content:

Theorem 5 *Every Petrov-Bel type D space-time that admits a $2 + 2$ conformal Killing tensor also admits a conformal Killing-Yano tensor.*

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