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Type I vacuum solutions with aligned Papapetrou fields: An intrinsic characterization

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We show that Petrov type I vacuum solutions admitting a Killing vector whose Papapetrou field is aligned with a principal bivector of the Weyl tensor are the Kasner and Taub metrics, their counterpart with timelike orbits and their associated windmill-like solutions, as well as the Petrov homogeneous vacuum solution. We recover all these metrics by using an integration method based on an invariant classification which allows us to characterize every solution. In this way we obtain an intrinsic and explicit algorithm to identify them. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2363258]

I. INTRODUCTION

If ξ is a Killing vector, the *Killing 2-form* $\nabla\xi$ is closed and, in the vacuum case, it is a solution of the source-free Maxwell equations. Because this fact was pointed out by Papapetrou,[1] the covariant derivative $\nabla\xi$ has also been called the *Papapetrou field*.[2] In the Kerr geometry the principal directions of the Killing 2-form associated with the timelike Killing vector coincide with the two double principal null (Debever) directions of the Weyl tensor.[2] This means that the Killing 2-form is a Weyl principal bivector. This fact has been remarked upon by Mars[3] who has also shown that it characterizes the Kerr solution under an asymptotic flatness behavior.

A question naturally arises: can all the vacuum solutions with this property of the Kerr metric be determined? In other words, is it possible to integrate Einstein vacuum equations under the hypothesis that the spacetime admits an isometry whose Killing 2-form is a principal bivector of the Weyl tensor? Some partial results are known about this question. Thus, we have studied the case of Petrov type D spacetimes elsewhere[4] and we have shown that the Kerr-NUT solutions are the type D vacuum metrics with a timelike Killing 2-form aligned with the Weyl geometry.

Metrics admitting an isometry were studied by considering the algebraic properties of the associated Killing 2-form,[5,6] and this approach was extended to the spacetimes with an homothetic motion.[7,8] More recently Fayos and Sopuerta[2,9] have developed a formalism that improves the use of the Killing 2-form and its underlined algebraic structure for analyzing the vacuum solutions with an isometry. They consider two new viewpoints that permit a more accurate classification of these spacetimes: (i) the differential properties of the principal directions of the Killing 2-form, and (ii) the degree of alignment of the principal directions of the Killing 2-form with those of the Weyl tensor. The Fayos and Sopuerta approach uses the Newman–Penrose formalism and several extensions have been built for homothetic and conformal motions[10,11] and for nonvacuum solutions.[12]

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Some of the conditions on the Killing 2-form imposed in the literature quoted above could be very restrictive. Thus, in a previous paper[13] we have shown that the Petrov type I vacuum spacetimes admitting an isometry whose Killing 2-form is aligned with a Weyl principal bivector belong to two classes of metrics which admit a three-dimensional group of isometries of Bianchi types I or II. In the present work we show that a close relation between the Weyl principal directions and the isometry group exists in these classes. This fact allows us to achieve an integration of the vacuum equations with the aid of an invariant classification and, in this way, to obtain an intrinsic and explicit characterization of all the Petrov type I vacuum solutions that admit an aligned Papapetrou field. It is worth remarking that the integration method used here could be suitable in order to obtain other type I solutions.

The vacuum homogeneous Petrov solution[14] was found to be the only one satisfying: (i) vacuum, and (ii) existence of a simply transitive group G_4 of isometries. Although these two conditions characterize the Petrov metric, it is quite difficult to know when a metric tensor (given in an arbitrary coordinate system) satisfies them. Indeed, the first condition is intrinsic because it imposes a restriction on a metric concomitant, the Ricci tensor. Nevertheless, the second one imposes equations that mix up, in principle, elements other than the metric tensor (Killing vectors of the isometry group) and, consequently, it cannot be verified by simply substituting the metric tensor. In Ref. 13 we have changed this last nonintrinsic condition for an intrinsic one: the Weyl tensor is Petrov type I with constant eigenvalues. Moreover, as the Ricci and Weyl tensors are concomitants of the metric tensor, $\mathrm{Ric} \equiv \mathrm{Ric}(g)$, $\mathcal{W} \equiv \mathcal{W}(g)$, we have finally obtained the following *intrinsic* and *explicit* characterization of the Petrov solution:[13] *the necessary and sufficient conditions for g to be the Petrov homogeneous vacuum solution are*

$$\mathrm{Ric} = 0, \quad 6(\mathrm{Tr}\mathcal{W}^2)^3 \neq (\mathrm{Tr}\mathcal{W}^3)^2, \quad \mathrm{dTr}\mathcal{W}^2 = \mathrm{dTr}\mathcal{W}^3 = 0. \tag{1}$$

A whole intrinsic and explicit characterization of a metric or a family of metrics is quite interesting from a conceptual point of view and from a practical one because it can be tested by direct substitution of the metric tensor in arbitrary coordinates. Thus, it is an approach to the metric equivalence problem alternative to the usual one. This and other advantages have been pointed out elsewhere[15] where this kind of identification has been obtained for the Schwarszchild spacetime as well as for all the other type D static vacuum solutions. A similar study has been fulfilled for a family of Einstein–Maxwell solutions that include the Reissner–Nordström metric.[16]

In order to obtain intrinsic and explicit characterizations, as well as having an intrinsic labelling of the metrics, we need to express these intrinsic conditions in terms of explicit concomitants of the metric tensor. When doing this, the role played by the results on the covariant determination of the eigenvalues and eigenspaces of the Ricci tensor[17] and the principal 2-forms and principal directions of the Weyl tensor[18,19] is essential.

In this work we solve vacuum equations under the hypothesis that the spacetime is Petrov type I and there is a Killing vector whose associated Papapetrou field is a eigenbivector of the Weyl tensor. In this way, we recover the Petrov homogeneous vacuum solutions as well as the Kasner and Taub metrics, their counterpart with timelike orbits and their associated windmill-like solutions. Our integration method is based on an invariant classification which allows us to characterize the solutions intrinsically and explicitly. For every solution some properties of the isometry group and the aligned Killing 2-forms are given in terms of the Weyl principal directions.

The article is organized as follows. In Sec. II we present the Cartan formalism adapted to the Weyl principal frame that a Petrov type I spacetime admits. In Sec. III we summarize some results needed here about type I vacuum metrics admitting aligned Papapetrou fields. In Sec. IV we write vacuum Einstein equations for the families of Petrov type I metrics that, having a nonconstant Weyl eigenvalue, admit aligned Papapetrou fields. Sections V and VI are devoted to integrate these equations in different invariant subcases, as well as to determine, for every solution, the Killing vectors with an aligned Killing 2-form. In Sec. VII we present a similar study when all the Weyl eigenvalues are constant. Finally, in Sec. VIII, we summarize the results in an algorithmic form in order to make the intrinsic and explicit character of our results evident.

II. CARTAN FORMALISM IN THE WEYL FRAME OF A TYPE I SPACETIME

The algebraic classification of the Weyl tensor was first tackled by Petrov[20] considering the number of the invariant subespaces of the Weyl tensor regarded as an endomorphism of the 2-forms space. This classification was completed by Géhéniau[21] and Bel[22] considering also the eigenvalue multiplicity. In this framework appears the notion of *Weyl principal bivector* that we use here and which was widely analyzed by Bel[23] for the different algebraic types. In the 1960s many other authors presented alternative approaches to this classification, and in more recent studies[19,24] a wide bibliography on this subject can be found. For short, we refer the different classes of the Weyl tensor as the Petrov types. An algebraically general Weyl tensor is Petrov type I.

In a Petrov type I spacetime the Weyl tensor W determines four orthogonal principal directions which define the *Weyl principal frame* $\{e_\alpha\}$.[19,23] Then, the bivectors (self-dual 2-forms) $\mathcal{U}_i = 1/\sqrt{2}(U_i - i * U_i)$, with $U_i = e_0 \wedge e_i$, are eigenbivectors of the self-dual Weyl tensor $\mathcal{W} = 1/2(W - i * W)$, * being the Hodge dual operator. These bivectors satisfy $2\mathcal{U}_i \times \mathcal{U}_i = g$, where $\times$ denotes the contraction of adjacent index in the tensorial product. The tern $\{\mathcal{U}_i\}$ constitutes an orthonormal frame in the bivector space which has the induced orientation given by

$$\mathcal{U}_i \times \mathcal{U}_j = -\frac{i}{\sqrt{2}}\epsilon_{ijk}\mathcal{U}_k, \quad i \neq j. \tag{2}$$

If α_i is the eigenvalue associated with the eigenbivector $\mathcal{U}_i$, the self-dual Weyl tensor takes the canonical form

$$\mathcal{W} = -\sum_{i=1}^{3} \alpha_i \mathcal{U}_i \otimes \mathcal{U}_i. \tag{3}$$

The Cartan formalism can be referred to the Weyl principal frame $\{e_\alpha\}$ or, equivalently, to the *frame of eigenbivectors* $\{\mathcal{U}_i\}$. So, the six connection 1-forms ω_α^β defined by $\nabla e_\alpha = \omega_\alpha^\beta \otimes e_\beta$ can be collected into three complex ones Γ_i^j ($\Gamma_i^j = -\Gamma_j^i$), and the first structure equations take the expression

$$\nabla \mathcal{U}_i = \Gamma_i^j \otimes \mathcal{U}_j, \quad \Gamma_i^j = \omega_i^{\ j} - \epsilon_{ijk}\omega_0^k. \tag{4}$$

The second structure equations for a vacuum type I spacetime follow by applying the Ricci identities $\nabla_{[\alpha}\nabla_{\beta]}\mathcal{U}_{i\epsilon\delta} = \mathcal{U}_{i\epsilon}{}^{\mu}R_{\mu\delta\beta\alpha} + \mathcal{U}_i{}^{\mu}{}_{\delta}R_{\mu\epsilon\beta\alpha}$, and in terms of the eigenbivectors $\{\mathcal{U}_i\}$ they can be written as

$$\mathrm{d}\Gamma_i^k - \Gamma_i^j \wedge \Gamma_j^k = i\sqrt{2}\epsilon_{ikm}\alpha_m\mathcal{U}_m. \tag{5}$$

If we make the product of each of these second structure Eq. (5) with $\mathcal{U}_m$ we can obtain the following three complex scalar equations:

$$\nabla \cdot \lambda_i = \lambda_i^2 - (\lambda_j - \lambda_k)^2 - \alpha_i \quad (i,j,k \neq), \tag{6}$$

where $\lambda_i = -\mathcal{U}_i(\nabla \cdot \mathcal{U}_i)$, and we have denoted $\nabla \cdot \equiv \mathrm{Tr}\nabla$ and $\lambda_i^2 = g(\lambda_i, \lambda_i)$. The three complex 1-forms λ_i contain the 24 independent connection coefficients as the Γ_i^j do. In fact, by using Eq. (2) and the first structure Eqs. (4), both sets $\{\Gamma_i^j\}$ and $\{\lambda_i\}$ can be related by

$$\lambda_i \equiv -\mathcal{U}_i(\nabla \cdot \mathcal{U}_i) = -\frac{i}{\sqrt{2}}\epsilon_{ijk}\mathcal{U}_k(\Gamma_i^j). \tag{7}$$

And the inverse of these expressions say that for different i,j,k,

$$\mathcal{U}_k(\Gamma_i^j) = \frac{i}{\sqrt{2}}\epsilon_{ijk}(\lambda_i + \lambda_j - \lambda_k), \quad (i,j,k \neq). \tag{8}$$

The Bianchi identities in the vacuum case state that the Weyl tensor is divergence-free $\nabla \cdot \mathcal{W}=0$, and from Eq. (3) they can be written as

$$\mathrm{d}\alpha_i = (\alpha_j - \alpha_k)(\lambda_j - \lambda_k) - 3\alpha_i\lambda_i \quad (i,j,k \neq). \tag{9}$$

Equations (9) show the relation that exists between the gradient of the Weyl eigenvalues and the 1-forms λ_i in the vacuum case. This fact has suggested a classification of Petrov type I spacetimes taking into account the dimension of the space that $\{\lambda_i\}$ generate. More precisely,[13]

Definition 1: We say that a Petrov type I spacetime is of class I_a ($a=1,2,3$) if the dimension of the space that $\{\lambda_i\}$ generate is a.

Differential conditions of this kind were imposed by Edgar[25] on the type I spacetimes, and he showed that in the vacuum case his classification also has consequences on the functional dependence of the Weyl eigenvalues. We have slightly modified the Edgar approach in order to obtain a classification that is symmetric in the principal structures of the Weyl tensor. We remark the invariant nature of this classification: it is based on the vector Weyl invariants λ_i.

We have been studied elsewhere the symmetries of the vacuum metrics of class I_1 and we have shown:[13]

Lemma 1: A vacuum metric of class I_1 admits at least a (simply transitive) group G_3 of isometries. It admits a G_4 if, and only if, it has constant eigenvalues.

III. ALIGNED KILLING 2-FORMS AND TYPE I VACUUM METRICS

If ξ is a (real) Killing vector its covariant derivative $\nabla\xi$ is named Killing 2-form or Papapetrou field.[1,2] The Papapetrou fields have been used to study and classify spacetimes admitting an isometry or an homothetic or conformal motion (see Refs. 2–12). In this way, some classes of vacuum solutions with a principal direction of the Papapetrou field aligned with a (Debever) null principal direction of the Weyl tensor have been considered.[9] Also, the alignment between the Weyl principal plane and the Papapetrou field associated with the timelike Killing vector has been shown in the Kerr geometry.[3,9]

Is it possible to determine all the vacuum solutions having this property of the Kerr metric? Elsewhere[4] we give an affirmative answer to this question for the case of Petrov type D spacetimes by showing that *the type D vacuum solutions with a timelike Killing 2-form aligned with the Weyl geometry are the Kerr-NUT metrics*. In this work we accomplish this study for the Petrov type I spacetimes by obtaining all the vacuum solutions with this property and by determining the Killing vectors with an aligned Killing 2-form. Moreover, we show the close relation between the Weyl tensor geometry and the geometries of ξ and $\nabla\xi$.

In order to clarify what kind of alignment between the Killing 2-form and the Weyl tensor is analyzed in this work we give the specific definition. If $\{\mathcal{U}_i\}$ is an orthonormal basis of the self-dual 2-forms space, the Papapetrou field $\nabla\xi$ associated to a Killing vector ξ has, generically, three independent complex components Ω_i:

$$\nabla\xi = \sum_{i=1}^{3} \Omega_i \mathcal{U}_i + \sum_{i=1}^{3} \tilde{\Omega}_i \tilde{\mathcal{U}}_i \tag{10}$$

where ~ means complex conjugate. Then:

Definition 2: We say that a Papapetrou field $\nabla\xi$ is aligned with a bivector $\mathcal{U}$ if both 2-forms have the same principal 2-planes, that is, $\nabla\xi = \Omega\mathcal{U} + \tilde{\Omega}\tilde{\mathcal{U}}$.

We say that a Papapetrou field $\nabla\xi$ is aligned (with the Weyl tensor) if it is aligned with a Weyl principal bivector.

When a Killing 2-form is aligned with a bivector of an orthonormal frame of invariants bivectors $\mathcal{U}_i$, the Killing vector is strongly restricted by the connection 1-forms. Thus, for type I metrics we have:[13]

Lemma 2: In a Petrov type I spacetime with a Killing vector ξ, the Papapetrou field $\nabla\xi$ is aligned with a Weyl principal 2-form $\mathcal{U}_i$ if, and only if, ξ is orthogonal to the two complex connection 1-forms Γ_i^j *(*defined by the Weyl principal frame $\{\mathcal{U}_i\}$*)*.

The alignment between a Killing 2-form and a Weyl principal bivector of a type I vacuum solution has been partially analyzed[13] and the following necessary condition has been obtained:

Lemma 3: A vacuum Petrov type I spacetime which admits a Killing field with an aligned Papapetrou field belongs to class I_1.

As a consequence of Lemmas 1 and 3, we obtain that *a vacuum Petrov type I spacetime which admits a Killing field with an aligned Papapetrou field admits, at least, a three-dimensional group of isometries.* This means that a unique symmetry with an aligned Papapetrou field implies that other symmetries exist.

These results imply that in order to find all the type I vacuum solutions admitting an aligned Papapetrou field, we must analyze the vacuum solutions of class I_1. We shall start with the case where a non constant eigenvalue α_1 exists. After that we shall finish by dealing with the case of all the eigenvalues being constant.

IV. VACUUM EQUATIONS FOR THE CLASS I_1

As we know that every vacuum solution of class I_1 admits a (simply transitive) G_3 group of isometries, a real function τ exists such that $\alpha_i \equiv \alpha_i(\tau)$. Moreover, as we are in class I_1, it must be $\lambda_i \wedge \lambda_j = 0$ and so, from Bianchi identities (9), we obtain $\lambda_i \wedge \mathrm{d}\alpha_1 = 0$. So, taking into account Eq. (7) and that a G_3 is admitted, three functions $\varphi_i(\tau)$ exist such that

$$\Gamma_i^j = \mathrm{i}\epsilon_{ijk}\varphi_k \mathcal{U}_k(\mathrm{d}\tau). \tag{11}$$

On the other hand, it has also been shown in Ref. 13 that $\mathrm{d}\alpha_1$ cannot be a null vector and so, $(\mathrm{d}\tau)^2 \neq 0$. Thus, $\{\mathrm{d}\tau, u^i\}$, with $u^i = \mathcal{U}_i(\mathrm{d}\tau)$, is an orthogonal frame such that $2(u^i)^2 = -(\mathrm{d}\tau)^2$. Then, we can write the bivectors $\{\mathcal{U}_i\}$ as

$$\mathcal{U}_i = -\frac{1}{(\mathrm{d}\tau)^2}\left(\mathrm{d}\tau \wedge u^i + \frac{i}{\sqrt{2}}\epsilon_{ijk}u^j \wedge u^k\right). \tag{12}$$

We can use this expression to eliminate $\mathcal{U}_i$ in the second structure Eqs. (4) and then they become an exterior system for the orthonormal frame $\{\mathrm{d}\tau, u^i\}$,

$$\mathrm{d}u^i = \mu_i(\tau)\mathrm{d}\tau \wedge u^i + \nu_i(\tau)u^j \wedge u^k \tag{13}$$

for every cyclic permutation i,j,k, and where the functions μ_i and ν_i are given by

$$\mu_i = -(\ln \varphi_i)' + \frac{\sqrt{2}\alpha_i}{\varphi_i(\mathrm{d}\tau)^2}, \quad \nu_i = -\mathrm{i}\left(\frac{2\alpha_i}{\varphi_i(\mathrm{d}\tau)^2} + \frac{\varphi_j\varphi_k}{\varphi_i}\right), \tag{14}$$

where $'$ stands for the derivative with respect to the variable τ. But $\mathrm{d}\tau$ is proportional to the invariant 1-form $\mathrm{d}\alpha_1$ and a G_3 exists. Thus, it follows that $(\mathrm{d}\tau)^2$ and $\Delta\tau$ depend on τ. This fact allows us to choose τ such that $\Delta\tau = 0$. Then, τ is fixed up to an affine transformation $\tau \hookrightarrow \alpha\tau + \beta$. In terms of this harmonic function, the Eqs. (6) become

$$((\varphi_j + \varphi_k)' - \sqrt{2}\varphi_j\varphi_k)(\mathrm{d}\tau)^2 = 2\sqrt{2}\alpha_i \tag{15}$$

for every cyclic permutation of i,j,k. The Bianchi identities (9) can be stated as

$$\alpha_1' = \frac{1}{\sqrt{2}}(\varphi_3(\alpha_2 - \alpha_1) - \varphi_2(2\alpha_1 + \alpha_2)), \quad \alpha_2' = \frac{1}{\sqrt{2}}(\varphi_3(\alpha_1 - \alpha_2) - \varphi_1(\alpha_1 + 2\alpha_2)). \tag{16}$$

At this point, it is clear that the integration of the system (13) depends strongly on the number of the $u^i = \mathcal{U}_i(\mathrm{d}\tau)$ that are integrable 1-forms. Thus, it seems suitable to give a classification of type I_1 spacetimes that takes into account these restrictions. But these conditions lead to an invariant classification because of u^i is proportional to $\mathcal{U}_i(\mathrm{d}\alpha_1)$:

Definition 3: We will say that a Petrov type I_1 vacuum metric with $\mathrm{d}\alpha_1 \neq 0$ is of class I_{1A} $(A=0,1,2,3)$ if there are exactly A integrable 1-forms in the set $\{\mathcal{U}_i(\mathrm{d}\alpha_1)\}$.

We have studied elsewhere[13] the symmetries that the different classes I_{1A} admit, as well as necessary conditions for the alignment of the associated Killing 2-forms with the Weyl tensor. Here we will make use of the following result:

Lemma 4: If a vacuum Petrov type I spacetime admits a Killing field with an aligned Papapetrou field then either it is the Petrov solution (that has constant eigenvalues) or it is of class I_{12} or I_{13}. These classes admit an isometry group G_3 of Bianchi types II and I, respectively.

Thus, in order to find the vacuum solutions with aligned Papapetrou fields, we must consider the Petrov solution that admits a G_4 or the classes I_{12} and I_{13}. Now we obtain the vacuum solutions for these two classes with non constant eigenvalues. To accomplish this goal, we will integrate the Bianchi identities (16) and the scalar Eqs. (15) taking into account that in class I_{13} all the functions ν_i given in Eq. (14) are zero and two of them vanish in class I_{12}. Finally, the second structure Eqs. (13) will be integrated to obtain the 1-forms u^i in terms of real coordinates. After that, the metric tensor will be obtained as

$$g = \frac{1}{(\mathrm{d}\tau)^2}\left[\mathrm{d}\tau \otimes \mathrm{d}\tau - 2\sum_{i=1}^{3} u^i \otimes u^i\right] \tag{17}$$

It is worth pointing out that in the spacetimes of type I_1 that we are studying here there exist two outlined coframes, namely, the Weyl principal coframe $\{\theta^\alpha\}$ and that defined by $\{\mathrm{d}\tau, u^i\}$. We will see in following sections the close relation between both coframes for the spacetimes in classes I_{12} and I_{13}. This fact allows us to give intrinsic conditions that label every type I vacuum solution admitting an aligned Papapetrou field.

V. VACUUM SOLUTIONS OF CLASS I_{13}

In class I_{13} all of the 1-forms u^i are integrable. Then Eqs. (13) hold with $\nu_i = 0$. Taking into account Eq. (14), we can solve the Eqs. (15) and (16) to obtain

$$\alpha_1 = b e^{2a\beta\tau}, \quad \alpha_2 = k\alpha_1, \quad (\mathrm{d}\tau)^2 = \frac{b e^{2a\beta\tau}}{a^2 k(k+1)}, \tag{18}$$

$$\varphi_3 = \sqrt{2}ka, \quad \varphi_2 = -\sqrt{2}a(k+1), \quad \varphi_1 = -\sqrt{2}k(k+1)a, \tag{19}$$

where a, b and $\beta = 1+k+k^2$ are nonzero constants and k is different from 1, -2, and $-1/2$ because g is not of Petrov type D, and different from -1 and 0 because none of the Weyl eigenvalues vanishes as a consequence of the Szekeres–Brans theorem.[26,27] The second structure Eqs. (13) constitute an exterior system for the 1-forms $u^i \equiv \mathcal{U}_i(\mathrm{d}\tau)$. It implies that three complex functions $\{x^i\}$ exist such that

$$u^1 = e^{a\tau}\mathrm{d}x^1, \quad u^2 = e^{ak^2\tau}\mathrm{d}x^2, \quad u^3 = e^{a(1+k)^2\tau}\mathrm{d}x^3. \tag{20}$$

From here and Eq. (17), we can obtain the metric tensor g in complex coordinates. In order to get real coordinates, another fact is needed. As τ is a real function, it follows that $\mathrm{d}\tau$, $(\mathrm{d}\tau)^2$ and $\nabla \mathrm{d}\tau$ must also be real. If we compute $\nabla \mathrm{d}\tau$ by using Eqs. (17)–(19) we obtain that, necessarily, either all of the coefficients are real and $\mathrm{d}\tau$ coincides with one of the principal directions θ^α, or two of the coefficients are conjugate and $\mathrm{d}\tau$ takes the direction of one of the bisectors $\theta^i \pm \theta^j$ of a spacelike principal plane. We shall analyze every case, but we must take into account that as $\mathrm{d}\tau \wedge \mathrm{d}\alpha_1 = 0$, these conditions can also be written in terms of the Weyl eigenvalue.

A. $d\alpha_1$ Is a Principal Direction

In this case a and k are real constants. We must remark that if $d\tau$ is a principal direction, then $u^i \equiv \mathcal{U}_i(d\tau)$ are so. But when $d\tau$ coincides with the timelike principal direction θ^0, every u^i is a real direction and, if $d\tau$ is a spacelike principal direction, some of them are purely imaginary. Now we will analyze each case in detail.

(i) Case $d\alpha_1 \wedge \theta^0 = 0$. We have $d\tau = e_0(\tau)\theta^0$. Then $u^i = (1/\sqrt{2})e_0(\tau)\theta^i$, and so dx^i of Eq. (20) are real. If we take into account that the harmonic coordinate τ is defined up to affine transformation, the metric tensor (17) in real coordinates takes the form of the Kasner metric

$$g = -e^{-2\tau}d\tau^2 + e^{2(1/\beta-1)\tau}(dx^1)^2 + e^{2(k^2/\beta-1)\tau}(dx^2)^2 + e^{2(((1+k)^2/\beta)-1)\tau}(dx^3)^2. \tag{21}$$

The coordinate transformation $e^{-\tau}=t$ changes the harmonic time to the proper time and gives us the usual expression for this solution.[24,28]

We must check whether there is a Killing field with an aligned Papapetrou field. We have established[13] that this condition is equivalent to a Killing field to be orthogonal to two of the complex connection 1-forms Γ_i^j (see Lemma 2). The real Killing fields of this metric are $\xi = k_1\partial_{x^1} + k_2\partial_{x^2} + k_3\partial_{x^3}$. As the connection 1-forms Γ_i^j are collinear with u^i it follows that every Killing field ∂_i satisfies this condition, and so, we have three Killing fields such that their Papapetrou fields are aligned with the three principal 2-forms.

(ii) Case $d\alpha_1 \wedge \theta^1 = 0$. Now, $d\tau = e_1(\tau)\theta^1$, and so $(d\tau)^2 > 0$. In order to get real coordinates we must take into account that in this case $\sqrt{2}u^1 = -e_1(\tau)\theta^0$, $\sqrt{2}u^2 = ie_1(\tau)\theta^3$ and $\sqrt{2}u^3 = ie_1(\tau)\theta^2$. And so, the coordinates adapted to u^2 and u^3 are purely imaginary $x^a = iy^a$ $(a=2,3)$, y^a being real functions. Then, for the metric tensor g we get a similar expression to the one in the previous case, the only change being the causal character of the gradient of the Weyl eigenvalue

$$g = e^{-2\tau}d\tau^2 - e^{2(1/\beta-1)\tau}(dx^1)^2 + e^{2(k^2/\beta-1)\tau}(dy^2)^2 + e^{2(((1+k)^2/\beta)-1)\tau}(dy^3)^2. \tag{22}$$

This is the static Kasner metric.[24]

This solution admits three Killing fields ∂_{x^1}, ∂_{y^2}, and ∂_{y^3} such that their Papapetrou fields are aligned with the three principal bivectors of the Weyl tensor. This finishes the study of the cases in which the gradient of the invariant $\alpha_1(\tau)$ is collinear with a principal direction of the Weyl tensor. The following proposition summarizes the main results.

Proposition 1: The Kasner metrics (21) and (22) are the only Petrov type I_{13} vacuum solutions where the gradient of the Weyl eigenvalue is a principal direction of the Weyl tensor.

The metrics of this family admit three Killing fields ξ_i which are collinear with the three principal directions $\mathcal{U}_i(d\alpha_1)$, such that their Papapetrou fields $\nabla\xi_i$ are aligned with the three principal bivectors $\mathcal{U}_i$ of the Weyl tensor.

B. $d\alpha_1$ Is Not a Principal Direction

As we have commented below, in this case $d\tau$ must take the direction of one of the bisectors of a spacelike principal plane, say $\theta^2+\theta^3$, $d\tau \propto \theta^2+\theta^3$. Then, $u^1 \propto \theta^3-\theta^2$, $u^2 \propto \theta^0 + i\theta^1$ and $u^3 \propto \theta^0 - i\theta^1$. Moreover, $\nabla d\tau$ is real if, and only if, a is real and $2k=-1+in$, n being a nonzero real constant because the metric is not of type D and $n^2 \neq 3$ because β cannot be zero. Then, the coordinate x^1 of Eq. (20) must be purely imaginary, $x^1 = ix$, and x^2 and x^3 must be conjugated functions, that is $x^2 = y - iz$, $x^3 = y + iz$. Thus we get a real coordinate system $\{\tau, x, y, z\}$ and, from Eq. (17), we find the following expression of the metric tensor:

$$g = \frac{1}{4}(3-n^2)^2 e^{(1/2)(3-n^2)\tau}d\tau^2 + e^{-(1/2)(1+n^2)\tau}dx^2 + e^{\tau}[\cos(n\tau)[dz^2 - dy^2] - 2\sin(n\tau)dy\,dz]. \tag{23}$$

This is the so called *windmill solution*.[24,29]

To see if an aligned Killing 2-form can exist in this spacetime, we must look for a Killing field to be orthogonal to two of the connection 1-forms. The Killing fields of this solution are $\xi = k_1\partial_x + k_2\partial_y + k_3\partial_z$ and, as every connection 1-form is parallel to one of the directions u^i, the only Killing field which is orthogonal to a pair of connection 1-forms is ∂_x, that can be characterized as the Killing field that takes the direction of the bisector $\theta^2-\theta^3$. Moreover, the Weyl tensor has just a real eigenvalue α_1 and if $\mathcal{U}_1$ is the associated eigenbivector, then $\mathcal{U}_1(\mathrm{d}\alpha_1)$ is collinear with the Killing field ∂_x. We can collect these results in the following:

Proposition 2: The windmill solution (23) is the only Petrov type I_{13} vacuum solution where the gradient of the Weyl eigenvalue α_1 is not a principal direction of the Weyl tensor.

In such spacetime a unique real eigenvalue α_1 exists. Then, if $\mathcal{U}_1$ is the associated eigenbivector, the field $\mathcal{U}_1(\mathrm{d}\alpha_1)$ is collinear with a Killing field that has a Papapetrou field aligned with $\mathcal{U}_1$.

VI. VACUUM SOLUTIONS OF CLASS I_{12}

Let us suppose now that only two directions, let us say u^2 and u^3, are integrable. So, we can take $\nu_2=\nu_3=0$ in the second structure Eqs. (13). Taking into account the definition of ν_i from Eq. (14) we obtain

$$\varphi_3 = \frac{k}{\sqrt{2}} - \varphi_1, \quad \varphi_2 = \frac{a^2}{\sqrt{2}k} - \varphi_1, \quad \varphi_1 = \frac{a}{\sqrt{2}}\frac{be^{-a\tau}+1}{be^{-a\tau}-1}, \tag{24}$$

where a, b, and k are complex constants, $a^2 \neq k^2$. Then, by also using the Bianchi identities (16) we obtain

$$(\mathrm{d}\tau)^2 = \frac{-2\sqrt{2}c}{a} e^{-((a^2+ak+k^2)/k)\tau}(b^2e^{-2a\tau}-1)^{-1}, \tag{25}$$

where c is another complex constant.

As in the previous section, the only possibilities for $\nabla\mathrm{d}\tau$ to be real are that either $\mathrm{d}\tau$ is a principal direction θ^α or it is the bisector $\theta^i+\theta^j$ of a spacelike principal plane.

A. $\mathrm{d}\alpha_1$ Is a Principal Direction

In this case we have that k, a^2 and $\varphi_3'/\varphi_3 - \sqrt{2}\varphi_3$ are real. From Eq. (24) we obtain

$$\frac{\varphi_3'}{\varphi_3} - \sqrt{2}\varphi_3 = -a\frac{b^2+e^{2\sqrt{2}\tau}}{b^2-e^{2\sqrt{2}\tau}}. \tag{26}$$

So we can conclude that a and b^2 must be real constants. Now we shall go on the integration of Eq. (13). As in the previous section it will be useful to distinguish the cases of $\mathrm{d}\tau$ to be the timelike principal direction θ^0 or a spacelike principal direction θ^i. We will study these cases separately.

(i) Case $\mathrm{d}\alpha_1 \wedge \theta^0 = 0$. Here we have $\mathrm{d}\tau = e_0(\tau)\theta^0$, and so u^i must be real for every i. Consequently, if we take into account Eq. (13) with $\nu_2=\nu_3=0$, real coordinates $\{x,y,z\}$ can be found such that

$$u^2 = \frac{e^{-(k/2)\tau}}{\sqrt{2}}\mathrm{d}x, \quad u^3 = \frac{e^{-(a^2/2k)\tau}}{\sqrt{2}}\mathrm{d}y, \quad u^1 = -\frac{\mathrm{i}\sqrt{2}abe^{-a\tau}}{b^2e^{-2a\tau}-1}e^{-((a^2+k^2)/2k)\tau}(\mathrm{d}z + x\mathrm{d}y). \tag{27}$$

As u^1 is real, we find that b is purely imaginary, $b=-i\beta$. Then, from Eq. (25) we can calculate $(\mathrm{d}\tau)^2$, and taking into account the freedom of an affine transformation in choosing the harmonic coordinate τ, we can take $\beta=1$ and we can write the metric in the usual form of the Taub[30] metric

$$g = \frac{\cosh(a\tau)}{a}(-e^{((a^2+k^2)/k)\tau}\mathrm{d}\tau^2 + e^{(a^2/k)\tau}\mathrm{d}x^2 + e^{k\tau}\mathrm{d}y^2) + \frac{a}{\cosh(a\tau)}(\mathrm{d}z + x\mathrm{d}y)^2. \tag{28}$$

To see if a Killing field with an aligned Papapetrou field exists, we must look for a Killing field which is orthogonal to two of the connection 1-forms. The Killing fields of the Taub metric (28) are $\xi = k_1\partial_x + k_2\partial_y + (k_3 - k_1 y)\partial_z$ and, taking into account that the connection 1-forms Γ_i^j are collinear with u^k, from Eq. (27) we find that the only Killing field that is orthogonal to a pair of connection 1-forms is $\xi = \partial_z$, and it is orthogonal to Γ_1^2 and Γ_1^3. So, the principal 2-form aligned with a Papapetrou field is $\mathcal{U}_1$, and it is characterized by the fact that $\mathcal{U}_1(\mathrm{d}\tau)$ is not integrable.

(ii) Case $\mathrm{d}\alpha_1 \wedge \theta^1 = 0$. Now, $\mathrm{d}\tau = e_1(\tau)\theta^1$ and we have $\sqrt{2}\,u^1 = -e_1(\tau)\theta^0$, $\sqrt{2}u^2 = ie_1(\tau)\theta^3$ and $\sqrt{2}\,u^3 = ie_1(\tau)\theta^2$. So, we can consider real coordinates $\{x, y, z\}$ such that

$$u^2 = \frac{\mathrm{i}e^{-(k/2)\tau}}{\sqrt{2}}\mathrm{d}x, \quad u^3 = \frac{ie^{-(a^2/2k)\tau}}{\sqrt{2}}\mathrm{d}y, \quad u^1 = \frac{-\sqrt{2}\beta a e^{-a\tau}}{\beta^2 e^{-2a\tau} + 1} e^{-((a^2+k^2)/2k)\tau}(\mathrm{d}z - x\mathrm{d}y). \tag{29}$$

Then, the same analysis of the previous case leads to the counterpart with timelike orbits of the Taub metric[24]

$$g = \frac{\cosh(a\tau)}{a}(e^{((a^2+k^2)/k)\tau}\mathrm{d}\tau^2 + e^{(a^2/k)\tau}\mathrm{d}x^2 + e^{k\tau}\mathrm{d}y^2) - \frac{a}{\cosh(a\tau)}(\mathrm{d}z - x\mathrm{d}y)^2. \tag{30}$$

The same property of the Taub metric concerned with the aligned Papapetrou fields holds in this case. We can summarize these results for the case that $\mathrm{d}\alpha_1$ is collinear with a principal direction in the following:

Proposition 3: The Taub metric (28) that has spacelike orbits, and its counterpart with timelike orbits (30) are the only type I_{12} vacuum solutions where the gradient of the Weyl eigenvalue α_1 is collinear with a principal direction of the Weyl tensor.

Both metrics admit a principal 2-form $\mathcal{U}_i$ such that $\mathcal{U}_i(\mathrm{d}\alpha_1)$ is not integrable. Then, the Killing field collinear with $\mathcal{U}_i(\mathrm{d}\alpha_1)$ is the only one whose Papapetrou field is aligned (with the principal 2-form $\mathcal{U}_i$).

B. $\mathrm{d}\alpha_1$ Is Not a Principal Direction

In this case $\mathrm{d}\tau$ must take the direction of one of the bisectors of a spacelike principal plane, say $\theta^2 + \theta^3$, $\mathrm{d}\tau \propto \theta^2 + \theta^3$. Then, a similar analysis to the one in the previous cases, leads to the metric

$$g = e^{2m\tau}\left[\frac{\cosh(a\tau)}{a}\mathrm{d}\tau^2 + \frac{a}{\cosh(a\tau)}e^{-2m\tau}(\mathrm{d}z - u\mathrm{d}v)^2 + \frac{\cosh(a\tau)}{a}e^{-m\tau}[\cos(n\tau)(\mathrm{d}v^2 - \mathrm{d}u^2) - 2\sin(n\tau)\mathrm{d}u\mathrm{d}v]\right], \tag{31}$$

where $a^2 = m^2 + n^2$, $n \neq 0$. This is an equivalent windmill-like metric for the Taub solution.

The real Killing fields of this metric in the previous coordinate system are $\xi = k_1\partial_u + k_2\partial_v + (k_1 v + k_3)\partial_z$. As the complex connection 1-forms Γ_i^j are collinear with $u^k \equiv \mathcal{U}_k(\mathrm{d}\tau)$, we conclude that there is only one Killing field ∂_z which is orthogonal to two connection 1-forms, more precisely, to Γ_1^2 and Γ_1^3. So, this Killing field has a Papapetrou field which is aligned with the principal bivector $\mathcal{U}_1$. We summarize these results in the following:

Proposition 4: The metric (31) is the only vacuum solution of class I_{12} where the gradient of the Weyl eigenvalue α_1 is not a Weyl principal direction.

This solution admits a principal 2-form $\mathcal{U}_i$ such that $\mathcal{U}_i(\mathrm{d}\alpha_1)$ is not integrable. Then, the Killing field collinear with $\mathcal{U}_i(\mathrm{d}\alpha_1)$ is the only one whose Papapetrou field is aligned (with the principal 2-form $\mathcal{U}_i$).

VII. TYPE I VACUUM SOLUTIONS WITH CONSTANT EIGENVALUES

Elsewhere[13] we have shown that the only Petrov type I vacuum solution with constant eigenvalues is the homogeneous Petrov solution.[14,24] In real coordinates this metric writes as

$$k^2 g = \mathrm{d}x^2 + e^{-2x}\mathrm{d}y^2 + e^x(\cos\sqrt{3}x(\mathrm{d}z^2 - \mathrm{d}t^2) - 2\sin\sqrt{3}x\mathrm{d}z\mathrm{d}t). \tag{32}$$

The eigenvalues of this metric are proportional to the three cubic roots of -1, $\alpha_i = k^2\sqrt[3]{-1}$. So, a real eigenvalue, let us say α_3, exists. From the metric expression (32) we get that $\Gamma_1^3 \wedge \mathcal{U}_1(\Gamma_1^2) = 0$ and $\Gamma_2^3 \wedge \mathcal{U}_2(\Gamma_1^2) = 0$. Moreover, a straightforward calculation shows that dx takes the direction of one of the bisectors of the plane $*U_3$, $\mathrm{d}x = e_1(x)(\theta^1 + \theta^2)$, and that the complex connection 1-forms Γ_i^j are given by

$$\Gamma_1^2 = e^{-x}\mathrm{d}y, \quad \mathcal{U}_3(\Gamma_1^2) = -\frac{i}{\sqrt{2}}\mathrm{d}x,$$

$$\mathcal{U}_1(\Gamma_1^2) = \frac{1}{2}e^{(1/2)(1+i\sqrt{3})x}(\mathrm{d}t - i\mathrm{d}z), \quad \mathcal{U}_2(\Gamma_1^2) = \frac{1}{2}e^{(1/2)(1-i\sqrt{3})x}(\mathrm{d}t + i\mathrm{d}z).$$

The Killing fields of this solution are $\{\partial_t, \partial_z, \partial_y, \partial_x + y\partial_y + 1/2(\sqrt{3}t - z)\partial_z - 1/2(t + \sqrt{3}z)\partial_t\}$ and so, it easily follows:

Proposition 5: The Petrov homogeneous vacuum solution (32) admits just a Killing field such that its Papapetrou field is aligned with a principal bivector. If α_3 is the real eigenvalue, this Killing field is proportional to Γ_1^2 and its Papapetrou field is aligned with $\mathcal{U}_3$.

VIII. SUMMARY IN ALGORITHMIC FORM

In this article we have found all the Petrov type I vacuum solutions admitting a Killing field whose Papapetrou field is aligned with a principal bivector of the Weyl tensor. We knew[13] that these solutions admit either a simply transitive group G_4 of isometries and then the metric must be the homogeneous Petrov solution (32), or a simply transitive G_3 group of isometries and then the spacetime belongs to one of the classes I_{13} and I_{12} in definition 3. Here we have shown that these necessary conditions given in Ref. 13 are also sufficient conditions.

The solutions can be characterized by a condition on the normal direction to the orbits group: for class I_{13}, (i) if it is a timelike principal direction we reach the Kasner metric (21), (ii) if it is a spacelike principal direction we reach the static Kasner metric (22), and (iii) if it is not a principal direction we obtain the windmill Kasner metric (23); for class I_{12}, under similar conditions, we obtain (i) the Taub metric (28), (ii) the timelike counterpart of the Taub metric (30), and (iii) the windmill-like metric for the Taub solution (31).

It is worth pointing out that the integration procedure is based on intrinsic conditions imposed on algebraic and differential concomitants of the Weyl tensor. On the other hand, these Weyl invariants can be obtained directly from the components of the metric tensor g in arbitrary local coordinates and without solving any equations.[18,19] Consequently, we get an intrinsic and explicit labeling of every solution (similar to that given for the Petrov metric in Ref. 13). Table I summarizes these results and enables us to obtain the directions of the Killing fields having aligned Papapetrou field. In the table we find the Weyl tensor invariants

$$\alpha_i \equiv \alpha_i(g), \quad \theta^\alpha \equiv \theta^\alpha(g), \quad \mathcal{U}_i \equiv \mathcal{U}_i(g), \tag{33}$$

$$\lambda_i \equiv \lambda_i(g) = -\mathcal{U}_i(\nabla \cdot \mathcal{U}_i), \tag{34}$$

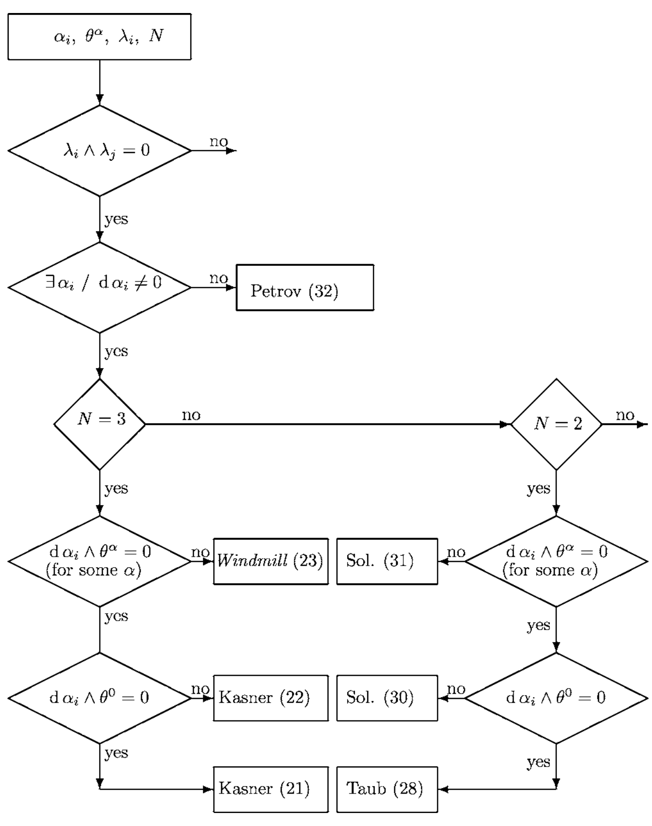
$$N \equiv N(g), \text{ number of integrable directions in the set } \{\mathcal{U}_j(\mathrm{d}\alpha_i)\}. \tag{35}$$

The metric concomitants (33) are, respectively, the Weyl eigenvalues $\alpha_i(g)$, the Weyl principal coframe $\theta^\alpha(g)$ and the unitary Weyl principal bivectors $\mathcal{U}_i(g)$. The explicit expressions of these Weyl invariants in terms of the Weyl tensor can be found elsewhere.[18,19]

TABLE I. Type I vacuum solutions with aligned Papapetrou fields.

SOLUTION	Intrinsic characterization $\lambda_i\wedge\lambda_j=0$, Ric$=0$	Killing vectors with aligned Papapetrou field
Kasner (21)	$d\alpha_1\neq 0$, $N=3$ $d\alpha_1\wedge\theta^0=0$	$\xi_i\propto\mathcal{U}_i(d\alpha_1)$, $i=1,2,3$ $\nabla\xi_i$ aligned with $\mathcal{U}_i$
Kasner (22)	$d\alpha_1\neq 0$, $N=3$ $d\alpha_1\wedge\theta^j=0$ for some j	
Windmill (23)	$d\alpha_1\neq 0$, $N=3$ $d\alpha_1\wedge\theta^\alpha\neq 0$ $\forall\,\alpha$	$\exists!$ α_{i_0} real, $\xi\propto\mathcal{U}_{i_0}(d\alpha_1)$ $\nabla\xi$ aligned with $\mathcal{U}_{i_0}$
Taub (28)	$d\alpha_1\neq 0$, $N=2$ $d\alpha_1\wedge\theta^0=0$	$\exists!$ $i_0/\mathcal{U}_{i_0}(d\alpha_1)$ is not integrable $\xi\propto\mathcal{U}_{i_0}(d\alpha_1)$ $\nabla\xi$ aligned with $\mathcal{U}_{i_0}$
Taub (30)	$d\alpha_1\neq 0$, $N=2$ $d\alpha_1\wedge\theta^j=0$ for some j	
Windmill (31)	$d\alpha_1\neq 0$, $N=2$ $d\alpha_1\wedge\theta^\alpha\neq 0$ $\forall\,\alpha$	
Petrov (32)	$d\alpha_i=0$ $\forall\,i$	$\exists!$ α_3 real, $\xi\propto\Gamma_1^2$ $\nabla\xi$ aligned with $\mathcal{U}_3$

Finally, to underline the intrinsic nature of our results we present a flow diagram that characterizes, among all the type I vacuum solutions, those having an aligned Papapetrou field. This operational algorithm can be useful from a computational point of view and also involves the Weyl invariants (33)–(35).



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