The Newtonian Point Particle*

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Abstract

Local inverse questions for Newtonian gravitation remain unanswered. Restricted to one particle, these questions are: i) What are the *local necessary and sufficient* conditions for an acceleration field to be the force field of a sole point particle?, and ii) What are the *mass* and the *position* of the particle corresponding to such an acceleration field? Their answer is given for both, inertial and accelerated observers. For the last ones, this is made through a *characterization of inertial acceleration fields*. The results are *covariant*, *intrinsic* and *constructive*, i.e., they are coordinate-free, expressed in terms of the sole acceleration field, and may be checked by direct substitution of the field and its derivatives.

1 Introduction

The questions considered here concern also the domains of electromagnetism and general relativity. The extreme dificulties to analyse them for Einstein's and for Maxwell's equations has led us to first consider these questions in Newtonian gravity. For simplicity, we shall here limit their setting to Newton's gravity.

Our starting point is the following *standard* situation: an arbitrary and unknown distribution of masses creates a gravitational field, which is measured

*Proc. Spanish Relativistic Meeting, ERE 1997, Sept. 1997, Mallorca, Spain

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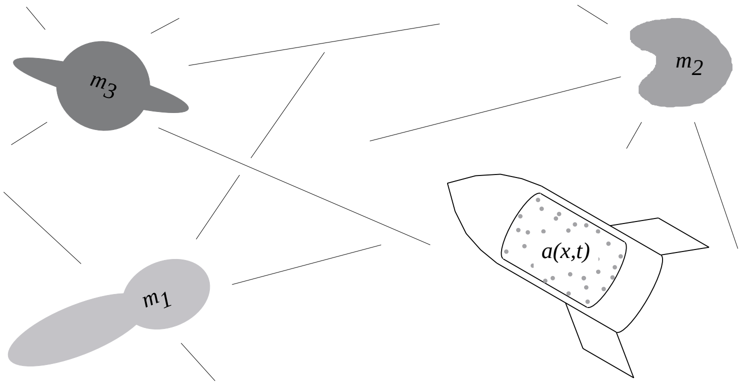

Figure 1: An unknown distribution of masses creates a gravitational field, which is measured *only* in a *local* domain (the interior of the rocket). What information about the masses may be *constructively* extracted from this local knowledge?

only in a *local* domain or laboratory (the interior of the rocket in Figure 1). We are here concerned with the informations that can be extracted from this local knowledge.

All what Newton's theory says about the gravitational fields measured in this local domain, is that they verify the gravitational equations:

$$\Delta V = 0 \Leftrightarrow \begin{cases} da = 0 \\ \delta a = 0 \end{cases} , \tag{1}$$

a being the *acceleration field*, $a = dV$, and V the gravitational potential.

But these field equations are *incomplete*: although they are supposed to contain *all* physical fields [1], it is well known that, conversely, *almost all* their solutions are unphysical fields [2]. We are thus led to ask our first question:

Question 1 *Is it possible to find a* complete *set of local field equations for gravity, that is to say, a set of local field equations such that their solutions be only the physical ones?*

In general, it would be interesting to know how to find, for *any* local solution of equations (1), the position and "charges" of its singularities, but we shall here restrinct our question to the sole physical solutions. The second question is thus:

Question 2 *Does there exist a method allowing to find the* masses *and* positions *corresponding to a local physical gravitational field?*

In theoretical physics, questions may be answered in many different, non equivalent forms. The above two questions have been analysed in Newtonian theory by simplicity, but also because we hope that the Newtonian answers may help to understand some aspects of the corresponding relativistic questions. In part for this aim, we are interested in finding covariant (i.e. coordinate-free) and intrinsic (i.e. involving only the data, namely the local acceleration field) expressions. Also, for practical reasons, we would need these expressions to be *constructive*, that is to say, such that they may be *explicitly* verified by direct substitution of the data and of differential concomitants of them. Our third question is thus:

Question 3 *Do there exist* covariant, intrinsic *and* constructive *answers to the above two questions?*

Even for Newtonian theory, the *general* answers to these questions seem, at least for the moment, out of reach. Here we shall humbly present the results corresponding to the case of a sole Newtonian point particle. Their proofs and a deeper analysis of them will be given elsewhere [3].

The results for Galilean observers are given in Section 2, theorems 1 and 2. To treat accelerated observers, Section 3 presents the covariant, intrinsic and constructive characterization of inertial fields in theorem 3; its constructive character being completed by propositions 1 and 2. The results for accelerated observers are presented in Section 4, theorem 4 and its corollary. Finally, Section 5 is devoted to some comments.

2 Gravitational Field of a Point Particle for Galilean Observers

Suppose that a local Galilean observer measures, in the domain of its laboratory, an acceleration field $\gamma(x,t)$. The restriction of question 1 to *the gravitational fields created by a sole particle* may be equivalently asked in the following terms: what are the necessary and sufficient conditions that an acceleration field $\gamma(x,t)$ must verify in order that it correspond to the field associated to a massive point particle? Or, in other words, how to characterize exclusively all the situations correspondig to the scheme of Figure 2? The answer is given by the following theorem:

Theorem 1 *A (local) acceleration vectorfield* $\gamma(x,t)$ *is the* gravitational field *of a point particle if, and only if, it verifies the equation*

$$\nabla\gamma = f\left(g - 3\,\frac{\gamma\otimes\gamma}{|\gamma|^2}\right) , \tag{2}$$

where f *is a negative function,* $f<0$ *, being* g *and* ∇ *respectively the 3-dimensional space metric and the associated covariant derivative.*

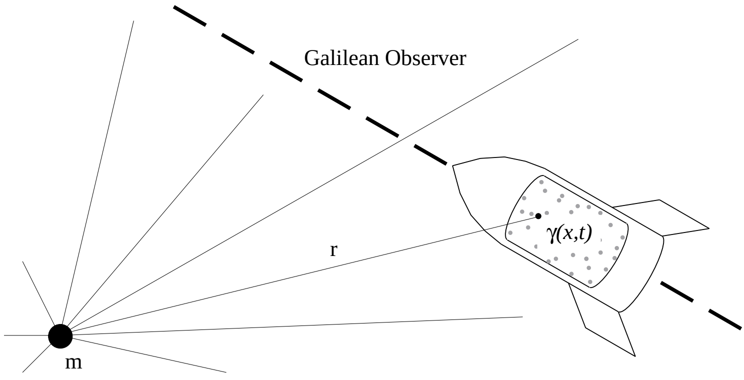

Figure 2: A Galilean observer measures a local gravitational field $\gamma(x,t)$. How to deduce that it is created by a spherically symmetric mass, and how to know and locate this mass?

Expresion (2) is a well known *consequence* of a inverse-square field; it is its *sufficient* character which is a less trivial result and, we believe, a new one.

Once known that the acceleration field $\gamma(x,t)$ is due to a sole massive particle, question 2 naturaly arises: where it is located and what is its mass? The answer is as follows:

Theorem 2 *Let γ be the (local) acceleration field of a gravitational point particle. Then, its* mass m *and its* position r *are given by*

$$m = \frac{4|\,\gamma\,|^5}{(\mathcal{L}(\gamma)\,|\,\gamma\,|)^2} \qquad , \qquad r = -\frac{2|\,\gamma\,|}{\mathcal{L}(\gamma)\,|\,\gamma\,|}\;\gamma\;, \tag{3}$$

where $\mathcal{L}(\gamma)$ stands for the Lie derivative along γ.

Of course, the integrability conditions of equation (2) implies that the mass given by equation (3) is constant. Its possitive character is insured by the condition $f < 0$ of theorem 1.

3 Inertial Acceleration Field for Accelerated Observers

Suppose now that a local accelerated observer measures, in the domain of its laboratory, an acceleration field $\alpha(x,t)$. This acceleration field is, in general, the superposition of its proper accelerated motion, and of the exterior gravitational fields. Can he know when exterior gravitational fields are absent, as Figure 3 shows? or, in other words, can he be sure that he is not submitted but to *inertial* forces?

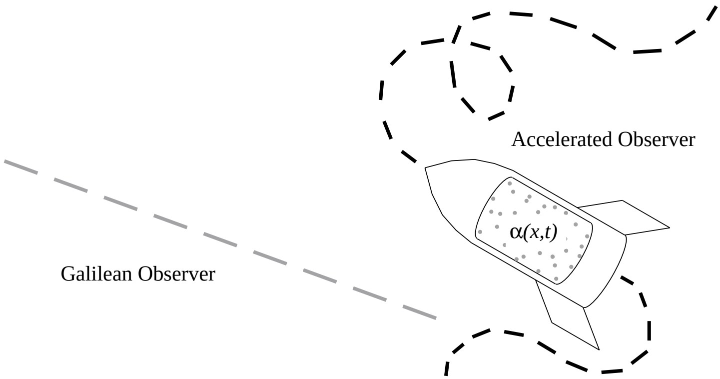

Figure 3: An accelerated observer measures locally an acceleration field $\alpha(x,t)$. How to know that he is not submitted but to *inertial* forces?

According to question 3, we are looking for constructive answers. In order to obtain them, we need two results concerning square roots of tensors. More precisely, we need to know *when* a symmetric tensor L admits an antisymmetric square root A, $A^2 = L$, and, in that case, *what* is its expression in terms of L. The corresponding results are given by propositions 1 and 2:

Proposition 1 *A second order symmetric tensor L admits an antisymmetric square root if, and only if, it verifies*

$$L^2 - \frac{1}{2}\,(\mathrm{tr}\, L)\, L = 0 \ . \tag{4}$$

Proposition 2 *For such a second order symmetric tensor L, its antisymmetric square root $\sqrt[\wedge]{L}$ is given by*

$$\sqrt[\wedge]{L} = \frac{1}{\sqrt{i^2(x)\overline{L}}}\ * i(x)\overline{L}\ , \tag{5}$$

where

$$\overline{L} \equiv L - \frac{1}{2}(\mathrm{tr}\, L)\, g\ , \tag{6}$$

x is an arbitrary regular *vectorfield for L, i.e. such that $i(x)\,\overline{L} \neq 0$, $i(.)$ denotes the interior product and $*$ the Hodge dual operator.*

Note that, in spite of the arbitrary character of the vector x appearing in equation (5), $\sqrt[\wedge]{L}$ is *unique* and *independent* of the chosen regular x. We shall not consider here the operator $\sqrt[\wedge]{\cdot}$ on symmetric tensors that would be undefined on tensors not verifying equation (4), but we shall rather consider $\hat{\sqrt{L}}$ as the notation of the existing antisymmetric tensor obtained from a tensor L that verifies this equation.

We are now in position to give the *covariant*, *intrinsic* and *constructive* characterization of inertial fields:

Theorem 3 *An acceleration field $\alpha(x,t)$ is an* inertial acceleration field *corresponding to an accelerated observer if, and only if, it verifies:*

$$\left.\begin{aligned} \nabla\mathcal{L}(\alpha)\,g = 0 \\ d\alpha = \left(\hat{\sqrt{2\,\mathcal{L}(\alpha)\,g}}\right)^{\bullet} \end{aligned}\right\} \quad , \tag{7}$$

where $(\ )^{\bullet}$ *stands for the time derivative.*

4 Gravitational Field of a Point Particle for Accelerated Observers

We now consider the general situation in which a local accelerated observer is under the influence of both, the gravitational field $\gamma(x,t)$ of a point particle and its proper inertial field $\alpha(x,t)$, as shown in Figure 4, where $a(x,t)$ denotes the *total* acceleration field measured in the local laboratory.

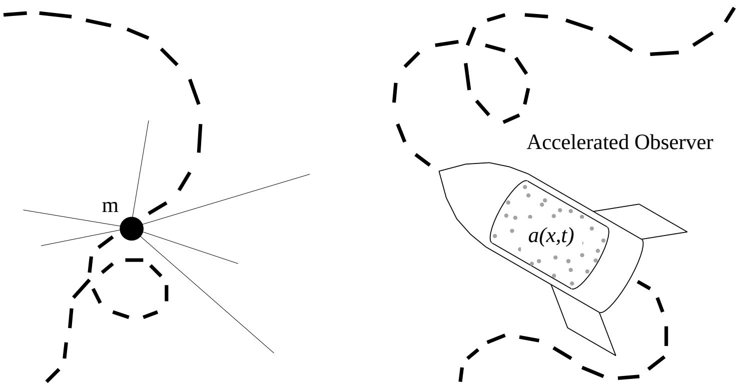

Figure 4: An accelerated observer, submitted to the gravitational field of a spherically symmetric mass, measures locally an acceleration field $a(x,t)$. How to extract from it $\gamma(x,t)$ and $\alpha(x,t)$, i.e. its gravitational and inertial components?

In order to present the answers to our above three basic questions, it is convenient to previously introduce some differential concomitants of an arbitrary vectorfield v.

An adequate measure of the modulus of the hessian of the vectorfield v, is the *scalar* $\Phi(v)$ given by

$$\Phi(v) \equiv \frac{1}{\sqrt{\mid \nabla\nabla v \mid}} \, , \tag{8}$$

whose gradient allows to define the *1-form* $\Gamma(v)$ by

$$\Gamma(v) \equiv \sqrt{\frac{8}{45}} \, \frac{1}{\mid d\Phi(v) \mid^3} \, d\Phi(v) \; . \tag{9}$$

On the other hand, theorem 1 suggest us to introduce for any v the *gravitational* differential concomitant $\mathcal{G}(v)$ associated to equation (2):

$$\mathcal{G}(v) \equiv \nabla v + \lambda^2 \left(g - 3\frac{v \otimes v}{\mid v \mid^2} \right) \; . \tag{10}$$

Similarly, theorem 3 suggests to introduce the doble *inertial* differential concomitant $\mathcal{I}(v)$ associated to equations (7):

$$\mathcal{I}(v) \equiv \left\{ \begin{array}{c} \nabla \mathcal{L}(v) g \\ \\ dv - \left(\hat{\sqrt{2\,\mathcal{L}(v)\, g}} \right)^{\bullet} \end{array} \right\} \; . \tag{11}$$

With the aid of these concomitants, the *complete* characterization of the acceleration fields in question is the following one:

Theorem 4 *A local acceleration field $a(x,t)$ is the* total acceleration field *of an accelerated observer inmersed in the gravitational field of one point particle if, and only if, it verifies the equations*

$$\mathcal{G}\left(\Gamma(a)\right) = 0 \qquad , \qquad \mathcal{I}\left(a - \Gamma(a)\right) = 0 \; . \tag{12}$$

Taking into account the meaning of equations (2) and (7), explained respectively by theorems 1 and 3, from equations (12) and theorem 2 one easily obtains:

Corollary 1 *For a* total acceleration field $a(x,t)$, *the* gravitational acceleration γ *of the particle and the* inertial acceleration α *of the observer are given by*

$$\gamma = \Gamma(a) \qquad , \qquad \alpha = a - \Gamma(a) \; , \tag{13}$$

and the mass m *and* position r *of the particle by*

$$m = \frac{4 \mid \Gamma \mid^5}{\left(\mathcal{L}(\Gamma) \mid \Gamma \mid \right)^2} \qquad , \qquad r = -\frac{2 \mid \Gamma \mid}{\mathcal{L}(\Gamma) \mid \Gamma \mid} \; \Gamma \; . \tag{14}$$

5 Conclusions

We would like to conclude by noting briefly a few points (a detailed analysis will be given elsewhere [3]):

i) The above results show that, at least for *one particle*, a *complete gravitational theory* including *location* of masses is *possible*, and has been constructed.

ii) Even for *one particle*, this theory is far from being trivial.

iii) The method may, in principle, be extended to a *finite* number of particles (and perhaps, with a limiting process, to an infinite number).

iv) It seems to show that a *complete* theory of fields is necessarily related to a *hierarchy* of equations, and not to a unic, universal set of equations, as we used to think up to now.

v) The present results are first elements of a non usual way of thinking field theory. For some applications they may be too hard to use, but for some others they (and their plausible extensions) constitute the shortest and clear answers.

vi) Although Newtonian, these results are also heuristically interesting for some relativistic problems, for example the physical interpretation of static space-times. We shall consider these problems elsewhere.

References

[1] By *physical* gravitational fields we understand here fields that may be created by means of point particles of *positive* mass.

[2] It is a direct consequence of the linear structure of the space of solutions and the positivity of the mass for physical gravitational fields.

[3] B. Coll and J.J. Ferrando, Newtonian Gravitation of One Point Particle, to be submitted to the *Journal of Mathematical Physics*.