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An intrinsic characterization of the Schwarzschild metric

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Abstract. An intrinsic algorithm that exclusively involves conditions on the metric tensor and its differential concomitants is presented to identify every type-D static vacuum solution. In particular, the necessary and sufficient explicit and intrinsic conditions are given for a Lorentzian metric to be the Schwarzschild solution.

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1. Introduction

'The unique spherically symmetric vacuum solution of Einstein equations is the Schwarzschild metric'. This popular result, known as the Birkhoff theorem [1], evidently gives a complete characterization of the Schwarzschild spacetime by requiring two conditions on the metric tensor g. The first one (C1) implies that its Ricci tensor is zero, $\mathrm{Ric}(g) = 0$, and the second one (C2) states that g admits an isometry group G_3 acting on spacelike two-dimensional orbits with positive curvature. We are interested in underlining the different nature of each condition: while C1 is *explicit* in the metric tensor (in one of its differential concomitants, the Ricci tensor), condition C2 is *implicit* since the equations that yield it mix up, in principle, other elements than the metric tensor (isometry maps or Killing vectors of the spherical symmetry).

Since the *intrinsic* (depending solely on the metric tensor) and *explicit* conditions may be verified by direct substitution of the metric tensor, it is evident that getting an *intrinsic* and fully *explicit* characterization of a spacetime (a family of spacetimes) is of interest. So, after the above comments on the Birkhoff theorem, a question naturally arises: is it possible to express, solely in terms of the metric tensor and its differential concomitants, the necessary and sufficient conditions for a spacetime to be the Schwarzschild solution? We show here that the answer is affirmative, by obtaining the equations defining the Schwarzschild metric explicitly.

In order to obtain this kind of description it is certainly useful to get a complete algebraic study of the curvature tensor (the Ricci and Weyl tensors) and, in particular, to acquire the covariant determination of its characteristic directions and 2-planes [2, 3]. These results have shown their usefulness, for instance, in building the Rainich theory for the thermodynamic perfect fluid [4] and in giving the intrinsic characterization of the type-I spacetimes admitting isotropic radiation [5].

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For the purpose of determining the type-D static vacuum solutions, Ehlers and Kundt [6] applied the $1+3$ formalism adapted to the hypersurface-orthogonal timelike Killing vector, and tried an intrinsic description. Although they got quite far, they were two steps short of arriving: the static condition was not written intrinsically, and some invariant scalars they used to discriminate different solutions depended on the Weyl principal 2-form, which was not explicitly expressed in terms of the Weyl tensor. Thus, they obtained a complete *invariant* (but non-explicit) way to distinguish (among the static metrics) between two different type-D vacuum solutions. We have overcome these shortcomings by obtaining the metric concomitants and the intrinsic conditions allowing us to identify (among all metrics) every type-D static vacuum solution.

These solutions were later recovered by Kinnersley [7], who acquired all type-D vacuum spacetimes using the Newman–Penrose formalism adapted to the double Debever directions. Here we present an alternative method based on the $2+2$ Weyl principal almost-product structure, which is more suitable for accomplishing the goal of this article: the intrinsic and explicit characterization of the solutions.

This paper is organized as follows. In section 2 we introduce the formalism used and recapitulate some basic results about $2+2$ almost-product structures. In section 3 we give a canonical expression for all type-D metrics whose Weyl tensor is divergence-free and has real eigenvalues (proposition 1). Vacuum equations for these metrics (proposition 2) are considered in section 4, and the static type-D metrics by Ehlers and Kundt arise in a natural way and its intrinsic characterization follows (proposition 3). In section 5 we study the explicit conditions that label the different types of these metrics (proposition 4). We insist on, and emphasize, the equations that define the Schwarzschild solution in section 6 and give an algorithm to determine the timelike Killing vector and the Schwarzschild mass (theorem 1). Finally, in section 7 we present a flow diagram that summarizes the main results.

2. $2+2$ almost-product structures

On the spacetime (V_4, g) of signature $(-+++)$, an almost-product structure is defined by a plane field V and its orthogonal complement H. Taking into account the integrable (hypersurface orthogonal), minimal, umbilical or geodesic character of each plane, an invariant classification of the almost-product structures follows [8]. In the $2+2$ case it may be useful to work with the *canonical* unitary 2-form U, the volume element of the timelike plane V. Then $v = U^2$ and $h = -(*U)^2$ are the respective projectors and $v+h = g$. Moreover, U may be written as $U = l_- \wedge l_+$, with $l_\pm$ as its principal directions. With this notation, and writing $\delta = -\mathrm{tr}\nabla$, it is easy to prove

Lemma 1. Let (V, H) be a $2+2$ almost-product structure and let U be its canonical 2-form. Then the following conditions hold:

(i) V (respectively, H) integrable $\Longleftrightarrow i(\delta * U)U = 0$ (respectively, $i(\delta U) * U = 0$).
(ii) V (respectively, H) minimal $\Longleftrightarrow i(\delta * U) * U = 0$ (respectively, $i(\delta U)U = 0$).
(iii) V, H both umbilical $\Longleftrightarrow$ principal directions of U are shear-free null geodesics.

When both planes are integrable, minimal, umbilical or geodesic, we call the almost-product structure accordingly. A product metric is a metric $\tilde{g}$ admitting a product structure (V, H), i.e. an integrable and geodesic (minimal and umbilical) structure. Then, and only then, local coordinates (x^A, x^i), $A = 0, 1$, $i = 2, 3$, exist such that $\tilde{g} = \sigma^- + \sigma^+$, with $\sigma^- = \sigma^-_{AB}(x^C)\,\mathrm{d}x^A\,\mathrm{d}x^B$ and $\sigma^+ = \sigma^+_{ij}(x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j$.

Integrable and umbilical properties are conformal invariants. However, this does not take place for the minimal character of a structure. In fact, if $\tilde{g}$ is a product metric and $g = e^{2\lambda}\tilde{g}$, we have

$$i(\delta U)U - i(\delta * U) * U = -2d\lambda. \tag{1}$$

3. Divergence-free type-D metrics with real eigenvalues

Let us consider the Petrov type-D spacetimes with Weyl real eigenvalues. In this case, the self-dual Weyl tensor is

$$\mathcal{W} \equiv \tfrac{1}{2}(W - \imath * W) = 3\rho\,\mathcal{U} \otimes \mathcal{U} + \rho\mathcal{G}, \tag{2}$$

with $\mathcal{G} = \frac{1}{2}(G - \imath\eta)$ and $\mathcal{U} = \frac{1}{\sqrt{2}}(U - \imath * U)$, where $G = \frac{1}{2}g \wedge g$, η and U are, respectively, the canonical metric on the space of the 2-forms, the metric volume element and the canonical 2-form associated to the Weyl principal structure, the double eigenvalue ρ being the real function $\rho = -\left(\frac{1}{12}\,\mathrm{tr}\,W^3\right)^{1/3} \neq 0$.

Moreover, if $S \equiv \frac{1}{3\rho}(W - \rho G)$, from (2) we obtain $S = U \otimes U - \overset{*}{U} \otimes \overset{*}{U}$ and then $S^2 + S = 0$. Conversely, if $S^2 + S = 0$, the self-dual tensor $\mathcal{S} = \frac{1}{2}(S - \imath * S)$ satisfies $\mathcal{S}^2 + \mathcal{S} = 0$ and the Weyl tensor has therefore a degree-2 minimal polynomial and, since $\rho \neq 0$, the spacetime is of type D, with ρ being the double eigenvalue. So we can state

Lemma 2. A Weyl tensor is of Petrov type-D with real eigenvalues if, and only if,

$$\rho \equiv -\left(\tfrac{1}{12}\,\mathrm{tr}\,W^3\right)^{(1/3)} \neq 0, \qquad S^2 + S = 0$$

with $S \equiv \frac{1}{3\rho}(W - \rho G)$.

In order to impose vacuum equations in the next section, let us now consider the Bianchi identities for the vacuum case: $\delta\mathcal{W} = 0$. From the Weyl canonical expression (2), this divergence-free condition for the Weyl tensor becomes

$$\mathrm{d}\rho - 3\rho\; i(\delta\mathcal{U})\,\mathcal{U} = 0 \tag{3}$$

$$i(\delta\mathcal{H}_+)\,\mathcal{H}_+ = i(\delta\mathcal{H}_-)\,\mathcal{H}_- = 0 \tag{4}$$

where $\mathcal{H}_\pm$ are the self-dual null 2-forms whose fundamental directions $l_\pm$ are the U principal directions. Condition (4) states exactly the geodesic and shear-free character of $l_\pm$, in agreement with the Golberg and Sachs result [9] and, after lemma 1, the umbilical nature of the Weyl principal structure. Taking lemma 1 into account, the imaginary part of (3) gives its integrability, and from (1) and the real part of (3) it follows that g is conformal to a product metric, with conformal factor $e^{2\lambda} = \kappa^2\rho^{-2/3}$, κ being an arbitrary constant.

Furthermore, from Gauss–Codazzi equations it follows that a $2+2$ product metric $\tilde{g} = \sigma^- + \sigma^+$ is of Petrov type-D with a double Weyl eigenvalue $\tilde{\rho} = -\frac{1}{6}(X_- + X_+)$, where X_ϵ is the Gaussian curvature of σ^ϵ ($\epsilon = \pm$), and that it degenerates to type 0 when $X_- = -X_+ =$ constant. Moreover, the Weyl eigenvalues change by a conformal transformation, $g = e^{2\lambda}\tilde{g}$, as $\rho = e^{-2\lambda}\tilde{\rho}$. Then, combining all these results, we can state

Proposition 1. The metric tensor of a Petrov type-D spacetime with real Weyl eigenvalues and divergence-free Weyl tensor may be written as

$$g = \frac{1}{(X_- + X_+)^2}\left(\sigma^- + \sigma^+\right) \tag{5}$$

where $\sigma^- = \sigma^-_{AB}(x^C)\,\mathrm{d}x^A\,\mathrm{d}x^B$ and $\sigma^+ = \sigma^+_{ij}(x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j$ are arbitrary two-dimensional metrics, σ^- hyperbolic and σ^+ elliptic, with Gaussian curvatures X_- and X_+, respectively.

4. Vacuum solutions

Until now, we have partially imposed the vacuum condition through Bianchi identities, $\delta W = 0$, and we have arrived at the canonical form (5). Now let us impose the complete vacuum equations $\mathrm{Ric}(g) = 0$ on these metrics. Taking into account the relationship between the Ricci tensors of two conformal metrics and recalling that for a product metric one has $\mathrm{Ric}(\sigma^- + \sigma^+) = X_-\sigma^- + X_+\sigma^+$, we prove

Proposition 2. All the type-D vacuum solutions with real eigenvalues are given by (5), where σ^- and σ^+ are bidimensional metrics, hyperbolic and elliptic respectively, which satisfy

$$\mathrm{D}^\epsilon\, \mathrm{d}X_\epsilon = -\frac{1}{2}(X_\epsilon^2 + a)\sigma^\epsilon, \qquad \sigma^\epsilon(\mathrm{d}X_\epsilon, \mathrm{d}X_\epsilon) = \epsilon f(\epsilon X_\epsilon) \tag{6}$$

with $f(X) = -\frac{1}{3}X^3 - aX + b$, and where X_ϵ is the Gaussian curvature of σ^ϵ, a and b are arbitrary constants and D^ϵ is the covariant derivative associated to each metric.

Thus, vacuum equations for the family of metrics given by (5) turn out to be the (only coupled by the common constants a and b) equations (6) for two bidimensional metrics σ^ϵ. If a solution σ^- and σ^+ is known, a vacuum solution g is given by (5). The above equation (6) admits the solution of constant curvature, $\mathrm{d}X_\epsilon = 0$; then $a = -X_\epsilon^2$ and $b = -\frac{2}{3}\epsilon X_\epsilon^3$. Otherwise, if $\mathrm{d}X_\epsilon \neq 0$, X_ϵ can be taken as a coordinate and a straightforward calculation shows that the solution of (6) in this case is

$$\sigma^\epsilon = \frac{1}{\epsilon f(\epsilon X_\epsilon)}\, \mathrm{d}X_\epsilon^2 + f(\epsilon X_\epsilon)\, \mathrm{d}Z^2. \tag{7}$$

Thus the analysis of vacuum equations leads us to distinguishing between the following cases:

(i) $\mathrm{d}X_+ = \mathrm{d}X_- = 0$. Then $X_+ + X_- = 0$ and the spacetime is flat.
(ii) $\mathrm{d}X_+ = 0$, $\mathrm{d}X_- \neq 0$. Then the elliptic metric σ^+ has constant curvature X_+, and σ^- is given by (7) with $a = -{X_+}^2$, $b = -\frac{2}{3}X_+^3$. Depending on X_+ being positive, negative or null, we have, respectively, the A_1, A_2 or A_3 metrics by Ehlers and Kundt [6].
(iii) $\mathrm{d}X_+ \neq 0$, $\mathrm{d}X_- = 0$. Then the hyperbolic metric σ^- has constant curvature, and σ^+ is given by (7) with $a = -{X_-}^2$, $b = \frac{2}{3}X_-^3$. The B_1, B_2 and B_3 spacetimes [6] appear, respectively, by choosing X_- as positive, negative or null.
(iv) $\mathrm{d}X_+ \neq 0$, $\mathrm{d}X_- \neq 0$. Then σ^+ and σ^- are given by (7), and this case allows us to obtain the C-metrics [6].

From these considerations it follows that the vacuum solutions of Petrov type-D with real Weyl eigenvalues are the type-D static vacuum solutions studied by Ehlers and Kundt [6]. Thus, we have an intrinsic characterization of these spacetimes that, after lemma 2, we can explicitly give in terms of the metric tensor:

Proposition 3. Type-D static vacuum solutions are characterized by conditions

$$\rho(g) \neq 0, \qquad S^2(g) + S(g) = 0, \qquad \mathrm{Ric}(g) = 0 \tag{8}$$

with $\rho(g) \equiv -\left(\frac{1}{12}\operatorname{tr} W^3(g)\right)^{1/3}$ and $S(g) \equiv \frac{1}{3\rho}(W(g) - \frac{1}{2}\rho(g)\ g \wedge g)$, $W(g)$ and $\mathrm{Ric}(g)$ being the Weyl and the Ricci tensors associated to the metric tensor g.

5. The intrinsic label

Proposition 3 gives us an intrinsic and explicit description of the family of the type-D static vacuum metrics. Now let us study an intrinsic and explicit way to identify every metric of this family (classes A_i, B_i and C by Ehlers and Kundt [6]). In terms of the Gaussian curvatures of the two-dimensional metrics σ^ϵ, the double Weyl eigenvalue of the metric (5) becomes $\rho = -\frac{1}{6}(X_- + X_+)^3$ (see section 3). Therefore, the metric σ^- (respectively, σ^+) is of constant curvature, $\mathrm{d}X_- = 0$ (respectively, $\mathrm{d}X_+ = 0$), if and only if the gradient of ρ is on the spacelike (timelike, respectively) principal 2-plane or, equivalently, $i(\mathrm{d}\rho)U = 0$ ($i(\mathrm{d}\rho) * U = 0$, respectively), U being the principal 2-form of the Weyl tensor.

On the other hand, from the expressions of g and ρ, a straightforward calculation shows that if $\mathrm{d}X_\epsilon = 0$, then $X_\epsilon = (6\rho)^{-2/3}(\frac{1}{9}g(\mathrm{d}\ln\rho, \mathrm{d}\ln\rho) - 2\rho)$. Thus we can state the following

Lemma 3. Let U and ρ be the (real) principal 2-form and the double Weyl eigenvalue of a type-D static vacuum metric, and X_ϵ the two-dimensional Gaussian curvatures of the canonical form (5). Then

(i) $\mathrm{d}X_- = 0$ (respectively, $\mathrm{d}X_+ = 0$) if, and only if, $i(\mathrm{d}\rho)U = 0$ (respectively, $i(\mathrm{d}\rho)\overset{*}{U} = 0$).
(ii) If $\mathrm{d}X_\epsilon = 0$ then $X_\epsilon = (6\rho)^{-2/3}(\frac{1}{9}g(\mathrm{d}\ln\rho, \mathrm{d}\ln\rho) - 2\rho)$.

The first statement of this lemma gives us a way of identifying the precedent cases (A-, B- or C-metrics), and the second one provides the invariant scalar that allows us to distinguish the three types of A-metrics or B-metrics. Consequently, we can identify different kinds of type-D static vacuum metrics by applying invariant conditions. Now we must write these conditions as functions of Weyl concomitants in order to obtain an *intrinsic* and *explicit* description.

Let us consider the canonical expression of a type-D Weyl tensor with real eigenvalues in terms of the (real) principal 2-form U, $W = 3\rho(U \otimes U - *U \otimes *U) + \rho G$. From here we get $*W = 3\rho\, U \,\tilde{\otimes}\, *U + \rho\eta$ and $S \equiv \frac{1}{3\rho}(W - \rho G) = U \otimes U - *U \otimes *U$. Thus, $*W(\mathrm{d}\rho, \cdot, \mathrm{d}\rho, \cdot) = 3\rho\; i(\mathrm{d}\rho)U \,\tilde{\otimes}\, i(\mathrm{d}\rho) * U$ and, after lemma 3, we have that condition $*W(\mathrm{d}\rho, \cdot, \mathrm{d}\rho, \cdot) \neq 0$ gives an explicit characterization of the C-metrics.

Otherwise, if $*W(\mathrm{d}\rho, \cdot, \mathrm{d}\rho, \cdot) = 0$, we have that either $i(\mathrm{d}\rho)U$ or $i(\mathrm{d}\rho) * U$ must vanish. When this happens, the quadratic form

$$Q \equiv S(\mathrm{d}\rho, \cdot, \mathrm{d}\rho, \cdot) = i(\mathrm{d}\rho)U \otimes i(\mathrm{d}\rho)U - i(\mathrm{d}\rho) * U \otimes i(\mathrm{d}\rho) * U \qquad (9)$$

is either non-negative or non-positive; this corresponds, after lemma 3, to the A-metrics and B-metrics, respectively. Moreover, Q is non-negative (non-positive, respectively) in accordance with the sign of its trace with an elliptical metric associated to g.

Taking into account all the above considerations, we can state the following

Proposition 4. Let g be a type-D static vacuum solution (characterized in proposition 3). Let us take the metric concomitants

$$\alpha \equiv \frac{1}{9}g(\mathrm{d}\ln\rho, \mathrm{d}\ln\rho) - 2\rho \qquad P \equiv *W(\mathrm{d}\rho, \cdot, \mathrm{d}\rho, \cdot) \qquad Q \equiv S(\mathrm{d}\rho, \cdot, \mathrm{d}\rho, \cdot)$$

and let x be an arbitrary unitary timelike vector. Then

(i) g is a C-metric if, and only if, $P \neq 0$.
(ii) g is an A-metric if, and only if, $P = 0$ and $2Q(x, x) + \mathrm{tr}\, Q > 0$.
Furthermore, it is of type A_1, A_2 or A_3 if $\alpha > 0$, $\alpha < 0$ or $\alpha = 0$, respectively.

(iii) g is a B-metric if, and only if, $P = 0$ and $2Q(x, x) + \operatorname{tr} Q < 0$.

Furthermore, it is of type B_1, B_2 or B_3 if $\alpha > 0$, $\alpha < 0$ or $\alpha = 0$, respectively.

6. The Schwarzschild characterization

Let us consider a more detailed analysis of the A_1-metrics. In the local coordinates that we have used to integrate vacuum equations, the metric tensor is

$$g = -\frac{1}{3}(2X_+ - X_-)\,\mathrm{d}Z_-{}^2 + \frac{3}{(X_- + X_+)^4(2X_+ - X_-)}\,\mathrm{d}X_-{}^2 + \frac{1}{X_+(X_- + X_+)^2}\,\mathrm{d}\Omega^2 \quad (10)$$

where $\mathrm{d}\Omega^2$ is the metric on the unitary sphere and X_+ is a positive constant. In fact, with the transformation $t = X_+Z_-$, $r = 1/(X_- + X_+)\sqrt{X_+}$, the metric (10) becomes the Schwarzschild metric in Schwarzschild coordinates. Now, it is easy to see that the Schwarzschild mass is given by $m = 6X_+{}^{-3/2}$, and the modulus of the timelike Killing vector by $\|\partial_t\| = \sqrt{\frac{1}{3}(2X_+ - X_-)}$, and so we can obtain them as functions of scalar invariants. These considerations and propositions 3 and 4 lead us to the following

Theorem 1. Let $\operatorname{Ric}(g)$ and $W \equiv W(g)$ be the Ricci and the Weyl tensors of a spacetime metric g, and let us take the metric concomitants

$$\rho \equiv -\left(\tfrac{1}{12}\operatorname{tr} W^3\right)^{1/3}, \qquad \alpha \equiv \tfrac{1}{9}g(\mathrm{d}\ln\rho,\, \mathrm{d}\ln\rho) - 2\rho, \qquad S \equiv \frac{1}{3\rho}(W - \tfrac{1}{2}\rho\, g \wedge g), \quad (11)$$

$$P \equiv *W(\mathrm{d}\rho, \cdot, \mathrm{d}\rho, \cdot), \qquad Q \equiv S(\mathrm{d}\rho, \cdot, \mathrm{d}\rho, \cdot) \quad (12)$$

The necessary and sufficient conditions for g to be the Schwarzschild metric are

$$\operatorname{Ric}(g) = 0, \qquad \rho \neq 0, \qquad S^2 + S = 0, \quad (13)$$

$$P = 0, \qquad 2Q(x, x) + \operatorname{tr} Q > 0, \qquad \alpha > 0 \quad (14)$$

where x is an arbitrary unitary timelike vector. Moreover, the Schwarzschild mass is given by $m = \rho/(\alpha^{3/2})$ and the timelike Killing vector by $\xi = \rho^{-4/3}(Q(x)/\sqrt{Q(x, x)})$.

7. A summary in algorithmic form

Finally, in order to emphasize the algorithmic nature of our results, we present them as a flow diagram (opposite) that identifies, among all metrics, every type-D static vacuum solution. This operational algorithm involves an arbitrary unitary timelike vector x and some metric concomitants that may be obtained from the components of the metric tensor g in arbitrary local coordinates: the Ricci tensor $\operatorname{Ric}(g)$ and the Weyl tensor invariants ρ, α, S, P and Q given in (11) and (12).

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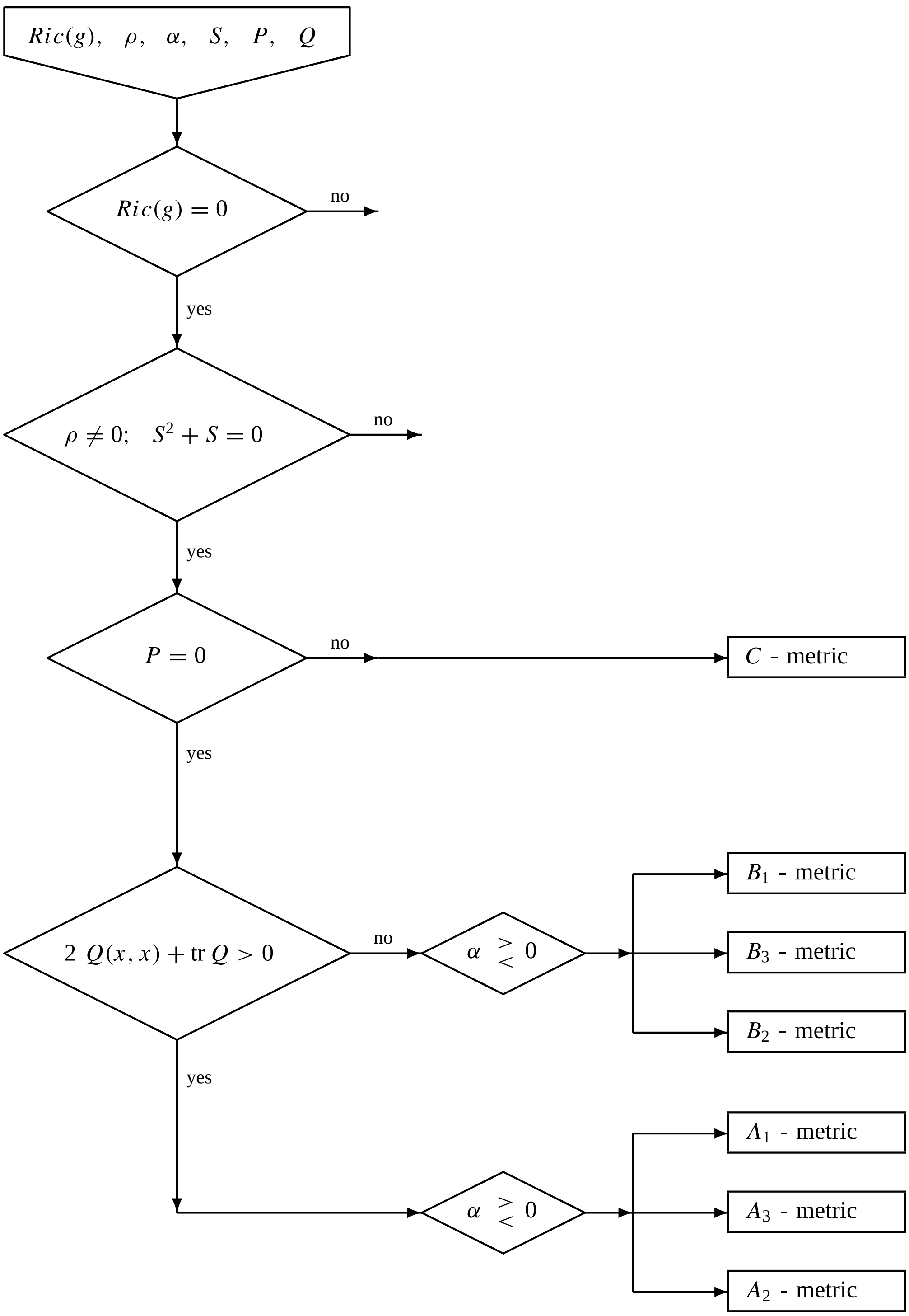
$Ric(g), \quad \rho, \quad \alpha, \quad S, \quad P, \quad Q$
$Ric(g) = 0$
no
yes
$\rho \neq 0; \quad S^2 + S = 0$
no
yes
$P = 0$
no
C - metric
yes
$2\ Q(x,x) + \mathrm{tr}\, Q > 0$
no
$\alpha \gtrless 0$
B_1 - metric
B_3 - metric
B_2 - metric
yes
$\alpha \gtrless 0$
A_1 - metric
A_3 - metric
A_2 - metric

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