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RESEARCH ARTICLE

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On the Weyl transverse frames in type I spacetimes

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Abstract We apply a covariant and generic procedure to obtain explicit expressions of the transverse frames that a type I spacetime admits in terms of an arbitrary initial frame. We also present a simple and general algorithm to obtain the Weyl scalars Ψ_2^T, Ψ_0^T and Ψ_4^T associated with these transverse frames. In both cases it is only necessary to choose a particular root of a cubic expression.

Keywords Weyl frames · Radiation scalars

1 Introduction

The components Ψ_a of the Weyl tensor in a complex null tetrad $\{l, k, m, \bar{m}\}$ have a specific physical meaning [1]. If an observer lying on the time-like plane $\{l, k\}$ analyzes the deviation of test free particles he can conclude that the components Ψ_0 and Ψ_4 describe, respectively, incoming and outgoing transverse waves, whereas Ψ_1 and Ψ_3 are incoming and outgoing longitudinal wave components. On the other hand, Ψ_2 is the Coulomb part of the gravitational field [1].

Depending on the Petrov-Bel type, special frames (and observers) exist for which the Ψ_a take particular simple forms [1, 2]. Thus, in algebraically general spacetimes we can consider two different types of adapted frames. If we take the real null vectors l and k lying on one of the three Weyl principal planes then $\Psi_1 = \Psi_3 = 0$, the transversal wave components Ψ_0 and Ψ_4 being non zero: we

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have then the *Weyl transverse frames* [3]. On the other hand, the frames with l and k aligned with two of the four Debever directions satisfy $\Psi_0 = \Psi_4 = 0$, the longitudinal wave components Ψ_1 and Ψ_3 being non zero: they are the *Weyl longitudinal frames* or *Debever frames*.

As a consequence of the peeling-off theorem [4], one of the transverse components, Ψ_0 or Ψ_4, is dominant in the wave zone of a radiative gravitational field. On the other hand, the Teukolsky [5] formalism for studying gravitational radiation in a Kerr black hole is built using a transverse frame, and the transverse components are the essential variables. These and other similar facts focus attention on the transverse frames and some *radiation scalars* have been associated with them [3].

The role played by the frames intrinsically associated with the curvature tensor in the metric equivalence problem is well known [2]. Likewise, both the transverse and Debever frames, can be of interest in dealing with type I spacetimes. A detailed analysis on the transformations leading to the standard canonical form of the Weyl tensor for the different Petrov-Bel types has been given in [6], but some cases involving the solution of a quartic equation have not been specified. In a recent paper [7], a general procedure is presented to obtain the transverse scalars in a generic type I spacetime in which all the initial Weyl scalars are non-vanishing. The method avoids solving the quartic equation but the expressions are quite extended and complicated. Moreover this aforementioned work does not present explicit expressions for the transverse tetrad.

In this paper we present a general algorithm to determine *all* the elements associated with *every* transverse frame in a *generic* type I spacetime and starting from an *arbitrary* initial tetrad. The procedure only uses a particular solution of a cubic equation and it affords, for each of the three principal planes: (i) the transverse scalars Ψ_0^T, Ψ_2^T, Ψ_4^T, (ii) the transverse base $\{\mathbf{W}^T, \mathbf{U}^T, \mathbf{V}^T\}$ of the self-dual bivector space, and (iii) the transverse null tetrad $\{l^T, n^T, m^T, \bar{m}^T\}$.

It is worth remarking that obtaining the three principal transverse frames in a type I spacetime can be suitable because, in dealing with the equivalence problem of two metrics, it could be necessary to compare a transverse frame of one metric with each of the three transverse frames of the other. The algorithmic obtaining of these frames could also be a necessary mathematical tool for studying gravitational radiation in numerical relativity as Beetle and Burko have pointed out in [3]. Some recent alternative approaches emphasize this aspect [8, 9].

The results in this paper are based on a previous paper [10] which offers a covariant algorithm to determine the type I Weyl canonical frames. This approach was further developed in [11] where a complete algebraic analysis of the Weyl tensor is presented. In this aforementioned paper *every* Weyl geometric element (invariant scalars, principal directions or principal planes) associated with *every* Petrov-Bel type is determined in a covariant way. This means that, given a metric in an arbitrary coordinate system or in an arbitrary tetrad, if we know the Weyl eigenvalues, we can obtain all these Weyl geometric elements without solving any other equation. The Weyl eigenvalues are the roots of the (cubic) characteristic equation of the self-dual Weyl endomorphism, whose coefficients are the symmetric algebraic invariant scalars of the Weyl tensor. Thus, this covariant approach could also be useful in looking for longitudinal scalars and Debever frames, as well as, in dealing with algebraically special Petrov-Bel types.

This work is organized as follows. In Sect. 2 we summarize some of the results in [10, 11] and, in Sect. 3 and 4, we use them to obtain the elements associated with the transverse frames. We will finish with a short discussion.

2 The covariant procedure

In an oriented spacetime (V_4, g) of signature $(-+++)$ the algebraic classification of the Weyl tensor C can be obtained by studying the linear map defined on the self-dual 2-form space by the self-dual Weyl tensor $\mathcal{C} = \frac{1}{2}(C - i * C)$. The restriction on this self-dual space of the metric of the 2–form space G is given by $\mathcal{G} = \frac{1}{2}(G - i\eta)$, where η is the spacetime volume element. In terms of the complex invariant scalars, $I = \frac{1}{2}\operatorname{tr}\mathcal{C}^2$, $J = \frac{1}{6}\operatorname{tr}\mathcal{C}^3$, the characteristic equation takes the form:

$$x^3 - Ix - 2J = 0 \tag{1}$$

and its roots are, for $k = 0, 1, 2$,

$$\alpha_{k+1} = \beta e^{\frac{2\pi k}{3}\,\mathrm{i}} + \frac{I}{3\beta} e^{-\frac{2\pi k}{3}\,\mathrm{i}}, \quad \beta = \sqrt[3]{\left(J + \sqrt{J^2 - I^3/27}\right)}. \tag{2}$$

The Weyl tensor is Petrov-Bel type I if (1) admits three different roots $\{\alpha_i\}$, which is equivalent to the condition $27J^2 \neq I^3$. In this case an orthonormal frame $\{\mathbf{W}_i\}$ exists which is built up with eigenvectors of $\mathcal{C}$. The self-dual 2–forms $\mathbf{W}_i$ are the *principal* 2*-forms* of the Weyl tensor [12]. Then, the self-dual Weyl tensor takes the canonical expression

$$\mathcal{C} = -\sum_{i=1}^{3} \alpha_i \, \mathbf{W}_i \otimes \mathbf{W}_i \tag{3}$$

In [10] we have determined the projection map on the eigenspace associated with every eigenvalue α_i and, consequently, we have acquired a covariant way to obtain the principal 2-forms $\{\mathbf{W}_i\}$ in terms of the Weyl tensor. More precisely, we can find in [10] the following result which we present here in a slightly different version:

Proposition 1 *Let $\mathcal{C}$ be the self-dual Weyl tensor of a type I space-time. The principal* 2*-form $\mathbf{W}_i$ corresponding to the eigenvalue α_i may be obtained as*

$$\mathbf{W}_i = \frac{\mathcal{P}_i(\mathbf{X})}{\sqrt{(I - 3\alpha_i^2)\mathcal{P}_i(\mathbf{X}, \mathbf{X})}} \tag{4}$$

with $\mathcal{P}_i \equiv \mathcal{C}^2 + \alpha_i \mathcal{C} + (\alpha_i^2 - I)\mathcal{G}$, and where $\mathbf{X}$ is an arbitrary self-dual 2-form such that $\mathcal{P}_i(\mathbf{X}) \neq 0$.

The principal 2-forms of a type I Weyl tensor are given by (4) and are determined up to sign and permutation. Thus, we can consider 24 oriented eigen-frames $\{\mathbf{W}_i\}$: for every permutation, the sign of two of them gives us 4 possibilities, and the third can be obtained as [10] $\mathbf{W}_3 = \mathrm{i}\sqrt{2}\mathbf{W}_1 \times \mathbf{W}_2$, where $\times$ stands for the contraction of adjacent indexes, $(A \times B)_{\alpha\beta} = A_\alpha^{\ \lambda} B_{\lambda\beta}$.

The three principal 2–forms $\mathbf{W}_i$ determine six Weyl principal 2–planes that cut in four orthogonal directions: one *time-like Weyl principal direction* and three *space-like Weyl principal directions* [10, 12], which have associated the *Weyl canonical orthonormal frames* $\{e_\alpha\}$. It has been shown in [10] that an oriented and orthochronous canonical orthonormal frame $\{e_\alpha\}$ corresponds biunivocally to every oriented eigen-frame $\{\mathbf{W}_i\}$. Its explicit expression is given in the following proposition [10]:

Proposition 2 *The Weyl canonical frames* $\{e_\alpha\}$ *of a type I space-time may be determined as*

$$e_0 = \frac{-P_0(x)}{\sqrt{P_0(x,x)}}, \quad P_0 \equiv -\frac{1}{2}\left(\frac{1}{2}g + \sum_{i=1}^{3} \mathbf{W}_i \times \overline{\mathbf{W}}_i\right); \quad e_i = \sqrt{2}\mathbf{W}_i(e_0) \quad (5)$$

where $\mathbf{W}_i$ *are the principal 2-forms given in Proposition 1, and where* x *is an arbitrary future-pointing vector and an overline stands for the complex conjugate.*

In the following sections we will see that the results in propositions 1 and 2 enable us to obtain the transverse scalars and the transverse frames starting from an arbitrary frame. The covariant method presented in [10] could also be used in looking for the longitudinal scalars and frames because the Debever directions of a type I spacetime have also been obtained in [10] explicitly. It is worth mentioning that this study has been extended in [11] for an arbitrary Petrov-Bel type in such a way that *given a particular cubic root* β of the expression (2), all the Weyl geometric elements (*principal 2–forms*, *principal and Debever directions* and *canonical frames*) can be obtained without solving any other equation.

3 Transverse scalars

An arbitrary null tetrad $\{l, k, m, \bar{m}\}$ has the following associated null base $\{\mathbf{W}, \mathbf{U}, \mathbf{V}\}$ of bivectors[1]

$$\mathbf{W} = \frac{1}{\sqrt{2}}[l \wedge k + m \wedge \bar{m}], \quad \mathbf{U} = -\frac{1}{\sqrt{2}} l \wedge \bar{m}, \quad \mathbf{V} = \frac{1}{\sqrt{2}} k \wedge m \qquad (6)$$

Let $\Psi_a = \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ be the usual components [2] of the Weyl tensor in the base of the traceless double bivectors, i. e.:

$$\frac{1}{2}\mathcal{C} = \Psi_0 \mathbf{U} \otimes \mathbf{U} + \Psi_1 \mathbf{U} \overset{\sim}{\otimes} \mathbf{W} + \Psi_2(\mathbf{U} \overset{\sim}{\otimes} \mathbf{V} + \mathbf{W} \otimes \mathbf{W}) + \Psi_3 \mathbf{W} \overset{\sim}{\otimes} \mathbf{V} + \Psi_4 \mathbf{V} \otimes \mathbf{V}$$

[1] The notation that we use here is similar to that used in reference [2], but the self-dual bivectors $\{\mathbf{W}, \mathbf{U}, \mathbf{V}\}$ differ from those in [2] by the factor $1/\sqrt{2}$, and the self-dual Weyl tensor by the factor $1/2$.

In terms of the Weyl scalars Ψ_a, the invariants I, J are [2]:

$$I = I[\Psi_\alpha] \equiv \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2 \tag{7}$$

$$J = J[\Psi_\alpha] \equiv \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix} \tag{8}$$

Then, the Weyl eigenvalues can be computed by (2) where β stands for a particular non null cubic root of (2) which, from now, we can suppose that it is given in terms of Ψ_a as a consequence of (7) and (8):

$$\beta = \beta[\Psi_a] \equiv \sqrt[3]{J + \sqrt{J^2 - I^3/27}} \tag{9}$$

The Weyl *transverse bivector bases* $\{\mathbf{W}^T, \mathbf{U}^T, \mathbf{V}^T\}$ are those for which $\Psi_1^T = \Psi_3^T = 0$. Thus, the Weyl tensor can be written:

$$\frac{1}{2}\mathcal{C} = \Psi_0^T\, \mathbf{U}^T \otimes \mathbf{U}^T + \Psi_2^T\, (\mathbf{U}^T \,\tilde{\otimes}\, \mathbf{V}^T + \mathbf{W}^T \otimes \mathbf{W}^T) + \Psi_4^T\, \mathbf{V}^T \otimes \mathbf{V}^T \tag{10}$$

Then, $\mathbf{U}^T$ and $\mathbf{V}^T$ can be parameterized to satisfy $\Psi_0^T = \Psi_4^T$. If we compare (10) with the canonical expression (3) we obtain that the tern $\{\mathbf{W}^T, -\mathrm{i}(\mathbf{V}^T + \mathbf{U}^T), \mathbf{V}^T - \mathbf{U}^T\}$ is an orthonormal eigenframe of the Weyl tensor if $\{\mathbf{W}^T, \mathbf{U}^T, \mathbf{V}^T\}$ is a transverse bivector base. Conversely, if $\{\mathbf{W}_i\}$ is an orthonormal Weyl eigenframe, for every choice of an eigenvalue α_i, we can take:

$$\mathbf{W}_i = \mathbf{W}_i\,, \quad \mathbf{U}_i = \frac{\mathrm{i}}{2}(\mathbf{W}_j + \mathrm{i}\,\mathbf{W}_k)\,, \quad \mathbf{V}_i = \frac{\mathrm{i}}{2}(\mathbf{W}_j - \mathrm{i}\,\mathbf{W}_k), \tag{11}$$

where i, j, k take the different values of a cyclic permutation. Then the tern $\{\mathbf{W}_i, \mathbf{U}_i, \mathbf{V}_i\}$ is an oriented bivector transverse frame that we name *principal transverse bivector base*. Moreover, for every i, the non null *principal transverse scalars* $\Psi_a^{(i)}$ are given in terms of the eigenvalues of the Weyl tensor by:

$$\Psi_2^{(i)} = -\frac{1}{2}\alpha_i, \quad \Psi_0^{(i)} = \Psi_4^{(i)} = \frac{\alpha_j - \alpha_k}{2} \tag{12}$$

where i, j, k take the different values of a cyclic permutation.

Consequently, three oriented *principal transverse bases* $\{\mathbf{W}_i, \mathbf{U}_i, \mathbf{V}_i\}$ exist basically (the others can be obtained by changing the sign of two elements or by changing one sign and interchanging $\mathbf{U}_i \leftrightarrow \mathbf{V}_i$). For each of these frames, we can give the corresponding *principal transverse scalars* $\Psi_a^{(i)}$ by using (12). Indeed, taking into account the expression of the eigenvalues (2) in terms of I, J and β, we have:

Theorem 1 *Let Ψ_a be the components of a type I Weyl tensor in an arbitrary frame. The principal transverse scalars are given by (12), with:*

$$\alpha_1 = \alpha_1[\Psi_a] \equiv \left(\beta + \frac{I}{3\beta}\right), \tag{13}$$

$$\alpha_2 = \alpha_2[\Psi_a] \equiv -\frac{1}{2}\left((1 - \mathrm{i}\sqrt{3})\beta + (1 + \mathrm{i}\sqrt{3})\frac{I}{3\beta}\right), \tag{14}$$

$$\alpha_3 = \alpha_3[\Psi_a] \equiv -\frac{1}{2}\left((1 + \mathrm{i}\sqrt{3})\beta + (1 - \mathrm{i}\sqrt{3})\frac{I}{3\beta}\right). \tag{15}$$

where $I \equiv I[\Psi_a]$, $J \equiv J[\Psi_a]$ and $\beta \equiv \beta[\Psi_a]$ are given by (7), (8) and (9), respectively.

If $\{\mathbf{W}_i, \mathbf{U}_i, \mathbf{V}_i\}$ is a principal transverse bivector base, then the tern:

$$\mathbf{W}^T = \mathbf{W}_i\,, \quad \mathbf{U}^T = z\mathbf{U}_i\,, \quad \mathbf{V}^T = z^{-1}\mathbf{V}_i \tag{16}$$

where z is a complex function, is a transverse bivector base because it satisfies the transverse condition, $\Psi_1^T = \Psi_3^T = 0$. Moreover, the transverse components are:

$$\Psi_2^T = \Psi_2^{(i)}\,, \quad \Psi_0^T = z^{-2}\Psi_0^{(i)}\,, \quad \Psi_4^T = z^2\Psi_4^{(i)}\,. \tag{17}$$

It is worth pointing out that the transverse principal components (given by (12) and Theorem 1) are invariant scalars, but for a generic (non necessarily principal) transverse frame only the Coulomb component Ψ_2^T is invariant. Nevertheless, it follows from (12) and (17) that the product of the transverse components does not depend on the complex Lorentz rotation z:

$$\xi^{(i)} \equiv \Psi_0^T\Psi_4^T = \left(\Psi_0^{(i)}\right)^2 = \left(\Psi_4^{(i)}\right)^2 \tag{18}$$

The invariant scalars $\xi^{(i)}$ are not but the Beetle-Burko radiation scalars which have been proposed in [3] as containing information about the field gravitational radiation.

Our simple method to determine the transverse components starting from an arbitrary frame presented in Theorem 1 improves some previous results [7] that offer quite complicated expressions of the transverse components in a (not necessarily principal) transverse frame.

The transformations leading from an initial configuration to the transverse frames have been studied in [6], but these transformation have not been obtained explicitly in the more regular cases. This problem is analyzed and solved in the following section by offering explicit expressions for the transverse frames.

4 Obtaining transverse frames from an arbitrary frame

In order to determine the transverse frames starting from an arbitrary frame, we begin by obtaining the principal 2–forms $\{\mathbf{W}_i\}$ in terms of the initial bivector base $\{\mathbf{W}, \mathbf{U}, \mathbf{V}\}$ and the initial Weyl scalars Ψ_a. We will use Proposition 1 and, consequently, we must pick out a self-dual 2-form $\mathbf{X}$. If we take $\mathbf{X} = \mathbf{U}$, and we compute $\mathcal{P}_i(\mathbf{U})$, where the projector $\mathcal{P}_i$ is given in Proposition 1, we obtain:

$$\mathcal{P}_i(\mathbf{U}) = A_i\ \mathbf{U} + B_i\ \mathbf{W} + C_i \mathbf{V}, \quad \mathcal{P}_i(\mathbf{U}, \mathbf{U}) = \frac{1}{2} C_i\ , \tag{19}$$

where the scalars A_i, B_i and C_i are the following functions of the Weyl components Ψ_a:

$$A_i = A_i[\Psi_a] \equiv \Psi_0\Psi_4 + \Psi_2^2 - 2\Psi_1\Psi_3 + \alpha_i\Psi_2 + \left(\alpha_i^2 - I\right) \tag{20}$$

$$B_i = B_i[\Psi_a] \equiv \Psi_1\Psi_4 - \Psi_2\Psi_3 + \Psi_3\alpha_i \tag{21}$$

$$C_i = C_i[\Psi_a] \equiv 2\Psi_2\Psi_4 - 2\Psi_3^2 + \alpha_i\Psi_4 \tag{22}$$

Then, if we take into account expression (4) for the principal 2–forms, we can state:

Theorem 2 *Let Ψ_a be the components of a type I Weyl tensor in a non transverse bivector base $\{\mathbf{W}, \mathbf{U}, \mathbf{V}\}$. The (unitary) principal 2–forms $\{\mathbf{W}_i\}$ are given by:*

$$\mathbf{W}_i = \frac{1}{\sqrt{D_i C_i}}(A_i \mathbf{U} + B_i \mathbf{W} + C_i \mathbf{V})\ , \tag{23}$$

where the scalars A_i, B_i and C_i are the functions of the Weyl scalars Ψ_a given in (20), (21) and (22), and

$$D_i = D_i[\psi_a] \equiv \frac{1}{2}\left(I - 3\alpha_i^2\right), \tag{24}$$

with $\alpha_i = \alpha_i[\Psi_a]$ given in Theorem 1 and where $I \equiv I[\Psi_a]$, $J \equiv J[\Psi_a]$ and $\beta \equiv \beta[\Psi_a]$ are given by (7), (8) and (9), respectively.

It should be mentioned that if we start from a non transverse frame as Theorem 2 states, the bivector $\mathbf{U}$ can not be orthogonal to any principal 2–form $\mathbf{W}_i$ and, consequently, the condition $\mathcal{P}_i(\mathbf{U}) \neq 0$ required in proposition 1 holds.

Now, if we consider the relation (11) between a null base and an orthonormal base in the bivector space, the determination of the principal transverse bivector bases is a simple consequence of Theorem 2:

Corollary 1 *Let Ψ_a be the components of a type I Weyl tensor in an arbitrary bivector base $\{\mathbf{W}, \mathbf{U}, \mathbf{V}\}$. The principal transverse bivector bases $\{\mathbf{W}_i, \mathbf{U}_i, \mathbf{V}_i\}$ can be obtained as (11) where $\{\mathbf{W}_i\}$ is given in Theorem 2.*

Theorem 2 and Corollary 1 give us explicit expressions for the three principal transverse bivector bases. The non principal transverse bivector bases $\{\mathbf{W}^T, \mathbf{U}^T, \mathbf{V}^T\}$ can be obtained from the principal ones as (16) by considering arbitrary values for the complex function z.

Once the transverse bivector bases $\{\mathbf{W}^T, \mathbf{U}^T, \mathbf{V}^T\}$ are known, we can look for the transverse frames $\{l^T, n^T, m^T, \bar{m}^T\}$ associated with them by (6). To obtain them, we could apply to the bivectors $\{\mathbf{W}^T, \mathbf{U}^T, \mathbf{V}^T\}$ the covariant method to determine the principal directions of a 2–form [13] (see also [10, 11]), but here we opt by an alternative procedure based on Proposition 2: starting from an arbitrary null tetrad $\{l, k, m, \bar{m}\}$ we will obtain the Weyl orthonormal frame $\{e_\alpha\}$ and, from it, we derive the null transverse frames $\{l^T, k^T, m^T, \bar{m}^T\}$.

From (23) we can obtain the e_0-projector P_0 given in Proposition 2. If we take $x = l$ and compute $P(l)$, we have:

$$-4P(l) = al + bk + cm + \bar{c}\bar{m}, \quad 4P(l, l) = b \tag{25}$$

where

$$\begin{aligned} a &= a[\Psi_a] \equiv 2 + \frac{c\bar{c}}{b} \\ b &= b[\Psi_a] \equiv \sum \frac{|C_i|}{|D_i|} \\ c &= c[\Psi_a] \equiv -\sum \frac{\bar{B}_i C_i}{|D_i||C_i|} \end{aligned} \tag{26}$$

On the other hand, if we take into account the expression for the e_i given in Proposition 2, we can state:

Theorem 3 *Let Ψ_a be the components of a type I Weyl tensor in a non transverse null frame $\{l, k, m, \bar{m}\}$. The orthonormal Weyl canonical frame can be obtained as:*

$$e_0 = \frac{1}{2\sqrt{b}}\,(al + bk + cm + \bar{c}\bar{m}), \quad e_i = \sqrt{2}\mathbf{W}_i(e_0) \tag{27}$$

where $\mathbf{W}_i$ are given in (23), a, b, c are the functions of the Weyl scalars Ψ_a given in (26). The scalars A_i, B_i, C_i and D_i depend on Ψ_a as (20), (21), (22) and (24), where α_i is given in Theorem 1 and I, J and β are given by (7), (8) and (9), respectively.

The principal transverse frames $\{l_{(i)}, k_{(i)}, m_{(i)}, \bar{m}_{(i)}\}$ can be determined as a consequence of Theorem 3:

Corollary 2 *Let Ψ_a be the components of a type I Weyl tensor in an arbitrary null frame $\{l, k, m, \bar{m}\}$. We can obtain the principal transverse frames $\{l_{(i)}, k_{(i)}, m_{(i)}, \bar{m}_{(i)}\}$ as:*

$$l_{(i)} = \frac{1}{\sqrt{2}}(e_0 + e_i)\,, \quad k_{(i)} = \frac{1}{\sqrt{2}}(e_0 - e_i)\,, \quad m_{(i)} = \frac{1}{\sqrt{2}}(e_j + \mathrm{i} e_k) \tag{28}$$

where i, j, k take the different values of a cyclic permutation and e_α are given in Theorem 3.

From each one of the three principal transverse frames $\{l_{(i)}, k_{(i)}, m_{(i)}, \bar{m}_{(i)}\}$ determined in Corollary 2 we can obtain the other (non principal) transverse null frames $\{l^T, k^T, m^T, \bar{m}^T\}$ by means a special boost ϕ on the time-like plane $\{l_{(i)}, k_{(i)}\}$ and a rotation θ on the space-like plane $\{m_{(i)}, \bar{m}_{(i)}\}$:

$$l^T = e^{\phi} l_{(i)}, \quad k^T = e^{-\phi} k_{(i)}, \quad m^T = e^{-\mathrm{i}\theta} m_{(i)}. \tag{29}$$

The real functions ϕ and θ are related to the complex Lorentz transformation z in (16) by $z = e^{\phi + \mathrm{i}\theta}$.

In this section we have obtained all the transverse frames of a type I spacetime when we know the Weyl scalars in a non transverse frame. But, if we initially have a transverse frame $\{l^T, n^T, m^T, \bar{m}^T\}$, we could be interested in knowing all the other ones. Let Ψ_2^T, Ψ_0^T, Ψ_4^T be the initial transverse scalars. Then, as a consequence of (17), the complex Lorentz transformation $z = e^{\phi + \mathrm{i}\theta}$ leading to a principal transverse frame $\{l_{(i)}, k_{(i)}, m_{(i)}, \bar{m}_{(i)}\}$ by means of (29) can be obtained as:

$$z = \left(\frac{\Psi_0^T}{\Psi_4^T}\right)^{1/4} \tag{30}$$

Then, the transformations (28) and their inverses allow us to obtain the orthonormal canonical Weyl frame $\{e_\alpha\}$ and the other principal transverse frames.

5 Summary and discussion

In this paper we have presented a general algorithm to determine *all* the elements associated with *every* transverse frame in a *generic* type I spacetime and starting from an *arbitrary* initial tetrad. Our procedure affords, for each of the three principal planes: (i) the *principal transverse scalars* $\Psi_2^{(i)}$, $\Psi_0^{(i)}$, $\Psi_4^{(i)}$ (Theorem 1) and the associated (non principal) transverse scalars Ψ_2^T, Ψ_0^T, Ψ_4^T (expression (17)), (ii) the *principal bivector transverse base* $\{\mathbf{W}_i, \mathbf{U}_i, \mathbf{V}_i\}$ (Theorem 2 and Corollary 1) and the associated (non principal) transverse bases $\{\mathbf{W}^T, \mathbf{U}^T, \mathbf{V}^T\}$ (expression (16)), and (iii) the *principal transverse null tetrad* $\{l_{(i)}, n_{(i)}, m_{(i)}, \bar{m}_{(i)}\}$ (Theorem 3 and Corollary 2) and the associated (non principal) transverse null tetrads $\{l^T, n^T, m^T, \bar{m}^T\}$ (expression (29)).

We would like to state that the above results improve previous ones on the same subject. We can quote an interesting work [6] where a detailed analysis has been made of the transformations leading to the standard canonical form of the Weyl tensor for the different Petrov-Bel types. In the mentioned study the authors present the transformations depending on the initial configuration of the Weyl components Ψ_a, but "certain cases involving the solution of a quartic equation have not been specified" (see Table 2 in [6]). It is worth mentioning, e.g. the case of type I spacetimes with non-vanishing initial transverse components. This shortcoming has partially been overcome in a recent paper [7] where a procedure is presented "to arrive at a transverse frame from the general case of all scalars non-zero" in a given type I spacetime. The method avoids solving the quartic equation but "the expressions for Ψ_2, Ψ_0 and Ψ_4 are quite complicated" according to the authors. Our paper overcomes this shortcoming and we offer simple and clear expressions for the transverse scalars. Moreover, we also obtain the transverse components in the three principal transverse frames (Theorem 1) and in an arbitrary transverse frame (expression (17)). On the other hand, obtaining the transformations that were not studied in [6] is a problem that has not been considered in [7] either. This question is analyzed and solved in this work by offering explicit expressions for all the transverse frames.

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