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On Weyl-electric and Weyl-magnetic spacetimes

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Abstract
The concepts of purely electric and purely magnetic Weyl tensors are extended and the intrinsic characterization of the new wider classes is given. The solutions v to the equations $W(v; v) = 0$ or $*W(v; v) = 0$ are determined for every Petrov type, and the new electric or magnetic type I cases are studied in more detail.

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1. Introduction

The electric and magnetic parts of the Weyl tensor were introduced by Matte [1] when searching for gravitational quantities E and H, attached to any observer and playing an analogous role to the electric and magnetic fields. He showed that, for high frequency gravitational waves, these tensors satisfy Maxwell equations, remarking consequently on the analogy between electromagnetic and gravitational fields. These results were improved by Bel [2], who also pointed out the use of such concepts in defining states of intrinsic gravitational radiation.

In a similar way, electric and magnetic parts of the curvature tensor can be attached to any observer. The gravitational fields admitting an observer u with purely electric curvature were studied by Trümper [3] who showed that the Weyl tensor is also purely electric and u is an eigenvector of the Ricci tensor. These properties, which characterize the Riemann-electric spacetimes, do not hold for the purely magnetic case. Indeed, Riemann-magnetic gravitational fields have been considered [4, 5], and Riemann-magnetic (non-vacuum) solutions are known with a non-purely-magnetic Weyl tensor and vice versa [6–8].

It is known that if a spacetime admits a timelike shear-free and vorticity-free congruence u, the Weyl tensor is purely electric with respect to u and, consequently, the metric is of Petrov type I, D or O with real eigenvalues, u being a principal direction of the Weyl tensor. This result by Trümper [3] shows that the family of Weyl-electric spacetimes is wide and interesting from a physical point of view. Indeed, all the static gravitational fields and some known solutions of Einstein equations, for example the perfect fluid Barnes class [9], have a purely electric Weyl tensor. This fact has motivated the search for and the study of new solutions with this property; for instance, we can quote results about Weyl-electric perfect fluids [10, 11] and a wide bibliography about the so-called silent universes (see [12] and references therein). Nevertheless, one knows few solutions with a purely magnetic Weyl tensor [13–15]. Both, Weyl-electric and Weyl-magnetic spacetimes, have been analysed and classified in a work by McIntosh *et al* [7].

The result by Trümper quoted above [3] can be useful in different domains. It permits, for example, the Killing vector to be detected in static type I gravitational fields and these spacetimes to be explicitly identified [16]. We have also used it for an intrinsic characterization of type I metrics, admitting isotropic radiation for a vorticity-free observer u [17]; these spacetimes can be of interest when one is looking for cosmological models, u playing the role of a generalized cosmological observer [18].

Two levels can be distinguished in the proof of the result by Trümper: at first level, Ricci identities are used to prove that the magnetic part $*W(u;u)$ of the Weyl tensor vanishes; from here, the Petrov matrix for u is a real symmetric matrix in a three-dimensional Euclidean space and always diagonalizes. On the other hand, if we consider a spacelike shear-free and vorticity-free congruence s, the first step in the proof by Trümper still holds true, that is, $*W(s;s) = 0$. However, as the space orthogonal to s is not a Euclidean space, the Petrov matrix can be non-diagonal and so we have no restriction on the Petrov type. This fact alone motivates us to study in this work the general solution to the equations $*W(v;v) = 0$ and $W(v;v) = 0$ for every Petrov type and independent of the causal character of the solutions. Our results provide a generalization of the purely electric and purely magnetic concepts, a generalization that can be advantageous in different situations. For instance, they allow the hypersurface-orthogonal Killing vectors to be identified in an arbitrary non-conformally flat spacetime and so the gravitational fields admitting these kind of symmetries can be characterized. In this respect, the case of a Petrov type III spacetime has been studied recently [16].

In order to consider 'degenerate' type I spacetimes, McIntosh and Arianrhod [19] used the adimensional complex scalar $M = \frac{a^3}{b^2} - 6$, with $a = \mathrm{Tr}\,\mathcal{W}^2$ and $b = \mathrm{Tr}\,\mathcal{W}^3$ being, respectively, the quadratic and the cubic Weyl scalar invariants. The scalar M is related to the Penrose cross-ratio invariant [20] and it governs the geometry defined by the Debever null directions: $M = 0$ in Petrov type D and, in type I, M is real positive or infinite when the four Debever directions span a 3-plane [19]; the case M real negative occurs when the Penrose–Rindler [21] disphenoid associated with the Debever directions has four equal edges [22]. Elsewhere we have presented an alternative approach for analysing this Debever geometry using the complex angle between the principal bivectors and the unitary Debever bivectors [16]; the permutability properties with respect to the metric tensor of a frame [23] built with Debever null vectors permit a re-interpretation of the case M negative [24].

In the work by McIntosh *et al* [7], where Weyl-electric and Weyl-magnetic spacetimes are studied and classified, a characterization is also given in terms of some Weyl complex scalar invariants: M must be real positive or infinite, and a must be real, positive in the electric case and negative in the magnetic case. Consequently, the Debever directions of a type I purely electric and purely magnetic Weyl tensor span a 3-plane. We will show here that our generalized Weyl-electric and Weyl-magnetic spacetimes also permit the scalar M to take real negative values, and therefore, the new classes of gravitational fields that we consider, admit a partially symmetric frame built with Debever vectors.

This paper is organized as follows. In section 2 we introduce some definitions and notations and generalize the concepts of purely electric and purely magnetic spacetimes. In section 3 we study, for every Petrov type, the equations $*W(v;v) = 0$ and $W(v;v) = 0$, and determine the solutions independent of their causal character. Some direct consequences of this study are presented in section 4, pointing out the families that these results define and acomplishing intrinsic characterization theorems for them. Finally, in section 5 we specifically analyse the type I new classes by studying their Debever null vectors and other related matters; some examples are also presented. Some of the results in this paper were communicated without proof at the Spanish Relativity Meeting, 2000 [25].

2. Purely electric and purely magnetic Weyl tensor

Let (V_4, g) be an oriented and time-oriented spacetime of signature $\{-, +, +, +\}$, and let W be its Weyl tensor. We can associate with any unitary vector field v ($v^2 = \epsilon, \epsilon = \pm 1$) the electric and magnetic Weyl fields

$$E = E[v] \equiv W(v; v), \qquad H = H[v] \equiv *W(v; v) \tag{1}$$

where we denote $W(v; v)_{\alpha\gamma} = W_{\alpha\beta\gamma\delta} v^\beta v^\delta$. The electric and magnetic fields (1) fully determine the Weyl tensor. This fact was pointed out years ago for the timelike case [1, 2], and can be inferred, taking into account the expression

$$W = -v \wedge E \wedge v + *(v \wedge E \wedge v) * + * (v \wedge H \wedge v) + (v \wedge H \wedge v) * \tag{2}$$

where $(v \wedge T)_{\alpha\beta\gamma} = v_\alpha T_{\beta\gamma} - v_\beta T_{\alpha\gamma}$ and $(T \wedge v)_{\alpha\beta\gamma} = T_{\alpha\beta} v_\gamma - T_{\alpha\gamma} v_\beta$. This formula can be easly shown by applying twice the known expression that gives a 2-form in terms of an observer and its relative electric and magnetic fields [26]. It has been known for years in the timelike case [27] but also holds when v is a spacelike vector.

When v is a null vector, we can also define the electric and magnetic fields (1) but, in this case, they do not determine the Weyl tensor. Indeed, as we will see later, a non-conformally-flat spacetime can admit a null congruence with both the electric and magnetic associated fields zero. Nevertheless, in order to generalize and analyse the purely electric and purely magnetic concepts, we also consider this case here. Then the following definitions can be useful.

Definition 1. *A metric is Weyl-electric (WE) at a point of spacetime when there is a vector v which is solution of the equation $W(v; v) = 0$ at this point. We will say that the metric is timelike Weyl-electric (TWE), spacelike Weyl-electric (SWE) or null Weyl-electric (NWE) if the solution v is, respectively, a timelike, a spacelike or a null vector. Similarly we define Weyl-magnetic (WM), timelike Weyl-magnetic (TWM), spacelike Weyl-magnetic (SWM) and null Weyl-magnetic (NWM) metrics.*

Definition 2. *We say that v is a Weyl-electric (Weyl-magnetic) vector if $*W(v; v) = 0$ ($W(v; v) = 0$).*

It could also be of interest to consider similar definitions for Riemann-electric and Riemann-magnetic spacetimes, but in this work we only analyse these conditions for the Weyl tensor.

From here we work in the self-dual complex formalism. A self-dual 2-form is a complex 2-form $\mathcal{F}$ such that $*\mathcal{F} = \mathrm{i}\mathcal{F}$. We can associate biunivocally with every real 2-form F the self-dual 2-form $\mathcal{F} = \frac{1}{\sqrt{2}}(F - \mathrm{i} *F)$. We refer here to a self-dual 2-form as a *bivector*. The endowed metric on the three-dimensional complex space of the bivectors is $\mathcal{G} = \frac{1}{2}(G - \mathrm{i}\eta)$, G being the usual metric on the 2-form space, $G = \frac{1}{2} g \wedge g$, $(g \wedge g)_{\alpha\beta\mu\nu} = 2(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu})$, and η being the metric volume element. A $\mathcal{G}$-unitary bivector $\mathcal{U} = \frac{1}{\sqrt{2}}(U - \mathrm{i} *U)$ corresponds to every timelike unitary simple 2-form U $((U, *U) = 0,\ (U, U) = -1)$, and $\mathcal{H} = \frac{1}{\sqrt{2}}(H - \mathrm{i} * H)$ is a null bivector for $\mathcal{G}$ when H is singular $((H, H) = (H, *H) = 0)$.

A unitary bivector $\mathcal{U}$ defines a timelike 2-plane with volume element U and its orthogonal spacelike 2-plane with volume element $*U$. We denote these *principal 2-planes* as their volume element. The null directions $l_{\pm}$ in the 2-plane U are the (real) eigendirections of $\mathcal{U}$ and they are called *principal directions*. These principal directions may be parametrized in such a way that $U = l_- \wedge l_+$. On the other hand, a null bivector $\mathcal{H}$ defines two null *fundamental 2-planes*, with volume elements H and $*H$, which cut in the unique (real) eigendirection l that $\mathcal{H}$ admits. Just a parametrization of the null vector l exists such that it is future-pointing and $H = l \wedge e_2$, where e_2 is a spacelike unitary vector orthogonal to l, and fixed up to change $e_2 \hookrightarrow e_2 + \mu l$. With this parametrization we name l the *fundamental vector* of $\mathcal{H}$.

The algebraic classification of the Weyl tensor W can be obtained by studying the traceless linear map defined by the self-dual Weyl tensor $\mathcal{W} = \frac{1}{2}(W - \mathrm{i} * W)$ on the bivectors space. We can associate with the Weyl tensor the complex scalar invariants

$$a \equiv \operatorname{Tr} \mathcal{W}^2 = \rho_1^2 + \rho_2^2 + \rho_3^2, \qquad b \equiv \operatorname{Tr} \mathcal{W}^3 = \rho_1^3 + \rho_2^3 + \rho_3^3 = 3\rho_1\rho_2\rho_3 \tag{3}$$

where ρ_i are the eigenvalues. It will also be useful to consider the adimensional scalar invariant [7, 19]:

$$M \equiv \frac{a^3}{b^2} - 6 = \frac{2(\rho_1 - \rho_2)^2(\rho_2 - \rho_3)^2(\rho_3 - \rho_1)^2}{9\rho_1^2\rho_2^2\rho_3^2}. \tag{4}$$

The invariant M is well defined for the Petrov types I, D or II if we permit it to be infinite in the case of a type I metric with $b = 0$. In types D and II, M is identically zero, and we extend its validity by considering that it also takes the null value for type N and type III metrics.

In terms of the invariants a and b, the characteristic equation reads

$$x^3 - \tfrac{1}{2}ax - \tfrac{1}{3}b = 0. \tag{5}$$

Then, the Petrov classification follows taking into account both the eigenvalue multiplicity and the degree of the minimal polynomial. The algebraically regular case (type I) occurs when $6b^2 \neq a^3$ and so the characteristic equation admits three different roots. If $6b^2 = a^3 \neq 0$, there are a double root and a simple one and the minimal polynomial distinguishes between types D and II. Finally, if $a = b = 0$, all the roots are equal and hence zero, and the Weyl tensor is of type O, N or III depending on the degree of the minimal polynomial.

The electric and magnetic Weyl fields (1) associated with a unitary vector field v give, respectively, the real and imaginary parts of the Petrov matrix $\mathcal{W}(v; v)$:

$$2\mathcal{W}(v; v) = W(v; v) - \mathrm{i} * W(v; v) \equiv E[v] - \mathrm{i}H[v]. \tag{6}$$

On the other hand, there are four scalars built with the electric and magnetic Weyl fields which are independent, up to sign, of the unitary vector v $(v^2 = \epsilon)$ [1, 2]. In fact, they are the real and imaginary parts of the complex scalar invariants a and b:

$$a = (\operatorname{Tr} E^2 - \operatorname{Tr} H^2) - 2\,\mathrm{i} \operatorname{Tr}(E \cdot H), \tag{7}$$

$$b = -\epsilon[(\operatorname{Tr} E^3 - 3 \operatorname{Tr}(E \cdot H^2)) + \mathrm{i}(\operatorname{Tr} H^3 - 3 \operatorname{Tr}(E^2 \cdot H))]. \tag{8}$$

The TWE and TWM spacetimes have been widely considered in the literature. Their intrinsic characterization and the determination of the Weyl-electric and Weyl-magnetic observers can be infered starting from the concepts and expressions given above. For explicit proof of these results we refer to the works by Trümper [3] and McIntosh *et al* [7]. Here we state them in a slightly modified version.

Theorem 1. *A non-conformally-flat spacetime is TWE (TWM) if, and only if, the Weyl tensor is Petrov type I or D and satisfies one of the following equivalent conditions:*

(i) The eigenvalues ρ_i are real (imaginary).
(ii) The adimensional invariant M is a non-negative real or infinite, and b is real (imaginary).
(iii) The adimensional invariant M is a non-negative real or infinite, and a is real and positive (negative).

Moreover, the Weyl-electric (Weyl-magnetic) observer u is a principal direction of the Weyl tensor.

In a TWE (TWM) spacetime the Petrov matrix (6) for an observer u is real (imaginary) and so it always diagonalizes because the space orthogonal to u is Euclidean. This fact implies the result by Trümper [3] on the principal nature of the Weyl-electric (magnetic) observers and the real (imaginary) character of the eigenvalues, and it plays an important role in the proof of theorem 1. But for a spacelike vector s, this property does not hold and, at this point, we can only state a necessary condition that follows from (7) and (8).

Proposition 1. *A SWE (SWM) spacetime has real invariants a and b $(i\,b)$. So, the invariant M is real or infinite.*

Nevertheless, it is possible now to extend a result by Trümper [3] about the Weyl-electric character of shear-free and vorticity-free observers to every non-null direction. Indeed, for any unitary vector v, $v^2 = \epsilon$, we can consider the usual invariant decomposition

$$\nabla v = \epsilon v \otimes \dot{v} + \omega + \sigma + \tfrac{1}{3}\theta\gamma. \tag{9}$$

Then, putting (9) in the Ricci identities for the vector v, and taking $\sigma = \omega = 0$, we show that the Weyl magnetic field associated with v is zero. Thus, we can state

Proposition 2. *Any shear-free and vorticity-free unitary vector v defines a Weyl-electric direction, that is, $*W(v; v) = 0$.*

We are interested in obtaining intrinsic characterizations for SWE (SWM) spacetimes and also for NWE (NWM) ones, similar to those given in theorem 1 for the TWE (TWM) spacetimes. In order to get them and to complete the partial results of proposition 1, we need a previous study of the general solution to the equations $*W(v; v) = 0$ and $W(v; v) = 0$. This is the goal of the following section. This study will allow us to detect the shear-free and vorticity-free directions in an arbitrary non-conformally-flat spacetime and, as a consequence, to find the hypersurface-orthogonal Killing or conformal Killing vectors. These kinds of results have been used elsewhere in order to characterize static type I spacetimes and Petrov type III metrics with a hypersurface-orthogonal Killing vector [16].

3. Weyl-electric and Weyl-magnetic directions

Here we determine, for every Petrov type, the conditions for a spacetime to be WE or WM and, if so, we will calculate the general solution to the equations $*W(v; v) = 0$ and $W(v; v) = 0$. We will express these solutions in terms of the canonical frames or other geometric elements associated with the Weyl tensor. The richness of these frames depends on the algebraic class. So, we consider every Petrov type separately. An exahustive study on Weyl geometry can be found in [16] where all these Weyl elements are explicitly and covariantly obtained.

In every case, we start from the canonical expression of the self-dual Weyl tensor $\mathcal{W}$ in terms of eigenbivectors and other distinctive bivectors. We summarize the spacetime geometry that these *canonical bivectors* determine, and we remark on the relationship with the canonical frames that they define. Then, using all these elements, we set and solve equations

$*W(v; v) = 0$ and $W(v; v) = 0$. In the next section we summarize these results and present them in a more suitable version that allows us to state theorems that characterize the WE and WM spacetimes intrinsically. It is worth pointing out that the results of this section and those given in [16] provide a covariant and explicit method to obtain all the Weyl-electric and Weyl-magnetic directions.

Conditions for a (real) vector v to satisfy $*W(v; v) = 0$ can be stated as $\mathcal{W}(v; v)$ to be real. In the same way, solutions v of equation $W(v; v) = 0$ are those for which $\mathcal{W}(v; v)$ is imaginary or, equivalently, $\mathrm{i}\,\mathcal{W}(v; v)$ is real. So the congruences that have zero electric part follow from those that have zero magnetic part by changing $\mathcal{W} \hookrightarrow \mathrm{i}\,\mathcal{W}$.

3.1. Type N

In Petrov type N, a unique null bivector $\mathcal{H}$ exists such that the self-dual Weyl tensor is written as [16]

$$\mathcal{W} = \mathcal{H} \otimes \mathcal{H}. \tag{10}$$

The *canonical bivector* $\mathcal{H}$ determines the *fundamental 2-planes* H and $*H$ and the *fundamental vector* l of a type N Weyl tensor. The direction l is the quadruple null Debever direction that a type N spacetime admits. Taking the imaginary part of (10) it follows that

$$*W = H \mathbin{\tilde{\otimes}} *H$$

where the tilde denotes symmetrization. So, the equation $*W(v; v) = 0$ becomes for a given vector v:

$$H(v) \mathbin{\tilde{\otimes}} *H(v) = 0.$$

Thus, the necessary and sufficient condition for a direction v to be Weyl-electric is $H(v) = 0$ or $*H(v) = 0$, that is, v lies in one of the fundamental (null) planes that $\mathcal{H}$ defines. So it holds that

Proposition 3. *Every type N spacetime is WE. The solutions to the equation* $*W(v; v) = 0$ *are the directions in the fundamental planes, that is,* $v \in H$ *or* $v \in *H$*. The only null Weyl-electric direction is the quadruple Debever direction l, and all the other Weyl-electric directions are spacelike. So, every type N spacetime is NWE and SWE.*

To get the congruences satisfying $W(v; v) = 0$ we can take into account that $\mathrm{i}\,\mathcal{W} = (\sqrt{\mathrm{i}}\mathcal{H}) \otimes (\sqrt{\mathrm{i}}\mathcal{H})$. So, the solutions to $W(v; v) = 0$ follow from the proposition above just changing $H \hookrightarrow H + *H$. Thus, we have

Proposition 4. *Every type N spacetime is WM. The solutions to the equation* $W(v; v) = 0$ *are the directions in the (null) planes* $H + *H$ *and* $H - *H$*, that is,* $v \in H + *H$ *or* $v \in H - *H$*. The only null Weyl-magnetic direction is the quadruple Debever direction l, and all the other Weyl-magnetic directions are spacelike. So, every type N spacetime is NWM and SWM.*

3.2. Type III

In type III, a unitary bivector $\mathcal{U}$ and a null bivector $\mathcal{H}$ exist such that the self-dual Weyl tensor is written as [16]

$$\mathcal{W} = \mathcal{U} \mathbin{\tilde{\otimes}} \mathcal{H}. \tag{11}$$

The *canonical null bivector* $\mathcal{H}$ determines the *fundamental 2-planes* H and $*H$ and the *fundamental vector l*. The direction l is the triple null Debever direction that a type III spacetime admits. The bivector $\mathcal{H}$ defines the unique eigendirection of the Weyl tensor, but in the invariant subspace orthogonal to $\mathcal{H}$ the *canonical unitary bivector* $\mathcal{U}$ is outlined.

The canonical bivectors $\mathcal{H}$ and $\mathcal{U}$ define an oriented and ortochronous null real frame $\{l, l', e_2, e_3\}$ such that $U = \pm l \wedge l'$, $H = l \wedge e_2$. This frame is characterized by l to be the triple Debever direction, l' the simple one, and e_2 (resp. e_3) to be the intersection of the planes $*U$ and H (resp. $*H$) [16]. Taking the imaginary part of (11), we have

$$*W = U \overset{\sim}{\otimes} *H + *U \overset{\sim}{\otimes} H.$$

So, for a given vector v the equation $*W(v; v) = 0$ reads

$$U(v) \overset{\sim}{\otimes} *H(v) + *U(v) \overset{\sim}{\otimes} H(v) = 0. \tag{12}$$

Writing $v = \mu l + \nu l' + \gamma e_2 + \lambda e_3$, a straightforward calculation shows that the only solutions to (12) are the directions l and e_3. Thus we have

Proposition 5. *Every type III spacetime is WE. The only solutions to the equation* $*W(v; v) = 0$ *are the triple null Debever direction* l *and the spacelike direction* e_3*, which are the intersections of the 2-planes* U *and* H*, and* $*U$ *and* $*H$*, respectively. So, every type III spacetime is NWE and SWE.*

The vectors v that satisfy $W(v; v) = 0$ follow by taking the real part of $\mathcal{W}(v; v)$ to be zero, or equivalently, the imaginary part of $\mathrm{i}\,\mathcal{W}(v; v)$ to be zero. From (11) we have

$$\mathrm{i}\,\mathcal{W} = \mathcal{U} \overset{\sim}{\otimes} (\mathrm{i}\,\mathcal{H}).$$

So, the solutions to the equation $W(v; v) = 0$ follow by changing $H \hookrightarrow *H$ in the proposition above. That is,

Proposition 6. *Every type III spacetime is WM. The only solutions to the equation* $W(v; v) = 0$ *are the triple Debever null direction* l *and the spacelike direction* e_2*, which are the intersection of the 2-planes* U *and* $*H$*, and* $*U$ *and* H*, respectively. So, every type III spacetime is NWM and SWM.*

3.3. Type D

The self-dual Weyl tensor of a Petrov type D spacetime is written as

$$\mathcal{W} = 3\rho\,\mathcal{U} \otimes \mathcal{U} + \rho\,\mathcal{G} \tag{13}$$

where $\mathcal{U}$ is the *canonical bivector* and $\rho = -\frac{b}{a}$ is the double eigenvalue [16]. Thus, in this case, the *principal 2-planes* U and $*U$ are outlined. The principal directions $l_\pm$ of U are the double Debever directions that the type D admits. Writing $\rho = \alpha + \mathrm{i}\,\beta$ and taking the imaginary part of (13), we have

$$*W = 3\beta(U \otimes U - *U \otimes *U) - 3\alpha U \overset{\sim}{\otimes} *U + \beta G - \alpha\eta.$$

So, for a given vector v the equation $*W(v; v) = 0$ reads

$$3\beta[U(v) \otimes U(v) - *U(v) \otimes *U(v)] - 3\alpha U(v) \overset{\sim}{\otimes} *U(v) + \beta(v^2 g - v \otimes v) = 0. \tag{14}$$

The principal 2-planes U and $*U$ are complementary orthogonal spaces. So, every vector v can be written as $v = v_- + v_+$ in terms of principal directions $v_- \in U$ and $v_+ \in *U$. Then, expression (14) implies that a congruence v satisfying $*W(v; v) = 0$ exists if, and only if, ρ is real ($\beta = 0$). Moreover, when this condition holds, the solutions to $*W(v; v) = 0$ are those satisfying $v_- \overset{\sim}{\otimes} v_+ = 0$, that is,

Proposition 7. *A type D spacetime is WE if, and only if, the Weyl tensor has real eigenvalues. The solutions to the equation* $*W(v; v) = 0$ *are the principal directions, that is,* $v \in U$ *or* $v \in *U$.

The only null Weyl-electric directions are the double Debever directions $l_{\pm}$. *In the 2-plane U, there are also timelike and spacelike Weyl-electric directions. Every* $v \in *U$ *is a spacelike Weyl-electric direction. So, every type D spacetime with real eigenvalues is TWE, SWE and NWE.*

To obtain the solutions to the equation $W(v; v) = 0$ we can use

$$\mathrm{i}\,\mathcal{W} = (\mathrm{i}\,\rho)\mathcal{U} \otimes \mathcal{U} + (\mathrm{i}\,\rho)\mathcal{G}.$$

Then, these solutions follow from the proposition above, just changing $\rho \hookrightarrow \mathrm{i}\,\rho$. That is,

Proposition 8. *A type D spacetime is WM if, and only if, the Weyl tensor has imaginary eigenvalues. The solutions to the equation* $W(v; v) = 0$ *are the principal directions, that is,* $v \in U$ *or* $v \in *U$.

The only null Weyl-magnetic directions are the double Debever directions $l_{\pm}$. *In the 2-plane U there are also timelike and spacelike Weyl-magnetic directions. Every* $v \in *U$ *is a spacelike Weyl-magnetic direction. So, every type D spacetime with imaginary eigenvalues is TWM, SWM and NWM.*

3.4. Type II

The self-dual Weyl tensor of a Petrov type II spacetime takes the canonical form

$$\mathcal{W} = 3\rho\mathcal{U} \otimes \mathcal{U} + \rho\mathcal{G} + \mathcal{H} \otimes \mathcal{H}, \tag{15}$$

$\mathcal{U}$ being the unitary eigenbivector associated with the simple eigenvalue $\rho = -\frac{b}{a}$, and $\mathcal{H}$ being the only eigendirection associated with the double eigenvalue [16]. The *fundamental vector l* of the *null canonical bivector* $\mathcal{H}$ gives the double Debever direction that a type II Weyl tensor admits. We have also outlined the *fundamental 2-planes H* and $*H$ of $\mathcal{H}$, and the 2-planes U and $*U$ defined by the *unitary canonical bivector* $\mathcal{U}$.

These geometric elements define an oriented and orthochronous null real frame $\{l, l', e_2, e_3\}$ such that $U = \pm l \wedge l'$ and $H = l \wedge e_2$. This frame is defined by l to be the double Debever direction, and e_2 (resp. e_3) to be the intersection of the 2-planes $*U$ and H (resp. $*H$) [16]. If we write $\rho = \alpha + \mathrm{i}\,\beta$, the imaginary part of (15) can be written as

$$*W = 3\alpha(U \otimes U - *U \otimes *U) + 3\beta U \mathbin{\tilde{\otimes}} *U + \alpha G + \beta\eta - H \mathbin{\tilde{\otimes}} *H.$$

Then, for a vector v, the equation $*W(v; v) = 0$ reads

$$\begin{aligned} &3\alpha[U(v) \otimes U(v) - *U(v) \otimes *U(v)] + 3\beta U(v) \mathbin{\tilde{\otimes}} *U(v) \\ &\qquad + \alpha(v^2 g - v \otimes v) - H(v) \mathbin{\tilde{\otimes}} *H(v) = 0. \end{aligned} \tag{16}$$

If we consider $v = \mu l + \nu l' + \gamma e_2 + \lambda e_3$, equation (16) has a non-trivial solution if, and only if, $\beta = 0$, that is, if the Weyl tensor has real eigenvalues. Moreover, the solutions are l, e_2 and e_3. This result can be stated as

Proposition 9. *A type II spacetime is WE if, and only if, the Weyl tensor has real eigenvalues. The only solutions to the equation* $*W(v; v) = 0$ *are the double Debever direction l, which is the intersection of the planes U and H, and the spacelike directions* e_2 *and* e_3, *which are the intersections of* $*U$ *with H and* $*H$, *respectively. So, every type II spacetime with real eigenvalues is SWE and NWE.*

To get the solutions to $W(v; v) = 0$, we must take into account that, from (15), we have

$$\mathrm{i}\,\mathcal{W} = 3(\mathrm{i}\,\rho)\mathcal{U} \otimes \mathcal{U} + (\mathrm{i}\,\rho)\mathcal{G} + (\sqrt{\mathrm{i}}\mathcal{H}) \otimes (\sqrt{\mathrm{i}}\mathcal{H}).$$

So, the congruences with zero electric part follow from the proposition above, just changing $\rho \hookrightarrow \mathrm{i}\,\rho$, $\mathcal{H} \hookrightarrow \sqrt{\mathrm{i}}\mathcal{H}$, or equivalently, $H \hookrightarrow H + *H$. So we have

Proposition 10. *A type II spacetime is WM if, and only if, the Weyl tensor has imaginary eigenvalues. The only solutions to the equation $W(v; v) = 0$ are the double Debever direction l, which is the intersection of the planes U and H, and the spacelike directions $e_2 \pm e_3$, which are the intersections of $*U$ with $H \pm *H$. So, every type II spacetime with imaginary eigenvalues is SWM and NWM.*

3.5. Type I

The self-dual Weyl tensor of a type I spacetime takes the canonical form

$$\mathcal{W} = \sum_{j=1}^{3} \rho_j \mathcal{U}_j \otimes \mathcal{U}_j \tag{17}$$

where $\mathcal{U}_i$ are the unitary eigenbivectors associated with the simple eigenvalues ρ_i [16]. The *canonical bivectors* $\mathcal{U}_i$ define six *principal 2-planes* U_i and $*U_i$ which cut in the four orthogonal *principal directions* that a type I metric admits.

The unitary principal vectors define the Weyl canonical frame $\{e_\alpha\}$ that satisfies $U_i = e_0 \wedge e_i$ [16]. Writing $\rho_j = \alpha_j + \mathrm{i}\,\beta_j$ and taking the imaginary part of equation (17), we get

$$*W = \sum [\beta_j(U_j \otimes U_j - *U_j \otimes *U_j) - \alpha_j U_j \,\tilde{\otimes}\, *U_j].$$

Then, for a vector v, the equation $*W(v; v) = 0$ reads

$$\sum [\beta_j(U_j(v) \otimes U_j(v) - *U_j(v) \otimes *U_j(v)) - \alpha_j U_j(v) \,\tilde{\otimes}\, *U_j(v)] = 0. \tag{18}$$

If we put $v = \sum v^\alpha e_\alpha$, a strightforward calculation shows that equation (18) has non-trivial solutions if, and only if, one of the following conditions hold: (i) the eigenvalues are all real ($\rho_j = \bar{\rho}_j$), or (ii) two of them are conjugated, say $\rho_2 = \bar{\rho}_3$.

In case (i), the solutions to the equation $*W(v; v) = 0$ are the principal directions e_α. In the second case, the solutions are the directions $e_2 \pm e_3$. This result can be stated as

Proposition 11. *A type I spacetime is WE if, and only if, one of the following conditions hold:*

(1) The eigenvalues are real.
(2) Two of the eigenvalues are conjugated.

If condition (1) holds, the Weyl-electric directions are the timelike Weyl principal direction e_0 and the spacelike Weyl principal directions e_i. So, the spacetime is TWE and SWE.

If condition (2) holds, say $\rho_2 = \bar{\rho}_3$, the Weyl-electric directions are the spacelike directions $e_2 \pm e_3$, which are the intersections of $*U_1$ with $U_2 \pm U_3$. So, the spacetime is SWE.

The congruences satisfying $W(v; v) = 0$ follow easily just taking into account that $\mathrm{i}\,\mathcal{W} = \sum_{j=1}^{3}(\mathrm{i}\,\rho_j)\mathcal{U}_j \otimes \mathcal{U}_j$, and so we get the same conditions as in the proposition above, just making the substitution $\rho_j \hookrightarrow \mathrm{i}\,\rho_j$. So, we have

Proposition 12. *A type I spacetime is WM if, and only if, one of the following conditions hold:*

(1) The eigenvalues are imaginary.
(2) Two of the eigenvalues are anticonjugated.

If condition (1) holds the solutions to $W(v; v) = 0$ *are the timelike Weyl principal direction* e_0 *and the spacelike Weyl principal directions* e_i*. So, the spacetime is TWM and SWM.*

If condition (2) holds, say $\rho_2 = -\bar{\rho}_3$*, the solutions are the spacelike directions* $e_2 \pm e_3$*, which are the intersections of* $*U_1$ *with* $U_2 \pm U_3$*. So, the spacetime is SWM.*

4. Intrinsic characterization of Weyl-electric and Weyl-magnetic spacetimes

Once the Weyl-electric and Weyl-magnetic directions have been found for an arbitrary Weyl tensor by considering the different Petrov types, in this section we summarize and analyse the results and we present them by means of a new structure which allows us to characterize the spacetime families that this study has outlined.

We begin analysing links between the different Weyl-electric and Weyl-magnetic concepts. A simple overview of the propositions from the previous section leads to the following statements.

Corollary 1. *If a metric is TWE (TWM) then it is SWE (SWM). If a metric is NWE (NWM) then it is SWE (SWM).*

Then, as a consequence of these results, the following equivalences hold.

Corollary 2. *Every WE (WM) spacetime is SWE (SWM).*

On the other hand, the relation of the Weyl-electric and Weyl-magnetic directions with the Weyl tensor geometry is also explained in the results of section 3. From these we can easily show

Corollary 3. *The timelike Weyl-electric (Weyl-magnetic) directions are principal directions of the Weyl tensor. The spacelike and null Weyl-electric (Weyl-magnetic) directions of a TWE (TWM) metric are also principal directions.*

Corollary 4. *Every null Weyl-electric (or Weyl-magnetic direction) is a multiple Debever direction.*

The quadruple (triple) Debever direction of a type N (III) spacetime is always a Weyl-electric and a Weyl-magnetic direction.

A double Debever direction of a type D or type II spacetime is a Weyl-electric (Weyl-magnetic) direction if, and only if, the Weyl tensor has real (imaginary) eigenvalues.

These results, and in particular corollary 4, imply that, in the end, three families of purely electric (magnetic) spacetimes can be considered. We have the family of all the WE (WM) metrics (or, equivalently, the SWE (SWM) metrics), and two outlined subfamilies of it: the TWE (TWM) metrics and the NWE (NWM) metrics. An important remark is that these subfamilies have non vacuum intersection. More precisely, we have

Proposition 13. *A metric is simultaneously TWE and NWE (and so SWE) if, and only if, it defines a type D spacetime with real eigenvalues. The explicit statement of this characterization is written in terms of the Weyl tensor:*

$$S^2 + S = 0, \qquad S \equiv \frac{1}{3\rho}(W - \rho G), \qquad \rho \equiv -\left(\frac{1}{12} tr\, W^3\right)^{\frac{1}{3}} \neq 0.$$

A metric is simultaneously TWM and NWM (and so SWM) if, and only if, it defines a type D spacetime with imaginary eigenvalues. The explicit statement of this characterization is written in terms of the Weyl tensor:

$$S^2 + *S = 0, \qquad S \equiv \frac{1}{3\lambda}(W - \lambda\eta), \qquad \lambda \equiv -\left(\frac{1}{12} tr * W^3\right)^{\frac{1}{3}} \neq 0.$$

On the other hand, these subfamilies do not cover all the WE (WM) metrics. This means that there are *strict spacelike Weyl-electric (magnetic)* metrics, that is, SWE (SWM) metrics without timelike or null Weyl-electric (magnetic) directions. More specifically, we have

Proposition 14. *The only strict SWE (SWM) metrics are the type I spacetimes with two conjugated (anticonjugated) eigenvalues, say ρ_2 and ρ_3. The explicit statement in terms of the Weyl tensor of this condition is: M is real negative or infinite and b is real (imaginary).*

The spacelike Weyl-electric (Weyl-magnetic) directions are the bisectors $e_2 \pm e_3$ of the principal directions associated with the conjugated (anticonjugated) eigenvalues.

Now we look for an intrinsic characterization of the above mentioned families, namely the WE (WM), the TWE (TWM) and the NWE (NWM) spacetimes. The case of the TWE (TWM) metrics is already known [3, 7] and a characterization theorem has been stated in section 2 (theorem 1). Here, we can improve the knowledge of this case with a property that follows from sections 3.3 and 3.5 and corollary 3.

Proposition 15. *A type I or a type D spacetime is TWE (TWM) if, and only if, there exists a Weyl-electric (Weyl-magnetic) principal direction. Then, every principal direction is a Weyl-electric (Weyl-magnetic) direction.*

Also for the widest family we know a necessary condition (proposition 1). It is not difficult to prove that this condition is also sufficient, and other equivalent conditions can also be stated. Indeed, a direct analysis of the results in the previous section for the different Petrov types shows that the WE (WM) condition is equivalent to the eigenvalues being either real (imaginary) or two of them conjugated (anticonjugated). Moreover, a is real and b is real (imaginary) as stated in proposition 1. Condition a, if real can be changed for M if real or infinity. Finally, if a is real and b is real (imaginary), the characteristic equation (5) admits, at least, a real (imaginary) eigenvalue, the other two being then either real (imaginary) too or conjugated (anticonjugated). Thus, we can state:

Theorem 2. *A non-conformally flat spacetime is WE (WM) if, and only if, the Weyl tensor satisfies one of the following equivalent conditions:*

(i) Either the eigenvalues are real (imaginary) or two of them are conjugated (anticonjugated).
(ii) The scalar invariants a and b (ib) are real.
(iii) The adimensional invariant M is real or infinite, and b is real (imaginary).

Moreover, the WE or WM metrics can also be characterized at once by M to be real or infinite and a to be real.

On the other hand, the characterization for the NWE (NWM) metrics follows from corollary 4.

Theorem 3. *A spacetime is NWE (NWM) if, and only if, the Weyl tensor is algebraically special with real (imaginary) eigenvalues, or explicitly in terms of the Weyl tensor, if and only if,* $a^3 = 6b^2$*, and b is real (imaginary).*

The null Weyl-electric (Weyl-magnetic) directions are the multiple Debever directions.

Let us observe that spacetimes exist with both purely electric and purely magnetic properties. This fact is evident for type N and type III metrics. But we can also remark on the case of type I metrics with real (imaginary) eigenvalues, one of them being null ($b = 0$). Then there are two anticonjugated (conjugated) eigenvalues. Consequently, we can state:

Proposition 16. *The necessary and sufficient conditions for a spacetime to be both WE and WM are* $b = 0$ *and* $a \in \mathbb{R}$.

When $a = 0$*, that is in type N or type III metrics, the multiple Debever direction is a Weyl-electric and a Weyl-magnetic direction.*

When a is real positive, that is, the eigenvalues are real and two of them satisfy $\rho_2 = -\rho_3$*, the principal directions* e_α *are Weyl-electric directions and* $e_2 \pm e_3$ *are Weyl-magnetic directions.*

When a is real negative, that is, the eigenvalues are imaginary and two of them satisfy $\rho_2 = -\rho_3$*, the principal directions* e_α *are Weyl-magnetic directions and* $e_2 \pm e_3$ *are Weyl-electric directions.*

In order to improve the understanding of the purely electric and purely magnetic spacetimes, and to put in place the new families introduced here with respect to the older ones, it could be worth analysing a scalar invariant diagram. From theorem 2 it follows that the WE or WM metrics have, at most, two non-null real Weyl scalar invariants. Thus, a bidimensional diagram can be built. If we look for two homogeneous invariants we can choose b^2 and $a^3 - 6b^2$, the adimensional invariant M determining then the straightlines which focus on the origin (see figure 1). The first and the third quadrant ($M > 0$) correspond, respectively, to the older purely electric and magnetic concepts, that is, the TWE and TWM spacetimes. On the other hand, on the second and the fourth quadrants ($M < 0$) lie, respectively, the new strict (s) SWM and SWE spacetimes. On the horitzontal axis ($M = 0$) we can find algebraically special metrics, the NWE metrics on the positive part and the NWM on the negative part. Type N and type III metrics are both NWE and NWM and they are placed on the origin. Type D or type II metrics with real (imaginary) eigenvalues cover the positive (negative) horizontal axis outside the origin. This axis belongs only partially to first or third quadrants, because type II metrics are not TWE or TWM. Finally, every point outside the origin on the vertical axis ($M = \infty$) can be considered as belonging to the two adjoining quadrants. The positive branch is TWE and strict SWM, and the negative branch is TWM and strict SWE.

5. Weyl-electric and Weyl-magnetic type I metrics

In a Petrov type I spacetime, four simple Debever null directions exist. They can be put together two by two in three different ways, thus defining three pairs of *unitary Debever bivectors* [16]. Let $\mathcal{V}_\epsilon$, $\epsilon = \pm 1$ be one of these pairs. Elsewhere we have shown that $\mathcal{V}_\epsilon$ lay on the 2-plane orthogonal to a canonical bivector $\mathcal{U}_i$, say $i = 1$. In terms of the canonical bivectors $\mathcal{U}_2$ and $\mathcal{U}_3$ they are written as [17]

$$\mathcal{V}_\epsilon = \cos\Omega\, \mathcal{U}_2 + \epsilon \sin\Omega\, \mathcal{U}_3, \qquad \cos 2\Omega = \frac{3\rho_1}{\rho_3 - \rho_2}. \tag{19}$$

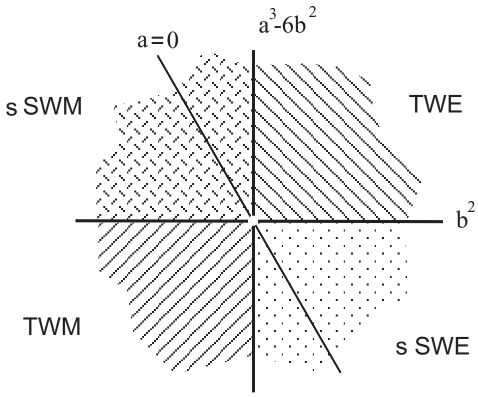

Figure 1. Scalar invariants diagram.

Thus, the complex invariant Ω is the angle in the bivectors space between the unitary Debever bivectors $\mathcal{V}_\epsilon$ and the canonical bivector $\mathcal{U}_2$. If we put $\Omega \equiv \phi - \mathrm{i}\psi$, we have

$$\cos 2\Omega = \cos 2\phi \cosh 2\psi - \mathrm{i} \sin 2\phi \sinh 2\psi. \tag{20}$$

Then, a general expression for the Debever null vectors in terms of the type I Weyl canonical frame $\{e_\alpha\}$ follows from (19) and (20) [17]:

$$l_{\epsilon\pm} = \cosh\psi\, e_0 + \epsilon \sinh\psi\, e_1 \pm \cos\phi\, e_2 \pm \epsilon \sin\phi\, e_3, \qquad (\epsilon = \pm 1). \tag{21}$$

Now we analyse the properties of the Debever directions of a WE or WM type I spacetime. We start with the TWE or TWM metrics, that is, when the invariant M is positive or infinite (the first and third quadrants in figure 1). This case corresponds to a Weyl tensor with real or imaginary eigenvalues as a consequence of theorem 1. Then, a shorter eigenvalue exists, say ρ_1, and the ratios $\frac{\rho_i}{\rho_j}$ are real or infinite. Consequently, taking into account the second expression in (19), $\cos 2\Omega$ is a real number with modulus smaller than the unity. So, from (20), $\psi = 0$ and the Debever directions (21) are written as

$$l_{\epsilon\pm} = e_0 \pm \cos\phi\, e_2 \pm \epsilon \sin\phi\, e_3, \qquad (\epsilon = \pm 1). \tag{22}$$

One can notice that the four Debever directions lie on the 3-plane orthogonal to e_1 and the invariant $\Omega = \phi$ may be interpreted as the angle between the principal direction e_2 and the projection on the space orthogonal to e_0 of every Debever direction. From this analysis we recover a known result [3, 4, 7].

Proposition 17. *In a TWE or TWM type I spacetime the Debever directions span the 3-plane orthogonal to the principal direction associated with the shortest eigenvalue of the Weyl tensor.*

It is worth remarking on the two possible limits, which may be considered in the family of the TWE or TWM metrics. From one side, if we force ϕ to take the values 0 or $\pi/2$, the four Debever directions become two double ones, and the metric is Petrov type D (horizontal axis in figure 1). On the other side, in the case $\phi = \pi/4$, the shortest eigenvalue becomes zero and so $b = 0$ (vertical axis in figure 1).

Let us go on now to the strict SWE or SWM (type I) metrics. Then, it follows from proposition 4 that the Weyl tensor has two conjugated or anticonjugated eigenvalues, say ρ_2 and ρ_3, and the invariant M is negative or infinite (second and fourth quadrants in figure 1).

Then, ρ_1 is real or imaginary, and the second expression in (19) shows that $\cos 2\Omega$ is imaginary. So from (20) $\phi = \pi/4$ and, consequently, the four Debever vectors are written as

$$l_{\epsilon\pm} = \cosh\psi\, e_0 + \epsilon \sinh\psi\, e_1 \pm \frac{1}{\sqrt{2}}(e_2 + \epsilon e_3). \tag{23}$$

In this case the projections of the Debever null vectors on the principal 2-plane $\{e_2, e_3\}$ are the bisectors of the principal directions e_2 and e_3. This fact implies that the frame built with the four Debever vectors (23) has an interesting property of symmetry. Indeed, the products two by two of these vectors give

$$\begin{aligned} &(l_{++}, l_{+-}) = (l_{-+}, l_{--}) = -2 \\ &(l_{++}, l_{-+}) = (l_{+-}, l_{--}) = (l_{++}, l_{--}) = (l_{+-}, l_{-+}) = -\cosh 2\psi. \end{aligned} \tag{24}$$

So, the vectors $l_{\epsilon+}$ and $l_{\epsilon-}$ are metrically indistinguishable. Moreover, when $a = 0$, that is, on the straightline $M = -6$ in figure 1, in addition one has $\cosh 2\psi = 2$. So, from (24), the four Debever vectors are metrically equivalent and they define a null symmetric frame [23] given by

$$l_{\epsilon\pm} = \sqrt{3}e_0 \pm (e_1 + \epsilon e_2) + \epsilon e_3. \tag{25}$$

These considerations lead to the following statement.

Proposition 18. *In a strict SWE or SWM (type I) spacetime the Debever null vectors define a partially symmetric frame, being metrically equivalent two by two. Moreover, they define a symmetric frame when $a = 0$.*

Finally let us consider now some examples of WE and WM metrics. In the introduction we have commented on the wide family of known TWE solutions which contains the static ones. One non-static example is the Kasner metric, general solution for vacuum equations with a three-dimensional Abelian group of isometries on the spacelike orbits S_3 [28, 29]:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{2\alpha}\,\mathrm{d}x^2 + t^{2\beta}\,\mathrm{d}y^2 + t^{2\gamma}\,\mathrm{d}z^2 \tag{26}$$

where the constant a, b and c are submitted to the restrictions $\alpha^2 + \beta^2 + \gamma^2 = \alpha + \beta + \gamma = 1$. The analysis of the Weyl invariants by McIntosh *et al* [7] shows that (26) is a TWE metric. Here we only want to point out that $\partial_x, \partial_y, \partial_z$ are the hypersurface-orthogonal Killing vectors that determine the three spacelike principal directions. So, they must be Weyl-electric directions accordingly with proposition 2 and, taking into account corollary 3 and proposition 15, the metric is TWE and the timelike principal vector ∂_t determines the Weyl-electric observer.

We have also observed in the introduction that there are few known TWM solutions, and some results suggest the non-existence under determined conditions [7, 13]. A summary of the known cases can be found in [15]. Here we glance at the stiff matter solution by Lozanowski and McIntosh [14]:

$$\mathrm{d}s^2 = (ty)^{1-r^2}[-\mathrm{d}t^2 + \mathrm{d}y^2] + ty[(ty)^r\,\mathrm{d}x^2 + (ty)^{-r}\,\mathrm{d}z^2] \tag{27}$$

where $r = \sqrt{2/3}$. The invariant M is infinite ($b = 0$) and a is real. Consequently, it is a TWM (TWE) metric in the region where $a < 0$ ($a > 0$) as the authors claim, but from propositions 11, 12 and 16, it is also a strict SWE (SWM) in the same region.

An example of Riemann-magnetic spacetime given by Misra *et al* has as line element [6, 7]

$$\mathrm{d}s^2 = \mathrm{d}x^2 - \sqrt{2}\,\mathrm{e}^{k(x-y)}\,\mathrm{d}y\,\mathrm{d}z - \sqrt{2}\,\mathrm{e}^{-k(x-y)}\,\mathrm{d}y\,\mathrm{d}w - 2\,\mathrm{d}z\,\mathrm{d}w \tag{28}$$

where k is a real constant. It has been pointed out [7] that in spite of its Riemann-magnetic character, the metric (28) does not define a TWM spacetime because the invariant M is negative. But, as in this case b is a real invariant, (28) is a strict SWE metric as a consequence of proposition 14.

The general type I vacuum solution admitting a four-dimensional group of isometries is the Petrov metric [30, 29]:

$$k^2\,\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{e}^{-2x}\,\mathrm{d}y^2 + \mathrm{e}^x[\cos\sqrt{3}x\,(\mathrm{d}z^2 - \mathrm{d}t^2) - 2\sin\sqrt{3}x\,\mathrm{d}z\,\mathrm{d}t] \tag{29}$$

where k is a real constant. This metric has been characterized elsewhere [24] as the only type I vacuum solution with constant eigenvalues. The Weyl tensor has eigenvalues proportional to the three cubic roots of -1:

$$\rho_j = -k^2\,\mathrm{e}^{\mathrm{i}\frac{\pi}{3}(1+2\mathrm{j})}, \qquad j = 1, 2, 3. \tag{30}$$

Consequently the Weyl invariants are $a = 0$ and $b = -3k^6 < 0$. So, it is a strict SWE metric, placed in figure 1 at the fourth quadrant and on the line $a = 0$. From (29), it is clear that ∂_y is a vorticity-free Killing field and so it defines a spacelike Weyl-electric direction collineal with one of the bisectors $e_2 \pm e_3$.

To finish we would like to mention a class of metrics given by Ariarnhod and McIntosh as having degenerate Debever null directions [22]. The Weyl invariants a and b are real, and so M is also real. Then, theorem 2 implies that these metrics define WE spacetimes. Moreover, arbitrary functions on the coordinates appear in the metric canonical form. Then, it seems the different classes of WE metrics (TWE or strict SWE) are allowed depending on the sign of M. We have shown in this section that all the WE or WM spacetimes have degenerate Debever vectors, but it is worth mentioning that the converse property does not hold. Indeed, the Debever degeneracy is characterized by the invariant M to be real, and this is possible with a and b taking no real values.

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