JOURNAL OF MATHEMATICAL PHYSICS VOLUME 45, NUMBER 2 FEBRUARY 2004

On the classification of type D space–times

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(Received 5 November 2002; accepted 12 November 2003)

We give a classification of the type D space–times based on the invariant differential properties of the Weyl principal structure. Our classification is established using tensorial invariants of the Weyl tensor and, consequently, besides its intrinsic nature, it is valid for the whole set of the type D metrics and it applies on both, vacuum and nonvacuum solutions. We consider the Cotton-zero type D metrics and we study the classes that are compatible with this condition. The subfamily of space–times with constant argument of the Weyl eigenvalue is analyzed in more detail by offering a canonical expression for the metric tensor and by giving a generalization of some results about the nonexistence of purely magnetic solutions. The usefulness of these results is illustrated in characterizing and classifying a family of Einstein–Maxwell solutions. Our approach permits us to give intrinsic and explicit conditions that label every metric, obtaining in this way an operational algorithm to detect them. In particular a characterization of the Reissner–Nordström metric is accomplished.
[DOI: 10.1063/1.1640795]

I. INTRODUCTION

Type D space–times have been widely considered in literature and we can point out not only the large number of known families of exact solutions but also the interest of these solutions from the physical point of view. Let us quote, for example, the Schwarszchild or the Kerr metrics which model the exterior gravitational field produced, respectively, by a nonrotating or a rotating spherically symmetric bounded object. Or also the related metrics in the case of a charged object, the Reissner–Nordström or the Kerr–Newman solutions. However, although some classes of type D metrics have been considered taking into account algebraic properties of the Weyl eigenvalue or differential conditions on the null Weyl principal directions, a classification of the type D solutions involving all the first-order differential properties of the Weyl tensor geometry is a task which has not been totally accomplished yet. In this work we present this classification of the type D metrics and we show the role that it can play in studying geometric properties of known space–times, in looking for new solutions of Einstein equations or in offering new elements which allow us to give intrinsic and explicit characterizations of all these space–times.

At an algebraic level, a type D Weyl tensor determines a complex scalar invariant, the eigenvalue, and a $2+2$ almost-product structure defined by its principal 2–planes. Some classes of type D metrics can be considered by imposing the real or imaginary nature of the Weyl eigenvalue. In this way we find the so-called purely electric or purely magnetic space–times. The purely electric character often appears as a consequence of usual geometric or physical restrictions.[1] This is the case of the static type D vacuum spacetimes found by Ehlers and Kundt,[2] or the Barnes degenerate perfect fluid solutions with shear-free normal flow.[3] On the other hand, some restrictions are

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known on the existence of purely magnetic solutions.[4,5] A wide bibliography about Weyl-electric and Weyl-magnetic space–times can be found in a recent work[6] where these concepts have been generalized.

The most usual approaches to look for exact solutions of the Einstein equations work in frames or local coordinates adapted to some outlined direction of the curvature tensor. For example, in the case of perfect fluid solutions or static metrics the $3+1$ formalism adapted, respectively, to the fluid flow or to the normal timelike Killing vector can be useful. Sometimes one considers that some of the kinematic coefficients associated with the unitary vector are zero. This means that one is searching for new solutions belonging to a class of metrics that are defined by first-order differential conditions imposed on the curvature tensor. A similar situation appears when local coordinates adapted to the multiple Debever direction are considered when looking for algebraically special solutions. Indeed, if the hypotheses of the generalized Goldberg–Sachs theorem hold, the multiple Debever direction defines a shear-free geodesic null congruence. In this case, or when considering nondiverging or nontwisting restrictions on a Debever direction, we are imposing differential conditions on the Weyl tensor.

It is worth pointing out that the kinematic coefficients associated with a unitary vector completely determine the first-order differential properties of the $1+3$ almost-product structure that it defines. Nevertheless, the conditions usually imposed on the two double Debever directions of a type D space–time do not cover all the differential properties of the principal $2+2$ almost-product structure of the Weyl tensor exhaustively. The first goal of this work is to offer a classification of the type D metrics based on all the first-order differential properties of the principal structure, and to reinterpret under this view the usual conditions that can be found in the literature. This classification is not based on the scalar invariants, but on tensorial invariants of the Weyl tensor. These invariants are well adapted to the generic type D metrics, where a Weyl canonical frame is not univocally determined, and where the eigenvalues and the $2+2$ principal structure are the only invariants associated with the Weyl tensor.

The (proper) Riemannian almost-product structures have been classified according the invariant decomposition of their structure tensor,[7] and the classes have been interpreted in terms of the foliation, minimal and umbilical properties.[8] This classification can be generalized to the space–time structures by also considering the causal character of the planes.[9] Almost-product structures have shown their usefulness in studying the underlying geometry of some physical fields. The $1+3$ structures are frequently used in relativity and sometimes the properties of a physical field can be expressed in terms of the kinematic properties of a unitary vector.[10,11] On the other hand, the $2+2$ structure associated with a regular solution of Maxwell equations[12] is a basic concept in building the "already unified theory" for the electromagnetic field.[13] It has also allowed a geometric interpretation[14] of the Teukolsky–Press relations[15] used in analyzing incident electromagnetic waves on a Kerr black hole.

In General Relativity we can also find almost-product structures attached to the geometric or physical properties of the spacetime. Indeed, some energy contents (for example, in the Einstein–Maxwell or perfect fluid solutions) define underlying structures that restrict, via Einstein equations, the Ricci tensor. On the other hand, the Weyl tensor also defines almost-product structures associated with its principal bivectors depending on the different Petrov types.[16] These structures determine the Weyl canonical frames.[17] In the type D case, only the *principal structure* is outlined.

Until now we have mentioned two different ways of classifying the type D space–times: The first one is strictly algebraic and takes into account the real or imaginary character of the Weyl eigenvalues; the second one, which we will present here, involves differential conditions of the $2+2$ principal structure, that is, on the Weyl eigenvectors. Nevertheless, there is a third natural manner to impose restrictions on the type D metrics: To take into account the relative position between the principal 2–planes and the gradient of the Weyl scalar invariants. This is a mixed classification, differential in the eigenvalues and algebraic in the principal structure, which affords 16 different classes of type D metrics. In this work we will show the marked relation that exists between this classification and the two previous ones.

A classification of type D space–times taking into account the properties of the $2+2$ principal structure shows quite interesting advantages. Indeed, the integration of the static type D vacuum equations using an alternative approach based on the Weyl principal structure has allowed us to complete the results by Ehlers and Kundt[2] in order to accomplish an algorithmic and intrinsic identification of the solutions and, in particular, to obtain the equations that define the Schwarzschild space–time explicitly.[18] Moreover, our classification affords a geometric interpretation of the other families of vacuum type D solutions. Starting from this approach two Killing vectors can be determined in terms of Weyl concomitants,[19] a result which shows that a commutative bidimensional group of isometries exists. Although all the type D vacuum solutions were found by Kinnersley[20] a integration method based on our classification permits their intrinsic label, as well as a geometric interpretation of the NUT and acceleration parameters.[21]

In this work we apply our classification to the study of space–times with zero Cotton tensor. For them, the Bianchi identities impose the same restrictions on the Weyl tensor as the vacuum condition. We interpret these restrictions in terms of geometric properties of the principal structure and we show that the compatible classes can be characterized in terms of the relative position between the gradient of some invariant scalars and the principal 2–planes. From a physical point of view these metrics have two interesting properties. Firstly, the two double Debever directions define shear-free geodesic null congruences and, secondly, the principal structure is Maxwellian. This result can be of interest in order to generalize the Teukolsky–Press relations[14,15] and their applications to type D nonvacuum solutions.

In order to show the usefulness of this approach in analyzing properties of known metrics, in integrating Einstein equations and in labeling the solutions, here we study the space–times with the two properties quoted above for the particular classes with integrable structure. In this case, the space–time metric turns out to be conformal to a product metric. Then, as a first consequence, we extend the result by Hall[4] (see also McIntosh *et al.*[5]) concerning the nonexistence of purely magnetic type D vacuum solutions in a double sense: The family of solutions where the new result applies is wider than the vacuum metrics, and the purely magnetic restriction is weakened to an arbitrary constant argument. Elsewhere[22] we have acquired a similar extension for some results concerning the purely magnetic type I solutions. Moreover, starting from a canonical form we begin on the integration of the Einstein–Maxwell equations for the compatible classes, and we recover the charged counterpart of the A, B, C vacuum metrics by Ehlers and Kundt. The integration method at once provides an algorithm to detect the solutions with intrinsic and explicit conditions and, in particular, it offers a characterization of the Reissner–Nordström space–time. The classification of the Kinnersley rotating type D vacuum solutions will be considered elsewhere.[21]

The paper is organized as follows. In Sec. II we introduce some definitions and notations and we give some results about $2+2$ almost-product structures. In Sec. III we present the classification of the type D metrics based on the first-order differential properties of the Weyl principal structure, as well as the mixed classification involving the eigenvalues gradient and the principal structure. The Cotton-zero type D metrics are analyzed in Sec. IV, and we show that the principal 2–planes define an umbilical structure and, consequently, we only have 16 compatible classes which coincide precisely with those defined by the mixed classification. The four classes with integrable structure are studied in detail in Sec. V: We present a canonical form for them and generalize a result about the nonexistence of purely magnetic solutions. Finally, in Sec. VI, we apply our results to recover a family of Einstein–Maxwell solutions, to give an operational algorithm to detect them and to explicitly and intrinsically characterize the Reissner–Nordström space–time. Some of the results in this paper were communicated without proof at the Spanish Relativity Meeting–96.[19]

II. SPACE–TIME ALMOST-PRODUCT STRUCTURES

On a Riemannian manifold (M,g) an almost-product structure is defined by a p-plane field V and its orthogonal complement H. Let v and $h=g-v$ the respective projectors, and let Q_v be the (2,1)-tensor:

$$Q_v(x,y)=h(\nabla_{vx}vy), \quad \forall\ x,y. \tag{1}$$

Let us consider the invariant decomposition of Q_v into its antisymmetric part A_v and its symmetric part $S_v \equiv S_v^T + (1/p)\, v \otimes \mathrm{Tr} S_v$, where S_v^T is a traceless tensor:

$$Q_v=A_v+\frac{1}{p}v\otimes \mathrm{Tr}\ S_v+S_v^T. \tag{2}$$

The plane V is foliation if, and only if, $A_v=0$. In this case $Q_v=S_v$ and it coincides with the second fundamental form of the integral manifolds of the foliation V.[23] Moreover V is minimal, umbilical or geodesic if, and only if, $\mathrm{Tr}\ S_v=0$, $S_v^T=0$ or $S_v=0$, respectively. Then one can generalize these geometric concepts for plane fields which are not necessarily foliation:

Definition 1: *A plane field V is said to be geodesic, umbilical or minimal if the symmetric part* S_v *of its (generalized) second fundamental form* Q_v *satisfies, respectively*, $S_v=0$, $S_v^T=0$ *or* $\mathrm{Tr}\ S_v=0$.

The (proper) Riemannian almost-product structures (V,H) have been classified taking into account the invariant decomposition (2) of the tensors Q_v and Q_h or, equivalently, according with the foliation, minimal, umbilical, or geodesic character of each plane.[7,8] Some of these properties have also been interpreted in terms of invariance along vector fields.[24] A generalization for the spacetime structures follows taking into account the causal character of the planes. We will say that a structure is integrable when both planes are foliation and we will say that it is minimal, umbilical or geodesic if both of the planes are so.

This way, on an oriented space–time (V_4,g) of signature $(-+++)$ we have generically $2^6=64$ different classes of (almost-product) structures depending on the first-order geometric properties. Nevertheless, when $p=1$, V is always an umbilical foliation and, consequently, only 16 possible classes exist. In this case Q_v and Q_h depend on the kinematic coefficients associated with a unitary vector u, and the classes are defined by the vanishing or nonvanishing of the acceleration, rotation, shear, and expansion. Elsewhere this kinematical interpretation has been extended to the $2+2$ space–time structures and, as a consequence, the Maxwell–Rainich equations have been expressed in terms of kinematical variables.[9]

In order to be used in next sections, we now analyze the space–time $2+2$ almost-product structures in detail by giving the characterization of their properties in terms of their canonical 2–form U, and by showing their relation with other usual approaches, the Newmann–Penrose and the self-dual formalisms. We also study the change of these properties for a conformal transformation and we summarize some results about Maxwellian structures.

A. 2+2 structures

In the case of a $2+2$ space–time structure it is useful to work with the *canonical* unitary 2-form U, volume element of the time-like plane V. Then, the respective projectors are $v=U^2$ and $h=-(*U)^2$, where $U^2=U\times U=\mathrm{Tr}_{23}\ U\otimes U$ and $*$ is the Hodge dual operator.

The tensors Q_v and Q_h determine the derivatives of the volume elements U and $*U$ by means of

$$\nabla_\alpha U_{\beta\lambda}=(Q_v)_{\alpha\mu,[\beta}U^\mu{}_{\lambda]}+(Q_h)_{\alpha[\beta,}{}^{\mu}\ U_{\lambda]\mu},$$
$$\nabla_\alpha *U_{\beta\lambda}=(Q_h)_{\alpha\mu,[\beta}*U^\mu{}_{\lambda]}+(Q_v)_{\alpha[\beta,}{}^{\mu}*U_{\lambda]\mu}. \tag{3}$$

Then, if we denote $\delta=-\mathrm{Tr}\ \nabla$, a straightforward calculation leads to

$$\delta U=i(\mathrm{Tr}\ S_h)U-2(U,A_v) \qquad \delta *U=i(\mathrm{Tr}\ S_v)*U-2(*U,A_h), \tag{4}$$

where $2(U,A_v)^\mu=U^{\alpha\beta}(A_v)_{\alpha\beta}{}^{\mu}$. So, the minimal and the foliation character of the planes can be stated in terms of the projections of δU and $\delta *U$ onto V and H. On the other hand, let us consider

$$G_\perp = U\otimes U - *U\otimes *U + G; \quad \eta_\perp = U\tilde{\otimes} *U + \eta, \tag{5}$$

where η is the metric volume element of the space–time, $G=\frac{1}{2}g \owedge g$ is the metric on the 2–forms space, and $\owedge$ denotes the double-forms exterior product, $(A\owedge B)_{\alpha\beta\mu\nu}=A_{\alpha\mu}B_{\beta\nu}+A_{\beta\nu}B_{\alpha\mu}-A_{\alpha\nu}B_{\beta\mu}-A_{\beta\mu}B_{\alpha\nu}$. The tensors (5) satisfy $G_\perp(U)=G_\perp(*U)=0$, $\eta_\perp(U)=\eta_\perp(*U)=0$ and they can be calculated as

$$G_\perp = v\owedge h, \quad \eta_\perp = U\owedge *U. \tag{6}$$

Then, from expressions (3) and (4) we get

$$(2\nabla U - K)_{\lambda\alpha\beta} = (S_v^T)_{\lambda\mu,[\alpha} \; U^\mu{}_{\beta]} + (S_h^T)_{\lambda\mu,[\alpha} \; *U^\mu{}_{\beta]}, \tag{7}$$

$$K \equiv i(\delta U)G_\perp - i(\delta *U)\eta_\perp, \tag{8}$$

and so, the umbilicity of each plane is equivalent to the vanishing of the respective projections of the first member of (7). We summarize these results in the following lemma:

Lemma 1: *Let (V,H) be a $2+2$ almost-product structure and let U be its canonical 2-form. Then, the following conditions hold*:

(1) V (resp. H) is foliation $\Leftrightarrow i(\delta U)*U=0$ (resp. $i(\delta *U)U=0$);
(2) V (resp. H) is minimal $\Leftrightarrow i(\delta *U)*U=0$ (resp. $i(\delta U)U=0$);
(3) V is umbilical $\Leftrightarrow U\times\{2\nabla U-[i(\delta U)G_\perp - i(\delta *U)\eta_\perp]\}=0$
H is umbilical $\Leftrightarrow *U\times\{2\nabla U-[i(\delta U)G_\perp - i(\delta *U)\eta_\perp]\}=0$.

A $2+2$ structure is also determined by the two null directions $l_\pm$ on the plane V. A family of complex null bases $\{l_+, l_-, m, \overline{m}\}$ exists such that $U=l_-\wedge l_+$. This family is fixed up to change $l_\pm \hookrightarrow e^{\pm\phi}l_\pm$, $m\hookrightarrow e^{i\theta}m$. Then, conditions of lemma 1 can be interpreted in terms of the Newman–Penrose coefficients[25] as

Lemma 2: *Let $U=l_-\wedge l_+$ be the canonical 2–form of a $2+2$ structure. It holds*:

(1) The plane V is umbilical iff $\kappa=0=\nu$;
(2) the plane H is umbilical iff $\lambda=0=\sigma$;
(3) the plane V is minimal iff $\overline{\pi}=\tau$;
(4) the plane H is minimal iff $\rho+\overline{\rho}=0=\mu+\overline{\mu}$;
(5) the plane V is a foliation iff $\overline{\pi}=-\tau$;
(6) the plane H is a foliation iff $\rho-\overline{\rho}=0=\mu-\overline{\mu}$.

Taking into account the significance of the NP coefficients[25] this lemma implies that the umbilical nature of a $2+2$ structure means that its principal directions $l_\pm$ define shear-free geodesic null congruences. The minimal or foliation character of the spacelike 2-plane have also a kinematical interpretation and state, respectively, that both principal directions are expansion-free or vorticity-free. Elsewhere[9] all the geometric properties have been interpreted in terms of kinematic coefficients associated with every direction in a 2–plane (not only the null ones) with respect to the other 2–plane.

When both planes have a specific differential property, it is more convenient to introduce the self-dual unitary 2–form $\mathcal{U}\equiv(1/\sqrt{2})(U-i*U)$ associated with U. We have

$$2\,\mathrm{Re}[i(\delta\mathcal{U})\mathcal{U}] = i(\delta U)U - i(\delta *U)*U \equiv \Phi(U),$$

$$2\,\mathrm{Im}[i(\delta\mathcal{U})\mathcal{U}] = -i(\delta U)*U - i(\delta *U)U \equiv \Psi(U). \tag{9}$$

So, the complex 1–form $\delta\mathcal{U}$ collects the information about the minimal and foliation character of the structure. On the other hand, if $\mathcal{G}=\frac{1}{2}(G-\mathrm{i}\eta)$ is the metric on the self-dual 2–forms space, and $\mathcal{K}\equiv(1/\sqrt{2})(K-\mathrm{i}*K)$ is the self-dual 2-form associated to the vector valued 2–form K given in (8), we have

$$\mathcal{K}=i(\delta\mathcal{U})[\mathcal{U}\otimes\mathcal{U}+\mathcal{G}]. \tag{10}$$

Consequently, from lemma 1 and Eqs. (9) and (10), we have

Lemma 3: *Let us consider the* $2+2$ *structure defined by* $\mathcal{U}=(1/\sqrt{2})(U-\mathrm{i}*U)$. *It holds*:

(1) The structure is minimal if, and only if, $\mathrm{Re}[i(\delta\mathcal{U})\mathcal{U}]=0$;
(2) the structure is integrable if, and only if, $\mathrm{Im}[i(\delta\mathcal{U})\mathcal{U}]=0$;
(3) the structure is umbilical, if, and only if, $\nabla\mathcal{U}=i(\delta\mathcal{U})[\mathcal{U}\otimes\mathcal{U}+\mathcal{G}]$.

B. Almost-product structures and conformal transformations

If (V,H) is a $p+q$ almost-product structure for a metric g, then (V,H) is also an almost-product structure for every conformal metric $\hat{g}=e^{2\lambda}g$, and the projectors are related by the conformal factor: If $g=v+h$, then $\hat{g}=\hat{v}+\hat{h}$, where $\hat{v}=e^{2\lambda}v$, $\hat{h}=e^{2\lambda}h$. The generalized second fundamental form change as

$$Q_{\hat{v}}=e^{2\lambda}\,(Q_v-v\otimes h(\mathrm{d}\lambda)). \tag{11}$$

So, the foliation and the umbilical character are conformal invariants, but the minimal character is not. Indeed, taking the trace of the expression above, we have

$$\mathrm{Tr}\ Q_{\hat{v}}=\mathrm{Tr}\ Q_v-p h(\mathrm{d}\lambda). \tag{12}$$

These expressions immediately lead to the following result.

Lemma 4: *Let* (V,H) *be a* $p+q$ *almost-product structure for a metric* $g=v+h$. *The structure* (V,H) *is minimal for a conformal metric* $\hat{g}=e^{2\lambda}g$ *if, and only if,*

$$\frac{1}{p}\mathrm{Tr}\ Q_v+\frac{1}{q}\mathrm{Tr}\ Q_h=\mathrm{d}\lambda. \tag{13}$$

If $p=q$ (as happens for the space–time $2+2$ structures), we conclude that the necessary and sufficient condition for a structure to be minimal for a conformal metric is the sum of the traces of the second fundamental forms to be a closed 1-form, $\mathrm{d}(\mathrm{Tr}\ Q_v+\mathrm{Tr}\ Q_h)=0$. Thus, taking into account (4) and the expression (9) for $\Phi(U)$, lemma 4 can be stated for the $2+2$ case as

Lemma 5: *Let U be the canonical 2–form of a* $2+2$ *structure for the space–time metric* g. *The structure is minimal for a conformal metric if, and only if,* $\mathrm{d}\Phi(U)=0$. *More precisely, when this condition hold, let* λ *be such that* $2\mathrm{d}\lambda=\Phi(U)$. *Then, the structure is minimal for the conformal metric* $\hat{g}=e^{2\lambda}g$.

The most degenerated class of almost-product structures are the product ones, which means, those that satisfy $Q_v=0=Q_h$. A metric that admits a product structure is called a product metric. Then, and only then, local coordinates (x^A,x^i), $A=0,1$, $i=2,3$, exist such that $\tilde{g}=\sigma^-+\sigma^+$, being $\sigma^-=\sigma^-_{AB}(x^C)\mathrm{d}x^A\mathrm{d}x^B$ and $\sigma^+=\sigma^+_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j$ bidimensional metrics, hyperbolic and elliptic, respectively. Then, if $\tilde{g}$ is a *2+2* product metric and $g=e^{-2\lambda}\tilde{g}$, lemma 5 and expression (11) lead to the following result.

Lemma 6: *The necessary and sufficient condition for a metric g to be conformal to a product metric* $\tilde{g}$, *is that an integrable and umbilical almost-product structure U exists such that* $\mathrm{d}\Phi(U)=0$. *More precisely, if* $2\mathrm{d}\lambda=\Phi(U)$, *then* $\tilde{g}=e^{2\lambda}g$ *is a product metric*.

C. Maxwellian structures

A regular 2-form F takes the canonical expression $F = e^{\phi}[\cos\psi U + \sin\psi {*U}]$, where U defines the 2+2 associated structure, ϕ is the *energetic index* and ψ is the *Rainich index*. When F is solution of the source-free Maxwell equations, $\delta F = 0$, $\delta {*F} = 0$, one says that U defines a *Maxwellian structure*. In terms of the canonical elements (U, ϕ, ψ), Maxwell equations become:[12,14]

$$\mathrm{d}\phi = \Phi(U) \equiv i(\delta U)U - i(\delta{*U}){*U}, \tag{14}$$

$$\mathrm{d}\psi = \Psi(U) \equiv -i(\delta U){*U} - i(\delta{*U})U. \tag{15}$$

Then, from (14) and (15) the Rainich theorem[12] follows:

Lemma 7: *A unitary 2-form U defines a Maxwellian structure if, and only if, it satisfies*:

$$\mathrm{d}\Phi(U) = 0; \qquad \mathrm{d}\Psi(U) = 0. \tag{16}$$

The Maxwell–Rainich equations (14) and (15) have a simple expression in the self-dual formalism. Indeed, the self-dual 2–form $\mathcal{F} = (1/\sqrt{2})(F - \mathrm{i}{*F})$ may be written as $\mathcal{F} = e^{\phi + \mathrm{i}\psi}\,\mathcal{U}$. Then, from Maxwell equations, $\delta\mathcal{F} = 0$, and taking into account that $2\,\mathcal{U}^2 = g$,

$$\mathrm{d}(\phi + \mathrm{i}\psi) = 2i(\delta\mathcal{U})\mathcal{U}. \tag{17}$$

This last equation is equivalent to (14) and (15) if we take into account (9). Moreover, from here we recover the complex version of (16) easily

$$\mathrm{d}i(\delta\mathcal{U})\mathcal{U} = 0. \tag{18}$$

III. CLASSIFYING TYPE D SPACE–TIMES

The self–dual Weyl tensor $\mathcal{W} = \frac{1}{2}(W - \mathrm{i}{*W})$ of a type D space–time takes the canonical expression[17]

$$\mathcal{W} = 3\alpha\,\mathcal{U}\otimes\mathcal{U} + \alpha\mathcal{G}, \tag{19}$$

where $\alpha = -\,\mathrm{Tr}\,\mathcal{W}^3/\mathrm{Tr}\,\mathcal{W}^2$ is the double eigenvalue and $\mathcal{U}$ is the self-dual principal 2–form. This principal 2–form defines a $2+2$ almost-product structure which is called the *principal structure* of a type D space–time. In terms of the canonical 2–form U of the principal structure the self-dual 2–form $\mathcal{U}$ becomes $\mathcal{U} = (1/\sqrt{2})(U - \mathrm{i}{*U})$. So, at the algebraic level, a type D Weyl tensor only determines the complex scalar α and the principal structure U. Consequently, any generic classification of the type D metrics must depend on these invariants associated with the Weyl tensor.

The families of purely electric or purely magnetic type D spacetimes are defined, at first glance, by means of alternative conditions, namely, the nullity of the magnetic or the electric Weyl fields associated with an observer u. But, actually, they admit a simple intrinsic characterization in terms of the Weyl scalar invariant: The eigenvalue is real or imaginary.[5] In spite of these strong conditions, the family of Weyl-electric type D space–times contains quite interesting solutions. We can quote, for example, the static vacuum metrics[2] or the degenerate perfect fluids with shear-free normal flow.[3] All the type D silent universes are also known[26,27] as well as other families of purely electric type D perfect fluid solutions.[28,30] Nevertheless, few Weyl-magnetic type D solutions have been found,[29] and some restrictions about their existence are known. Indeed, there are not vacuum metrics with purely magnetic type D Weyl tensor.[4,5] The classification that we present below allows us to give an extension of this result in Sec. 5. On the other hand, the generalization of the purely electric or magnetic concepts to the spacelike or null directions does not afford new classes in the type D case.[6]

But the purely electric or magnetic properties define very narrow subsets of the generic type D metrics because they impose one of the two real scalar invariants to be zero. The large family of known solutions of the Einstein equation recommends us to consider other classifications, based on less restrictive properties, which afford new intrinsic elements that increase the knowledge of the metrics and permit their explicit characterization. Besides the *intrinsic* nature, the classification must be *generic*, that is, valid for the whole set of the type D metrics. Consequently, it will be independent of the energy content and it will have to be built on the intrinsic geometry associated with a type D Weyl tensor.

The first classification that we propose is based on the geometric properties of the principal 2–planes, that is, it is induced by the geometric classification of the principal structure. Every principal 2–plane can be submitted or not to three properties, so $2^6=64$ classes can be considered.

Definition 2: *Taking into account the foliation, minimal, or umbilical character of each principal 2–plane we distinguish* 64 *different classes of type D space–times*.

We denote the classes as D^{pqr}_{lmn}, *where the superscripts* p,q,r *take the value 0 if the time-like principal plane is, respectively, a foliation, a minimal or an umbilical distribution, and they take the value 1 otherwise. In the same way, the subscripts* l,m,n *collect the foliation, minimal or umbilical nature of the space-like plane*.

The most degenerated class that we can consider is D^{000}_{000} which corresponds to a type D product metric, and the most regular one is D^{111}_{111} which means that neither V nor H are foliation, minimal or umbilical distributions. We will put a dot in place of a fixed script (1 or 0) to indicate the set of metrics that cover both possibilities. So, for example, the metrics of type $\mathrm{D}^{111}_{11\cdot}$ are the union of the classes D^{111}_{111} and D^{111}_{110}; or a metric is of type $\mathrm{D}^{0\cdot\cdot}_{\cdot\cdot\cdot}$ if the timelike 2–plane is a foliation.

Taking into account lemma 1, every class is defined by means of first-order differential equations imposed on the canonical 2–form U. On the other hand, U can be written explicitly in terms of the Weyl tensor[17] and, consequently, every class admits an intrinsic and explicit characterization.

The above classification depends on the derivatives of the principal 2–form U. An alternative classification at first order in the Weyl eigenvalues can also be considered by taking into account the four 1-forms defined by the principal 2–planes and the gradient of the modulus and the argument of the eigenvalue. So, we will have $2^4=16$ classes.

Definition 3: *Let* $\alpha=e^{\frac{3}{2}(\rho+\mathrm{i}\theta)}$ *be the Weyl eigenvalue. Taking into account the relative position between the gradients* $d\theta, d\rho$ *and each principal 2–plane we distinguish* 16 *different classes of type D* space–times.

We denote the classes $D[pq,rs]$ *where* p,q,r,s *take the values* 0 *or* 1 *to indicate, respectively, that one of the 1-forms* $U(d\theta), U(d\rho), *U(d\theta), *U(d\rho)$ *is zero or nonzero*.

The most degenerated class D[00;00] is occupied by the type D metrics with constant eigenvalues, and the most general one D[11;11] by those type D space–times for which both, the modulus and the argument of the Weyl eigenvalue, have nonzero projection onto the principal planes. As above, a dot means that a condition is not fixed. So, for example, we write $\mathrm{D}[0\cdot;\cdot\cdot]$ to indicate the type D metrics for which the argument of the eigenvalues have zero projection onto the timelike principal 2–plane.

The type D metrics with constant modulus, $d\rho=0$, correspond to the classes $D[\cdot 0;\cdot 0]$, and those with constant argument, $d\theta=0$, are the metrics of type $D[0\cdot;0\cdot]$. This last family contains the Weyl-electric and the Weyl-magnetic space–times because a real or imaginary eigenvalue means that the argument takes the constant value 0, π or $\pi/2$, $3\pi/2$, respectively.

In the next section we will show the marked relation between the two classifications given in definitions 2 and 3 when some usual restrictions are imposed on the Ricci tensor.

IV. TYPE D METRICS WITH ZERO COTTON TENSOR

The space–time Cotton tensor P is a vector valued 2–form which depends on the Ricci tensor as

$$P_{\mu\nu,\beta}\equiv\nabla_{[\mu}Q_{\nu]\beta}, \qquad 2Q\equiv Ric-\tfrac{1}{6}(\mathrm{Tr}\ Ric)g. \tag{20}$$

The Bianchi identities equal the Cotton tensor with the divergence of the Weyl tensor. Indeed, if $\mathcal{W}$ is the self-dual Weyl tensor and $\mathcal{P}=\frac{1}{2}(P-\mathrm{i}*P)$ is the self-dual 2–form associated with the Cotton tensor, Bianchi identities become

$$\mathcal{P}=-\delta\mathcal{W}. \tag{21}$$

So, the vanishing of the Cotton tensor is equivalent to the Weyl tensor to be divergence free, $\delta\mathcal{W}=0$. Taking into account the canonical expression of a type D Weyl tensor (19), a straightforward calculation leads to the following:

Proposition 1: *Let $\mathcal{U}$ and $\alpha=-\operatorname{Tr}\mathcal{W}^3/\operatorname{Tr}\mathcal{W}^2$ be the principal 2–form and the double eigenvalue of a type D Weyl tensor. Then, the space–time Cotton tensor is zero if, and only if,*

$$\nabla\mathcal{U}=i(\delta\mathcal{U})[\mathcal{U}\otimes\mathcal{U}+\mathcal{G}] \quad ; \quad i(\delta\mathcal{U})\mathcal{U}=\tfrac{1}{3}\mathrm{d}\ln\alpha. \tag{22}$$

From the results of the previous section, we know that the first condition means that the principal structure is umbilical, that is, the principal directions are shear free null geodesics accordingly to the Goldberg–Sachs theorem. Consequently, every type D space–time with zero Cotton tensor is of type $\mathrm{D}^{\cdot\cdot 0}_{\cdot\cdot 0}$. The second equation in (22) shows that the principal structure is Maxwellian and the electromagnetic invariant scalars depend on the Weyl eigenvalue. If we take the real and the imaginary parts of this equation and write $\rho+\mathrm{i}\theta=\frac{2}{3}\ln\alpha$, we get

$$\Phi(U)=\mathrm{d}\rho \ ; \qquad \Psi(U)=\mathrm{d}\theta. \tag{23}$$

So the modulus and the argument of the Weyl eigenvalue govern, respectively, the minimal and the foliation character of the principal planes. This relation establishes a bijection between the classes of the two classifications that we have presented. More precisely, we have:

Theorem 1: *Every type D spacetime with zero Cotton tensor is of type $D^{\cdot\cdot 0}_{\cdot\cdot 0}$. Moreover, it is of class D^{pq0}_{lm0} if, and only if, it is of class $D[lm,pq]$.*

So we have just 16 classes of type D space–times with zero Cotton tensor and each one is characterized by the vanishing or not of the projections of the gradient of the Weyl eigenvalue onto the principal planes. The second condition in (22) implies that a solution of the Maxwell equations exists that has $\mathcal{U}$ as its associated structure. Then, taking into account the results of Sec. II C, it holds:

Proposition 2: *The principal structure of a type D space–time with zero Cotton tensor is Maxwellian. More precisely, if $\mathcal{U}$ and $\alpha=-\operatorname{Tr}\mathcal{W}^3/\operatorname{Tr}\mathcal{W}^2$ are the principal 2–form and the double eigenvalue of the Weyl tensor, the self–dual 2–form*

$$\mathcal{F}_M=\alpha^{2/3}\mathcal{U}, \tag{24}$$

is a solution of the source-free Maxwell equations, $\delta\mathcal{F}_M=0$.

In the following D(M) denotes the type D space–times with Maxwellian principal structure, and $\mathrm{D(M)}^{pqr}_{lmn}$ expresses the type D(M) space–times of class D^{pqr}_{lmn}. With this notation, from theorem 1 and proposition 2 it follows: *Every type D space–time with zero Cotton tensor is of type $D(M)^{\cdot\cdot 0}_{\cdot\cdot 0}$.*

It is worth pointing out that the family of type D metrics admitting a conformal Killing–Yano tensor attached to its principal structure are those of type $\mathrm{D(M)}^{\cdot\cdot 0}_{\cdot\cdot 0}$,[31] and this family includes the Cotton-zero type D metrics.

The results of this section have been used elsewhere[21] in offering a new approach to the Kinnersley type D vacuum solutions. An integration of the Einstein vacuum equations based on the classification given above permits the explicit and intrinsic labeling of the solutions as well as to put over interesting geometric properties of these space–times.

V. SOME RESULTS ABOUT TYPE $D(M)_{0\cdot 0}^{0\cdot 0}$ SPACE–TIMES

Now, in this section, we restrict our study to the type D metrics with Maxwellian, integrable and umbilical structure, that is, those of type $D(M)_{0\cdot 0}^{0\cdot 0}$. We can easily obtain a canonical form for these metrics. Indeed, lemma 6 states that the metric is conformal to a product one with a conformal factor determined by the potential of the closed 1–form $\Phi(U)$. More precisely, the metric can be written as

$$g=\frac{1}{\Omega^2}[\sigma_{AB}^{-}(x^C)\ dx^A dx^B+\sigma_{ij}^{+}(x^k)\ dx^i dx^j], \tag{25}$$

where Ω satisfies

$$2\ \mathrm{d}\ln\Omega=\Phi(U)\equiv i(\delta U)U-i(\delta * U)*U. \tag{26}$$

Conversely, we can analyze the Petrov type of the metric (25) by studying a product metric $\tilde{g}=\sigma^{-}+\sigma^{+}$. Let X_- and X_+ be the Gaussian curvatures of the arbitrary bidimensional metrics, σ^- and σ^+, hyperbolic and elliptic, respectively. The Gauss–Codazzi equations show that the Riemann and the Ricci tensors of $\tilde{g}$ are

$$Riem(\tilde{g})=\tfrac{1}{2}X_-\sigma^- \owedge \sigma^- + \tfrac{1}{2}X_+\sigma^+ \owedge \sigma^+;\quad Ric(\tilde{g})=X_-\sigma^-+X_+\sigma^+. \tag{27}$$

So, the Weyl tensor of a product metric is Petrov-type O precisely when $X_-+X_+=0$, and then both curvatures are constant. On the other hand, when $X_-+X_+\neq 0$, the space–time is type D. Moreover U determines the principal structure and the double eigenvalue is given by

$$\tilde{\alpha}=-\tfrac{1}{6}(X_-+X_+). \tag{28}$$

So, we have

Lemma 8: *Every $2+2$ product metric $\sigma^-+\sigma^+$ is of type D (or O) with real eigenvalues, and the double eigenvalue is given by (28), where X_- and X_+ are the Gaussian curvatures of σ^- and σ^+, respectively. Moreover, it is of type O if, and only if, $X_-=-X_+=constant$.*

A conformal transformation $\tilde{g}=\Omega^2 g$ preserves the Petrov type and the Weyl eigenvalues change as $\tilde{\alpha}=\Omega^{-2}\ \alpha$. Consequently, from Eq. (26) and taking into account lemmas 1 and 8, we can conclude:

Proposition 3: *A space–time is of type $D(M)_{0\cdot 0}^{0\cdot 0}$ if, and only if, there exist local coordinates such that the metric g takes the expression (25) with $X_-+X_+\neq 0$, where X_- and X_+ are the Gaussian curvatures of σ^- and σ^+, respectively. Moreover, it is of class D_{010}^{010}, D_{010}^{000}, D_{000}^{010}, or D_{000}^{000} if, and only if, $\sigma^-(d\Omega)\neq 0\neq\sigma^+(d\Omega)$, $\sigma^+(d\Omega)=0\neq\sigma^-(d\Omega)$, $\sigma^+(d\Omega)\neq 0=\sigma^-(d\Omega)$, or $d\Omega=0$, respectively.*

Furthermore, taking into account the expressions (27) for the Ricci and (28) for the eigenvalue of a product metric, and considering the change of these metric concomitants for a conformal transformation, we can state:

Proposition 4: *The Weyl eigenvalue of the metric (25) is real and it is given by*

$$\alpha=-\tfrac{1}{6}\Omega^2(X_-+X_+). \tag{29}$$

The Ricci tensor of this metric is

$$Ric(g)=\frac{2}{\Omega}\nabla d\Omega+X_-\sigma^-+X_+\sigma^+ +\left[\frac{1}{\Omega}\Delta\Omega-\frac{3}{\Omega^2}\tilde{g}(d\Omega,d\Omega)\right]\tilde{g}, \tag{30}$$

where $\nabla=\nabla_{\sigma^-}+\nabla_{\sigma^+}$ is the connection of the product metric $\tilde{g}=\sigma^-+\sigma^+$.

Let us consider metrics with zero Cotton tensor again. If they have a constant argument, theorem 1 implies that the principal structure is integrable and so, the space–times are of type $\mathrm{D(M)}^{0\cdot 0}_{0\cdot 0}$. Consequently, from proposition 4 the Weyl tensor has real eigenvalues. So we can state:

Theorem 2: *The Weyl eigenvalues of a type D space–time with zero Cotton tensor have constant argument if, and only if, they are real*.

This result generalizes a previous one by Hall[4] (see also McIntosh *et al.*[5]). He showed that there are no purely magnetic Type D vacuum metrics. But the purely magnetic case occurs when the eigenvalue argument is $\frac{3}{2}\theta=\pm\pi/2$, that is to say, a particular value of constant argument. So, from theorem 2 it follows:

Corollary 1: *There is no purely magnetic Type D metric with zero Cotton tensor*.

This corollary shows that not only the purely magnetic vacuum solutions are forbidden, but also the Weyl-magnetic space–times with zero Cotton tensor. On the other hand the Hall result is also generalized in the sense that theorem 2 excludes all the constant arguments that differ from 0 or π. Although this approach could be of interest in studying the existence of purely magnetic type I solutions, the recent results on this subject have been obtained by using the $1+3$ formalism.[22,32,33]

From the results above it is easy to recover the canonical form for the metrics with zero Cotton tensor and real Weyl eigenvalues. Indeed, expressions (23) and (26) show that the conformal factor and the Weyl eigenvalue are related by $\Omega^2=c^2e^{\rho}=c^2\alpha^{2/3}$, c being an arbitrary constant. On the other hand they also satisfy expression (29) and, consequently, Ω coincides, up to a constant factor, with X_-+X_+. So we have

Proposition 5: *Every type D metric with real eigenvalues and zero Cotton tensor may be written*

$$g=\frac{1}{(X_-+X_+)^2}(\sigma^-+\sigma^+),$$

where $\sigma^-=\sigma^-_{AB}(x^C)dx^Adx^B$, $\sigma^+=\sigma^+_{ij}(x^k)dx^idx^j$, *are two arbitrary bidimensional metrics,* σ^- *hyperbolic and* σ^+ *elliptic, with Gaussian curvatures* X_- *and* X_+, *respectively*.

This canonical expression was obtained in a previous work[18] where it was used to integrate the Einstein vacuum equations, in this way getting an intrinsic algorithm to identify every A, B, and C-metric of Ehlers and Kundt.[2] In the following section, starting from the propositions 3 and 4 we present a similar study for the charged counterpart of these vacuum solutions.

VI. ALIGNED EINSTEIN–MAXWELL SOLUTIONS OF TYPE $\mathrm{D}^{0\cdot 0}_{0\cdot 0}$

If (v,h) is the principal structure of the Weyl tensor, the aligned Einstein–Maxwell solutions satisfy

$$Ric(g)=\chi(v-h)=\kappa\ (\sigma^--\sigma^+), \tag{31}$$

where the second equality is satisfied for the type $\mathrm{D}^{0\cdot 0}_{0\cdot 0}$ metrics as a consequence of proposition 3: $\chi=\kappa\Omega^2$, $\sigma^-=\Omega^2 v$, $\sigma^+=\Omega^2 h$. Moreover, as the principal structure is integrable, it is Maxwellian and the associated Rainich index is a constant. So, (31) is a necessary and sufficient condition for the metric (25) to be an aligned solution of the Einstein–Maxwell equations. Taking into account the expression (30) for the Ricci tensor, condition (31) becomes

$$\Omega=\lambda_-(x^A)+\lambda_+(x^i), \tag{32}$$

$$\nabla \mathrm{d}\lambda_\epsilon=\beta_\epsilon\sigma^\epsilon, \tag{33}$$

$$\frac{\Omega^2}{6}(X_-+X_+)+\Omega\ \ (\beta_-+\beta_+)=\sigma^-(\mathrm{d}\lambda_-,\mathrm{d}\lambda_-)+\sigma^+(\mathrm{d}\lambda_+,\mathrm{d}\lambda_+), \tag{34}$$

$$\mathrm{d}\beta_\epsilon+X_\epsilon \mathrm{d}\lambda_\epsilon=0. \tag{35}$$

Equations (35) are the integrability conditions of (33). Moreover, if we differentiate (34), project on σ_-, differentiate again and take into account (35), we have

$$2(X_- + X_+)\mathrm{d}\lambda_- \wedge \mathrm{d}\lambda_+ + \Omega[\mathrm{d}X_+ \wedge \mathrm{d}\lambda_- - \mathrm{d}X_- \wedge \mathrm{d}\lambda_+] = 0. \tag{36}$$

Then a simple analysis of the expressions (32)–(36) leads to

Lemma 9: *The following conditions are equivalent*: (i) $dX_\epsilon = 0$, (ii) $\beta_\epsilon = 0$, (iii) $d\lambda_\epsilon = 0$, (iv) $\sigma^\epsilon(d\Omega) = 0$. *Moreover, these conditions hold if* $\sigma^\epsilon(d\lambda_\epsilon, d\lambda_\epsilon) = 0$ *everywhere*.

A. The solutions: A, B, and C charged metrics

Proposition 3 states that the classes D_{0m0}^{0q0} can be discriminated using the vectors $\sigma^\epsilon(\mathrm{d}\Omega)$. Then, lemma 9 implies that, as happens in the vacuum case,[18] the four classes can be characterized by σ^- or σ^+ to be bidimensional metrics that have constant curvature or not.

If g is in class D_{010}^{010}, lemma 9 implies that λ_ϵ can be taken as coordinate in the plane σ^ϵ. Then, Eqs. (34)–(36) say that β_ϵ, X_ϵ, and $\sigma^\epsilon(d\lambda_\epsilon, d\lambda_\epsilon)$ depend just on λ_ϵ, and that $X_\epsilon = -\beta'_\epsilon$. Then, from (34) we have $\beta'''_-(\lambda_-) + \beta'''_+(\lambda_+) = 0$ and, consequently, β_ϵ is a polynomial in λ_ϵ of degree less than or equal to three. But lemma 9 also states that $d\lambda_\epsilon$ is not a null vector everywhere. Then, Einstein–Maxwell equations (32)–(36) finally lead to

$$\sigma^\epsilon = \frac{1}{\epsilon f(\epsilon\lambda_\epsilon)}\mathrm{d}\lambda_\epsilon^2 + f(\epsilon\lambda_\epsilon)dZ^2, \tag{37}$$

with $f(\lambda)$ a fourth degree polynomial. Then, putting (37) and (32) into (25) we recover the known expression of the charged C-metrics.[25]

If g is in class D_{010}^{000}, lemma 9 implies that λ_- can be taken as a coordinate in the plane σ^- and, moreover, σ^+ must be of constant curvature. Thus, a redefinition of Ω and σ^- allows us to consider $X_+ \in \{-1,0,1\}$ and $\Omega = \lambda_-$. Then, if we introduce the coordinate transformation $r = -1/\lambda_-$, a similar procedure that leads in the general case to the charged counterpart of the A_i-metrics:

$$g = -a(r)\mathrm{d}t^2 + \frac{1}{a(r)}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2, \qquad a(r) \equiv X - \frac{C}{r} + \frac{D}{r^2}, \tag{38}$$

$\mathrm{d}\sigma^2$ being the bidimensional elliptic metric of constant curvature X, with $X = 1, -1, 0$ depending on the A_1, A_2, or A_3 case.

If g is in the class D_{000}^{010}, in a similar way λ_+ can be taken as a coordinate in the plane σ^+, and σ^- must be of constant curvature $X_- \in \{-1,0,1\}$. Then, the coordinate transformation $r = -1/\lambda_+$, leads to the charged counterpart of the B_i-metrics:

$$g = r^2\mathrm{d}\sigma^2 + a(r)\mathrm{dz}^2 + \frac{1}{a(r)}\mathrm{d}r^2, \qquad a(r) \equiv X - \frac{C}{r} + \frac{D}{r^2}, \tag{39}$$

$\mathrm{d}\sigma^2$ being the bidimensional hyperbolic metric of constant curvature X, with $X = 1, -1, 0$ depending on the B_1, B_2, or B_3 case.

Finally, in class D_{000}^{000} both bidimensional metrics have a constant curvature and Eq. (34) implies that $X_- + X_+ = 0$. This means that the space–times is conformally flat and the metric becomes $g = \sigma^- + \sigma^+$, where σ^ϵ are bidimensional metrics, hyperbolic, and elliptic, respectively, with a constant curvature ϵX. The metrics of this more degenerated class are the only ones that have zero Cotton tensor.

B. The intrinsic characterization

The metrics of type $\mathrm{D(M)}_{0\cdot 0}^{0\cdot 0}$, which take the canonical form (25), admit an intrinsic identification by means of conditions involving the principal 2–form $\mathcal{U}$. These characterization equations that we have given in previous sections could be written explicitly in terms of metric concomitants because $\mathcal{U}$ can be determined from the Weyl tensor.[17] Nevertheless, as a consequence of the Bianchi identities some of the above conditions can be satisfied identically taking into account the properties of the Ricci tensor. This is the case of vacuum metrics: As $Ric=0$ implies the nullity of the Cotton tensor, the principal planes always define an umbilical and Maxwellian structure as a consequence of the results in Sec. IV. Actually we want to characterize aligned Einstein–Maxwell solutions that are conformal to a product metric. So, the Weyl tensor must have real eigenvalues and the principal planes are the eigenspaces of the Ricci tensor, that is,

$$W=3\alpha(U\otimes U-*U\otimes *U)+\alpha G, \quad Ric=\chi(v-h), \tag{40}$$

where $v=U^2$, $h=-*U^2$. Then, taking into account the expressions in Sec. II about $2+2$ almost-product structures, a straightforward calculation shows that the Bianchi identities (21) can be written

$$(3\alpha+2\chi)\ Q_v=v\otimes h(d\alpha); \quad (3\alpha-2\chi)\ Q_h=h\otimes v(d\alpha), \tag{41}$$

$$v(d\chi)-2\chi i(\delta U)U=0; \quad h(d\chi)+2\chi i(\delta *U)*U=0. \tag{42}$$

From these expressions we find that, under the scalar restriction $(3\alpha)^2\neq(2\chi)^2$, the properties of the structure follow just by imposing that the Weyl and the Ricci tensor take expressions (40). On the other hand, the case $(3\alpha)^2=(2\chi)^2$ leads to the *exceptional* metrics considered by Plebański and Hacyan.[34] Nevertheless, it can easily be shown that $(3\alpha)^2\neq(2\chi)^2$ for the solutions recovered in the subsection above. So we get the following characterization:

Lemma 10: *The charged counterpart of the A, B, and C-metrics are the only aligned Einstein–Maxwell solutions of type D with real eigenvalues that satisfy* $(3\alpha)^2\neq(2\chi)^2$, *α and χ being, respectively, the Weyl and the Ricci eigenvalues*.

Elsewhere,[18] conditions for g to be of type D with real eigenvalues have been given in terms of Weyl concomitants. In order to impose the Ricci tensor to take the form (40) we can use the algebraic Rainich conditions.[12] But if the Weyl tensor is of type D with real eigenvalues, a part of these Rainich conditions hold identically when we impose the aligned restriction. From these considerations and lemma 10 we have:

Theorem 3: *The A, B, and C Einstein–Maxwell solutions can be characterized by conditions*

$$\alpha\neq 0; \quad S^2+S=0; \quad Ric(x,x)\geqslant 0,$$

$$\mathrm{Tr}\ Ric=0, \quad S[Ric]+Ric=0; \quad (3\alpha)^2-(2\chi)^2\neq 0.$$

$W\equiv W(g)$ and $Ric\equiv Ric(g)$ being the Weyl and Ricci tensors of the metric g, and where $\alpha=\alpha(g)\equiv-(\frac{1}{12}\,\mathrm{Tr}W^3)^{1/3}$, $\chi=\chi(g)\equiv-\frac{1}{2}(\mathrm{Tr}Ric^2)^{1/2}$, $S=S(g)\equiv\frac{1}{3}(\alpha^{-1}W-\frac{1}{2}g \owedge g)$, $S[Ric]_{\alpha\beta}=S_{\alpha\mu\beta\nu}R^{\mu\nu}$ and x is an arbitrary time-like vector.

This theorem offers an intrinsic and explicit description of the aligned Einstein–Maxwell solutions of type $\mathrm{D}^{0\cdot 0}_{0\cdot 0}$. Now we look for an intrinsic and explicit way to identify every metric of this family, that is, to distinguish the A_i, B_i, and C charged metrics. In a first step we must discriminate between the classes D^{0p0}_{0m0} and, as a consequence of proposition 3, this depends on the nullity of the vectors $v(d\Omega)$ and $h(d\Omega)$. But the expression (29) for the Weyl eigenvalue and lemma 9 imply that, equivalently, the vectors $v(d\alpha)$ and $h(d\alpha)$ determine these properties. So, the same scheme as in the vacuum case[18] can be used to distinguish between the classes.

The last step to obtain the intrinsic and explicit characterization of the solutions is to get an invariant that provides the sign of the bidimensional curvature when this is constant. A straightforward calculation shows that if X_ϵ is constant, then

$$X_\epsilon\Omega^2=\omega_\epsilon\equiv\tfrac{1}{9}(\mathrm{d}\ln(\alpha+\chi))^2-2\alpha-\epsilon\chi. \tag{43}$$

So, we have a characterization of the Einstein–Maxwell A, B, and C-metrics, and we recover the type D static vacuum solutions making $\chi=0$.

Theorem 4: *Let g be an aligned Einstein–Maxwell solution of type $D^{0\cdot 0}_{0\cdot 0}$ (characterized in theorem 3). Let us take the metric concomitants*

$$M \equiv *W(d\alpha,\cdot,d\alpha,\cdot) \quad N \equiv S(d\alpha,\cdot,d\alpha,\cdot),$$

and let x be an arbitrary unitary timelike vector. Then,

(i) *g is a charged C-metric if, and only if, $M \neq 0$;*
(ii) *g is a charged A-metric if, and only if, $M=0$ and $2N(x,x)+trN>0$.*
Furthermore, it is of type A_1, A_2 or A_3 if $\omega_+>0$, $\omega_+<0$ or $\omega_+=0$, respectively, where $\omega_+ \equiv \frac{1}{9}(\mathrm{d}\ln(\alpha+\chi))^2-2\alpha-\chi$;
(iii) *g is a charged B-metric if, and only if, $M=0$ and $2N(x,x)+trN<0$.*
Furthermore, it is of type B_1, B_2 or B_3 if $\omega_->0$, $\omega_-<0$ or $\omega_-=0$, respectively, where $\omega_- \equiv \frac{1}{9}(\mathrm{d}\ln(\alpha+\chi))^2-2\alpha+\chi$.

This theorem provides an algorithm to identify, in the set of all metrics, the charged counterpart of the Ehlers and Kundt[2] vacuum solutions. The particular case of the A_1 metrics corresponds to a charged spherically symmetric spacetime, that is, to the Reissner–Nordström solution. In this case the metric takes the form (38) with $X=1$, and the mass and the charge are related with the constants C and D, respectively. Moreover, these constants can be given in terms of Weyl and Ricci invariants. Then, from the last theorem and previous subsection it follows:

Theorem 5: *Let $Ric \equiv Ric(g)$ and $W \equiv W(g)$ be the Ricci and the Weyl tensors of a spacetime metric g, and let us take the metric concomitants*:

$$\alpha \equiv -\left(\tfrac{1}{12}\mathrm{Tr}\, W^3\right)^{1/3}, \quad \chi \equiv -\tfrac{1}{2}(\mathrm{Tr}\, Ric^2)^{1/2}, \quad \omega \equiv \tfrac{1}{9} g(d\ln\alpha, d\ln\alpha)-2\alpha-\chi, \tag{44}$$

$$S \equiv \frac{1}{3\alpha}\left(W-\tfrac{1}{2}\alpha g \owedge g\right), \quad M \equiv *W(d\alpha,\cdot,d\alpha,\cdot), \quad N \equiv S(d\alpha,\cdot,d\alpha,\cdot). \tag{45}$$

The necessary and sufficient conditions for g to be the Reissner–Nordström metric are

$$\alpha \neq 0, \quad S^2+S=0, \quad Ric(x,x) \geqslant 0,$$

$$\mathrm{Tr}\, Ric=0, \quad S[Ric]+Ric=0, \quad (3\alpha)^2-(2\chi)^2 \neq 0,$$

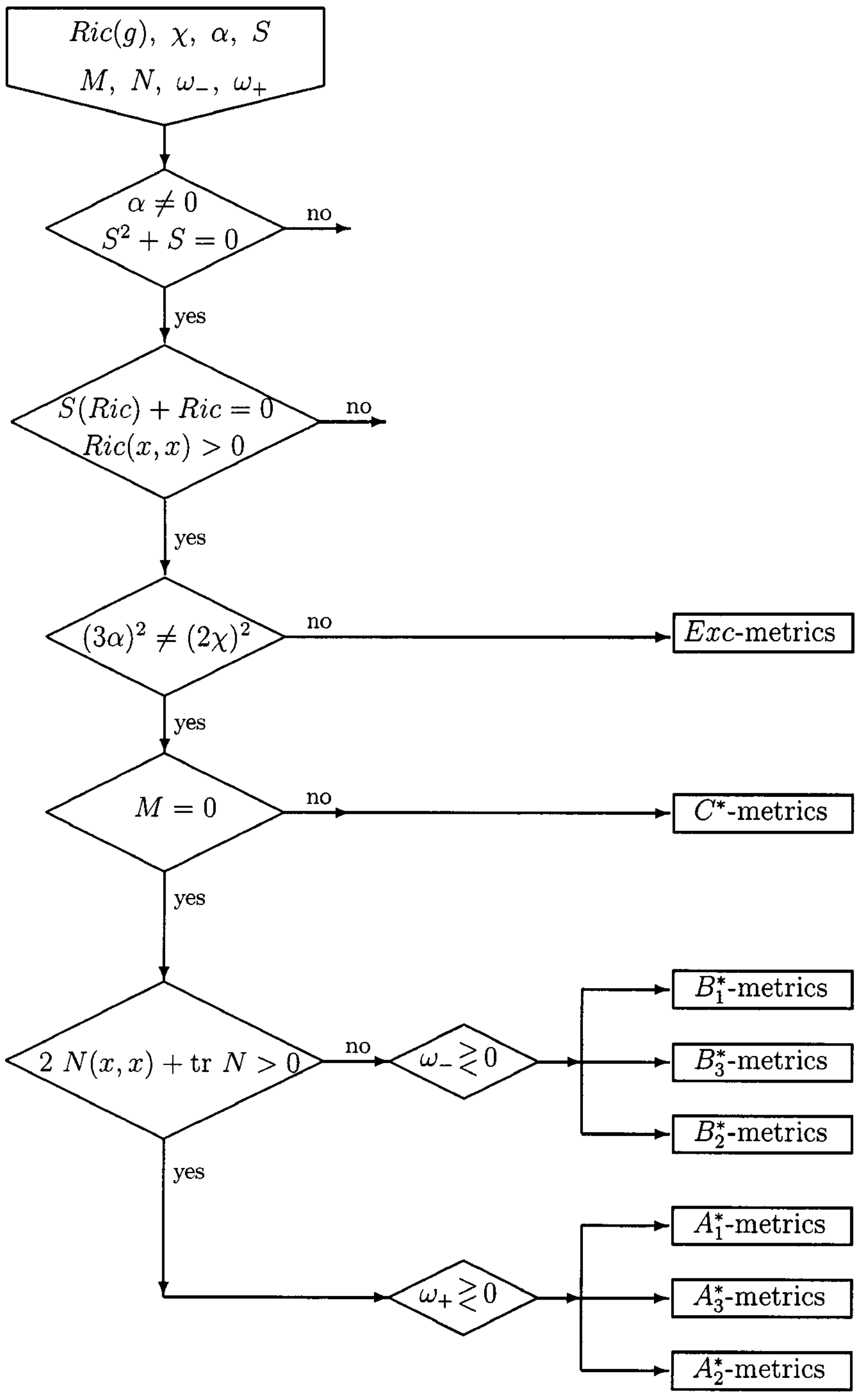
$$M=0, \quad 2N(x,x)+trN>0, \quad \omega>0,$$

where x is an arbitrary unitary time-like vector. Moreover, the mass m and the electric charge e are given, respectively, by $m=(\alpha+\chi)/\omega^{3/2}$ and $e^2=-\chi/\omega^2$, and the timelike Killing vector by $\xi=[\sqrt{\omega}(3\alpha+2\chi)]^{-1}[N(x)/\sqrt{N(x,x)}]$.

C. A summary in algorithmic form

Finally, in order to emphasize the algorithmic nature of our results, we present them as a flow diagram that identifies, among all metrics, every A, B, or C Einstein–Maxwell solution (in the following flow chart we denote them A^*, B^*, and C^*-metrics). The exceptional metrics studied by Plebański are also identified and they are denoted Exc-metrics. This operational algorithm involves an arbitrary unitary timelike vector, x, and some metric concomitants that may be obtained from the components of the metric tensor g in arbitrary local coordinates: The invariants α,

χ, ω_ϵ, S, M, and N are given in (43)–(45) in terms of the Ricci and Weyl tensors. Making $\chi=0$, we recover the vacuum solutions[18]



ACKNOWLEDGMENTS

The authors would like thank B. Coll and J. A. Morales for some useful comments. This work has been supported by the Spanish Ministerio de Ciencia y Tecnología, project AYA2000-2045.

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