Almost-product structures in Relativity*

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The physical fields usually considered in relativity (perfect fluids, electromagnetic fields) are related to particular almost-product structures.

The 1+3 almost-product structures associated to a perfect fluid may be classified according to well known kinematical quantities (acceleration, rotation, deformation, shear and expansion), providing a good physical interpretation, but no analog is known for almost-product structures others than the 1+3 ones.

In this paper we extend to arbitrary almost-product structures the above kinematical quantities and, then, we show how their vanishing is related to foliation, totally geodesic, umbilical or minimal geometrical character of the plane fields of the structure.

In particular, we apply these results to the kinematical characterization of the 2+2 almost-product structure associated to any vacuum regular electromagnetic field: it is very pleasant to find that the rather complicated and apparently geometrical meaningless Rainich equations when expressed in terms of our kinematical variables, become simple and meaningful.

Geometric Concepts

Let us consider a plane field V and its orthogonal complement H. Let $\mathbf{v}$ and $\mathbf{h} = Id - \mathbf{v}$ be the respective projectors. Let $\mathcal{Q}_V$ be the (1,2)-tensor $\mathcal{Q}_V \equiv (\mathcal{Q}^\alpha_{\beta\gamma})$: $\mathcal{Q}_V(x, y) = \mathbf{h}(\nabla_{\mathbf{v}x}\mathbf{v}y)$, $\forall(x, y)$, and let $\mathcal{S}_V$ and $\mathcal{A}_V$ be its symmetric and antisymmetric parts.

V is foliation iff $\mathcal{A}_V = 0$. In this case we have $\mathcal{S}_V = \mathcal{Q}_V$, second fundamental form of its integral manifolds,[1] and it can be shown[2] that V is geodesic, umbilical or minimal iff $\mathcal{S}_V \equiv 0$, $\mathcal{S}_V \equiv a \otimes \mathbf{v}$ or $\mathrm{tr}\mathcal{S}_V \equiv 0$. Consequently, for plane fields which are not necessarily foliations, we give the following definition:

Definition 1 *A plane field V is said to be geodesic, umbilical or minimal if its second fundamental form $\mathcal{S}_V$ verifies, respectively, $\mathcal{S}_V \equiv 0$, $\mathcal{S}_V \equiv a \otimes \mathbf{v}$ or $\mathrm{tr}\mathcal{S}_V \equiv 0$.*

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Almost-product structures (V,H) may be classified according to the invariant decomposition of the tensors $\mathcal{Q}_V$ and $\mathcal{Q}_H$, or equivalently, according to the foliation, geodesic, umbilical or minimal character of the plane fields V and H.[3,2] Some of these properties have been interpreted in terms of invariances along vector fields.[4]

Kinematic Concepts

Let us consider a 1+(n-1) almost-product structure. In this case, V is generated by a unit vector field u and H is the orthogonal hyperplane, $V = \{u\}$, $H = \{u\}^\perp$; the corresponding projectors are $\mathbf{v} = \varepsilon u \otimes u$ and $\mathbf{h} = Id - \mathbf{v}$, $\varepsilon = u^2$. It is known that the covariant derivative of a unit vector field u may be written

$$\nabla u = \varepsilon u \otimes a + \sigma + \Omega + \frac{1}{n-1}\theta \mathbf{h}$$

where

$$a = \nabla_u u, \qquad 2\Omega = \perp (du), \qquad \omega = *(u \wedge du),$$
$$2d = \perp (\mathcal{L}_u g), \qquad \theta = \mathrm{tr} d, \qquad \sigma = d - \tfrac{1}{n-1}\theta \mathbf{h},$$

are the kinematic coefficients associated to u. A simple calculation leads to: $\mathcal{S}_V = \mathcal{Q}_V = a \otimes u \otimes u$, $\mathcal{A}_V = 0$; $\mathcal{Q}_H = -\varepsilon u \otimes \perp (\nabla u)$, $\mathcal{S}_H = -\varepsilon u \otimes d$, $\mathcal{A}_H = -\varepsilon u \otimes \Omega$. Consequently, $V = \{u\}$ is always a foliation and umbilical field and it is geodesic (minimal) iff $a = 0$; on the other hand, $H = \{u\}^\perp$ is foliation, geodesic, umbilical or minimal iff Ω, d, σ or θ vanish respectively.

Thus, the geometric properties of a 1+(n-1) structure may be characterized by the kinematic quantities a, Ω, d, σ and θ. Can they be extended to arbitrary p+(n-p) structures?

In order to answer this question, it is important to note that whereas a is a quantity intrinsically associated to *every* integral curve of u, the quantities σ, Ω and θ are associated to *every* section orthogonal to u. Consequently, we are lead to give the following definition:

Definition 2 *Let V be a p-plane field with projector tensor $\mathbf{v}$. We call respectively vorticity $\Omega_x(V)$, rotation $\omega_x(V)$, deformation $d_x(V)$, expansion $\theta_x(V)$ and shear $\sigma_x(V)$ of V along the vector field $x \in V^\perp = H$, and acceleration $a_V(x)$ of x on the p-plane V, the tensors:*

$$\Omega_x(V) = \frac{1}{2} \perp (dx), \quad \omega_x(V) = *(\eta_H \wedge dx),$$
$$d_x(V) = \frac{1}{2} \perp (\mathcal{L}_x g), \qquad \theta_x(V) = \mathrm{tr} d_x(V), \qquad \sigma_x(V) = d_x(V) - \frac{1}{\mathrm{p}}\theta_x(V)\mathbf{v},$$
$$a_V(x) = \perp (\nabla_x x),$$

where $\perp$ is the projector operator on V and η_H is the volume element on H.

The above definitions become the usual kinematic coefficients associated to a unit vector field u when $x = u$ and $V = \{u\}^\perp$. In the following section we shall see that, for arbitrary structures, they play he role that we looked for.

Geometric-Kinematic Equivalence

Let (V, H) be a p+q almost-product structure and $\mathcal{Q}_V$, $\mathcal{S}_V$ and $\mathcal{A}_V$ the tensors defined above. One can see that

$$2\mathcal{S}_V(x,y) = \mathbf{h}(\{\mathbf{v}x, \mathbf{v}y\}),$$

$$2\mathcal{A}_V(x,y) = \mathbf{h}([\mathbf{v}x, \mathbf{v}y]),$$

where $\{x,y\} \equiv \nabla_x y + \nabla_y x$ and $[x,y] \equiv \nabla_x y - \nabla_y x$. On account of these expressions, the generalized kinematic quantities of Definition 2 may be expressed, $\forall x \in H$, $\forall z \in V$,

$$\Omega_x(V) = -i(x)\mathcal{A}_V, \quad d_x(V) = -i(x)\mathcal{S}_V,$$
$$\theta_x(V) = -i(x)\mathrm{tr}\mathcal{S}_V, \quad \sigma_x(V) = -i(x)\{\mathcal{S}_V - \tfrac{1}{\mathrm{p}}\mathrm{tr}\mathcal{S}_V \otimes \mathbf{v}\},$$

$$a_H(z) = \mathcal{S}_V(z,z)$$

From them, it follows:

Proposition 1 *The following equivalent conditions hold:*
i) V geodesic ($\mathcal{S}_V = 0$) $\Longleftrightarrow d_x(V) = 0$, $\forall x \in H \Longleftrightarrow a_H(z)$, $\forall z \in V$
ii) V minimal ($\mathrm{tr}\mathcal{S}_V$) $\Longleftrightarrow \theta_x(V)$, $\forall x \in H \Longleftrightarrow \theta(V) = \frac{1}{\mathrm{p}}\mathrm{tr}\mathcal{S}_V = 0$
iii) V umbilical ($\mathcal{S}_V = a \otimes \mathbf{v}$) $\Longleftrightarrow \sigma_x(V)$, $\forall x \in H \Longleftrightarrow a_H(z) = z^2\theta(V)$, $\forall z \in V$
iv) V foliation ($\mathcal{A}_V = 0$) $\Longleftrightarrow \Omega_x(V) = 0$ *(iff* $\omega_x(V) = 0$), $\forall x \in H \Longleftrightarrow$
$\Longleftrightarrow \omega(V) = *(\eta_H \wedge \mathcal{A}_V) = 0$, *(H integrable)*

In the above proposition we have also introduced the vector field $\theta(V)$ and the vector valued (p-2)-form $\omega(V)$ which are called, respectively, *expansion* and *rotation* of V.

In order to show the suitability of these quantities, we shall consider in the following section the Maxwell equations from the point of view of its 2+2 almost product structure.

Vacuum Maxwell Equations

Every regular two-form F defines a 2+2 almost-product structure characterized by the volume element G (*geometric component* of F) of the time-like two-plane. One has $F = \alpha G + \beta * G$, and the vacuum Maxwell equations for F, $\delta F = \delta * F = 0$, may be written

$$d\phi = *(\delta G \wedge *G + \delta * G \wedge G) = \Phi$$

$$d\psi = *(\delta G \wedge G + \delta * G \wedge *G) = \Psi$$

where the *energetic* and *Rainich* scalars ϕ and ψ are related to the eigenvalues α and β of F by the expressions: $\alpha = \exp\phi\cos\psi$, $\beta = \exp\phi\sin\psi$. Then, it follows the Rainich theorem:[5] *a unit simple two-form G is the geometric component of a Maxwell field F iff it verifies* $d\Phi = 0$, $d\Psi = 0$. The almost-product structures defined by a solution G to these last equations are called *maxwellian structures*; they play an important role in many relativistic domains.[6]

In terms of our kinematic definitions, we have:

Proposition 2 *The gradients $d\phi = \Phi$ and $d\psi = \Psi$ of the energetic and the Rainich scalars of a maxwelian structure are respectively the sum of the expansions and the rotations of the two 2-plane fields of the structure:*

$$\Phi \equiv \theta(G) + \theta(*G), \qquad \Psi \equiv \omega(G) + \omega(*G)$$

Consequently, a 2+2 structure G is maxwellian if, and only if,

$$d[\theta(G) + \theta(*G)] = 0, \qquad d[\omega(G) + \omega(*G)] = 0$$

and it follows:

Corollary 1 *A maxwell field F has constant eigenvalues if, and only if, it defines an integrable and minimal structure.*

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