On the geometry of Killing and conformal tensors

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The second order Killing and conformal tensors are analyzed in terms of their spectral decomposition, and some properties of the eigenvalues and the eigenspaces are shown. When the tensor is of type I with only two different eigenvalues, the condition to be a Killing or a conformal tensor is characterized in terms of its underlying almost-product structure. A canonical expression for the metrics admitting these kinds of symmetries is also presented. The space–time cases 1+3 and 2+2 are analyzed in more detail. Starting from this approach to Killing and conformal tensors a geometric interpretation of some results on quadratic first integrals of the geodesic equation in vacuum Petrov-Bel type D solutions is offered. A generalization of these results to a wider family of type D space–times is also obtained. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2207717]

I. INTRODUCTION

Killing tensors are associated with first integrals to the geodesic equation. In the second order case, they define quadratic first integrals and they play a central role in the theory of separability of the Hamilton-Jacobi equation. The relationship between separability and Killing tensors was shown by Eisenhart[1] and abundant literature exists regarding this property (for example, see Ref. 2 and references therein).

Within the relativistic framework the study of Killing tensors grew when Walker and Penrose[3] showed how the existence of a Killing tensor explains the Carter results[4] on the integrability by variable separation of the geodesic equation in the Kerr solution. Since then a lot of studies have been devoted to determining and classifying the space–times admitting Killing tensors and also to obtaining the Killing tensors of a given metric. A summary of known results on this subject can be found in Ref. 5.

The problem of finding the metrics admitting a quadratic integral of the geodesic equation was established by Eisenhart.[1] He wrote the *intrinsic Killing tensor equations*, i.e., the Killing equations in terms of the eigenvectors e_i and the eigenvalues ρ_i of a Killing tensor, and he pointed out that (see Ref. 1, p. 129): "the problem of finding all V_n admitting a quadratic integral consists in finding a tensor g and an orthogonal ennuple e_i that satisfy the conditions obtained by the elimination of the ρ's from the intrinsic Killing tensor equations. The general solution has not been obtained, but we shall consider two particular solutions of the problem." Later, he considered the

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trivial case when all the ρ's are equal, and the case with different eigenvalues and normal principal congruences, a case which led to the Stäckel form of the metric.[6,1]

The general solution to the problem set by Eisenhart is far from being solved, although a number of results are known for some classes of Einstein-Maxwell solutions or algebraically special space–times, as well as those for flat metrics.[5] Nevertheless, the usual way in which this subject is tackled differs from the Eisenhart conception. Indeed, the common approach consists of studying the integrability conditions of the Killing tensor equations, whereas the Eisenhart method involves the following: (i) to write the intrinsic Killing tensor equations, (ii) to determine the equivalent equations involving exclusively the eigenspaces and the metric tensor (the eigenvalues having been removed), and (iii) to study the integrability conditions of the aforementioned equations. Both procedures, the usual one and Eisenhart's, may be suitable depending on the different situations. In this work we adopt the Eisenhart approach and we will show how useful it is by considering the case of Killing tensors with two complementary eigenspaces.

The conformal extension of the Killing tensor equation determines the conformal tensors which define first integrals to the null geodesic equation. Here we also analyze the Eisenhart problem for the class of conformal tensors with two complementary eigenspaces.

In the problem of finding the Riemannian spaces admitting a Killing or a conformal tensor two different aspects can be considered. On the one hand, we can look for a general canonical expression for the metric tensors with these kinds of first integrals. In this case, we must also obtain the expression of the Killing or conformal tensors in terms of the elements appearing in this canonical form. This approach may be useful in working in spaces with these symmetries, the adapted coordinates allowing calculations to be simplified and throwing light on the geometric interpretation of the expressions we can find.

On the other hand, we can give explicit and intrinsic conditions that characterize the metric tensors, and then we must offer the expression of the Killing or conformal tensors in terms of metric concomitants (namely, the Riemann tensor and its covariant derivatives). This approach is helpful in analyzing when a metric, which is known in an arbitrary coordinate system, has these kinds of symmetries. Moreover, we can obtain these tensorial symmetries without solving the Killing or conformal equations.

In this work we analyze both viewpoints. Regarding the first one, we can quote several results previously obtained in the relativistic framework. Thus, canonical forms for the four-dimensional space–time metrics admitting a Killing or a conformal tensor of type 2+2 have been proposed in literature.[7,8] In this case the Killing or conformal tensor admits two complementary eigenplanes. Here we generalize these results by considering a general $p+q$ tensor (with two complementary eigenspaces of dimensions p and q, respectively) in a generic Riemannian space with arbitrary signature and dimension.

The second approach, the intrinsic characterization of the metrics admitting Killing and conformal tensors, has also been partially considered in relativity. Thus, it is known that every Petrov-Bel type D vacuum solution admits a conformal tensor of type 2+2 which may be obtained form the Weyl tensor.[5] Here we extend this result by characterizing all the Petrov-Bel type D metrics with conformal tensors. Moreover we also identify the type D solutions admitting a Killing tensor, thus generalizing some results that are known for the vacuum case.[5]

It is worth remarking that the Eisenhart approach used here allows the intrinsic and explicit labeling of the metrics to be obtained easily. Indeed, in this approach we give conditions for the underlying 2+2 structure of the Killing or conformal tensors. Moreover, for the Petrov-Bel type D metrics, this is the principal structure one of the Weyl tensor, and it is explicitly known in terms of the metric tensor.[9] The reason why it is of interest to obtain an explicit and intrinsic characterization of a space–time metric has been pointed out elsewhere[10] and the method used here has been useful in labeling the Schwarszchild[10] and Reissner-Nordström[11] solutions, the static Petrov type I space–times[9] and the Petrov type I space–times admitting isotropic radiation.[12]

Here we show that the eigenspaces of a Killing or a conformal tensor are umbilical planes. Moreover they are totally geodesic for a conformal metric. This geometric interpretation could be useful in clarifying the role played by the Killing tensor in the separability theory.

The paper is organized as follows. Some notation, definitions and properties related to regular Riemannian p planes are introduced in Sec. II. In Sec. III we study some properties of the eigenvectors and eigenvalues of a Killing or a conformal tensor. The type I case (when the tensor admits an orthonormal basis of eigenvectors) is analyzed in detail in Sec. IV and we write the Eisenhart intrinsic Killing tensor equations in a form that is more useful to our purposes. In Sec. V we use this new form for the Killing tensor equations to analyze the Eisenhart problem when the Killing or the conformal tensor has two complementary eigenspaces. A canonical form for the metrics admitting these kinds of first integrals is presented in Sec. VI. In Sec. VII we study the $1+(n-1)$ case and outline when these Killing or conformal tensors are not reducible. In the last two sections some results concerning the usual four-dimensional space–time are obtained. The $2+2$ space–time structures associated to a Killing or conformal tensor are analyzed in detail in Sec. VIII. Finally, Sec. IX is devoted to obtaining an intrinsic and explicit characterization of the Petrov-Bel type D metrics admitting Killing or conformal tensors attached to its principal structure, and we also present an algorithm to obtain these quadratic first integrals in a given type D space–time.

II. SOME NOTATION AND USEFUL CONCEPTS

On an n-dimensional Riemannian manifold (M,g) we shall refer to a (regular) p-dimensional distribution V as a p plane. Let v be the projector on V and $h=g-v$ the projector on the plane orthogonal to V. The generalized second fundamental form of V is defined as the $(2,1)$-tensor Q_v given by

$$Q_v(x,y) = h(\nabla_{v(x)}v(y)) \tag{1}$$

for every pair of vector fields x, y. We can consider the decomposition of Q_v into its antisymmetric part A_v and its symmetric part $S_v \equiv S_v^T + (1/p)v \otimes \mathrm{Tr}\, S_v$, where S_v^T is a traceless tensor:

$$Q_v = A_v + \frac{1}{p}v \otimes \mathrm{Tr}\, S_v + S_v^T. \tag{2}$$

The plane V is a foliation if, and only if, $A_v=0$. In this case $Q_v=S_v$ and it coincides with the second fundamental form of the integral manifolds of the foliation V.[13] Moreover V is minimal, umbilical or geodesic if, and only if, $\mathrm{Tr}\, S_v=0$, $S_v^T=0$ or $S_v=0$, respectively. Then one can generalize these geometric concepts for plane fields which are not necessarily a foliation.

Definition 1: A plane field V is said to be geodesic, umbilical or minimal if the symmetric part S_v of its (generalized) second fundamental form Q_v satisfies $S_v=0$, $S_v^T=0$ or $\mathrm{Tr}\, S_v=0$, respectively.

From these definitions, and defining $\{x,y\}=\nabla_x y+\nabla_y x$, a lemma easily follows.

Lemma 1: A plane field V is umbilical for the metric g if, and only if, a vector field $\boldsymbol{a}$ exists such that $h(\{x,y\})=g(x,y)\boldsymbol{a}$ for every $x,y \in V$, h being the projector on the plane orthogonal to V.

On a n-dimensional Riemannian manifold (M,g) an almost-product structure is defined by a p-plane field V and its orthogonal complement H. The almost-product structures can be classified taking into account the invariant decomposition of the covariant derivative of the *structure tensor* $\Pi=v-h$. Likewise, they can be classified according to the foliation, minimal, umbilical or geodesic character of each plane.[14,15] We will say that a structure (V,H) is integrable when both planes are foliations and we will say that it is minimal, umbilical or geodesic if both of the planes are so.

In an oriented four-dimensional space–time (V_4,g) of signature $(-\ +\ ++)$ a more accurate classification for the almost-product structures follows taking into account the causal character of the planes.[16] Elsewhere[11] we have classified the Petrov-Bel type D space–times in accordance with the class of the $2+2$ principal structure of the Weyl tensor.

III. SECOND ORDER KILLING AND CONFORMAL TENSORS

The quadratic first integrals of the geodesic equation are associated with second rank *Killing tensors*.[1] Indeed, if K is a solution to the generalized *Killing equation*

$$[K,g] = 0 \quad ([K,g]_{abc} = \nabla_{(a} K_{bc)}), \tag{3}$$

then the scalar $K(v,v)$ is constant along an affine parametrized geodesic with tangent vector v.

It is known[5] that if K is a Killing tensor, its traceless part $P = K - (1/n)\mathrm{Tr}\, Kg$ is a *conformal tensor*, i.e., it satisfies the *conformal equation*

$$[P,g] = \mathcal{S}\{g \otimes t\}, \tag{4}$$

where t is, up to a factor, the divergence of P, $t = [2/(n+2)]\nabla \cdot P$, and $\mathcal{S}\{B\}$ denotes the total symmetrization of a tensor B. Then, the scalar $P(v,v)$ is constant along an affinely parametrized null geodesic with tangent vector v. Moreover, Killing equation (3) implies

$$2n \nabla \cdot P + (n+2)\mathrm{d}\,\mathrm{Tr}\, K = 0. \tag{5}$$

Then, we have the following.

Lemma 2: If K is a second rank Killing tensor [solution to (3)] then its traceless part $P = K - (1/n)\mathrm{Tr}\, Kg$ is a conformal tensor [solutions to (4)] and it satisfies

$$\mathrm{d} \nabla \cdot P = 0. \tag{6}$$

Conversely if a traceless conformal tensor P satisfies (6), a scalar π exists such that $\mathrm{d}\pi = \nabla \cdot P$. Then, $K = P - [2/(n+2)]\pi g$ is a Killing tensor.

In this work we analyze some properties of the eigenvalues and eigenspaces of Killing and conformal tensors and we present some of their properties. We proceed by studying both classes of tensors simultaneously and we will comment on the differences when they exist. So, if we consider a second rank tensor T solution to (4) the consequences on its eigenspaces and eigenvalues apply to both, Killing and conformal tensors. We particularize the conformal case by taking T as a traceless tensor. If we add condition (6), then T is the traceless part of a Killing tensor. But we can also recover the Killing tensor case by taking the vector t to be zero. It is worth pointing out that if P is a traceless conformal tensor, then $P + \Phi g$ is a conformal tensor, and both define the same first integrals of the null geodesic equation. Nevertheless, here we will always work with the traceless representative.

We denote E_ρ the eigenspace of T corresponding to the eigenvalue ρ. Then, if $x,y \in E_\rho$, a straightforward calculation leads to

$$[T,g](x,y,\cdot) = x(\rho)y + y(\rho)x + g(x,y)\mathrm{d}\rho - (T - \rho g)\{x,y\}. \tag{7}$$

On the other hand,

$$\mathcal{S}\{g \otimes t\}(x,y,\cdot) = g(x,y)t + g(t,x)y + g(t,y)x. \tag{8}$$

So, for two eigenvectors $x,y \in E_\rho$, the conformal condition (4) implies

$$(T - \rho g)\{x,y\} = g(x,y)s + g(s,x)y + g(s,y)x, \quad s \equiv \mathrm{d}\rho - t. \tag{9}$$

On the other hand, if we consider three eigenvectors x,y,z corresponding to three different eigenvalues, a similar calculation leads to

$$T(x,\{y,z\}) + T(z,\{x,y\}) + T(y,\{z,x\}) = 0. \tag{10}$$

Thus, we can state the following.

Lemma 3: Let T be a Killing (respectively, conformal) tensor. Then we have the following.

(i)   *If $x,y \in E_\rho$ are eigenvectors associated with the eigenvalue ρ, Eq. (9) holds, where the*

vector t is zero (*respectively,* $t=[2/(n+2)]\nabla\cdot T$).

(ii)   *If x,y,z are eigenvectors corresponding to three different eigenvalues, Eq. (10) holds.*

A consequence of Lemma 3 follows by taking $x=y$ in Eq. (9). Indeed, if one makes a new product with x one obtains

$$x^2 g(\mathrm{d}\rho - t, x) = 0 \tag{11}$$

and so, if x,y are non-null vectors, Eq. (9) becomes

$$(T - \rho g)\{x,y\} = g(x,y)(\mathrm{d}\rho - t). \tag{12}$$

If E_ρ is a regular eigenspace of T, then a basis of E_ρ formed with non-null eigenvectors exists and, consequently, (12) holds even for the null eigenvectors. Moreover, taking into account (11) we have the following.

Lemma 4: Let E_ρ be a regular eigenspace of a Killing (respectively, conformal) tensor T. Then (12) with $t=0$ (respectively, $t=[2/(n+2)]\nabla\cdot T$) holds for every $x,y\in E_\rho$.

Moreover, $\mathrm{d}\rho \in E_\rho^\perp$ *[respectively,* $2\nabla\cdot T-(n+2)\mathrm{d}\rho \in E_\rho^\perp$ *].*

IV. EIGENVALUES AND EIGENVECTORS OF SECOND ORDER KILLING AND CONFORMAL TENSORS OF TYPE I

Let us now go to type I Killing and conformal tensors, that is, those admitting an orthonormal basis of eigenvectors. In this case every eigenspace is regular and then the Killing (or conformal) equation implies (10) and (12). Moreover a basis of eigenvectors exists and, consequently, these restrictions are also sufficient conditions for T to be a Killing (or conformal) tensor. Thus, we have the following.

Proposition 1: Let T be a symmetric 2-tensor of type I and let E_i be the eigenspaces corresponding to the eigenvalues ρ_i. Then, T is a Killing (respectively, conformal) tensor if, and only if

(i) $(T-\rho_i g)\{x,y\}=g(x,y)(\mathrm{d}\rho_i-t)$, *for every* $x,y \in E_i$, *where the vector t is zero* (*respectively,* $t=[2/(n+2)]\nabla\cdot T$).

(ii) $T(x,\{y,z\})+T(z,\{x,y\})+T(y,\{z,x\})=0$, *for x, y, z, eigenvectors with different eigenvalue.*

Let K be a Killing tensor of type I and let $\{e_a\}$ and $\{\rho_a\}$ be an orthonormal basis of eigenvectors and the corresponding eigenvalues. A straightforward calculation allows us to write the two conditions in Proposition 1 in terms of $\{e_a\}$ and $\{\rho_a\}$ obtaining, in this way

$$\rho_a s_{bca} + \rho_b s_{cab} + \rho_c s_{abc} = 0, \quad a,b,c \neq , \tag{13}$$

$$e_a^2 e_b(\rho_a) - (\rho_b - \rho_a)s_{aab} = 0, \quad a \neq b, \tag{14}$$

$$e_b(\rho_b) = 0, \tag{15}$$

where s_{abc} are the symmetrized rotation coefficients, $s_{abc}=g(e_c,\{e_a,e_b\})$. If we set Eqs. (13)–(15) in terms of the rotation coefficients we easily recover the *intrinsic Killing tensor equations* obtained by Eisenhart.[1] In order to study the metrics which admit a second order Killing tensor, Eisenhart[1] started from these intrinsic equations and he looked for a set of equivalent conditions involving the eigenvectors exclusively. He considered the case when all the eigenvalues are equal and the case with different eigenvalues and normal principal congruences.[1] In this work we solve this Eisenhart problem for both the Killing and conformal tensors, when the second order tensor admits two complementary eigenspaces. We could also start from Eqs. (13)–(15) and similar conditions for the conformal case, but we will choose an alternative approach that makes the geometric properties of the eigenspaces of the Killing and conformal tensors more evident.

Let ρ_i and h_i be the eigenvalue and the projector associated with the eigenspace E_i, and let p_i be its dimension. Then

$$T = \sum \rho_i h_i, \quad g = \sum h_i, \quad \text{Tr } h_i = p_i. \tag{16}$$

With this notation, the second statement of Lemma 4 becomes $h_i(\mathrm{d}\rho_i - t) = 0$ and, consequently,

$$t = \sum h_i(\mathrm{d}\rho_i). \tag{17}$$

On the other hand, by projecting condition (i) in Proposition 1 on every eigenspace E_j one obtains

$$(\rho_j - \rho_i)h_j(\{x,y\}) = g(x,y)h_j(\mathrm{d}\rho_i - t). \tag{18}$$

So, if v_i denotes the projection on the orthogonal space $E_i^\perp$, one has

$$v_i(\{x,y\}) = g(x,y)\sum_{j \neq i} \frac{1}{\rho_j - \rho_i} h_j(\mathrm{d}\rho_i - t) \tag{19}$$

for every $x, y \in E_i$. Then, according to Lemma 1 and taking into account that t is zero for a Killing tensor and it can be written as (17) for a conformal one, we arrive to the following:

Theorem 1: *Let T be a symmetric 2-tensor of type I and let h_i be the projector corresponding to the eigenvalue ρ_i. Then, T is a Killing or a conformal tensor if, and only if,*

(i)   *The eigenspaces are umbilical subspaces, that is, their second fundamental form can be written as $S_i = \frac{1}{2} h_i \otimes \boldsymbol{a}_i$.*

(ii)   *For every eigenspace the trace of its second fundamental form $\text{Tr } S_i = (p_i/2)\boldsymbol{a}_i$ satisfies*

$$\boldsymbol{a}_i = \sum_{j \neq i} \frac{1}{\rho_j - \rho_i} h_j(\mathrm{d}\rho_i), \quad h_i(\mathrm{d}\rho_i) = 0, \quad \textit{for a Killing tensor}, \tag{20}$$

$$\boldsymbol{a}_i = -\sum_{j \neq i} h_j(\mathrm{d}\ln|\rho_i - \rho_j|), \quad \sum p_i\rho_i = 0, \quad \textit{for a conformal tensor}. \tag{21}$$

(iii) $T(x,\{y,z\}) + T(z,\{x,y\}) + T(y,\{z,x\}) = 0$, *for x, y, z, eigenvectors with different eigenvalues.*

The first condition of this theorem gives a geometric property involving the eigenvectors exclusively: every eigenspace is an umbilical subspace. Thus, it offers a decoupled equation that partially solves the Eisenhart problem. In the next section we will analyze the other two conditions in Theorem 1 for the case of two complementary eigenspaces. The last condition makes no sense in this case and we will see that the second one can be easily decoupled.

V. GEOMETRY OF KILLING AND CONFORMAL TENSORS OF TYPE $p+q$

A particular case of type I second order tensors are those having two complementary eigenspaces of dimensions p and $q = n - p$. So, a $p+q$ almost-product structure (V, H) is associated with these tensors, and we say that they are of type $p+q$. If v and h are the projectors onto the eigenspaces and α and β are the eigenvalues, such a tensor takes the form $T = \alpha v + \beta h$. In this case the previous theorem can be stated concisely in terms of the canonical elements $(v, h; \alpha, \beta)$ as

Proposition 2: A symmetric 2-tensor $K = \alpha v + \beta h$ of type $p+q$ is a Killing tensor if, and only if, the following conditions hold:

(i)   *The eigenstructure (V, H) is umbilical, that is, the second fundamental forms can be written as*

$$S_v = \tfrac{1}{2} v \otimes \boldsymbol{a}, \quad S_h = \tfrac{1}{2} h \otimes \boldsymbol{b}. \tag{22}$$

(ii)   *The traces of the second fundamental forms, $\text{Tr } S_v = (p/2)\boldsymbol{a}$ and $\text{Tr } S_h = (q/2)\boldsymbol{b}$, and the eigenvalues α, β are related by*

$$a = \frac{1}{\beta - \alpha} \mathrm{d}\alpha, \quad b = \frac{1}{\alpha - \beta} \mathrm{d}\beta. \tag{23}$$

A similar result takes place for conformal tensors as the following proposition says.

Proposition 3: A traceless symmetric 2-tensor $P = \alpha(qv - ph)$ of type $p + q$ is a conformal tensor if, and only if, the following conditions hold:

(i) *The eigenstructure (V, H) is umbilical, that is, the second fundamental forms can be written as*

$$S_v = \tfrac{1}{2} v \otimes a, \quad S_h = \tfrac{1}{2} h \otimes b. \tag{24}$$

(ii) *The traces of the second fundamental forms, $\mathrm{Tr}\, S_v = (p/2)a$ and $\mathrm{Tr}\, S_h = (q/2)b$, and the scalar α are related by*

$$a + b = - \,\mathrm{d} \ln |\alpha|. \tag{25}$$

It is worth remembering that, for the space-time $2+2$ case, the umbilical nature of the structure is equivalent to the geodesic and shear-free character of its two null principal directions.[11] Consequently, the above propositions generalize some results for the space–time Killing and conformal tensors of type $2+2$ (see Ref. 5, theorem 35.4) to an arbitrary dimension n and an arbitrary type $p + q$. Now we want to remark that the covariant formalism used here allows us to accomplish the second step in the Eisenhart method: the characterization of the Killing and conformal tensors in terms of their eigenspaces.

The characterization of a $p + q$ Killing or conformal tensor presented in the propositions above involves the structure tensor [conditions (i) and (ii)] and the eigenvalues [condition (ii)]. The next step consists of removing the eigenvalues in order to obtain the conditions that an almost product structure must satisfy in order to be the eigenstructure of a Killing or a conformal tensor. Condition (ii) of Proposition 2 can be written as

$$(\alpha - \beta)a = - \,\mathrm{d}\alpha, \quad (\alpha - \beta)b = \mathrm{d}\beta. \tag{26}$$

Then we have $(\beta - \alpha)(a + b) = \mathrm{d}(\alpha - \beta)$. If we differentiate (26) and make the substitution of $\mathrm{d}(\alpha - \beta)$ we get

$$\mathrm{d}a + a \wedge b = 0, \quad \mathrm{d}b + b \wedge a = 0. \tag{27}$$

Conversely, if a, b satisfy Eqs. (27), two functions x, y exist such that

$$a + b = \mathrm{d}x, \quad a - b = e^x \mathrm{d}y.$$

Then, taking $\alpha = e^{-x} - y$ and $\beta = -e^{-x} - y$, Eq. (26) is satisfied and $K = \alpha v + \beta h$ is a Killing tensor provided that (22) holds. The freedom in choosing x and y leads to the family of Killing tensors $CK + Dg$, C and D being arbitrary constants.

In the same way, condition (25) for a conformal tensor implies that $\mathrm{d}(a + b) = 0$. Conversely, if $\mathrm{d}(a + b) = 0$, a function x exists such that $a + b = \mathrm{d}x$. Then, the traceless tensor $P = e^{-x}(qv - ph)$ is a conformal Killing tensor provided that (24) holds. The freedom in choosing x leads to the family CP, C being an arbitrary constant. Thus, we have obtained the following.

Theorem 2: *The necessary and sufficient conditions for a $p + q$ almost-product structure (V, H) to be the eigenstructure of a Killing or a conformal tensor are the following.*

(i) *(V, H) is umbilical, that is, the second fundamental forms take the expression*

$$S_v = \tfrac{1}{2} v \otimes a, \quad S_h = \tfrac{1}{2} h \otimes b. \tag{28}$$

(ii) *The traces, $\mathrm{Tr}\, S_v = (p/2)a$ and $\mathrm{Tr}\, S_h = (q/2)b$, of the second fundamental forms satisfy*

$$\mathrm{d}a + a \wedge b = 0, \quad \mathrm{d}b + b \wedge a = 0 \quad \text{for Killing tensors}, \tag{29}$$

$$\mathrm{d}(\boldsymbol{a} + \boldsymbol{b}) = 0 \quad \textit{for conformal tensors}. \tag{30}$$

If (28) *and* (29) *hold, two functions x, y exist such that* $\boldsymbol{a} + \boldsymbol{b} = \mathrm{d}x$, $\boldsymbol{a} - \boldsymbol{b} = e^x \, \mathrm{d}y$. *Then taking* $\alpha = e^{-x} - y$, $\beta = -e^{-x} - y$, $K = C(\alpha v + \beta h) + Dg$ *is a Killing tensor, C and D being two arbitrary constants.*

If (28) *and* (30) *hold, a function x exists such that* $\mathrm{d}x = \boldsymbol{a} + \boldsymbol{b}$. *Then*, $P = Ce^{-x}(qv - ph)$ *is a conformal Killing tensor, C being an arbitrary constant.*

This theorem offers the second step in solving the Eisenhart problem for Killing or conformal tensors with two complementary eigenspaces. In fact, once the eigenvalues have been removed, we have obtained necessary and sufficient conditions involving the sole eigenspaces. In Sec. VIII we will see that, for the space–time 2+2 case, these conditions can be written as tensorial conditions on the structure tensor (or on the canonical 2-form associated with the structure). This fact allows us to give an intrinsic and explicit characterization of the four-dimensional Petrov-Bel type D space–times admitting a Killing or a conformal tensor in Sec. IX.

VI. METRICS ADMITTING A KILLING OR A CONFORMAL TENSOR OF TYPE *p + q*

In this section we show that a metric admitting a Killing or a conformal tensor of type $p + q$ admits a canonical expression in terms of a particular conformal metric and a specific conformal factor. First we state a corollary which trivially follows on from Propositions 2 and 3.

Corollary 1: Let (V, H) *be a p+q almost-product structure for the metric tensor g. The following statements are equivalent:*

(i) (V, H) *is a p+q totally geodesic almost-product structure.*
(ii) $Cv + Dh$ *is a Killing tensor, C and D being arbitrary constants.*
(iii) $C(qv - ph)$ *is a conformal tensor, C being an arbitrary constant.*

This corollary states that the Riemannian spaces admitting a second order Killing tensor with constant eigenvalues are those admitting a $p + q$ totally geodesic structure (V, H). We will show now that these Riemannian spaces generate all the spaces admitting Killing or conformal tensors by using an adequate conformal transformation.

The umbilical property is known to be a conformal invariant.[15,11] Moreover, if we take into account the change of the second fundamental form through a conformal transformation,[11] condition (25) for a conformal tensor states that the eigenstructure (V, H) is minimal for the conformal metric $\tilde{g} = |\alpha|^{-1}g$. Consequently, the family of metrics that admit a $p + q$ conformal tensor are those that are conformal to a metric which admits a totally geodesic $p + q$ structure. More precisely, we have the following.

Proposition 4. The metrics g that admit a p+q conformal tensor are those that may be written as $g = |\alpha|\tilde{g}$, *where* $\tilde{g}$ *is a metric admitting a totally geodesic p+q structure* (V, H).

Then the conformal tensor for g is $P = C\alpha(qv - ph)$, *C being an arbitrary constant.*

This proposition and Corollary 1 generalize to an arbitrary dimension n and an arbitrary type $p + q$ a result by Hauser and Malhiot[8] concerning the 2+2 space–time case. Moreover we also recover another known result easily:[17] a (contravariant) conformal tensor for a metric is a conformal tensor for every conformally related metric.

A similar result holds for Killing tensors. In fact, the sum of expressions (29) says that $\mathrm{d}(a + b) = 0$, which is exactly the condition necessary for (V, H) to be the eigenstructure of a conformal tensor, and so the metric is conformal to a metric admitting a $p + p$ totally geodesic structure. But now, the conformal factor is not arbitrary because it must satisfy the two equations in (29). A detailed analysis of these conditions leads to the following.

Proposition 5: The metrics g that admit a p+q Killing tensor are those that may be written as $g = |\alpha - \beta|\tilde{g}$, *where* $\tilde{g}$ *is a metric admitting a totally geodesic p+q structure* (V, H), *and* α *and* β *are functions such that* $v(\mathrm{d}\alpha) = 0$, $h(\mathrm{d}\beta) = 0$.

Moreover, the Killing tensor for g is $K = C(\alpha v + \beta h) + Dg$, *C and D being arbitrary constants.*

The two propositions above imply that the study of the Riemannian spaces admitting a Killing

or a conformal tensor reduces to the study of the metrics $\widetilde{g}$ admitting a totally geodesic $p+q$ structure. As Proposition 4 states, for every metric $\widetilde{g}$ of this type we obtain a metric g admitting a conformal tensor by using an arbitrary conformal factor, $g=\Omega^2\widetilde{g}$.

Nevertheless, Proposition 5 states that the richness of metrics admitting a Killing tensor conformally related to a $\widetilde{g}$ of this type depends on the quantity of normal directions aligned with one of the planes of the structure. This fact induces a classification of the metrics admitting a totally geodesic $p+q$ structure.

In the more regular metrics no aligned normal direction exists and only constant conformal factors can be considered, the Killing tensor then have constant eigenvalues.

The more degenerate class corresponds to the product metrics $\widetilde{g}=\widetilde{v}+\widetilde{h}$, $\widetilde{v}_{AB}(x^C)$ and $\widetilde{h}_{ij}(x^k)$ being two arbitrary p and q dimensional metrics, respectively; then, the available conformal factors are $\Omega^2=|\alpha-\beta|$, $\alpha(x^k)$ and $\beta(x^C)$ being arbitrary functions depending on the product coordinates and they coincide with the Killing tensor eigenvalues.

An intermediate situation occurs when, for example, only one normal aligned direction exists on each plane. Then, through the adequate conformal transformation we can obtain a metric admitting a Killing tensor with nonconstant eigenvalues. In dealing with $2+2$ space–time Killing tensors this case leads to the Hauser and Malhiot[7,8] canonical form for the metric.

VII. KILLING AND CONFORMAL TENSORS OF TYPE $1+(n-1)$

Let us consider the case of a $1+(n-1)$ structure (V,H) defined by the unitary direction u ($u^2=\epsilon=\pm1$) and its orthogonal complement. Then $g=v+h$ where $v=\epsilon u\otimes u$ and $h=g-\epsilon u\otimes u$. In terms of the usual kinematic coefficients of u ($\nabla u=\epsilon u\otimes\dot{u}+[1/(n-1)]\theta h+\sigma+\Omega$) the (generalized) second fundamental forms are

$$Q_v=u\otimes u\otimes\dot{u},\quad Q_h=-\epsilon\left(\frac{1}{n-1}\theta h+\sigma+\Omega\right)\otimes u. \tag{31}$$

The condition for (V,H) to be an umbilical structure just states $\sigma=0$, and then

$$S_v=u\otimes u\otimes\dot{u},\quad S_h=-\epsilon\frac{1}{n-1}\theta h\otimes u. \tag{32}$$

Thus taking into account Theorem 2, we find that the necessary and sufficient condition for u to define the eigenstructure of a conformal tensor is

$$\sigma=0,\quad \mathrm{d}\left(\dot{u}-\frac{\theta}{n-1}u\right)=0. \tag{33}$$

But these conditions state that u defines the direction of a conformal Killing vector.[18] Thus, we have the following.

Proposition 6: A $1+(n-1)$ structure defined by the unitary direction u is the eigenstructure of a conformal Killing tensor if, and only if, u defines the direction of a conformal Killing vector, that is, it satisfies (33).

This proposition implies that every traceless conformal tensor of type $1+(n-1)$ is the traceless part of $\xi\otimes\xi$, ξ being a Killing conformal vector. In other words, *every $1+(n-1)$ conformal tensor is reducible*

A similar procedure allows us to characterize the fact that u defines the eigenstructure of a $1+(n-1)$ Killing tensor. But in this case we find that it is not, necessarily, reducible. Indeed, taking into account (32) the condition (29) of Theorem 2 is equivalent to

$$d\left(\dot{u} - \frac{\theta}{n-1}u\right) = 0, \quad \theta\, du + d\theta \wedge u + 2\epsilon\theta u \wedge \dot{u} = 0.$$

When $\theta = 0$ these equations hold if $d\dot{u} = 0$, that is, if u defines the direction of a Killing vector. On the contrary, if $\theta \neq 0$, the second equation implies $du \wedge u = 0$, and so $du = \epsilon u \wedge \dot{u}$. In this case, u defines the direction of a normal conformal Killing vector and the second equation can be written as

$$d(\theta^{1/3}u) = 0. \tag{34}$$

These results are summarized in the following.

Proposition 7: The $1 + (n-1)$ structure defined by the unitary direction u is the eigenstructure of a Killing tensor if, and only if, one of the following conditions hold:

(i)   *u defines the direction of a Killing vector, that is, it satisfies $\sigma = 0 = \theta$, $d\dot{u} = 0$.*
(ii)   *u defines the direction of a normal conformal Killing vector with integrant factor $\theta^{1/3}$, that is, it satisfies equations (33) and (34).*

This proposition shows that we can distinguish two classes of Killing tensors of type $1 + (n-1)$. On the one hand, we have the reducible ones, that is, those that can be written as $\xi \otimes \xi + Bg$, ξ being a Killing vector and B an arbitrary constant. On the other hand, a class of irreducible Killing tensors that can be obtained from normal conformal Killing vectors. This last class has been considered by Koutras[19] and Rani *et al.*[17]

The results in the preceding section allow us to give the canonical form for the metric tensors admitting irreducible Killing tensors of type $1 + (n-1)$. Indeed, as the eigenstructure is integrable, the metric will be conformally related to a $1 + (n-1)$ product metric. Moreover Proposition 5 gives the conformal factor. Finally, we can state the following

Proposition 8: The metrics admitting an irreducible Killing tensor of type $1 + (n-1)$ are those that may be written as

$$g = |\alpha(x^i) - \beta(x^0)|[\epsilon\, dx^0 \otimes dx^0 + \gamma(x^i)], \tag{35}$$

where $\gamma(x^i)$ is an arbitrary $(n-1)$-dimensional metric.

The Killing tensor is then given by $C|\alpha - \beta|[\epsilon\alpha\, dx^0 \otimes dx^0 + \beta\gamma(x^i)] + Dg$, C and D being arbitrary constants.

VIII. SPACE–TIME KILLING AND CONFORMAL TENSORS OF TYPE [(11) (11)]

Let T be a Killing or a conformal tensor of type $[(11)\,(11)]$ in an oriented four-dimensional space–time (V_4, g) of signature $(-+++)$. Then T has two eigenspaces: a timelike two-plane V and its spacelike orthogonal complement H. The almost-product eigenstructure (V, H) is determined by the *canonical* unitary 2-form U, volume element of the timelike plane V. Then, the respective projectors are $v = U^2$ and $h = -(*U)^2$, where $U^2 = U \times U = \mathrm{Tr}_{23} U \otimes U$ and $*$ is the Hodge dual operator.

In order to study the geometric properties of a $2 + 2$ structure it is useful to introduce the self-dual unitary 2-form $\mathcal{U} \equiv (1/\sqrt{2})(U - i*U)$ associated with U. The metric on the self-dual 2-forms space is $\mathcal{G} = \frac{1}{2}(G - i\eta)$, where η is the metric volume element of the space–time, $G = \frac{1}{2}g \wedge g$ is the metric on the 2-forms space, and $\wedge$ denotes the double-forms exterior product, $(A \wedge B)_{\alpha\beta\mu\nu} = A_{\alpha\mu}B_{\beta\nu} + A_{\beta\nu}B_{\alpha\mu} - A_{\alpha\nu}B_{\beta\mu} - A_{\beta\mu}B_{\alpha\nu}$. Then, we can consider some first order differential concomitants of U that determine the geometric properties of the structure. Indeed, if $i(\cdot)$ denotes the interior product and δ the exterior codifferential, $\delta = *d*$, we have the following lemma.[11]

Lemma 5: Let us consider the $2 + 2$ structure defined by $\mathcal{U} = (1/\sqrt{2})(U - i*U)$. Then

(i)   *The traces of the second fundamental forms take the expression*

$$\mathrm{Tr}\, Q_v = \boldsymbol{a}[U] \equiv -i(\delta * U) * U, \quad \mathrm{Tr}\, Q_h = \boldsymbol{b}[U] \equiv i(\delta U)U. \tag{36}$$

(ii)   *The structure is umbilical, if, and only if,*

$$\Sigma[U] \equiv \nabla \mathcal{U} - i(\delta \mathcal{U})\mathcal{U} \otimes \mathcal{U} - i(\delta \mathcal{U})\mathcal{G} = 0. \tag{37}$$

With this notation, we can write the intrinsic equations in Propositions 2 and 3 for the case of Killing or conformal tensors of type $[(11)\,(11)]$ by using the eigenvalues and the canonical 2-form U exclusively.

Proposition 9: The traceless symmetric tensor $P = \alpha[U^2 + (*U)^2]$ is a conformal tensor if, and only if, the canonical elements $\{\alpha, U\}$ satisfy (37) and

$$-\,\mathrm{d}\ln|\alpha| = \Phi[U] \equiv i(\delta U)U - i(\delta * U)*U. \tag{38}$$

Proposition 10: The symmetric tensor $K = \alpha U^2 + \beta(*U)^2$ is a Killing tensor if, and only if, the canonical elements $\{\alpha, \beta, U\}$ satisfy (37) and (38) and

$$\mathrm{d}\alpha = (\alpha - \beta)i(\delta * U)*U. \tag{39}$$

This last proposition is the tensorial version of the intrinsic equations for a Killing tensor that are known in Newmann-Penrose formalism (Ref. 5, Theorem 35.4). Now we can easily write the conditions in Theorem 2 in terms of the canonical 2-form U, that is, we obtain the characterization of the Killing and conformal tensor in the sole variable U.

Theorem 3: The $2+2$ *structure defined by the unitary simple 2-form U is the eigenstructure of a conformal tensor if, and only if, U satisfies*

$$\Sigma[U] \equiv \nabla \mathcal{U} - i(\delta \mathcal{U})\mathcal{U} \otimes \mathcal{U} - i(\delta \mathcal{U})\mathcal{G} = 0, \tag{40}$$

$$\mathrm{d}\Phi[U] \equiv \mathrm{d}[i(\delta U)U - i(\delta * U)*U] = 0. \tag{41}$$

If these conditions hold, a function α exists such that $\Phi[U] = -\mathrm{d}\ln|\alpha|$. Then, the conformal tensor is $P = C\alpha[U^2 + (*U)^2]$, C being an arbitrary constant.

Theorem 4: *The* $2+2$ *structure defined by the unitary simple 2-form U is the eigenstructure of a Killing tensor if, and only if, U satisfies*

$$\Sigma[U] \equiv \nabla \mathcal{U} - i(\delta \mathcal{U})\mathcal{U} \otimes \mathcal{U} - i(\delta \mathcal{U})\mathcal{G} = 0, \tag{42}$$

$$\mathrm{d}\Phi[U] \equiv \mathrm{d}[i(\delta U)U - i(\delta * U)*U] = 0, \tag{43}$$

$$\mathrm{d}i(\delta U)U = i(\delta U)U \wedge i(\delta * U)*U. \tag{44}$$

If these conditions hold, two functions α and β exist such that $\Phi[U] = -\mathrm{d}\ln|\alpha - \beta|$ and $\mathrm{d}(\alpha+\beta) = 2(\alpha-\beta)[i(\delta U)U + i(\delta * U)*U]$. Then, the Killing tensor is $K = C[\alpha U^2 - \beta(*U)^2] + Dg$, C and D being two arbitrary constants

It is worth pointing out that the first order differential properties of a $2+2$ structure admit a kinematical interpretation[16] and, in particular, the umbilical conditions (40) and (42) equivalently imply that the two principal null directions of the structure are geodesic and shear-free congruences.[11] Thus, we recover a known result obtained independently by Hauser and Malhiot[7] and by Collinson.[20]

On the other hand, condition (41) states that the structure is pre-Maxwellian.[21,22] Then, taking into account the study of these structures given in Ref. 22 we have the following.

Corollary 2: The $2+2$ traceless tensor $P = \alpha(v-h)$ is a conformal tensor if, and only if, $T = \alpha^{-2}(v-h)$ is a conservative Maxwell-Minkowski energy tensor and the principal directions of the associated electromagnetic field are geodesic and shear-free congruences.

IX. PETROV-BEL TYPE D SPACE–TIMES ADMITTING KILLING OR CONFORMAL TENSORS

The results in the preceding sections help us to characterize intrinsically and explicitly some families of metrics. More precisely, in this section: (i) we obtain necessary and sufficient conditions on the metric concomitants for a four-dimensional space–time to be a Petrov-Bel type D solution admitting a 2+2 Killing or conformal tensor and, when they hold, (ii) we give an algorithm to determine these tensors.

In the preceding section we have characterized the 2+2 Killing and conformal tensors in terms of the volume element U of their timelike eigenplane. Moreover, for the case of Petrov-Bel type D metrics, this 2-plane determines the Weyl principal structure and, consequently, U can be obtained from the Weyl tensor. The intrinsic and explicit characterization of type D solutions and the covariant obtaining of the Weyl canonical bivector have been given in Ref. 9. Consequently we can state the following invariant characterizations.

Proposition 11: A Petrov-Bel type D metric admits a conformal tensor if, and only if, the Weyl principal null directions define geodesic shear-free congruences and the Weyl canonical 2-form satisfies (43).

A Petrov-Bel type D metric admits a Killing tensor if, and only if, the Weyl principal null directions define geodesic shear-free congruences and the Weyl canonical 2-form satisfies (43) and (44).

Finally, taking into account the algebraic results for Petrov-Bel type D metrics quoted above (see Ref. 9), we obtain from Theorems 3 and 4 the explicit expression of the conditions in Proposition 11 and the algorithm for obtaining the conformal or Killing tensors.

Theorem 5: Let $\mathcal{W} \equiv \mathcal{W}(g) = \frac{1}{2}(W(g) - \mathrm{i} * W(g))$ and $\mathcal{G} \equiv \mathcal{G}(g) = \frac{1}{2}\left(\frac{1}{2}g \wedge g - \mathrm{i}\eta(g)\right)$ *the self-dual Weyl tensor and self-dual metric associated with a space–time metric* g, *and let us take the metric concomitants*

$$\rho \equiv -\frac{\mathrm{Tr}\mathcal{W}^3}{\mathrm{Tr}\mathcal{W}^2}, \quad \mathcal{S} \equiv \frac{1}{3\rho}(\mathcal{W} - \rho\mathcal{G}), \quad \mathcal{U} \equiv \frac{\mathcal{S}(\mathcal{X})}{\sqrt{\mathcal{S}(\mathcal{X},\mathcal{X})}}, \tag{45}$$

$$\Sigma \equiv \nabla\mathcal{U} - i(\delta\mathcal{U})\mathcal{U} \otimes \mathcal{U} - i(\delta\mathcal{U})\mathcal{G}, \tag{46}$$

$$U \equiv \sqrt{2}\,\mathrm{Re}\{\mathcal{U}\}, \quad \boldsymbol{a} \equiv -i(\delta * U) * U, \quad \boldsymbol{b} \equiv i(\delta U)U, \tag{47}$$

where $\mathcal{X}$ *is an arbitrary self-dual bivector.*

The necessary and sufficient conditions for g *to be a Petrov-Bel type D solution admitting a* 2+2 *conformal tensor are*

$$\rho \neq 0, \quad \mathcal{S}^2 + \mathcal{S} = 0, \quad \Sigma = 0, \quad \mathrm{d}(\boldsymbol{a} + \boldsymbol{b}) = 0. \tag{48}$$

When (48) hold, a function α *exists such that* $-\mathrm{d}\ln|\alpha| = \boldsymbol{a} + \boldsymbol{b}$. *Then, the conformal tensor is* $P = C\alpha[U^2 + (*U)^2]$, C *being an arbitrary constant.*

The necessary and sufficient conditions for g *to be a type D solution admitting a* 2+2 *Killing tensor are* 48 *and*

$$\mathrm{d}\boldsymbol{b} + \boldsymbol{b} \wedge \boldsymbol{a} = 0. \tag{49}$$

When (48) and (49) hold, two functions α *and* β *exist such that* $-\mathrm{d}\ln|\alpha - \beta| = \boldsymbol{a} + \boldsymbol{b}$ *and* $\mathrm{d}(\alpha + \beta) = 2(\alpha - \beta)[\boldsymbol{b} - \boldsymbol{a}]$. *Then, the Killing tensor is* $K = C[\alpha U^2 - \beta(*U)^2] + Dg$, C *and* D *being two arbitrary constants.*

For Petrov-Bel type D solutions with a vanishing Cotton tensor (the Weyl tensor is divergence-free) the Bianchi identities take the expression[11]

$$\nabla\mathcal{U} = i(\delta\mathcal{U})[\mathcal{U}\otimes\mathcal{U} + \mathcal{G}], \quad i(\delta\mathcal{U})\mathcal{U} = \tfrac{1}{3}\mathrm{d}\ln\rho, \tag{50}$$

where $\mathcal{U}$ is the Weyl canonical bivector and ρ the double Weyl eigenvalue. The real part of the second equation in (50) states

$$\tfrac{2}{3}\mathrm{d}\ln|\rho| = \Phi[U] \equiv i(\delta U)U - i(\delta * U) * U. \tag{51}$$

Thus, the principal structure of a type D divergence-free Weyl tensor is umbilical and pre-Maxwellian and, as a consequence of Theorem 3, it is the structure of a conformal tensor. Moreover, in this case the eigenvalue of the conformal tensor can be obtained algebraically from the Weyl eigenvalues if we take into account Proposition 9. Thus, we have the following.

Theorem 6: *Every Petrov-Bel type D solution with vanishing Cotton tensor admits a conformal tensor. Let ρ, $\mathcal{S}$, and $\mathcal{U}$ be the Weyl concomitants given in* (45). *Then we have the following.*

(i)   *These space–times are characterized by the conditions*

$$\rho \neq 0, \quad \mathcal{S}^2 + \mathcal{S} = 0, \quad \delta W = 0. \tag{52}$$

(ii)   *The conformal tensor is given by*

$$P = C|\rho|^{-2/3}\mathcal{U}\times\tilde{\mathcal{U}}. \tag{53}$$

This theorem generalizes the result about the existence of conformal tensors in Petrov-Bel type D vacuum solutions (see Ref. 5, Theorem 35.2).

We finish with two comments. The characterization of the Killing or conformal tensors in terms of their underlying structure has allowed us to give an explicit and intrinsic labeling of the Petrov-Bel type D space–times admitting Killing or conformal tensors, as well as to generalize some known results on the existence of these symmetries. Furthermore, our Eisenhart-type approach to the Killing and conformal tensor may also be useful in analyzing and extending other properties. For example, it is known that all type D vacuum solutions that admit a Killing tensor, also admit a Killing-Yano tensor.[20,23] Our result here and those given in Ref. 22 allow us to generalize this property. This question and other related topics will be considered elsewhere.[24]

Our study of the geometry of the Killing and conformal tensors and the canonical expressions of the metric tensor in terms of this geometry can be applied, in particular, to n-dimensional Lorentzian metrics. We know that, for four-dimensional Petrov-Bel type D space–times, this underlying geometry is closely related with the Weyl tensor and, this fact allows us to determine the 2+2 Killing and conformal tensors (see Theorems 5 and 6). The generalization of these results to higher dimensions is an open problem that could be fruitful in some classes of the Weyl tensor. But this study will require a further analysis of the Weyl classification in higher dimensions.[25,26]

ACKNOWLEDGMENT

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