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On the Leibniz bracket, the Schouten bracket and the Laplacian

Bartolomé Coll[a)]
Systèmes de référence relativistes, SYRTE, Obsevatoire de Paris-CNRS, 75014 Paris, France

Joan Josep Ferrando[b)]
Departament d'Astronomia i Astrofísica, Universitat de València, 46100 Burjassot, València, Spain

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The Leibniz bracket of an operator on a (graded) algebra is defined and some of its properties are studied. A basic theorem relating the Leibniz bracket of the commutator of two operators to the Leibniz bracket of them is obtained. Under some natural conditions, the Leibniz bracket gives rise to a (graded) Lie algebra structure. In particular, those algebras generated by the Leibniz bracket of the divergence and the Laplacian operators on the exterior algebra are considered, and the expression of the Laplacian for the product of two functions is generalized for arbitrary exterior forms. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1738188]

I. INTRODUCTION

In mathematical physics, some operators of interest are not derivations of the underlying algebraic structures. Their complement to the Leibniz rule of derivation defines then a product, called the Leibniz bracket. The Leibniz bracket of a linear operator on an algebra is thus a bilinear form that gives rise to a new algebra, called the Leibniz algebra. Leibniz algebras present interesting properties, and this work concerns them.

In particular, if the Leibniz bracket of an operator (its adjoint action) is a derivation, the operator is of degree odd and its square vanishes or is also a derivation, then the Leibniz bracket is a Lie bracket.

This is the case, for example, in the antibracket formalism context,[1] for the exterior derivative considered as a second order differential operator on the differential forms of finite codimension: the antibracket can then be defined as the corresponding Leibniz bracket, and some of its known properties are simple consequences of the general results obtained here.

A similar situation occurs for the divergence operator over the exterior algebra, for which the Leibniz bracket is nothing but the Schouten[2,3] bracket (in another different context, an equivalent result has been obtained by Koszul[4]). The expression obtained here relating the Schouten bracket to the divergence operator is of interest in mathematical physics. It allows, for example, to express Maxwell equations in terms of Schouten bracket and to study *proper variations* of Maxwell fields.[5,6] It has been also used to express the electromagnetic field equations in a non linear theory which solves, in part, an old problem concerning the existence and physical multiplicity of null electromagnetic fields in general relativity.[6,7]

The Leibniz bracket of the commutator of two operators admits a simple expression: It is the commutator of the Leibniz bracket of every one of them with respect to the operation defined by the Leibniz bracket of the other one. For the Laplacian operator, which *appears* as the (graded) commutator of the divergence and the exterior derivative, the above expression may be applied directly to it, giving the following interesting result: The Leibniz bracket of the Laplace operator

[a)]Electronic mail: bartolome.coll@obspm.fr
[b)]Electronic mail: joan.ferrando@uv.es

acting over the exterior algebra equals the Leibniz bracket of the exterior derivative acting over the Schouten algebra. This gives an interesting generalization to the exterior algebra of the well known expression for the Laplacian of a product of functions, and it has been applied in the analysis of harmonic coordinates in General Relativity.[8]

II. LEIBNIZ ALGEBRA OF A GRADED OPERATOR

(a) Let $\mathcal{E}=\oplus\mathcal{E}_a$ be a commutative graded group and $\circ:\mathcal{E}\times\mathcal{E}\rightarrow\mathcal{E}$ an operation verifying $\mathcal{E}_a \circ\mathcal{E}_b\subseteq\mathcal{E}_{a+b+k}$. Although it is always possible to regraduate $\mathcal{E}$ so that k vanishes, we shall retain the above graduation to avoid confusion when using different operations, as we shall do; such a k will be called the *degree* of the operation $\circ$ (with respect to this graduation).

The known properties (and concepts) on a graded group $\mathcal{E}$ concerning an operation $\circ$ of degree zero admit an equivalent form, depending generically on the degree k, when an arbitrary graduation is considered. Thus, the k-graded operation $\circ$ is *commutative* (resp. *anticommutative*) if it verifies $A\circ B=\epsilon(-1)^{(a+k)(b+k)}B\circ A$ with $\epsilon=1$ (resp. $\epsilon=-1$), and it is *associative* if $A\circ(B\circ C) =(A\circ B)\circ C$.

If $\mathcal{E}$ is a module and $\circ$ is bilinear, $(\mathcal{E},\circ)$ is said a k-*graded algebra*. A *derivation of degree* r is a r-graded endomorphism $\mathbf{D}$ on $\mathcal{E}$, $\mathbf{D}(\mathcal{E}_a)\subseteq\mathcal{E}_{a+r}$, verifying the Leibniz rule $\mathbf{D}(A\circ B)=\mathbf{D}A\circ B+(-1)^{(a+k)r}A\circ\mathbf{D}B$. An anticommutative k-graded algebra $(\mathcal{E},[,])$ verifying the Jacobi identity $\oint(-1)^{(a+k)(c+k)}[[A,B],C]=0$ is said a k-*graded Lie algebra*. Jacobi identity states, equivalently, that the (a+k)-graded endomorphism ad$A$, ad$A(B)=[A,B]$, is a derivation on $(\mathcal{E},[,])$. If $(\mathcal{E},\circ)$ is a k-graded associative algebra, the commutator defines a k-graded Lie algebra.

Let $\mathcal{E}$ be a graded group, $\circ$ a k-graded operation and $\mathbf{P}$ a p-graded operator. When $\mathbf{P}$ does not satisfy the Leibniz rule, its "deviation" interests us. So we give the following definition: the *Leibniz bracket* $\mathrm{L}_{\mathbf{P}}\langle\circ\rangle$ of $\mathbf{P}$ with respect to $\circ$ is the (p+k)-graded operation given by

$$\mathcal{L}_{\mathbf{P}}\langle\circ\rangle(A,B)=A\circ\mathbf{P}(B)+(-1)^{p(a+k)}[\mathbf{P}(A)\circ B-\mathbf{P}(A\circ B)]. \tag{1}$$

Of course, $\mathbf{P}$ verifies the Leibniz rule iff the Leibniz bracket $\mathcal{L}_{\mathbf{P}}\langle\circ\rangle$ vanishes identically.

The Leibniz bracket of a linear operator with respect to a bilinear operation is a bilinear operation, so that: *when* $(\mathcal{E},\circ)$ *is a k-graded algebra and* $\mathbf{P}$ *is a p-graded endomorphism*, $(\mathcal{E},\mathcal{L}_{\mathbf{P}}\langle\circ\rangle)$ *is a* $(k+p)$*-graded algebra*. We call it the *Leibniz algebra* of $\mathbf{P}$ on $(\mathcal{E},\circ)$. If $\mathbf{P}$ and $\mathbf{Q}$ are, respectively, p- and q-graded endomorphisms, their commutator $[\mathbf{P},\mathbf{Q}]=\mathbf{PQ}-(-1)^{pq}\mathbf{QP}$ is a (p+q)-graded endomorphism. Then, taking into account that $\mathcal{L}_{\mathbf{P}}\langle\circ\rangle$ and $\mathcal{L}_{\mathbf{Q}}\langle\circ\rangle$ are, respectively, (p+k)- and (q+k)-graded bilinear operations, and applying successively relation (1), one obtains the fundamental result:

Theorem 1: *In a k-graded algebra* $(\mathcal{E},\circ)$, *the Leibniz bracket of the commutator of two endomorphisms is related to the Leibniz bracket of every one of them by*

$$\mathcal{L}_{[\mathbf{P},\mathbf{Q}]}\langle\circ\rangle=\mathcal{L}_{\mathbf{Q}}\langle\mathcal{L}_{\mathbf{P}}\langle\circ\rangle\rangle-(-1)^{pq}\mathcal{L}_{\mathbf{P}}\langle\mathcal{L}_{\mathbf{Q}}\langle\circ\rangle\rangle. \tag{2}$$

In Marx's style:[9] The Leibniz bracket of the commutator $[\mathbf{P},\mathbf{Q}]$ of two endomorphisms $\mathbf{P}$ and $\mathbf{Q}$ on the algebra $(\mathcal{E},\circ)$ equals the graded difference between the Leibniz bracket of $\mathbf{Q}$ on the Leibniz algebra $(\mathcal{E},\mathcal{L}_{\mathbf{P}}\langle\circ\rangle)$ of $\mathbf{P}$ and the Leibniz bracket of $\mathbf{P}$ on the Leibniz algebra $(\mathcal{E},\mathcal{L}_{\mathbf{Q}}\langle\circ\rangle)$ of $\mathbf{Q}$.

In particular, as $\mathbf{P}^2=\mathbf{P}\cdot\mathbf{P}$ is a 2p-graded operator, it follows that *for any odd-graded operator* $\mathbf{P}$, *one has*

$$\mathcal{L}_{\mathbf{P}^2}\langle\circ\rangle=\mathcal{L}_{\mathbf{P}}\langle\mathcal{L}_{\mathbf{P}}\langle\circ\rangle\rangle\ , \tag{3}$$

Let us note that $\mathcal{L}_{\mathbf{P}}\langle x\rangle$ may be thought as *an operator* $\mathcal{L}_{\mathbf{P}}$ *over any operation x on* $\mathcal{E}$. In this sense, theorem 1 says that $\mathcal{L}_{[\mathbf{P},\mathbf{Q}]}\langle x\rangle=[\mathcal{L}_{\mathbf{Q}},\mathcal{L}_{\mathbf{P}}]\langle x\rangle$, and relation (3) says that $\mathcal{L}_{\mathbf{P}^2}\langle x\rangle=(\mathcal{L}_{\mathbf{P}})^2\langle x\rangle$.

Theorem 1 shows directly the well known result that if $\mathbf{P}$ and $\mathbf{Q}$ are derivations on $(\mathcal{E},\circ)$, so is $[\mathbf{P},\mathbf{Q}]$. Also, from (3), it follows:

Lemma 1: The square $\mathbf{P}^2$ *of an endomorphism* $\mathbf{P}$ *of odd degree is a derivation on* $(\mathcal{E},\circ)$ *iff* $\mathbf{P}$ *is a derivation on* $(\mathcal{E},\mathcal{L}_{\mathbf{P}}\langle\circ\rangle)$.

On the other hand, if the operation $\circ$ is commutative or anticommutative, i.e., $A\circ B=\epsilon(-1)^{(a+k)(b+k)}B\circ A$, one can find the following result:

$$\mathcal{L}_{\mathbf{P}}\langle\circ\rangle(A,B)=\epsilon(-1)^{(a+k+p)(b+k+q)+p}\mathcal{L}_{\mathbf{P}}\langle\circ\rangle(B,A), \tag{4}$$

that is to say, *for a k-graded commutative (resp. anticommutative) algebra* $(\mathcal{E},\circ)$, *the* $(k+p)$*-graded Leibniz algebra* $(\mathcal{E},\mathcal{L}_{\mathbf{P}}\langle\circ\rangle)$ *is commutative (resp. anticommutative) if* $\mathbf{P}$ *is even-graded, and it is anticommutative (resp. commutative) if* $\mathbf{P}$ *is odd-graded*.

Let us denote, for simplicity, $\{A,B\}_{\mathbf{P}}=\mathcal{L}_{\mathbf{P}}\langle\circ\rangle(A,B)$. Then, when $(\mathcal{E},\circ)$ is a k-graded associative algebra and $\mathbf{P}$ an endomorphism, one has

$$\{A,B\circ C\}_{\mathbf{P}}-\{A,B\}_{\mathbf{P}}\circ C=(-1)^{p(b+k)}[\{A\circ B,C\}_{\mathbf{P}}-A\circ\{B,C\}_{\mathbf{P}}]. \tag{5}$$

(b) Let $(\mathcal{F},\circ)$ be a 0-graded associative and commutative algebra generated by its submodule $\mathcal{F}_1$, $\{,\}_{\mathbf{P}}$ be the Leibniz bracket of the p-graded endomorphism $\mathbf{P}$ on $(\mathcal{F},\circ)$, $\{A,B\}_{\mathbf{P}}\equiv\mathcal{L}_{\mathbf{P}}\langle\circ\rangle(A,B)$, and $\mathrm{ad}\{A\}_{\mathbf{P}}$ be the adjoint of A in the Leibniz algebra $(\mathcal{F},\{,\}_{\mathbf{P}})$, i.e., $\mathrm{ad}\{A\}_{\mathbf{P}}(B)\equiv\{A,B\}_{\mathbf{P}}$.

From the commutativity of $(\mathcal{F},\circ)$ and relation (4), Eq. (5) may be written $\{C,B\}_{\mathrm{ad}\{A\}_{\mathbf{P}}}=(-1)^{ca}\{A,B\}_{\mathrm{ad}\{C\}_{\mathbf{P}}}$. Then, it follows: *for any p-graded endomorphism* $\mathbf{P}$ *in* $(\mathcal{F},\circ)$, *one has*

$$\mathrm{ad}\{C\}_{\mathrm{ad}\{A\}_{\mathbf{P}}}=(-1)^{ca+a+p}\mathrm{ad}\{A\}_{\mathrm{ad}\{C\}_{\mathbf{P}}}. \tag{6}$$

In particular, $\mathrm{ad}\{A\}_{\mathbf{P}}$ obeys the Leibniz rule on the set $\{C\}\times\mathcal{F}$ iff $\mathrm{ad}\{C\}_{\mathbf{P}}$ does it on the set $\{A\}\times\mathcal{F}$. Thus, iff $\mathrm{ad}\{X\}_{\mathbf{P}}$ is a derivation for every $X\in\mathcal{F}_1$, $\mathrm{ad}\{A\}_{\mathbf{P}}$ verifies the Leibniz rule on $\mathcal{F}_1\times\mathcal{F}$. But an endomorphism that verifies the Leibniz rule on $\mathcal{F}_1\circ\mathcal{F}$ is a derivation on $(\mathcal{F},\circ)$, so that one has:

Lemma 2: If $\mathrm{ad}\{X\}_{\mathbf{P}}$ *is a derivation on* $(\mathcal{F},\circ)$ *for any X of* $\mathcal{F}_1$, *then* $\mathrm{ad}\{A\}_{\mathbf{P}}$ *is a derivation on* $(\mathcal{F},\circ)$ *for every A of* $\mathcal{F}$.

If $\mathbf{P}$ is a derivation on its induced Leibniz algebra $(\mathcal{F},\{,\}_{\mathbf{P}})$, the Leibniz rule may be written $[\mathbf{P},\mathrm{ad}\{A\}_{\mathbf{P}}]=\mathrm{ad}\{\mathbf{P}(A)\}_{\mathbf{P}}$. Then, applying theorem 1 it follows that $\mathrm{ad}\{A\}_{\mathbf{P}}$ is a derivation on $(\mathcal{F},\{,\}_{\mathbf{P}})$ when $\mathrm{ad}\{A\}_{\mathbf{P}}$ and $\mathrm{ad}\{\mathbf{P}(A)\}_{\mathbf{P}}$ are derivations on $(\mathcal{F},\circ)$. From this result and lemma 2 one has:

Lemma 3: If $\mathrm{ad}\{X\}_{\mathbf{P}}$ *is a derivation on* $(\mathcal{F},\circ)$ *for any* $X\in\mathcal{F}_1$ *and if* $\mathbf{P}$ *is a derivation on* $(\mathcal{F},\{,\}_{\mathbf{P}})$, *then* $\mathrm{ad}\{A\}_{\mathbf{P}}$ *is a derivation on* $(\mathcal{F},\{,\}_{\mathbf{P}})$ *for any* $A\in\mathcal{F}$.

For p odd, lemma 1 states that $\mathbf{P}$ is a derivation on $(\mathcal{F},\{,\}_{\mathbf{P}})$ iff $\mathbf{P}^2$ do it on $(\mathcal{F},\circ)$. On the other hand, it follows from relation (4) that $(\mathcal{F},\{,\}_{\mathbf{P}})$ is a p-graded anticommutative algebra. But under this condition Jacobi identity says equivalently that $\mathrm{ad}\{A\}_{\mathbf{P}}$ is a derivation on $(\mathcal{F},\{,\}_{\mathbf{P}})$. All that and lemma 3 lead to the following result:

Theorem 2: *For p odd, if* $\mathbf{P}^2$ *and* $\mathrm{ad}\{X\}_{\mathbf{P}}$, *for any X in* $\mathcal{F}_1$, *are derivations on* $(\mathcal{F},\circ)$ *then the Leibniz algebra* $(\mathcal{F},\{,\}_{\mathbf{P}})$ *is a p-graded Lie algebra*.

III. SCHOUTEN BRACKET, DIVERGENCE OPERATOR, AND LAPLACIAN

(a) Let Λ^p (resp. $\Lambda^{\star p}$) be the set of p-forms (*resp.* p-tensors) over the differential manifold M, that is to say, the set of completely antisymmetric covariant (resp. contravariant) tensor fields. Then, $\Lambda=\oplus\Lambda^p$ (resp. $\Lambda^\star=\oplus\Lambda^{\star p}$) with the exterior product $\wedge$ is a 0-graded associative and commutative algebra over the function ring $\chi=\chi(M)$: the *exterior covariant algebra* (*resp. exterior contravariant algebra*). We shall denote by α, β, γ the elements of Λ, and by A,B,C those of $\Lambda^\star$, with corresponding degrees a,b,c.

Denote the interior product $i(A)\beta$, $[i(A)\beta]_{\underline{b-a}}=(1/a!)\,A^{\underline{a}}\beta_{a,\underline{b-a}}$ if $a\leqslant b$, by (A,β) and put $(\beta,A)=(-1)^{a(b-a)}(A,\beta)$. When $X\in\Lambda^1$, one has the usual interior product $i(X)$ which is a derivation of degree -1 on $(\Lambda,\wedge)$. Moreover, one has

$$(\gamma,A\wedge B)=((A,\gamma),B)+(-1)^{ab}((B,\gamma),A)\quad \text{if } c=a+b-1, \tag{7}$$

$$(A\wedge B,\gamma)=(B,(A,\gamma)),\quad (\gamma,A\wedge B)=((\gamma,B),A)\quad \text{if } c\geqslant a+b. \tag{8}$$

Suppose now that M is a n-dimensional and oriented manifold, and let η be a (covariant) volume element, $\eta^{\star}$ being its (contravariant) dual: $\eta_{p,n-p}\eta^{\star p',n-p}=\epsilon(n-p)^{-1}\delta_{p}^{p'}, \epsilon=\pm 1$. Then, the Hodge operators are given by $*A=(\eta,A)$, $*\alpha=(\eta^{\star},\alpha)$ and verify $**A=\epsilon(-1)^{a(n-a)}A$. Therefore, if $a+b\leqslant n$,

$$*(A\wedge B)=(*B,A),\qquad *A\wedge\beta=*(\beta,A)\ . \tag{9}$$

The set of real numbers $\mathcal{R}$ being a sub-ring of the set of functions χ, $(\Lambda, \wedge)$ and $(\Lambda^{\star},\wedge)$ are χ-algebras and $\mathcal{R}$-algebras. The exterior differentiation d is a 1-graded $\mathcal{R}$-derivation on $(\Lambda, \wedge)$, and the codifferentiation (divergence up to sign) is a (-1)-graded $\mathcal{R}$-endomorphism given by $\delta=\epsilon(-1)^{na}*d*$. Then, from (9) it follows,

$$\delta(A,\beta)=(\delta A,\beta)+(-1)^{r}(A,d\beta),\quad r=a-b>0. \tag{10}$$

(b) It is known that for $X,Y\in\Lambda^{\star 1}$, $\delta(X\wedge Y)=(\delta X)Y-(\delta Y)X-L_{X}Y$, where L_X denotes the Lie derivative operator with respect to the vector field X. So that the operator δ is not a derivation on $(\Lambda^{\star},\wedge)$. Thus, it is possible to consider the Leibniz bracket $\{\ ,\ \}_{\delta}$ of the codifferential operator on the exterior contravariant algebra $(\Lambda^{\star},\wedge)$,

$$(-1)^{a}\{A,B\}_{\delta}=\delta A\wedge B+(-1)^{a}A\wedge\delta B-\delta(A\wedge B)\ . \tag{11}$$

Taking into account relations (7), (8), and (10), it is not difficult to show that, for any (a+b−1)-form γ, one has

$$(-1)^{a}i(\{A,B\}_{\delta})\gamma=(d(\gamma,B),A)+(-1)^{ab}(d(\gamma,A),B)-(d\gamma,A\wedge B)\ . \tag{12}$$

The Schouten bracket $\{\ ,\ \}$ of two contravariant tensors[2] is a first order differential concomitant that generalize the Lie derivative.[3] For p-tensors (antisymmetric contravariant tensors) this bracket is defined by its action over the closed forms,[10] $i(\{A,B\})\gamma=(d(\gamma,B),A)+(-1)^{ab}(d(\gamma,A),B)$. Comparing this relation and (12), it follows $\{A,B\}=(-1)^{a}\{A,B\}_{\delta}$, and one has the following form of the Koszul[4] result:

Theorem 3: *The Schouten bracket is, up to a graded factor, the Leibniz bracket of the operator δ on the exterior contravariant algebra* $(\Lambda^{\star},\wedge)$: $\{,\}=(-1)^{a}\{,\}_{\delta}$. *Explicitly*:

$$\{A,B\}=\delta A\wedge B+(-1)^{a}A\wedge\delta B-\delta(A\wedge B), \tag{13}$$

This result justifies that we name *Leibniz–Schouten bracket* the Leibniz bracket $\{,\}_{\delta}$ of the operator δ on the exterior contravariant algebra. Is is worth pointing out that both, the Schouten bracket and the Leibniz–Schouten bracket, define on the exterior contravariant algebra two *equivalent* structures of (-1)-graded algebra, which we name, respectively, *Schouten algebra* and *Leibniz–Schouten algebra*. Althought equivalent, it is to be noted that the Schouten algebra does *not* satisfies the standard writing of a Lie algebra properties, meanwhile the Leibniz-Schouten algebra does. Let us see that.

It is not difficult to see that $\forall X\in\Lambda^{\star 1}$, $\forall A\in\Lambda^{\star p}$, one has $\{X,A\}=L_{X}A$; that shows how the Schouten bracket generalizes the Lie derivative. Let us write $\{A,B\}\equiv L_{A}B$, $\forall A,B\in\Lambda^{\star}$; as it is known, $L_X, X\in\Lambda^{\star 1}$, is a derivation and δ is a (-1)-graded endomorphism on the 0-graded associative and commutative algebra $(\Lambda^{\star p},\wedge)$ such that $\delta^{2}=0$. As a consequence, the Leibniz–Schouten bracket $\{,\}_{\delta}$ satisfies the hypothesis of theorem 2 and so *the Leibniz–Schouten algebra* $(\Lambda^{\star},\{,\}_{\delta})$ *is a* (-1)-*graded Lie algebra*, that is, $\{\Lambda^{\star a},\Lambda^{\star b}\}_{\delta}\subseteq\Lambda^{\star a+b-1}$ and

$$\{A,B\}_{\delta}=-(-1)^{(a-1)(b-1)}\{B,A\}_{\delta},\qquad \oint(-1)^{(a-1)(c-1)}\{\{A,B\}_{\delta},C\}_{\delta}=0. \tag{14}$$

The Schouten bracket $\{,\}$ also satisfies $\{\Lambda^{\star a},\Lambda^{\star b}\}\subseteq\Lambda^{\star a+b-1}$, and the properties of the Leibniz–Schouten Lie algebra (14) can equivalently be written in terms of the Schouten bracket as

$$\{A,B\}=(-1)^{ab}\{B,A\}, \qquad \oint(-1)^{ac}\{\{A,B\},C\}=0\ . \tag{15}$$

Let us note that these last relations (15) satisfied by the Schouten algebra do not reduce, by any regraduation, to the standard ones of a Lie algebra.

Jacobi identity equivalently states, the following generalization for the Lie derivative with respect to the Lie bracket:

$$L_{\{A,B\}}=-(-1)^{a}[L_A,L_B].$$

Also, from lemmas 1 and 2 and taking into account the properties of the codifferential operator, it follows that: *(i) The codifferential operator* δ *is a* $\mathcal{R}$*-derivation on the Leibniz–Schouten algebra*:

$$-\delta\{A,B\}=\{\delta A,B\}+(-1)^{a}\{A,\delta B\}.$$

(ii) The operator L_A *is a* $\mathcal{R}$*-derivation on the exterior contravariant algebra*:

$$L_A(B\wedge C)=L_AB\wedge C+(-1)^{b(a-1)}B\wedge L_AC.$$

The property (i) gives the generalization of the commutator of the codifferential and Lie derivative operators:

$$[\delta,L_A]\equiv\delta L_A+(-1)^{a}L_A\delta=-L_{\delta A}\,.$$

On the other hand, Eq. (10) may be written $[i(\beta),\delta]=i(d\beta)$. But $i(\omega)$ is a derivation on $(\Lambda^{\star},\wedge)$ for any 1-form ω. Then, taking into account theorem 1, we have $\mathcal{L}_{i(\omega)}\langle\{\ \}_\delta\rangle=\mathcal{L}_{i(d\omega)}\langle\wedge\rangle$. In particular, when ω is a closed 1-form, then $i(\omega)$ is a derivation on the Leibniz–Schouten algebra.

(c) Suppose now M endowed with a (pseudo-)Riemannian metric g, allowing to identify $(\Lambda,\wedge)$ and $(\Lambda^{\star},\wedge)$. The Laplacian operator is then the *graded* commutator of the differential and codifferential operators:

$$\Delta=[d,\delta]\equiv d\delta+\delta d\ .$$

It is known that Δ is not a derivation on the exterior algebra. From theorem 1 its Leibniz bracket is given by:

Theorem 4: *The Leibniz bracket of the Laplacian operator on the exterior algebra equals the Leibniz bracket of the exterior derivative on the Leibniz–Schouten algebra*: $\mathcal{L}_\Delta\langle\wedge\rangle=\mathcal{L}_d\langle\{\ \}_\delta\rangle$. *Explicitly*:

$$\Delta\alpha\wedge\beta+\alpha\wedge\Delta\beta-\Delta(\alpha\wedge\beta)=\{d\alpha,\beta\}+(-1)^{a}\{\alpha,d\beta\}+d\{\alpha,\beta\}, \tag{16}$$

where α *and* β *are arbitrary* a*- and* b*-forms, respectively*.

This theorem gives the generalization to the exterior algebra of the expression for the Laplacian of a product of functions: $\Delta f.h+f.\Delta h-\Delta(f.h)=2(df,dh)$.

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