Thermodynamic perfect fluid. Its Rainich theory

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The conditions for a relativistic perfect fluid to admit a thermodynamic scheme are considered, and the necessary and sufficient requirements for a metric to define a thermodynamic perfect fluid space-time are given.

I. INTRODUCTION

Let g be the metric tensor of (a region of) a space-time, S its Einstein tensor, and let (M,T) be the pair of the definition equations M of a medium and of its energy tensor T. We call here *Rainich theory of the medium* the set of necessary and sufficient conditions on g insuring the existence of the pair (M,T) such that the Einstein equations $S = T$ (Ref. 1) hold.

It is clear that this definition is nothing but a direct extension to other media of results developed by Rainich[2] for the regular electromagnetic field; in it, T is the Maxwell–Minkowski energy tensor and M is the set of the vacuum Maxwell equations.

Rainich worked out his theory about seven years after the Einstein paper on the foundation of the general theory of relativity,[3] where both media, the perfect fluid and the electromagnetic field, were explicitly considered. It seems rather paradoxical that the perfect fluid had not, up to now, been the object of a work analogous to Rainich's one on the electromagnetic field.[4] We would like to comment here on four of the factors that have contributed to this situation.

(i) The apparent simplicity of the barotropic case. A Rainich theory involves two sets of equations: a first, algebraic, set ensuring that S has the same algebraic structure as T, and a second, generally differential set translating in terms of g (and its differential concomitants) the definition equations M. In the barotropic case, the second set reduces to the expression of the functional dependence of the two algebraically independent invariant scalars of S, so that to complete the Rainich theory of the barotropic perfect fluid one only needs to know the algebraic characterization of the perfect fluid energy tensor. It is true that to obtain it is an easy task. Nevertheless, because of the Lorentzian character of the metric, it is not so easy a task as it has been evoked in the literature;[5] in addition to imposing T to have a triple eigenvalue and be of algebraic type I, one must give the condition insuring that to the simple eigenvalue corresponds a *timelike* eigenvector. For symmetric tensors, the general problems of finding the causal character of the eigenspace associated to a given eigenvalue, and its application to the perfect fluid, have been solved only very recently;[6] we will need here these results.

(ii) The apparent multiplicity of fluid thermodynamics. Both the equations of electromagnetism and relativistic continuous media have been largely analyzed, discussed, and criticized from the beginning of relativity. But, meanwhile, the matter for the electromagnetic field has been, in general, to find for it a nonlinear system.[7] For thermodynamic continuous media, the matter has been to *establish the basic system* of equations, playing the role analogous to the Maxwell ones. And, as it is well known, there are many proposed versions for this basic system. This situation would indicate that thermodymamics is not yet ripe to be incorporated in relativistic continuous media. Nevertheless, Marle's work[8] pointed out in the opposed sense: many of these versions[9] may be obtained from a unique relativistic kinetic theory, their differences corresponding essentially to the different methods used to approximate the Boltzmann equation.[10] Furthermore, any two arbitrary versions differ in at least one of the following three aspects: the form of the conserved quantities (stress energy, momentum), the thermodynamic closure (generalized Fourier law, entropy balance), and the physical definition of the variables appearing in the equations. What is important here for us is that, generically,[11] the proposed versions, when reduced to the thermodynamic perfect fluid, differ at most in the third aspect,[12] that is to say: the thermodynamic perfect fluid is generically unique, up to an eventual redefinition of some of its variables.

(iii) The apparent independence of the thermodynamics from the energy tensor. In the usual presentation of the thermodynamic perfect fluid, the thermodynamic scheme is obtained by adding to the standard energy tensor a conserved matter current, an entropy relation, and an equation of state. It would seem that the existence of these three elements could not be deduced from the metric and the energy tensor itself, so that a Rainich theory would not be possible. Nevertheless, we shall see that a unique condition from the energy tensor guarantees the existence of such a thermodynamic scheme.

(iv) The wideness of Rainich's work. The work developed by Rainich[2] to geometricize the electromagnetic field was, fortunately, superabundant. In particular, he revealed the (weighted) $(2 + 2)$ almost-product structure associated to the electromagnetic field[13] and obtained the necessary and sufficient equations that the volume element U of the structure must verify in order to have a solution of the Maxwell equations. As similarly, a perfect fluid has an associated (weighted) $(1 + 3)$ almost-product structure, the extension of the Rainich work to the perfect fluid would implicate correspondingly the obtainment of the necessary and sufficient equations that the volume element u (Ref. 14) of the structure must verify in order to have a solution of the hydrodynamic equations.[15] Rainich considered also the uniqueness of the Maxwell field, which he solved, but globally, the corresponding uniqueness of the thermodynamic scheme would need to introduce some rather artificial *ad hoc* conditions. It is to palliate these features that we have chosen our above definition of a *Rainich theory*, which includes only *a part* of Rainich's work.

The above analysis shows that a Rainich theory of the thermodynamic perfect fluid may be boarded. But, is it worthwhile? We think there are, at least, four reasons to construct it: (i) A general medium may not admit a Rainich theory. What are the media admitting it? According to Misner and Wheeler's geometrical point of view,[16] the existence of a Rainich theory would be a necessary condition for such a medium to be realistic. In any case, these media admit such a particular physical characterization [see (iv) below] that the question about the *existence* of a Rainich theory is already an interesting question. (ii) A Rainich theory offers an alternative method[17] of integration of the Einstein equations: the set of all unknowns being reduced to the *metric coefficients*, the completed system of equations (the Einstein ones plus those corresponding to the set M) is now an *overdetermined system* (unless $M=\varnothing$), and the corresponding methods of compatibility conditions may be applied. (iii) This last consideration may be of interest in the study of those conjectures about perfect fluids which do not restrict the space of solutions of the hydrodynamic (test) equations, but restrict the space-times with which they are coupled;[18] due to this fact, it seems plausible that the Rainich theory may help their analysis. (iv) In the penultimate phase of a Rainich theory, the set M is reduced to a system of equations on the energy tensor: a medium which admits a Rainich theory is a medium which may be completely described in terms of the sole energy tensor variables. This fact may be of interest for practical purposes;[19] it is certainly of interest for conceptual and epistemological analysis.[20]

In Sec. II we find a simple, necessary, and sufficient condition for a perfect fluid to admit a thermodynamic scheme (Theorem 1), and in Sec. III we give the equations of the Rainich theory for it (Theorem 4). The barotropic and polytropic particular cases are given explicitly (Corollaries 2 and 3).

The results without proof of this paper were communicated to the Spanish relativistic meeting E.R.E. 87.[21]

II. CHARACTERIZATION OF THE THERMODYNAMIC PERFECT FLUID

A. Thermodynamic scheme

The energy conservation equations $\delta T=0$ (Ref. 22) for a perfect fluid $T=(\rho+p)u\otimes u-pg$ (Ref. 23) may be written

$$dp=(\rho+p)a+\dot{p}u, \tag{1}$$

$$(\rho+p)\theta+\dot{\rho}=0, \tag{2}$$

where a and θ are, respectively, the *acceleration* and the *expansion* of u: $a\equiv\dot{u}$, $\theta\equiv-\delta u$.

From the evolution point of view, the system (1), (2) is open. A usual algebraic closure is obtained by imposing a *barotropic* condition $\rho=\rho(p)$; however acceptable in some cases, it is known that this condition is too restrictive in many other interesting physical situations.[24] The standard general closure to the energy conservation system is the differential closure consisting of a *thermodynamic scheme*.

Let r be the (Eckart) *matter density*[25] of the fluid; denoting by $E\equiv\rho-r$ the *internal energy density* and by $\epsilon\equiv E/r$ the *specific internal energy*, one has

$$\rho=r(1+\epsilon). \tag{3}$$

When an equation of state, depending only on the internal structure of the fluid, is known,

$$\epsilon=\epsilon(p,r), \tag{4}$$

the one-form $d\epsilon+p\,dv$ is integrable, $v=1/r$ being the *specific volume*. Then, the *temperature* Θ of the fluid may be identified, by a classical argument, with an integrant divisor, and the *specific entropy* s is given, up to an additive constant, by

$$\Theta\,ds=d\epsilon+p\,dv. \tag{5}$$

As far as creation or annihilation of baryons do not take place,[26] the equation of conservation of matter holds:

$$\delta(ru)=0. \tag{6}$$

The relation (5) allows us to write Eq. (2) in the form

$$\delta(ru)=[r\Theta/f]\dot{s}, \tag{7}$$

where $f\equiv1+\epsilon+pv$ is the *enthalpy index* of the fluid;[27] Eq. (7) shows the intimate relation existing between the local adiabatic motion and matter conservation.

It is interesting to note that, while in classical thermodynamics, because of the nonequivalence between mass and energy, the internal energy E_V of a given volume V is determined *up* to an additive constant; in relativistic thermodynamics this energy is *univocally* determined once the matter density is given. However, this fact does not imply that the zero of the internal energy E_V be fixed in relativity; because of its noninertial character, the matter density is only determined up to a constant factor and, as a consequence, there still exists indeterminacy of E_V by an additive constant. Thus, if M and $M'=kM$ denote two mass balances[28] of the particles contained in V one has $r=M/V$, $r'=M'/V$ and it results in $E'_V=(1-k)M+E_V$. This observation is pertinent, for example, in the study of reaction fronts, where it allows us to localize conveniently the binding specific energy of the reaction,[29] or in the study of those hot perfect gases for which the limit $\Theta\to0$ is meaningless.[30]

B. Characterization theorem

Einstein equations for the thermodynamic perfect fluid being not easy to solve, one often, in a first step, looks for a solution to the general perfect fluid and, once obtained, in a second step, considers the admissibility by this solution of a thermodynamic scheme.

The existence of a thermodynamic scheme for a perfect fluid verifying Eqs. (1) and (2) amounts to the existence of functions ϵ and r such that Eqs. (3), (4), and (6) hold. As a consequence of (6), the equation in the function F,

$$\dot{F} = \theta, \tag{8}$$

must admit at least one solution of the form

$$F(x) = F(\rho(x), p(x)). \tag{9}$$

If this is the case, the one-form $\Gamma \equiv d\rho + (\rho + p)dF$ is integrable, the variables r, ϵ, and f may be defined by $r \equiv e^{-F}$, $\epsilon = \rho e^F - 1$, and $f = (\rho + p)e^F$, and, to every integral factor $D > 0$ for Γ, it may be associated an absolute temperature $\Theta = e^F/D$ and a specific entropy s such that $ds = D\Gamma$. Thus we have the following.

Lemma 1: The necessary and sufficient condition for a perfect fluid to admit a thermodynamic scheme is the existence of solutions of the form $F = F(\rho,p)$ to the equation $\dot{F} = \theta$. Then, every pair $\{F,D\}$ where $D > 0$ is an integral factor of the one-form $d\rho + (\rho + p)dF$, determines a thermodynamic scheme.

For a thermodynamic perfect fluid, Eq. (9) may be written in the equivalent form

$$dF = h(\rho,p)d\rho + g(\rho,p)dp, \tag{10}$$

which implies

$$\dot{F} = h\dot{\rho} + g\dot{p}. \tag{11}$$

On the other hand, from (2) and (8) one obtains $\dot{\rho} + (\rho + p)\dot{F} = 0$, so that (11) becomes

$$h\dot{\rho} + g\dot{p} = -\dot{\rho}/(\rho + p). \tag{12}$$

Suppose $\dot{\rho} = 0$; from (12) it is either $\dot{p} = 0$ or $g = 0$. If $\dot{p} = 0$, every arbitrary function $F = F(\rho,p)$ verifies (8); meanwhile if $g = 0$, they are the functions of the form $F = F(\rho)$ which verify (8). Suppose $\dot{\rho} \neq 0$; then if $g = 0$, from (12) we have $h = -1/(\rho + p)$ and (10) implies that $p = p(\rho)$: the fluid is barotropic. Finally, if $g \neq 0$ we have from (12)

$$\dot{p}/\dot{\rho} = [-1/g(\rho,p)]\{1/(\rho + p) + h(\rho,p)\}, \tag{13}$$

which implies that $\dot{p}/\dot{\rho}$ is a function of state:

$$\dot{p}/\dot{\rho} \equiv \chi(\rho,p). \tag{14}$$

Conversely, if (14) is verified, we can consider the following first-order partial differential equation:

$$F_{\rho}' + \chi F_{p}' = -1/(\rho + p). \tag{15}$$

Then, because of (2) and (11), every solution $F(\rho,p)$ to it is a solution to (8). Differentiating (14) and multiplying by $\dot{\rho}^2$, we obtain an equivalent expression which is identically satisfied for $\dot{\rho} = 0$, and thus we have the following.

Theorem 1: The necessary and sufficient condition for a perfect fluid $T = (\rho + p)u \otimes u - pg$ to admit a thermodynamic scheme is

$$(\dot{\rho}d\dot{p} - \dot{p}d\dot{\rho}) \wedge d\rho \wedge dp = 0. \tag{16}$$

Let $\lambda = \lambda(\rho,p)$ and $\mu = \mu(\rho,p)$ be two independent thermodynamic variables, $J = J(\lambda,\mu;\rho,p) \neq 0$. We know that $d\lambda \wedge d\mu = J\, d\rho \wedge dp$ so that, evaluating $\dot{\rho}\, d\dot{p} - \dot{p}\, d\dot{\rho}$ up to terms in $d\lambda$ and $d\mu$, one easily finds the following.

Corollary 1: Let $T = (\rho + p)u \otimes u - pg$ be a perfect fluid and $\lambda = \lambda(\rho,p)$ and $\mu = \mu(\rho,p)$ two independent thermodynamic variables. T admits a thermodynamic scheme iff

$$(\dot{\lambda}\, d\dot{\mu} - \dot{\mu}\, d\dot{\lambda}) \wedge d\lambda \wedge d\mu = 0. \tag{17}$$

III. RAINICH THEORY FOR THE THERMODYNAMIC PERFECT FLUID

Remember that if S is the Einstein tensor of the metric g, and if $\{M,T\}$ is the pair of definition equations of a medium, with T the energy tensor and M the complementary equations, then we call *Rainich theory of the medium* the set of conditions on g and on its differential concomitants, which ensure the existence of the pair $\{M,T\}$ verifying the Einstein equations $S = T$.

As everyone knows, the genuine Rainich theory concerns the regular Einstein–Maxwell equations. The pair $\{M,T\}$ is constituted of the set M of the vacuum Maxwell equations, $\delta F = \delta {}^*F = 0$, and of the Minkowski energy tensor T, $2T = F^2 + ({}^*F)^2$. Let us write $\mathbf{R} = \mathrm{Ric}(g)$, $\mathbf{r} = \mathrm{tr}\,\mathbf{R}$, $\mathbf{s} = \mathrm{tr}\,\mathbf{R}^2$, and define the one-form $\psi = \mathbf{s}^{-1} \cdot {}^*(\nabla\mathbf{R} \times \mathbf{R})$ (Ref. 31); the Rainich theory of the regular Einstein–Maxwell space-times consists[2] of the algebraic equations $\mathbf{r} = 0$, $\mathbf{R}^2 = (1/4)\mathbf{s}g \neq 0$, and the differential equations $d\psi = 0$; any metric g verifying these conditions defines an Einstein–Maxwell space-time corresponding to a regular solution to the source-free Maxwell equations.

A. Algebraic conditions

Let us consider the thermodynamic perfect fluid space-times. The pair $\{M,T\}$ consists now of the set M of Eq. (16), ensuring the existence of a thermodynamic scheme, and of the energy tensor $T = (\rho + p)u \oplus u - pg$.

The algebraic set of equations characterizing the perfect fluid energy tensor T were partially given by Taub[5]; we will present here a slightly different form of his result.[32] Let tr and I be, respectively, the trace operator and the identity over the second rank tensors; consider the *trace-removing* operator $\mathbf{Q} \equiv I - (1/4)g\,\mathrm{tr}$, and, for any second rank tensor T write $\mathbf{t} \equiv \mathrm{tr}\, T$ and $\mathbf{s} \equiv \mathrm{tr}\, T^2$; then we have the following lemma.

Lemma 2 (Taub's lemma): A second rank symmetric tensor T is of algebraic type I and admits a strictly triple eigenvalue if, and only if, it satisfies the following relations:

$$\mathbf{Q}(T^2 - \chi T) = 0,$$

$$4\mathbf{s} > \mathbf{t}^2, \quad 2\chi \neq \mathbf{t}.$$

This result says nothing about the causal character of the associated eigenvectors. Regarding them, the following lemma has been shown elsewhere.[6]

Lemma 3: A necessary and sufficient condition for the eigenvector associated to the single eigenvalue to be timelike is that the expression

$$\epsilon\{2i^2(x)T-\chi\}$$

be positive for any timelike vector x, where ϵ denotes the sign of the quantity $\mathbf{t}^3 - 6\mathbf{ts} + 8 \text{ tr } T^3$.

We are assuming that the perfect fluids considered here correspond to a macroscopic level of description. For this reason it is plausible to submit them to the *Plebański energy conditions*, which state that, for any observer, the energy density is positive definite and the Poynting vector is nonspacelike.[33] In terms of ρ and p, the Plebański conditions for the perfect fluid are equivalent to the inequalities $-\rho<p\leqslant\rho$, which may in turn be expressed as $\epsilon=1$ and $\chi\geqslant 0$. Taking into account the above two lemmas, one obtains the following theorem.[6]

Theorem 2: In a space-time of signature -2, a second rank symmetric tensor T defines algebraically a perfect fluid submitted to the Plebański energy conditions if, and only if,

$$\mathbf{Q}(T^2-\chi T)=0,$$
$$4\mathbf{s}>\mathbf{t}^2, \quad \mathbf{t}<2\chi\geqslant 0, \tag{18}$$
$$2i^2(x)T>\chi,$$

where $\mathbf{t}=\text{tr } T$, $\mathbf{s}=\text{tr } T^2$, $\mathbf{Q}=I-(1/4)g \text{ tr}$, and x is any timelike unit vector.

The intrinsic decomposition of T may then be obtained according to the following result.[6]

Theorem 3: The total energy ρ, the pression p, and the direction of the unit velocity u of a perfect fluid energy tensor T are given by

$$\rho=1/2(3\chi-\mathbf{t}), \quad p=1/2(\chi-\mathbf{t}), \tag{19}$$
$$u\propto i(x)T+px,$$

where

$$\chi\equiv 1/2(\mathbf{t}+z), \quad z=[(4\mathbf{s}-\mathbf{t}^2)/3]^{1/2}, \tag{20}$$

and x is any timelike vector.

B. General case

Let us write $\mathbf{R}\equiv\text{Ric}(g)$, $\mathbf{r}\equiv\text{tr }\mathbf{R}$, and $\mathbf{s}\equiv\text{tr }\mathbf{R}^2$; from Einstein equations, we have (Ref. 1) $\mathbf{R}=T-1/2\mathbf{t}g$ so that $\mathbf{r}=-\mathbf{t}=-\text{tr } T$ and $\mathbf{s}=\text{tr } T^2$. Taking into account these values in definitions (20) and the expressions (19), the Jacobian of $\mathbf{r}$ and $\mathbf{s}$ with respect ρ and p is given by $J(\mathbf{r},\mathbf{s};\rho,p)=-6(2\chi+\mathbf{r})$, which does not vanish under the third of the assumptions (18). Thus according to Corollary 1, the perfect fluid admits a thermodynamic iff (17) holds for $\lambda=\mathbf{r}$ and $\mu=\mathbf{s}$.

If $\dot{\mathbf{r}}=0$, (17) holds trivially; if $\dot{\mathbf{r}}\neq 0$, (17) is equivalent to

$$d(\dot{\mathbf{s}}/\dot{\mathbf{r}})\wedge d\mathbf{r}\wedge d\mathbf{s}=0, \tag{21}$$

and we have to evaluate the scalar $\dot{\mathbf{s}}/\dot{\mathbf{r}}$ in terms of the concomitants $\mathbf{R}$, $\mathbf{r}$, and $\mathbf{s}$ of the space-time metric g. To do it, let us observe that the direction of the unit velocity u, as given by the third of the relations (19), is the image of the endomorphism U given by

$$U\equiv T+pg=\mathbf{R}+1/4(z-\mathbf{r})g, \tag{22}$$

so that $u=\lambda i(y)U$, where y is *any* vector field not belonging to the kernel of U: $i(y)U\neq 0$. Thus, for any function f we have $f=i(u)df=i(df)u=\lambda i(df)i(y)U$; in particular, taking $f=\mathbf{r}$ and $y=d\mathbf{r}$, we have $\dot{\mathbf{r}}=\lambda i^2(d\mathbf{r})U$, which vanishes only if $d\mathbf{r}$ belongs to the kernel of U. Also, for $f=\mathbf{s}$ we have $\dot{\mathbf{s}}=\lambda i(d\mathbf{r})i(d\mathbf{s})U$ and, consequently,

$$\dot{\mathbf{s}}/\dot{\mathbf{r}}=i(d\mathbf{r})i(d\mathbf{s})U/i^2(d\mathbf{r})U. \tag{23}$$

On the other hand, let us note that the three inequalities expressed by the second and the third of the relations (18) are equivalent to $4\mathbf{s}>\mathbf{r}^2$ and $z\geqslant\mathbf{r}$, which are nothing but $-2\mathbf{s}^{1/2}<\mathbf{r}\leqslant\mathbf{s}^{1/2}$, as it is not difficult to show.

Finally, taking into account this result, Theorem 2, and expressions (21) and (22), we have the following.

Theorem 4 (Rainich theory of the thermodynamic perfect fluid)**:** A metric g defines a thermodynamic perfect fluid space-time with the Plebański energy conditions if, and only if, it verifies

$$-2\mathbf{s}^{1/2}<\mathbf{r}\leqslant\mathbf{s}^{1/2},$$
$$\mathbf{R}^2-2\pi\mathbf{R}+1/4(2\pi\mathbf{r}-\mathbf{s})g=0,$$
$$i^2(x)\mathbf{R}>\pi,$$

and

$$i^2(d\mathbf{r})U=0$$

or

$$d[i(d\mathbf{r})i(d\mathbf{s})U/i^2(d\mathbf{r})U]\wedge d\mathbf{r}\wedge d\mathbf{s}=0,$$

where $\mathbf{R}\equiv\text{Ric}(g)$, $\mathbf{r}\equiv\text{tr}\mathbf{R}$, $\mathbf{s}\equiv\text{tr }\mathbf{R}^2$, $\pi\equiv 1/4\{\mathbf{r}+[(4\mathbf{s}-\mathbf{r}^2)/3]^{1/2}\}$, $U\equiv\mathbf{R}+(\pi-\mathbf{r}/2)g$ and x is an arbitrary unit timelike vector field.

As a corollary of Theorem 3, the total energy density ρ, the pression p, and the direction of the unit velocity u of the perfect fluid are then give by

$$\rho=3\pi-\mathbf{r}, \quad p=\pi, \quad u\propto i(x)\mathbf{R}+(\pi-\mathbf{r}/2)x.$$

C. Barotropic case

Let us note that in the barotropic case, since the Jacobian $J(\mathbf{r},\mathbf{s};\rho,p)$ does not vanish, the condition $d\rho\wedge dp=0$ is equivalent to $d\mathbf{r}\wedge d\mathbf{s}=0$. Thus we have the following corollary.

Corollary 2: A metric g is a barotropic perfect fluid space-time with the Plebański energy conditions if, and only if, it verifies the algebraic relations of Theorem 4 and the differential equation $d\mathbf{r}\wedge d\mathbf{s}=0$.

Also, in the case of a polytropic fluid of index γ, $p=(\gamma-1)\rho$, it is easy to show the following result.

Corollary 3: A metric g defines a polytropic perfect fluid space-time with the Plebański energy conditions if, and only if, it verifies the algebraic relations of Theorem 4 and the equation $d(\mathbf{s}/\mathbf{r}^2)=0$. Then, if $\mathbf{s}/\mathbf{r}^2=c$, the polytropic index is given by

$$\gamma=\{4c-1+[(4c-1)/3]^{1/2}\}/(3c-1).$$

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[1]In pertinent units.

[2]G. Y. Rainich, Trans. Am. Math. Soc. **27**, 106 (1925).

[3]A. Einstein, Ann. Phys. **49**, 769 (1916).

[4]At least, we have not been able to find it.

[5]The Taub conditions for a (1,1) tensor to be the energy tensor for a perfect fluid [A.H. Taub, "Relativistic Hydrodynamics," in *Lectures in Applied Mathematics* (Am. Math. Soc., Providence, RI), 1967, Vol. 8, p. 170] though not explicitly stated, apply *only in absence of a given metric*; see our Sec. III A.

[6]J. A. Morales, Ph.D. thesis, València, 1988; see also C. Bona, B. Coll, and J. A. Morales, "Caracterización algebraica de un 2-tensor simétrico," in *Actas de los E.R.E.86* (Pub. Univ. València, València, to be published).

[7]Including the Maxwell equations as linear approximation.

[8]C. Marle, Ann. Inst. H. Poincaré **10**, 67 (1969); **10**, 127 (1969).

[9]The Eckart and the Landau–Lifchitz ones, among others.

[10]Marle considers the relativistic versions of the Chapman–Enskog and the Grad classical, methods.

[11]Here, "generically" means "for *almost all* the versions that have been proposed in the literature." Of course, there are always some exceptions; for example, the Arzeliés fluids [H. Arzeliés, *Fluides Relativistes* (Masson, Paris, 1971)].

[12]For example, the Catteneo fluids [C. Catteneo, Rend. Accad. Naz. dei Lincei **46**, Sér. VIII, 699 (1969)].

[13]Rainich called it the *skeleton* of the electromagnetic field.

[14]Here, this volume element is nothing but the unit velocity of the fluid.

[15]This task is not easy. Restricted to the barotropic fluid, it induces an eightfold classification of the unit velocity (see B. Coll and J. J. Ferrando, "On the velocities of the barotropic perfect fluids," to be published).

[16]C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 525 (1957).

[17]Usually, the corresponding differential equations are presented in the form of Cauchy or underdetermined systems for the metric coefficients *and* some other additional unknowns (pression, electromagnetic field, etc.).

[18]This is the case for Lichnerowicz's conjecture on spherical symmetry under appropriate asymptotic conditions [see H. P. Kunzle, Commun. Math. Phys. **20**, 85 (1971) and references therein], or the Treciokas–Ellis conjecture on vorticity-free or expansion-free consequences under distortion-free conditions [see R. Treciokas and G. F. R. Ellis, Commun. Math. Phys. **23**, 1 (1971), or the more recent analysis by C. B. Collins, J. Math Phys. **26**, 2009 (1985)].

[19]As an application to the Maxwell case, see, for example, B. Coll, F. Fayos, and J. J. Ferrando, J. Math. Phys. **28**, 1075 (1987).

[20]Fortunately, the development of field theory began, historically, with *force* field variables and not with *energy* field variables. Otherwise Maxwell equations should remain undiscovered; to think so, a glance on the nonlinear Rainich complexion equations is largely sufficient.

[21]B. Coll and J. J. Ferrando, "Fluido perfecto termodinamico. Su teoria 'à la Rainich'," in *Actas de los E.R.E.87* (Pub. Inst. Astrof. de Canarias, La Laguna, Spain, 1988).

[22]δ, $i(u)$, $\perp(u)$, ∇, d, $*$, denote, respectively, the divergence, interior product, normal projection, covariant derivative, exterior differentiation, and Hodge dual operators. Newton's notation is used for timelike derivatives: $\dot{x} \equiv i(u)\nabla$, x being any tensorial quantity.

[23]Of course, u is the *proper unit velocity* of the fluid, p the *pression*, and ρ the *total energy density*.

[24]C. B. Collins, J. Math. Phys. **26**, 2009 (1985).

[25]Also called *rest mass density, proper mass density, baryonic* (average) *mass density* or, simply *density* of the fluid.

[26]The definition of r as a mass balance of the baryonic number allows us to include in this scheme the study of the propagation of chemical reactions fronts; see B. Coll, Ann. Inst. H. Poincaré **25**, 363 (1976).

[27]See A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics* (Benjamin, New York, 1967).

[28]Molecular, atomic, or baryonic mass balances.

[29]See the paper quoted in Ref. 26.

[30]In such a case, one does not have necessarily $\epsilon = C_v \Theta \to 0$, and every γ-law, $p = (\gamma - 1)\rho$, may be interpreted as a polytropic perfect gas [see B. Coll, C. R. Acad. Sci. Paris A **273**, 1185 (1971)].

[31]The symbol $\times$ denotes the *cross product*; contraction of the adjacent spaces of the tensor product. Of course, the operator $*$ selects the antisymmetric part of $\nabla \mathbf{R} \times \mathbf{R}$.

[32]J. A. Morales, Ref. 6.

[33]See J. Plebański, Acta Phys. Pol. **26**, 963 (1964), especially his prudent analysis (pages 1011 and 1012) on the validity of his two conditions. In *The large scale structure of space-time* (Cambridge U.P., Cambridge, 1973), S. W. Hawking and G. F. R. Ellis call them the *weak* and *dominant* energy conditions, seeming to be unaware of Plebański's work.