On the velocities of the barotropic perfect fluids

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The conditions for a unit vector field to be the velocity of a relativistic barotropic perfect fluid are given. These conditions induce an eightfold classification of such fluids; for every class, the admissible barotropic variables are found. Some special cases, in particular polytropic fluids, are analyzed separately.

I. INTRODUCTION

In relativity, a perfect fluid is characterized by an energy tensor T of the form $T = (\rho + p) u \otimes u - pg$, where ρ is the total energy density, p is the pressure, and u is the (unit) velocity of the fluid, and g is the space-time metric. The conservation of T leads to a system of equations in (u,ρ,p), open from the evolutive point of view, which is usually closed by the adjunction of a barotropic relation $\rho = \rho(p)$. So completed, this system is called the *fundamental system* of barotropic hydrodynamics.

Thus in a given domain of the space-time, a barotropic perfect fluid is a solution $s \equiv (u,\rho(p),p)$ to the fundamental system. Let us denote by $\mathbf{U}$ the set of unit vector fields u, by $\mathbf{R}$ the set of functions of a single variable $\rho = \rho(p)$, and by $\mathbf{F}$ the set of functions p over the given domain of the space-time. In the total space $\mathbf{U}\times\mathbf{R}\times\mathbf{F}$, the space of solutions $\{s\}$ to the fundamental system defines, by circumscription, a parallelepiped $\mathbf{U}_b \times \mathbf{R}_b \times \mathbf{F}_b$.

The Cauchy problem for the fundamental system shows that $\mathbf{R}_b = \mathbf{R}$ or, in other words, that locally, *any* function of a single variable $\rho(p)$ is an element of a solution $(u,\rho(p),p)$ to the fundamental system.[1] Nevertheless, it can be shown that $\mathbf{U}_b$ is a *proper* subset of $\mathbf{U}$, $\mathbf{U}_b \neq \mathbf{U}$, that is, there does not exist, in general, a barotropic perfect fluid having as the velocity field an arbitrary unit vector field of $\mathbf{U}$. Thus it is natural to ask the following question: Is it possible to intrinsically define $\mathbf{U}_b$ or, more precisely, is it possible to express, solely in terms of u and its derivatives, the necessary and sufficient conditions for u to be the velocity field of a barotropic perfect fluid?

The answer, as we shall show, is affirmative. The search for the conditions on u leads to a classification of the unit vector fields in eight classes. For each class, we obtain the necessary and sufficient conditions on u and its differential concomitants for insuring that u is the velocity field of some barotropic perfect fluid. Furthermore, we give the holonomy potentials that allow us to determine the corresponding barotropic relations.

Similar problems to that of the intrinsic characterization of $\mathbf{U}_b$, but restricted to particular forms of the barotropic relation or to particular evolution laws, may be also considered. As an illustration, here we obtain the intrinsic characterization of the unit vector fields u that are the velocity fields of (i) perfect fluids with constant pressure, (ii) perfect fluids with constant total energy density, (iii) barotropic perfect fluids with u-invariant (i.e., constant along the streamlines) pressure, and (iv) polytropic fluids. For these cases, the conditions on u are simpler than those corresponding to the generic barotropic case.

From a formal point of view, the differential system in u defining the set $\mathbf{U}_b$ is nothing but the *conditional system* in the variable u associated to the fundamental system of barotropic hydrodynamics. In other very different contexts, such as thermodynamic perfect fluids,[2] electromagnetic fields,[3] and almost-product structures or Killing tensors,[4] we have already shown the conceptual interest of conditional systems.

Now, what is the interest of an intrinsic characterization of the barotropic velocities in hydrodynamics? We think that such a characterization may be of interest in many domains, as, for example, in the following.

(i) Our conditional systems allow one to divide the task of integration of the fundamental (test) system into two clearly defined steps: a first step in which, after selecting the desired class of velocities from our eightfold classification of the unit vector fields, one looks for a solution u to the corresponding conditional system and, once it is obtained, a second step in which, with the aid of our results on the holonomy potentials, one constructs the barotropic relations $\rho = \rho(p)$ associated to this u.

(ii) In the usual approach to the integration of the Einstein equations for barotropic perfect fluid space-times, one considers directly the Einstein system and its first integrability conditions; the problems of compatibility that appear because of the relation $\rho(p)$ are well known. Our characterization of $\mathbf{U}_b$ guarantees the existence of such a relation and allows one to relegate to a last, third step its computation: In a first step, taking local charts adapted to u, one translates the chosen conditional system in u into a system in the components of the space-time metric g; in a second step, for the corresending constrained form of g, one evaluates its Ricci tensor and imposes that u be an eigenvector; and finally, in a third step, one considers the remaining Einstein equations with respect to the barotropic relation(s) computed from the g obtained in the first step.

(iii) One of the few known results on the restrictions that the Einstein equations impose to the space of solutions of the fundamental (test) system is the Treciokas–Ellis[5] conjecture, recently reconsidered by Collins.[6] The conjecture

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states that a distortion-free barotropic perfect fluid space-time is either vorticity-free or expansion-free.[7] Because of its purely kinematical character, our associated conditional systems in u are well adapted to the study of this conjecture.

(iv) In given (vacuum, Robertson–Walker, etc.) space-times, it is sometimes interesting to know if some particular congruences may be interpreted as the streamlines of barotropic test perfect fluids (e.g., weak accretion in the neighbors of a star). The answer to this follows directly from our results by a simple, direct computation.

(v) Whatever its barotropic equation $\rho = \rho(p)$, a (test) barotropic perfect fluid may always evolve following any (static or stationary) Killing direction of any space-time. Nevertheless, the analog statement for conformally Killing directions is false: In fact, the only barotropic perfect fluid that may evolve following any conformally Killing direction of any space-time is that of isotropic radiation $\rho = 3p$ in equilibrium with dust of constant energy density. Properties such as these may be easily obtained from our characterization of the barotropic velocities.

(vi) Every barotropic velocity may be endowed with a barotropic relation $\rho(p)$ and, of course, also with other more general thermodynamic relations. We think that in the study of nonbarotropic perfect fluids or nonperfect fluids (anisotropy, viscosity, heat conduction), the hypothesis that their velocities are barotropic may be useful in the study of the behavior of such fluids. Either this hypothesis is incompatible (the actual motion of the fluid cannot be reproduced by any barotropic test fluid) or it is acceptable (one can compare the ideal barotropic variables to the actual thermodynamic ones). Both results constitute an interesting complement of information; in particular, the latter result may help us to better understand the limitations involved in the Eckart and Landau thermodynamic schemes.

(vii) For the taxonomy of the solutions of the fundamental (test) system and the Einstein equations, the eight classes of velocity vector fields not only allow one to label the known solutions, but also to play an heuristic role in the search of new solutions.

The paper is organized as follows. In order to make the proofs of the main result easier, in Sec. II the case $p =$ const is separate from the generic one, for which the data are reduced to a unit vector field and a holonomy potential. Section III contains the main results of this paper: the eightfold classification of the unit vector fields (Definition 1), the characterization of the barotropic velocities corresponding to each of these classes (Theorem 1), and the associated equations for the holonomy potentials (Theorem 2). Finally, in Sec. IV we characterize the velocities corresponding to some particular cases often found in the literature: constant pressure or density, u-invariant pressure, and polytropic fluids.

A portion of the present results (those leading to Theorem 1) with a sketch of the proof has been published elsewhere.[8]

II. THE BAROTROPIC PERFECT FLUID

Let (V_4, g) be the space-time $\text{sig}(g) = -2$. Vector and tensor fields and the expressions that relate them, unless otherwise stated, are given in their covariant form. The symbols $i(u)$, $*$, d, ∇, and δ denote, respectively, the interior product, Hodge dual, exterior derivative, covariant derivative, and divergence operators.

In a domain of (V_4, g), the conservation $\delta T = 0$ of the energy tensor T of a perfect fluid amounts to the system

$$dp = (\rho + p)a + p^0 u, \quad \theta + \rho^0/(\rho + p) = 0, \qquad (1)$$

where $a = i(u)\nabla u$ is the *acceleration vector*, $\theta \equiv -\delta u$ is the *expansion*, and $f^0 \equiv \pounds(u) f$ for any function f.

A *barotropic* relation is a functional relation between ρ and p of the form

$$d\rho \wedge dp = 0. \qquad (2)$$

When such a relation takes place (1) is called the *fundamental system* of barotropic hydrodynamics.

In the particular case of constant pressure $p = \bar{p} =$ const, system (1) becomes

$$a = 0, \quad \theta + \rho^0/(\rho + \bar{p}) = 0. \qquad (3)$$

Given u (and consequently, θ), the second of Eqs. (3) associates one solution ρ to every $\bar{p}$ and to every u-invariant function f ($f^0 = 0$). Although simple, we explicitly state this result for completeness in the following proposition.

Proposition 1: Perfect fluids with constant pressure have geodesic velocities. Conversely, to every geodesic (unit) vector field u one can associate a family of perfect fluids with arbitrary constant pressure $\bar{p}$ and energy density $\rho = f\rho_0 + (f - 1)\bar{p}$, where ρ_0 is a given solution to $\theta + \rho^0/(\rho + \bar{p}) = 0$ and f is any u-invariant function.

From here on, unless otherwise stated, we have consider $dp \neq 0$. Perfect fluids with a barotropic relation such that $dp \neq 0$ with be called *barotropic fluids*. Because of (2), there exists a (local) function π verifying

$$dp = (\rho + p) d\pi. \qquad (4)$$

This function is called the *holonomy potential*.[9] For a nonconstant p one has

$$p = p(\pi), \quad p'(\pi) = \rho + p \neq 0, \qquad (5)$$

where $p' \equiv dp/d\pi$. From (4) and (5), the first of Eqs. (1) may be written as

$$d\pi = a + \pi^0 u \neq 0, \qquad (6)$$

the scalar $\rho^0/(\rho + p)$ adopts the form

$$\rho^0/(\rho + p) = [\rho'(\pi)/(\rho + p)]\pi^0 = [p''(\pi) - p'(\pi)]/p'(\pi),$$

and the second of Eqs. (1) becomes

$$\theta = g(\pi)\pi^0, \qquad (7)$$

where

$$g(\pi) = 1 - (\ln p'(\pi))'. \qquad (8)$$

Conversely, let π be a function verifying (6), let $p(\pi)$ be an arbitrary function of π, and define $\rho(\pi)$ by

$$\rho(\pi) = p'(\pi) - p(\pi).$$

We have $dp = p'(\pi)d\pi = (\rho + p)d\pi$, $p^0 = (\rho + p)\pi^0$ and the first of Eqs. (1) follows. If in addition, $p(\pi)$ is a solution to (8), where $g(\pi)$ is determined by (7), the second of Eqs. (1) also follows. Thus we have shown the following proposition.

Proposition 2: The fundamental system for the barotropic perfect fluid is strictly equivalent to the system

$$d\pi = a + \pi^0 u \neq 0, \quad \theta = g(\pi)\pi^0 \tag{9}$$

in the pair (u,π). Given such a pair, for every solution $p(\pi)$ to $g(\pi) = 1 - (\ln p'(\pi))'$ the triple (u,ρ,p) with $\rho = p'(\pi) - p$ is a barotropic fluid.

Let us note that if p and p_0 are two solutions to Eq. (8), one has $(\ln p(\pi)')' = (\ln p_0'(\pi))'$, so that $p_0 = k_1 p + k_2$ with $k_1 \geqslant 0$ and $\rho_0 = k_1\rho - k_2$. Thus if (u,ρ,p) is a barotropic fluid associated to the solution (u,π) to (9), all the other barotropic fluids associated to the same solution (u,π) are given by the biparametric family

$$(u,\ k_1 \cdot \rho - k_2,\quad k_1 \cdot p + k_2), \tag{10}$$

where k_1 and k_2 are constants and $k_1 \geqslant 0$.

If (u,π) is such that $\pi^0 = 0$ one has, from (9), $da = 0$, $\theta = 0$. Thus for the solutions (u,π) to (9) that verify either $da \neq 0$ or $\theta \neq 0$, one has $\beta \equiv \pi^0 \neq 0$. The first of Eqs. (9) is (locally) equivalent to the equation expressing the closed character of the one-form $b = a + \beta u$ and the second equation implies that θ/β is a function of π, so that we have the following proposition.

Proposition 3: A unit vector field u such that $\theta \cdot da \neq 0$ is the velocity of a barotropic fluid if and only if there exists a function $\beta \neq 0$ such that

$$db = 0, \quad d(\theta/\beta) \wedge b = 0, \tag{11}$$

where $b = a + \beta u$. For every such β, the holonomy potential π is determined, up to a constant, by $d\pi = b$.

III. CLASSIFICATION AND CHARACTERIZATION OF THE BAROTROPIC VELOCITIES

A vorticity-free unit vector field u is equivalently defined by $w \equiv *(u \wedge du) = 0$ or $du = u \wedge a$. If τ and σ are two integrating factors for u corresponding, respectively, to the potentials t and s,

$$u = \tau\, dt = \sigma\, ds, \tag{12}$$

then the quotient τ/σ is a function of t; conversely, if τ is an integrating factor and τ/σ is a function of t, then σ is an integrating factor as well. Now, by differentiation and the interior product by u of the first equality in (12), one obtains

$$d\boldsymbol{\tau} = a + \boldsymbol{\tau}^0 u, \quad \boldsymbol{\tau} \equiv -\ln \tau, \tag{13}$$

so that if π verifies the first of Eqs. (9), one has $d(\pi - \boldsymbol{\tau}) = (\pi^0 - \boldsymbol{\tau}^0)u$ or, equivalently, $d(\pi - \boldsymbol{\tau}) \wedge dt = 0$, that is, $\pi = \boldsymbol{\tau} + H(t)$: The function $\exp(-\pi)$ is an integrating factor. Thus we have obtained the following proposition.

Proposition 4: Let u be a vorticity-free unit vector field. The necessary and sufficient condition for π to verify $d\pi = a + \pi^0 u$ is that $\exp\{-\pi\}$ be an integrating factor for u.

Now, if $u = \sigma\, ds$ with $da = 0$, taking into account Proposition 3, one has $0 = d[(\ln \sigma)^0 u] = d[(\ln \sigma)'_s \cdot ds] \rightarrow (\ln \sigma)'_s = H(s) \rightarrow \sigma = h(s)\tau$, where τ is such that $\tau^0 = 0$: We have the following lemma characterization for the vector field u having a u-invariant integrating factor.

Lemma 1: A vorticity-free unit vector field admits a u-invariant integrating factor if and only if $da = 0$.

Let u be such that $w = 0$ and $\theta = 0$. Then according to Proposition 4, the holonomy potentials π that are solutions to $d\pi = a + \pi^0 u$ are determined by the integrating factors of u. Since $\theta = 0$, the second of Eqs. (9) is verified by taking $g(\pi) = 0$ (or $\pi^0 = 0$). Thus we have the following proposition.

Proposition 5: A unit vector fiend u verifying $w = 0$ and $\theta = 0$ is the velocity vector of the barotropic fluids having the holonomy potential π of the form $\pi = -\ln \tau$, where τ is an arbitrary integrating factor.

When $w = 0$ and $\theta \neq 0$, the hypothesis of Proposition 3 is verified. In the geodesic case $a = 0$, the integrating factors are constant, $u = dt$, and consequently, the first of Eqs. (11) reduces to $d\beta \wedge u = 0$ and, from it, the second equation becomes equivalent to $d\theta \wedge u = 0$: β and θ are of the form $\beta = \beta(t)$, $\theta = \theta(t)$. Then $d\pi = b = \beta\, dt$: The holonomy potential is a function of t as well. We have the following proposition.

Proposition 6: A unit vector field u verifying $w = 0$, $a = 0$, and $\theta \neq 0$ is the velocity of a barotropic fluid if and only if $d\theta \wedge u = 0$. The holonomy potentials π are the arbitrary functions $\pi(t)$ of the potential t of u, $u = dt$.

Let us now consider u verifying $w = 0$, $\theta \cdot a \neq 0$, and $æ \equiv a \wedge da = 0$; then one has

$$da = \alpha u \wedge a, \tag{14}$$

where α is the scalar $\alpha \equiv i(a_*)i(u)da$ and a_* is the vector field $a_* = (1/a^2)a$. With the hypothesis of Proposition 3 being verified by the interior and exterior products by u (resp., a_*) of the first (resp., second) of Eqs. (11), we obtain, for this u,

$$(\theta/\beta)^0(a + \beta u) - \beta d(\theta/\beta) = 0, \tag{15}$$

$$d(\theta/\beta) \wedge u \wedge a = 0, \tag{16}$$

$$i(a_*)da + \beta^* u + \beta i(a_*)du = 0, \tag{17}$$

$$d\beta \wedge u \wedge a = 0, \tag{18}$$

where for any function f, $f^* = \pounds(a_*)f$. On account of (18), (16) becomes

$$d\theta \wedge u \wedge a = 0 \tag{19}$$

and under our hypothesis, (17) is equivalent to

$$\beta^* = \beta + \alpha. \tag{20}$$

By (19), (15) may be written as $(\theta/\beta)^0 = \beta(\theta/\beta)^*$, which in turn becomes $\beta^0 = \beta^2(1 - \theta^*/\theta) + \beta(\alpha + \theta^0/\theta)$ via (20). Thus we have the following proposition.

Proposition 7: A unit vector field u such that $w = 0$, $\theta \cdot a \neq 0$, and $a \wedge da = 0$ is the velocity of a barotropic fluid if and only if there exists a function β such that

$$d\theta \wedge u \wedge a = 0, \quad d\beta \wedge u \wedge a = 0, \tag{21}$$

$$\beta^* = \beta + \alpha, \quad \beta^0 = \beta^2(1 - \Theta^*) + \beta(\alpha + \Theta^0), \tag{22}$$

where

$$\alpha \equiv (1/a^2)i(a)i(u)da, \quad \Theta \equiv \ln \theta.$$

For a function f verifying $df \wedge u \wedge a = 0$, one has $df = f^0 u + f^* a$ and thus

$$f^0 u \wedge a = df \wedge a, \quad f^* u \wedge a = -df \wedge u,$$
$$df^0 \wedge u \wedge a = 0, \quad df^* \wedge u \wedge a = 0, \tag{23}$$

so that if $du = u \wedge a$ and $da = \alpha u \wedge a$, one has

$$f^{0*} - f^{*0} = f^0 + \alpha f^*. \tag{24}$$

Moreover, because of (21) and (23), the result is that all the scalars in (22) verify relation (24). From relation (24) it follows that a necessary integrability condition for Eqs. (22) is

$$\beta^{0*} - \beta^{*0} - \beta^0 - \alpha\beta^* = 0, \tag{25}$$

which, according to (22), gives

$$\mu\beta^2 + \chi\beta + \gamma = 0, \tag{26}$$

where

$$\mu \equiv -\Theta^{**}, \quad \chi \equiv \Theta^{*0} + \alpha^* - \alpha\Theta^*, \quad \gamma \equiv \alpha\Theta^0 - \alpha^0.$$

Let u be such that it verifies the hypothesis of Proposition 7 with $\mu^2 + \chi^2 = 0$. Equation (26) then says that γ vanishes also and (25) becomes an identity. In this case, there always exists at least one solution to Eqs. (22); a simple way to see the solution is to consider an evolution problem with the constraint equation $L \equiv \beta^* - \beta - \alpha = 0$. Taking into account the second of Eqs. (22) and (25), one finds

$$L^0 = [2\beta(1 - \Theta^*) + \Theta^0]L + \mu\beta^2 + \chi\beta + \gamma,$$

so that since $\mu = \chi = \gamma = 0$, L^0 vanishes with L. Consequently, Eqs. (22) are in involution: If β is a solution of the second of Eqs. (22) in a neighborhood of a given instant and verifies the first of Eqs. (22) at that instant, then it is a solution to Eq. (22) in the neighborhood. Since the corresponding initial constraint admits a one-parametric family of solutions, we may state the following result.

Proposition 8: A unit vector field u such that $-w^2 + æ^2 + \mu^2 + \chi^2 = 0$ and $\theta \cdot a \neq 0$ is the velocity of a barotropic fluid if and only if it verifies $d\theta \wedge u \wedge a = 0$ and $\gamma = 0$. Equations (22) admit a one-parametric family of solutions $\beta_\lambda = \beta_\lambda[u]$: For each of them, the one-form $b_\lambda = a + \beta_\lambda$ is closed and the holonomy potential π_λ is determined, up to a constant, by $d\pi_\lambda = b_\lambda$.

Suppose now that u verifies the hypothesis of Proposition 7 with $\mu^2 + \chi^2 \neq 0$. If $\mu \neq 0$ the result is that from (26) a necessary condition for (22) to admit a solution is

$$\Delta \equiv \chi^2 - 4\mu\gamma \geqslant 0. \tag{27}$$

One then has $\beta = \beta_5$, where

$$\beta_5 = (1/2\mu)(-\chi \pm \Delta^{1/2}). \tag{28}$$

On the other hand, if $\mu = 0$ (and, therefore, $\chi \neq 0$,), the result is that $\beta = \beta_4$, where

$$\beta_4 = -\gamma/\chi. \tag{29}$$

Consequently, we have the following proposition.

Proposition 9: A unit vector field u such that $-w^2 + æ^2 = 0$, $\theta \cdot a \neq 0$, and $\mu^2 + \chi^2 \neq 0$ is the velocity of the barotropic fluid if and only if it verifies either $\mu \neq 0$, $\chi \geqslant 4\mu\gamma$, (18), and (22) for $\beta = \beta_5$ as given by (28) or $\mu = 0$, (18), and (22) for $\beta = \beta_4$ as given by (29). In each case, the corresponding one-form $b_i \equiv a + \beta_i\, u$, $(i = 4,5)$ is closed and the holonomy potential π_i is determined, up to a constant, by $d\pi_i = b_i$.

Let u be such that $w = 0$ and $\theta \cdot æ \neq 0$. In this case, taking into account that $du = u \wedge a$, the exterior product of Eqs. (11) by a implies that $a \wedge da + d\beta \wedge u \wedge a = 0$, $d(\theta/\beta) \wedge u \wedge a = 0$ and since $\theta \cdot \beta \neq 0$, it follows that

$$\beta\, d\theta \wedge u \wedge a = -\theta a \wedge da.$$

The one-form $z = -*(d\theta \wedge u \wedge a)$ does not vanish and is orthogonal to u. Consequently, $z^2 \neq 0$ and $\beta = \beta_6$, where

$$\beta_6 = (\theta/z^2) i(z) * (a \wedge da). \tag{30}$$

Therefore, we may state the following proposition.

Proposition 10: A unit vector field u such that $w = 0$ and $\theta \cdot æ \neq 0$ is the velocity of a barotropic fluid if and only if it verifies Eqs. (11) for $\beta = \beta_6$ as given by (30). Then the one-form $b_6 \equiv a + \beta_6 u$ is closed and the holonomy potential π_6 is determined, up to a constant, by $d\pi_6 = b_6$.

Consider now unit vector fields with $w \neq 0$ and $da = 0$. By differentiation and the exterior product by u of $d\pi = a + \pi^0 u$, one obtains $u \wedge da + \pi^0 u \wedge du = 0$, that is, $\pi^0 = 0$; thus on account of (7), $\theta = 0$. Conversely, since $da = 0$, let π be such that $d\pi = a$; then if $\theta = 0$, π is a solution to (9). Therefore, we have the following proposition.

Proposition 11: A unit vector field u such that $w \neq 0$ and $da = 0$ is the velocity of a barotropic fluid if and only if it verifies $\theta = 0$. Then the holonomy potential π is determined, up to a constant, by $d\pi = a$.

Finally, let us consider u such that $w \neq 0$ and $da \neq 0$. Since the hypothesis of Proposition 3 is verified, the result is that $u \wedge da + \beta u \wedge du = 0$ and since w is a nonvanishing spacelike vector field, one has $w^2 \neq 0$; consequently, $\beta = \beta_8$, where

$$\beta_8 \equiv -(1/w^2) i(w) * (u \wedge da). \tag{31}$$

Thus we have the following result.

Proposition 12: A unit vector field u such that $w \otimes da \neq 0$ is the velocity of a barotropic fluid if and only if it verifies Eqs. (11) for $\beta = \beta_8$ as given by (31). Then the one-form $b_8 = \alpha + \beta_8 u$ is closed and the holonomy potential π_8 is determined, up to a constant, by $d\pi_8 = b_8$.

In the above we have obtained conditional systems in u for the barotropic fluids. These systems depend on the nonvanishing of some differential quantities associated to u and do not admit a unique simple form valid for any unit field. On account of the above results, we are lead to introduce the following classification of unit vector fields.

Definition: A unit vector field u is said to be of class C_i $(i = 1,...,8)$ if it verifies the relations given in Table I, where we have written

$$w = *(u \wedge du), \quad a = i(u)\nabla u, \quad \theta = -\delta u,$$
$$\Theta = \ln \theta, \quad \alpha = (1/a^2) i(a) i(u) da,$$
$$\mu \equiv -\Theta^{**}, \quad \chi \equiv \Theta^{*0} + \alpha^* - \alpha\Theta^*,$$

and $f^0 = \pounds(u)f$, $f^* = (1/a^2)\pounds(a)f$ for any scalar f.

The results of this section may then be summarized in the following two theorems.

Theorem 1 (of characterization of barotropic velocities): A unit vector field u of class C_i $(i = 1,...,8)$ is the velocity of a barotropic perfect fluid if *and only if* it verifies the differential system B_i given in Table II, where the scalar β_j $(j = 4,5,6,8)$ is defined by

TABLE I. The eight classes of unit vector fields.

Class	Definition relations
C_1	$w=0, \theta=0$
C_2	$w=0, \theta\neq 0, a=0$
C_3	$w=0, \theta\neq 0, a\neq 0, a\wedge da=0, \mu^2+\chi^2=0$
C_4	$w=0, \theta\neq 0, a\neq 0, a\wedge da=0, \mu^2+\chi^2\neq 0, \mu=0$
C_5	$w=0, \theta\neq 0, a\neq 0, a\wedge da=0, \mu^2+\chi^2\neq 0, \mu\neq 0$
C_6	$w=0, \theta\neq 0, a\neq 0, a\wedge da\neq 0$
C_7	$w\neq 0, da=0$
C_8	$w\neq 0, da\neq 0$

TABLE III. Characterization of the holonomy potentials for a barotropic velocity.

Symbol	Characterization of π
P_1	$\pi=-\ln\tau+h(t), (u=\tau\, dt)$
P_2	$\pi=\pi(t), (u=\tau\, dt)$
P_3	$d\pi_\lambda=a+\beta_\lambda u$, where β_λ is the one-parametric family of solutions to the system $\beta^*=\beta+\alpha, \beta^0=\beta^2(1-\Theta^*)+\beta(\alpha+\Theta^0)$
P_4	$d\pi_4=a+\beta_4 u$
P_5	$d\pi_5=a+\beta_5 u$
P_6	$d\pi_6=a+\beta_6 u$
P_7	$d\pi=a$
P_8	$d\pi_8=a+\beta_8 u$

$$\beta_4\equiv(\alpha^0-\alpha\Theta^0)/\chi,\quad \beta_5\equiv(1/2\mu)(-\chi\pm\Delta^{1/2}),$$
$$\beta_6\equiv(\theta/z^2)i(z)*(a\wedge da),\quad \beta_8\equiv-(1/w^2)i(w)*(u\wedge da)$$

and we have written

$$\Delta\equiv\chi^2+4\mu(\alpha^0-\alpha\Theta^0),\quad z=-*(d\theta\wedge u\wedge a).$$

Theorem 2: The holonomy potential π associated to a barotropic velocity of class C_i $(i=1,...,8)$ is determined by the relations P_i given in Table III. Let $g(\pi)$ be the function such that $\theta=g(\pi)\pi^0$ and take

$$p(\pi)=\int\exp\left\{\int[1-g(\pi)]d\pi\right\}d\pi,\quad \rho(\pi)=p'(\pi)-p;$$

the triple (u,ρ,p) is then a barotropic perfect fluid.

IV. SOME SPECIAL BAROTROPIC MOTIONS: THE POLYTROPIC CASE

In many cases one may be interested in disclosing a more restricted character than that of barotropy. In this section, we study the following types of particular barotropic perfect fluids: (i) constant pressure $dp=0$; (ii) constant total energy density $d\rho=0$; (iii) u-invariant pressure (and density) $p^0=\rho^0=0$; and (iv) polytropic fluid, $p=(\lambda-1)\rho, \lambda\neq 1$.

We shall see that the characterization of these cases is easier than the general barotropic case.

Proposition 1 already characterized fluids of type (i); such fluids also belong to one of the types (ii)–(iv) if and only if $\theta=0$, so that (i) may be stated in form of the following proposition.

Proposition 13: The necessary and sufficient condition for a unit vector u to be the velocity of a perfect fluid with constant pressure and verifying one of the conditions (ii)–(iv) is that u be geodesic and expansion-free.

Now, let $dp\neq 0$. From Proposition 2, the barotropic relation $\rho=\rho(p)$ depends on the function $g(\pi)$ given by (7); indeed,

$$\rho'(p)=\rho'(\pi)/p'(\pi)=-g\{\pi(p)\}.$$

Thus one has $g(\pi)=$ const if and only if ρ is a linear function in p. It is then easy to see that cases (ii) and (iv) are characterized as in the following proposition.

Proposition 14: The necessary and sufficient condition for u to be the velocity of a barotropic fluid with $d\rho=0$ and $dp\neq 0$ is $\theta=0$ and $d\pi=a+\pi^0 u$ for some function π.

Proposition 15: The necessary and sufficient condition for u to be the velocity of a polytropic fluid with index λ is the existence of a function π such that $\{u,\pi\}$ is a solution to (9) with $g(\pi)=(1-\lambda)^{-1}$.

In case (iii), because of $\rho^0=p^0=0$, one has $\pi^0=0$, which by (9) leads to $\theta=0$ and $da=0$. Since the converse is also verified, one has the following proposition.

Proposition 16: The necessary and sufficient condition for u to be the velocity of a barotropic fluid with $\rho^0=p^0=0$ is $\theta=0$ and $da=0$. Then the holonomy index is determined, up to a constant, by $d\pi=a$.

When the conditions $\theta=0, da=0$ are verified for every function $p(\pi)$, the triple (u,ρ,p) with $\rho=p'(\pi)-p$ is a barotropic fluid verifying $\rho^0=p^0=0$. Consequently, every function $\rho=\rho(p)$ is admissible as a barotropic relation.

By additing suitable conditions to the systems B_i of Table II, one may associate barotropic relations of types (ii)–(iv) to unit vector fields of class C_i.

According to Proposition 16, the velocities of the classes C_1 and C_7 are of type (iii) if they verify $da=0$. Consequently, these velocities admit any function $\rho(p)$ as a barotropic relation and the velocities of class C_1 (with $da\neq 0$) and class C_7 (with $\theta=0$) are of constant energy density.

The velocities of classes C_8 (resp., C_6) with the additional conditions $\theta\neq 0$ and $\theta/\beta_8=$ const (resp., $\theta/\beta_6=$ const) admit polytropic barotropic relations.

The velocities of classes C_3, C_4, and C_5 admit a polytropic barotropic relation if $\beta=k\cdot\theta$ is a solution to the system (22), where k is a constant. One then has $\alpha/(\theta-\theta^0)=$ const.

TABLE II. Differential systems characterizing the barotropic velocities of class C_i.

Symbol	Necessary and sufficient conditions
B_1	ϕ
B_2	$d\theta\wedge u=0$
B_3	$d\theta\wedge u\wedge a=0, \alpha\Theta^0-\alpha^0=0$
B_4	$d\theta\wedge u\wedge a=0$ $\beta_4^*=\beta_4+\alpha, \beta_4^0=\beta_4^2(1-\Theta^*)+\beta_4(\alpha+\Theta^0)$
B_5	$d\theta\wedge u\wedge a=0, \Delta\geqslant 0$ $\beta_5^*=\beta_5+\alpha, \beta_5^0=\beta_5^2(1-\Theta^*)+\beta_5(\alpha+\Theta^0)$
B_6	$d(a+\beta_6 u)=0, d(\theta/\beta_6)\wedge(a+\beta_6 u)=0$
B_7	$\theta=0$
B_8	$d(a+\beta_8 u)=0, d(\theta/\beta_8)\wedge(a+\beta_8 u)=0$

Finally, the velocities of class C_2 admit any polytropic index because the holonomy potential is an arbitrary function of the potential t for u and one can always take it to be proportional, with an arbitrary constant, to a primitive of a given $\theta(t)$.

Propositions 1 and 16 characterize types (i) and (iii) in terms of u alone; meanwhile, Propositions 14 and 15 characterize types (ii) and (iv) in terms of u and π. Here we shall obtain the conditions in u ensuring the existence of π.

In case (ii) one has $\theta = 0$. When $w = 0$, Proposition 4 implies, for every integrant factor, the existence of a function π verifying $d\pi = a + \pi^0 u$. When $w \neq 0$ and $da = 0$, the potential π is such that $d\pi = a$ and if $w \neq 0$ and $da \neq 0$, according to the analysis given in Sec. III, u is a solution to $d(a + \beta_8 u) = 0$, where β_8 is given by (31). We thus have the following theorem.

Theorem 3: The necessary and sufficient conditions for u to be the velocity of a barotropic fluid with $dp \neq 0$ and $d\rho = 0$ are $\theta = 0$ and either $w = 0$ or $w \neq 0$ and $d(a + \beta_8 u) = 0$, where β_8 is given by $\beta_8 \equiv -(1/w^2) \times i(w) * (u \wedge da)$. In the first case, to every integrating factor τ corresponds a holonomy potential $\pi = -\ln \tau$; in the second case, the holonomy potential is determined, up to an additive constant, by $d\pi = a + \beta_8 u$. In both cases the triple (u, ρ_0, p) is a perfect fluid, where p is given by $p = k_0 \cdot \exp(\pi) - \rho_0$ and k_0 and ρ_0 are constants.

In case (iv), we know from Proposition 15 that u is the velocity of a polytropic fluid with index λ if and only if there exists a function π such that $d\pi = a + k\theta u$, $k = 1 - \lambda$; however, this is (locally) equivalent to

$$da + kd(\theta u) = 0, \tag{32}$$

so that $da = 0$ if and only if $d(\theta u) = 0$, where (32) then takes place for any constant k. If $da \neq 0$, for every two-form X such that $(X, da) \neq 0$, we have

$$k = -(X, da)/(X, d(\theta u)) \tag{33}$$

and by differentation

$$i(X) i'(X) \{ d(\theta u) \otimes {}^t\nabla\, da - da \otimes {}^t\nabla d(\theta u) \} + \{ i(d(\theta u)) i'(da) - i(da) i'(d(\theta u)) \} X \otimes {}^t\nabla X = 0.$$

Since this equation is verified for every X, the two expressions inside the curly braces vanish and conversely, if they vanish, there exists a constant k such that (32) is verified. We have thus shown the following therorem.

Theorem 4: A unit vector fluid u is the velocity of a polytropic fluid if and only if it verifies either $da = d(\theta u) = 0$ or $da \otimes d(\theta u) = d(\theta u) \otimes da \neq 0$ and $da \otimes \nabla d(\theta u) = d(\theta u) \otimes \nabla\, da$. In the first case, any polytropic index $\lambda \neq 1$ is admitted; in the second case, the polytropic index $\lambda = 1 - k$ is uniquely determined by (33), where X is any two-form nonorthogonal to da. In both cases, the one-form $b \equiv a + k\theta u$ is closed and the holonomy potential associated to every k is determined, up to an additive constant, by $d\pi = b$. The triple (u, ρ, p) is a polytropic fluid of index $\lambda = 1 - k$, where $p(\pi) = k_0 \exp\{\pi \cdot \lambda / (\lambda - 1)\}$ if $k \neq 1$ and $p(\pi) = k_2 \pi$ if $k = 1$.

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[1]Of course, here we are not considering the natural restrictions usually added to the fundamental system for physical interpretation, namely the Plebanski energy conditions and the Lichnerowicz compressibility conditions.

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[7]Of course, there are barotropic *test* fluids which do not verify the conjecture.

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[9]One has $\pi \equiv \ln F$, where F is the Synge *index function* or the Lichnerowicz *holonomy index*.